FMI, CS, Master I
Winter Term 2015/2016
Techniques of Combinatorial Optimization
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Seminar 3

(S3.1) Figure 1 represents a flow network $N = (D, c, s, t)$.

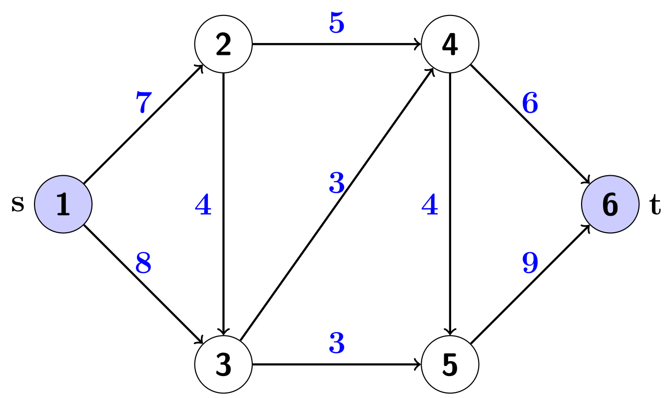

Figure 1: The flow network N

Write the corresponding digraph D and the capacity function c.

Proof. We have that $D = (V, A)$, where $V = \{1, 2, 3, 4, 5, 6\}$ and

$$A = \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4), (3, 5), (4, 5), (4, 6), (5, 6)\}.$$

Furthermore, $c : A \to \mathbb{Z}_+$, $c(1, 2) = 7, c(1, 3) = 8, c(2, 4) = 5, c(2, 3) = c(4, 5) = 4, c(3, 4) = c(3, 5) = 3, c(4, 6) = 6, c(5, 6) = 9$.

□

(S3.2) Find vectors b, d and a matrix B such that

$$\max\{\text{value}(f) \mid f \text{ is an } s - t \text{ flow for } N\} = \max\{d^T f \mid Bf \leq b\}.$$

Proof. Let M be the incidence matrix of D and for every $i = 1, \dots, 6$, let us denote by $\mathbf{m}_i$ the i-th line of M. Let M_0 be the matrix obtained from M by deleting the lines $\mathbf{m}_1$ and $\mathbf{m}_6$. Thus,

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccccc} (1,2) & (1,3) & (2,3) & (2,4) & (3,4) & (3,5) & (4,5) & (4,6) & (5,6) \end{array} \\ \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array},$$

$$M_0 = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccccccc} (1,2) & (1,3) & (2,3) & (2,4) & (3,4) & (3,5) & (4,5) & (4,6) & (5,6) \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \end{pmatrix} \end{array}.$$

Then (see Section 3.1.1 in the lecture notes),

$$\begin{aligned} \max\{\text{value}(f) \mid f \text{ is an } s-t \text{ flow}\} &= \max\{\mathbf{m}_6 f \mid M_0 f = \mathbf{0}, \mathbf{0} \le f \le c\} \\ &= \max\{d^T f \mid Bf \le b\}, \end{aligned}$$

where

$$d = \mathbf{m}_6^T, \quad B = \begin{pmatrix} M_0 \\ -M_0 \\ I \\ -I \end{pmatrix}, \quad b = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ c \\ \mathbf{0} \end{pmatrix}.$$

□

(S3.3) Figure 2 represents an s-t flow f for the network N.

(i) Verify that f is an s-t flow. What is the value of f?

(ii) Show that the set $\{(2,4), (3,4), (3,5)\}$ is an s-t cut and compute its capacity.

(iii) Prove that f is a maximum flow.

Proof. (i) $f : A \to \mathbb{Z}_+$ is defined by $f(1,2) = 7$, $f(1,3) = 4$, $f(2,3) = f(4,5) = 2$, $f(3,4) = f(3,5) = 3$, $f(2,4) = f(5,6) = 5$, $f(4,6) = 6$. It is obvious that $0 \le f \le c$. It remains to verify the flow conservation law at every vertex $v \ne 1, 6$:

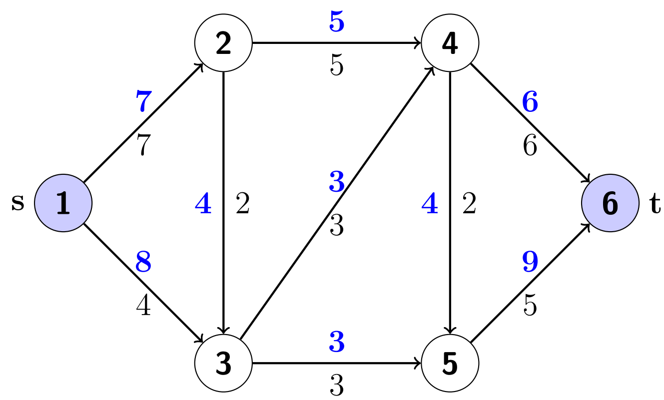

Figure 2: The flow network N with the flow f

(a) $v = 2$. $in_f(2) = 7, out_f(2) = 2 + 5 = 7$

(b) $v = 3$. $in_f(3) = 2 + 4 = 6, out_f(3) = 3 + 3 = 6$

(c) $v = 4$. $in_f(4) = 5 + 3 = 8, out_f(4) = 2 + 6 = 8$

(d) $v = 5$. $in_f(5) = 3 + 2 = 5, out_f(5) = 5$.

We have that value$(f) = out_f(s) - in_f(s) = out_f(s) = 7 + 4 = 11$.

(ii) Let $U = \{1, 2, 3\}$, hence $s \in U$, but $t \notin U$. We have that $\delta^{out}(U) = \{(2,4), (3,4), (3,5)\}$ and its capacity is $c(\delta^{out}(U)) = 5 + 3 + 3 = 11$.

(iii) Since $value(f) = c(\delta^{out}(U))$, we can apply Corollary 3.0.14 to conclude that f is a maximum flow.

□

(S3.4) Give two iterations of the Ford-Fulkerson algorithm for the flow network N, considering the path $P = 1246$ for the first augmentation and $Q = 1356$ for the second augmentation.

Proof. The initial flow is $f := 0$, hence the residual network coincides with N.

Let us consider the s-t path $P = 1246$ as an f-augmenting path. Then

$$\gamma = \min_{e \in A(P)} c_f(e) = \min\{7, 5, 6\} = 5.$$

Thus, the algorithm augments f along P with 5 units, i.e. we replace f with $f_1 := f_P^\gamma$.

After the first augmentation, we get the following residual graph D_{f_1} and residual capacities c_{f_1}:

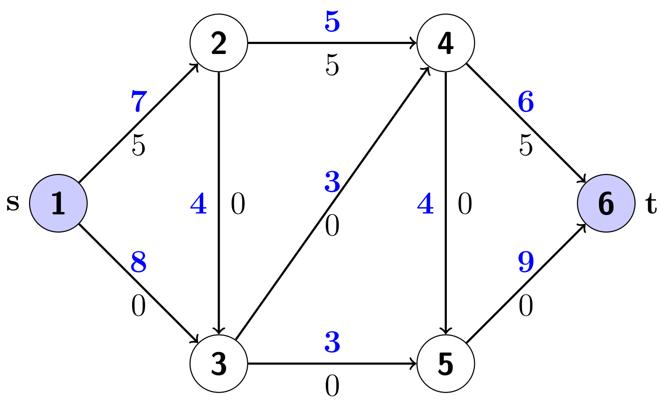

Figure 3: The flow network N with the flow f_1

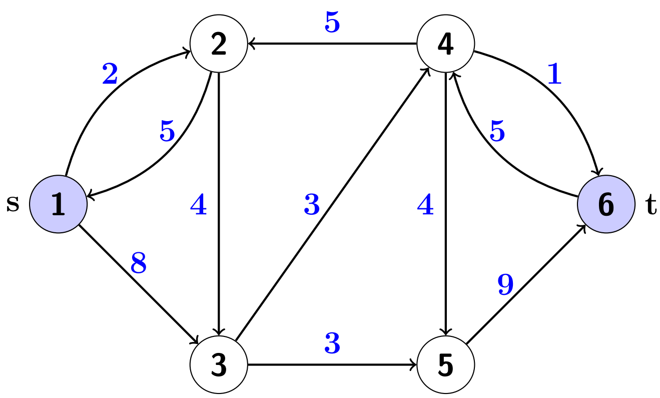

Figure 4: The residual graph D_{f_1}

Let us consider at the second iteration the s-t path $Q = 1356$ as an f_1-augmenting path. Then

$$\gamma = \min_{e \in A(P)} c_{f_1}(e) = \min\{8, 3, 9\} = 3.$$

Thus, the algorithm augments f along Q with 3 units, i.e. we replace f_1 with $f_2 := {f_1}^{\gamma}_{Q}$.

After the second augmentation, we get the following residual graph D_{f_2} and residual capacities c_{f_2}:

□

(S3.5) Figure 7 represents flow network N and an s-t flow f for N.

(i) Represent the residual graph D_f and the residual capacities c_f.

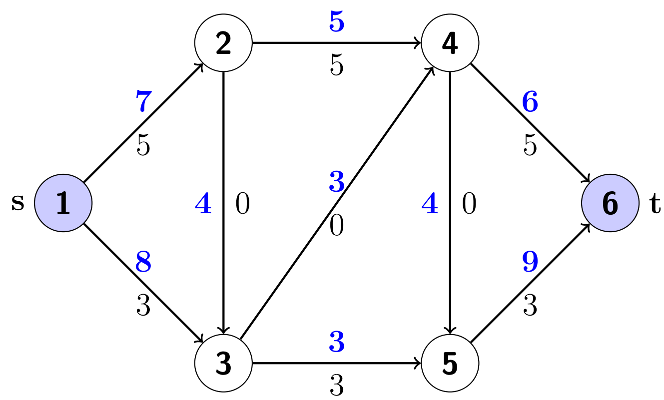

Figure 5: The flow network N with the flow f_2

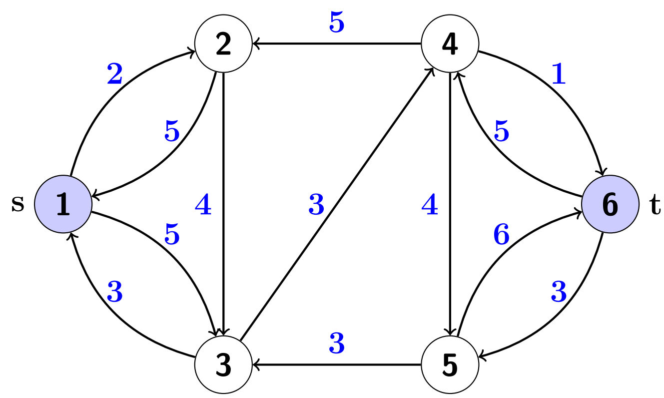

Figure 6: The residual graph D_{f_2}

(ii) Choose an f-augmenting path P of minimum length and compute the flow $g := f_P^\gamma$, where $\gamma = \min_{e \in A(P)} c_f(e)$.

(iii) Represent the residual graph D_g and the residual capacities c_g. Can you find an s-t path in D_g?

(iv) What is the maximum value of and s-t flow for N?

(v) Give an example of an s-t cut in N of minimum capacity.

Proof. (i) The residual graph D_f and residual capacities c_f are given in Figure 8.

(ii) The s-t path of minimum length is $P = 14256$, so we choose it as an f-augmenting

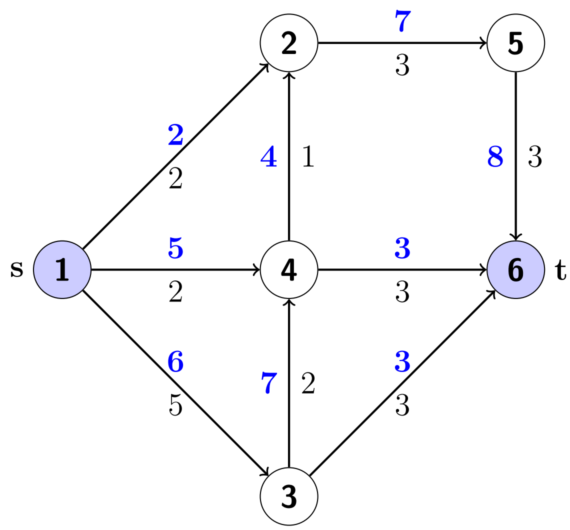

Figure 7: The flow network N with the flow f

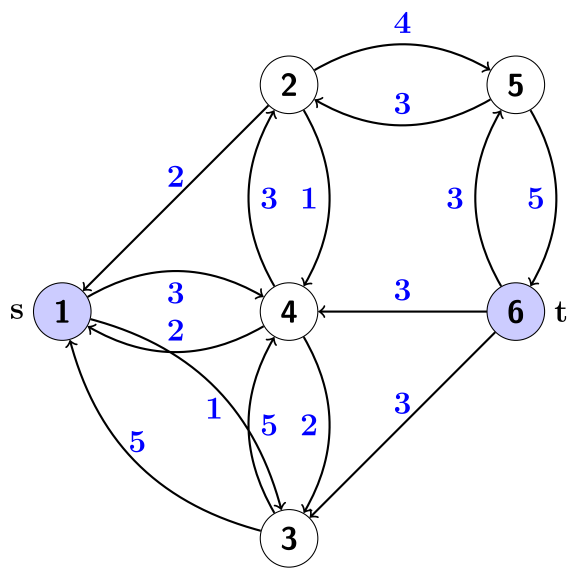

Figure 8: The residual graph D_f

path. Then

$$\gamma = \min_{e \in A(P)} c_f(e) = \min\{3, 3, 4, 5\} = 3.$$

The flow network N with the flow $g := f_P^\gamma$ is given in Figure 9.

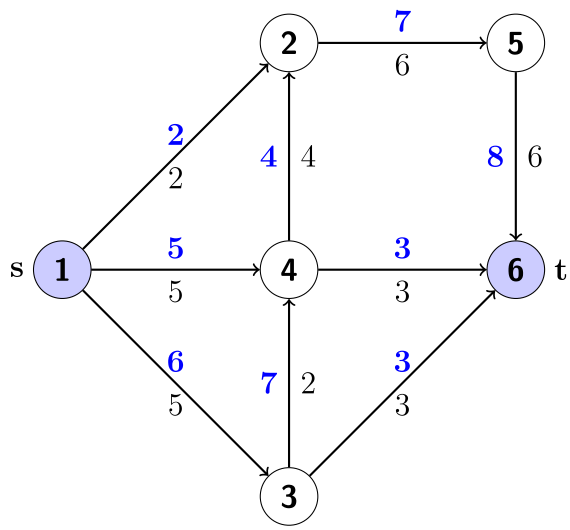

Figure 9: The flow network N with the flow g

(iii) The residual graph D_g and residual capacities c_g are given in Figure 10.

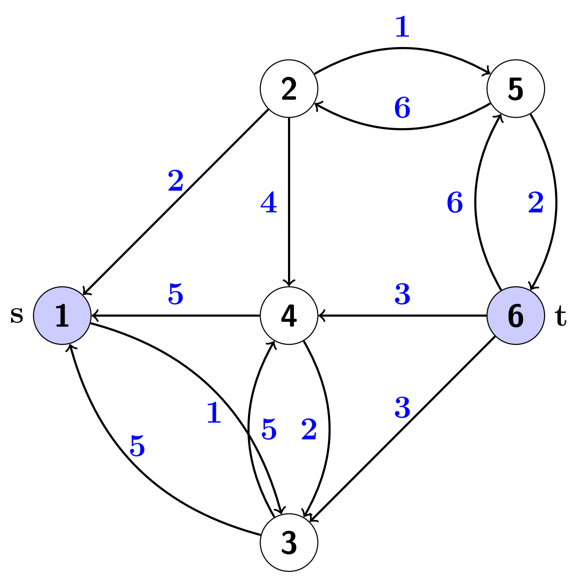

Figure 10: The residual graph D_g

It is obvious that there are no s-t paths in D_g.

(iv) By Theorem 3.2.4, it follows that g is a maximum flow. Hence, the maximal value of an s-t flow is value$(g) = 5 + 2 + 5 = 12$.

(v) By the Max-Flow Min-Cut Theorem 3.0.11, the minimum capacity of an s-t cut is 12. Apply Proposition 3.2.3. to get that an s-t cut having this capacity is

$$\{(1,2),(4,2),(4,6),(3,6)\} = \delta^{out}(U),$$

where $U = \{1,3,4\}$ is the set of vertices reachable in D_g from 1.

□