FMI, CS, Master I
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Techniques of Combinatorial Optimization
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Seminar 2

(S2.1) Let A be the incidence matrix of a cycle of length 5. Prove that A is a square matrix and compute its determinant.

Proof. Let $C_5 = v_0v_1v_2v_3v_4$ be the cycle of length 5, with vertices v_0, v_1, v_2, v_3, v_4 and edges $v_0v_1, v_1v_2, v_2v_3, v_3v_4, v_4v_0$. Then its incidence matrix is

$$A = \begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} v_0v_1 & v_1v_2 & v_2v_3 & v_3v_4 & v_4v_0 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

One can easily see that $det(A) = 2$. □

(S2.2) Verify if the following matrices are totally unimodular:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Proof. A is the incidence matrix of the bipartite graph $G = (X \cup Y, E)$, where

$$X = \{1, 2, 3\}, \quad Y = \{4, 5, 6\}, E = \{14, 15, 25, 26, 36, 34\}.$$

Thus, A is totally unimodular, by Theorem 2.0.5.

B is the incidence matrix of the graph $G = (V, E)$, where

$$V = \{1, 2, 3, 4, 5, 6, 7\}, \quad E = \{15, 16, 25, 27, 36, 37, 47, 75\}.$$

We remark that G contains the cycle $C = 2572$ of length 3. By Proposition 2.0.4, we get that G is not a bipartite graph. Aply Theorem 2.0.5 to conclude that A is not totally unimodular. □

(S2.3) Let $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Then

$$\max\{c^T x \mid x \geq \mathbf{0}, Ax \leq b\} = \min\{b^T y \mid y \geq \mathbf{0}, y^T A \geq c^T\}.$$

(assuming both sets are nonempty).
(This is Proposition 2.1.1.)

Proof. Remark that

$$\{x \in \mathbb{R}^n \mid x \geq \mathbf{0}, Ax \leq b\} = \{x \in \mathbb{R}^n \mid Cx \leq d\},$$

where $C = \begin{pmatrix} -I_n \\ A \end{pmatrix} \in \mathbb{R}^{(n+m)\times n}$ and $d = \begin{pmatrix} \mathbf{0} \\ b \end{pmatrix} \in \mathbb{R}^{n+m}$.

Hence, $\max\{c^T x \mid x \geq \mathbf{0}, Ax \leq b\}$ is the primal problem

$$(P) \qquad \max\{c^T x \mid Cx \leq d\}.$$

The dual problem associated to (P) is

$$(D) \qquad \min\{d^T z \mid z \geq \mathbf{0}, z^T C = c^T\}.$$

It suffices to prove that

$$\min\{d^T z \mid z \geq \mathbf{0}, z^T C = c^T\} = \min\{b^T y \mid y \geq \mathbf{0}, y^T A \geq c^T\}.$$

Let $P_1 := \{z \in \mathbb{R}^{n+m} \mid z \geq \mathbf{0}, z^T C = c^T\}$ and $P_2 := \{y \in \mathbb{R}^m \mid y \geq \mathbf{0}, y^T A \geq c^T\}$. We shall prove that $\{d^T z \mid z \in P_1\} = \{b^T y \mid y \in P_2\}$. We do this by showing that

(i) for all $z \in P_1$ there exists $y \in P_2$ such that $d^T z = b^T y$.

(ii) for all $y \in P_2$ there exists $z \in P_1$ such that $d^T z = b^T y$.

Let $z \in P_1$ Then $z = \begin{pmatrix} u \\ y \end{pmatrix}$, where $u \in \mathbb{R}^n, y \in \mathbb{R}^m$, $u \geq \mathbf{0}, y \geq \mathbf{0}$ and

$$c^T = z^T C = z^T \begin{pmatrix} -I_n \\ A \end{pmatrix} = u^T(-I_n) + y^T A = y^T A - u^T.$$

Thus, $y \geq \mathbf{0}$ and $y^T A = c^T + u^T \geq c^T$, hence $y \in P_2$. Moreover, $d^T z = (\mathbf{0}^T, b^T) \begin{pmatrix} u \\ y \end{pmatrix} = b^T y$.

Let now $y \in P_2$ and take $z := \begin{pmatrix} u \\ y \end{pmatrix}$, where $u^T := y^T A - c^T \geq \mathbf{0}$. Then $z \in P_1$ and $d^T z = b^T y$. □

(S2.4) Let G be a graph. Prove that

$$\max\{|M| \mid M \text{ is a matching of } G\} \leq \min\{|S| \mid S \text{ is a vertex cover of } G\}.$$

Show that the complete graph K_3 is an example of a graph where strict inequality holds.

Proof. Let $M = \{e_1, \ldots, e_n\}$ be an arbitrary matching and S be an arbitrary vertex cover of G. Then for every $i = 1, \ldots, n$, e_i intersects S in some $v_i \in V$. Since the edges in M are disjoint, it follows that for different i's we get different v_i's. Thus $|S| \geq n = |M|$. The conclusion follows.

We have that $K_3 = (\{1, 2, 3\}, \{12, 23, 31\}$. Then

$$\max\{|M| \mid M \text{ is a matching of } G\} = 1, \quad \text{while} \quad \min\{|S| \mid S \text{ is a vertex cover of } G\} = 2.$$

□