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Summary Ph. D. Thesis

STOCHASTIC ANALYSIS FOR PARTIAL DIFFERENTIAL EQUATIONS

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1 INTRODUCTION

In this thesis, we study problems of stochastic analysis for partial differential equations induced by the h-transform (in the sense of L. Doob) of a Markov process. Recall that if L is the weak generator of the given Markov process, then the generator of the h-transformed process is $g \mapsto \frac{1}{h}L(hg)$; see Corollary 3.5 (i) for a more precise statement.

In the study of the boundary behaviour of the harmonic functions in an arbitrary Euclidean domain, R.S. Martin [48] considered kernel functions given by a quotient of two superharmonic functions. This is the origin of the h-transform of Doob. An abundant literature was then

initiated from both analytic and probabilistic perspectives, we only mention the monographs of P. A. Meyer [49] and J. L. Doob [30].

The thesis is organised as follows.

In Chapter 3, entitled The h-transform of a Markov process, a transparent proof for the construction of the of the h-transformed process, X^h is given (in Proposition 3.1) and results are deduced on the excessive functions, the fine topology, the reduced function and the polar sets. The process X^h can be a tool for investigating the boundary behaviour of quotient of the form $\frac{u}{h}$, where $h > 0$ and u is a harmonic function on E.

In Section 3.4 it is proved that the h-transform and Bochner subordination are modifications of a Markov process which are compatible.

Section 3.5 is devoted to the h-transform in the frame of the L^p-spaces. It is shown that (Proposition 3.6) the h-transform of a Bochner subordinate C_0-semigroup of sub-Markovian operators on an L^p space is the transition function of a process obtained as the h-transform of the subordination of a right (Markov) process, on an enlarged state space.

The main result (Theorem 4.2) is proved in Chapter 4, entitled *The $\widehat{h}$-transform of a non-local branching process*. In particular, the restriction of a branching process to the set of all finite configurations of the complement of an absorbing set is obtained and show that it is still a branching process. As a consequence, the probabilistic representation it is deduced for the solution to the associated nonlinear evolution equation with bounded initial data (Corollary 4.3), completing an answer to a question of P. Hsu.

The Chapter 5, entitled *The h-transform of a nonlinear resolvent*, starts with some preliminaries on nonlinear sub-Markovian operators, and their properties as the (nonlinear) complete maximum principle in the sense developed by C. Dellacherie in [29].

In Section 5.1 the operator V^h is introduced, emphasising its main properties which will be used in studying the real Monge-Ampère operator and its h-transform.

In Section 5.2, entitled *The h-transform and exact functions*, a bounded function $k \in \mathcal{B}_+(E)$ is fixed and the semigroup $({}^kP_t)_{t\geq 0}$ given by the Feynman-Kac formula is considered,

$$ {}^kP_t f(x) = \mathbb{E}^x(e^{-\int_0^t k(X_s)ds} f(X_t)), \;\; x \in E, \; t \geq 0. $$

In Section 5.3 we consider the real Monge-Ampère operator studied [26], as an infimum of

a family of linear operators (5.2). Further, taking a bounded and strictly convex function h we define its h-transform, cf. [10].

Also in Chapter 5 the h-transform of Doob for the nonlinear operators is studied, where h is a strictly positive function. As an example we consider the Monge-Ampère operator on $\mathbb{R}^d$. More precisely, the h-transform of the resolvent associated to the Monge-Ampère operator is considered, and it is shown that it is the resolvent of an Monge-Ampère type operator (having the approximation with linear operators as in [26]). It is shown that the approximation with linear operators is preserved by the h-transform of the operator D^2, where D^2u is the Hessian matrix of the function u (Proposition 5.13).

2 PRELIMINARIES ON THE ANALYTICAL AND PROBABILISTIC POTENTIAL THEORY

The purpose of this chapter is to introduce the preliminary concepts that appear in this thesis. We present fundamental concepts and results regarding analytic and probabilistic potential theory, which we will further use in this thesis, such as kernels, resolvent of kernels, excessive functions, the fine topology, polar and semipolar sets.

3 h-TRANSFORM OF A MARKOV PROCESS

. Let E be a Lusin topological space (*i.e.,* E is homeomorphic to a Borel subset of a compact metric space) with Borel σ-algebra $\mathcal{B}(E)$. Let further $\mathcal{U} = (U_\alpha)_{\alpha\geq 0}$ be a sub-Markovian resolvent of kernels on $(E, \mathcal{B}(E))$.

We denote by $\mathcal{E}(\mathcal{U})$ the set of all $\mathcal{B}(E)$-measurable $\mathcal{U}$- excessive functions, i.e., $u \in \mathcal{E}(\mathcal{U})$ if and only if u is a nonnegative, numerical, $\mathcal{B}(E)$-measurable function, $\alpha U_\alpha u \leq u$ for all $\alpha\geq 0$, and $\lim_{\alpha\to\infty} \alpha U_\alpha u(x) = u(x)$, $x\in E$. Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x, \zeta)$ be the right Markov process on E having $(P_t)_{t\geq 0}$ as a transition function, $P_t f(x) = E^x(f(X_t), t < \zeta)$, $t \geq 0$, $f \in \mathcal{B}_+(E)$. The resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $(E, \mathcal{B}(E))$ associated with $(P_t)_{t\geq 0}$ is defined as $U_\alpha f := \int_0^\infty e^{-\alpha t} P_t f \, \mathrm{d}t, \quad f \in \mathcal{B}_+(E)$.

Assume that $h \in \mathcal{B}_+(E)$. We set

$$E_h := \{x \in E : 0 < h(x) < \infty\}$$

. If N is a kernel on $(E, \mathcal{B}(E))$ we consider the kernel N^h on E_h defined as

$$N^h f := \frac{1}{h} N(h\bar{f})|_{E_h},\ f \in \mathcal{B}_+(E),$$

where $\bar{f}$ is the extension of f from E_h to E with zero on $E \setminus E_h$.

Proposition 3.1. *Let X be a right process with state space E and transition function $(P_t)_{t\geq 0}$ and let $h \in \mathcal{E}(\mathcal{U})$, i.e., h is an excessive function with respect to X. Then, the following assertions hold.*

i) The family of kernels $(P_t^h)_{t\geq 0}$ on E_h, defined as

$$P_t^h f := \frac{1}{h} P_t(hf) \text{ for all } f \in \mathcal{B}(E_h) \text{ and } t \geq 0,$$

is the transition function of a right (Markov) process $X^h = (X_t^h, \Omega, \mathbb{P}^{x/h})$ with state space E_h, $P_t^h f(x) = \mathbb{E}^{x/h}(f(X_t^h), t < \zeta)$, $f \in \mathcal{B}_+(E_h)$, $x \in E_h$.

(ii) Let $\mathcal{U}^h = (U_\alpha^h)_{\alpha>0}$ be the resolvent family of $(P_t^h)_{t\geqslant 0}$ on E_h. If $\beta > 0$ then $\mathcal{E}(\mathcal{U}_\beta^h) = \frac{1}{h}\mathcal{E}(\mathcal{U}_\beta)|_{E_h}$.

(iii) Let $M \subset E_h$, $M \in \mathcal{B}(E)$ and $u \in \mathcal{E}(\mathcal{U}_\beta)$ with $\beta > 0$. Then ${}^hR_\beta^M(\frac{u}{h}|_{E_h}) = \frac{1}{h}R_\beta^M u|_{E_h}$, where ${}^hR_\beta^M$ denotes the β-level reduced operator on E_h w.r.t. X^h.

(iv) The fine topology on E_h with respect to X^h is the trace on E_h of the fine topology on E with respect to X. A subset of E_h is polar, respectively semipolar, totally thin with respect to the process X^h if and only if it has the same property w.r.t. X.

The process X^h given by assertion (i) of Proposition 3.1 is called the *h-transform of X*.

Remark 3.2. (i) *The first assertion of Proposition* 3.1 *follows also from [54], page. 298. The right process X^h is obtained there applying a result (Theorem (62.19)) on the transformation of X by means of a supermartingale multiplicative functional. However, the origin of this result is much older, see [43] and ch.* 9 *in [32], under various restrictive assumptions. We presented here a more direct, potential theoretical approach and we completed the existence of the process result with additional related aspects.*

(ii) *In [50] it is given the interpretation of the measure* $\mathbb{P}^{x/h}$ *as a "Fölmer measure" associated with a supermartingale.*

3.1 The tightness of the capacities under the h-transform

We adopt the notations from [23], Section 4.

Assume that $0 < h < \infty$, that is $E_h = E$. Let λ be a finite (nonnegative) measure on $(E, \mathcal{B})$ with $u_0 := U_\beta(hf_0)$, where $\beta > 0$ is fixed and $f_0 \in b\mathcal{B}_+(E)$, $f_0 > 0$, is such that hf_0 is also bounded. Then the functional $M \longmapsto c_\lambda(M), M \subset E$, defined as

$$c_\lambda(M) := \inf\{\lambda(R^G_\beta u_0) : \ G \ \text{ open}, M \subset G\}$$

is a Choquet capacity on E, see e.g., [7], Section 3.5.
Define analogously the Choquet capacity c^h_λ w.r.t. X^h,

$$c^h_\lambda(M) := \inf\{\lambda({}^hR^G_\beta u^h_0) : G \ \text{ open}, M \subset G\}, \text{where } u^h_0 := \frac{u_0}{h}$$

. Recall that the capacity c_λ is named *"tight"*, provided that there exists an increasing sequence $(K_n)_n$ of compact subsets of E, such that $c_\lambda(E \setminus K_n) \searrow 0$. The sequence $(K_n)_n$ is called "λ-nest" of compacts.

Corollary 3.3. *The process* X *possesses a "*λ*-nest" of compacts (i.e., the capacity* c_λ *is "tight") if and only if the process* X^h *has the same property (i.e., the capacity* c^h_λ *is "tight").*

3.2 The restriction to the complement of an absorbing set

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Let A be a (finely closed) absorbing set. We may suppose that v is bounded.

We define the transition function $(P'_t)_{t\geqslant 0}$ on the set $A^c := E \setminus A$ as:

$$P'_t f := P_t \bar{f}|_{A^c} \ \text{ for all } f \in \mathcal{B}_+(A^c),$$

where $\bar{f}$ is the extension of f from A^c to E, with zero on A, i.e., $\bar{f} = f$ on A^c and $\bar{f} = 0$ on A.

Proposition 3.4. *The following assertions hold.*

(i) The family of kernels $(P'_t)_{t\geqslant 0}$ is the transition function of a right (Markov) process X' with state space A^c. More precisely, we have $X' = X^h$, where $h = 1_{A^c}$.

(ii) If $\mathcal{U}$ (resp. $\mathcal{U}'$) is the resolvent of X (resp. of X') and $\beta > 0$, then $\mathcal{E}(\mathcal{U}'_\beta) = \mathcal{E}(\mathcal{U}_\beta)|_{A^c}$. A function $f \in \mathcal{B}_+(A^c)$ is finely continuous on A^c with respect to X' if and only if the function $\bar{f}$ is finely continuous on E (with respect to X).

We call that the process X' given by Proposition 3.4 the *restriction of X to A^c*.

3.3 The weak generator

Consider now the weak generator $(L, \mathcal{D}(L))$ of X; cf. [6]. More precisely, let

$$\mathcal{B}_o := \{f \in b\mathcal{B}(E) : f \text{ is finely continuous on } E\}$$

and $\mathcal{D}(L)$ be the set of all $f \in \mathcal{B}_o$, such that $\left(\frac{P_t f(x)-f(x)}{t}\right)_{t,x}$ is bounded for $x \in E$ and t in a neighbourhood of zero, there exists $\lim_{t\searrow 0} \frac{P_t f - f}{t}$ pointwise and the above limit is an element of $\mathcal{B}_o$. Define the linear operator $L : \mathcal{D}(L) \longrightarrow b\mathcal{B}(E)$ as:

$$Lf(x) := \lim_{t\searrow 0} \frac{P_t f(x) - f(x)}{t},\ f \in \mathcal{D}(L),\ x \in E.$$

The operator $(L, \mathcal{D}(L))$ is called the *weak generator* of X and $(L^h, \mathcal{D}(L^h))$ the weak generator of X^h.

Corollary 3.5. *Let $f \in b\mathcal{B}(E)$, $f = 0$ on $E\backslash E_h$. Then the following assertions hold.*

(i) If $f \in \mathcal{D}(L)$ (resp. $\frac{f}{h} \in \mathcal{D}(L^h)$) is such that $\frac{f}{h}$ and $\frac{1}{h}Lf$ are bounded functions (resp. $hL^h(\frac{f}{h})$) is a bounded function, then $\frac{f}{h} \in \mathcal{D}(L^h)$ (resp. $f \in \mathcal{D}(L)$) and $L^h(\frac{f}{h}) = \frac{1}{h}Lf$, and X^h solves the following martingale problem*: the process $\left(\frac{f}{h}(X^h_t) - \int_0^t (\frac{1}{h}Lf)(X^h_s)ds\right)_{t\geq 0}$ is a martingale under $\mathbb{P}^{x/h}$ with respect to the filtration of X^h for every $x \in E_h$.*

(ii) Let A be a (finely closed) absorbing set, consider the restriction X' of X to A^c and let $(L', \mathcal{D}(L'))$ be the weak generator of the process X'. Then $f \in \mathcal{D}(L')$ if and only if $\bar{f} \in \mathcal{D}(L)$. We have $L'f = L\bar{f}|_{A^c}$.

3.4 h-transform and Bochner subordination

We consider $X = (X_t)_{t\geq 0}$ a (Borel) right Markov process with state space E.

Let $\xi = (\xi_t)_{t\geqslant 0}$ be a positive real-valued stationary stochastic process with independent nonnegative increments (called subordinator), induced by a convolution semigroup $\mu = (\mu_t)_{t\geqslant 0}$ on $\mathbb{R}_+$.

Recall that the *subordinate process by means of* ξ is a right (Markov) process $X^\xi = (X_t^\xi)_{t\geqslant 0}$ with state space E, define as $X_t^\xi := X_{\xi_t},\ t \geqslant 0$. Its transition function is the family $(P_t^\mu)_{t\geqslant 0}$, $P_t^\mu = \int_0^\infty P_s\, \mu_t(\mathrm{d}s),\ t > 0$ and we have $\mathcal{E}(X) \subset \mathcal{E}(X^\xi)$.

The family $(P_t^\mu)_{t\geqslant 0}$ is called the *subordinate in the sense of Bochner* of $(P_t)_{t\geqslant 0}$.

(3.1)

Let $h \in \mathcal{E}(X)$. Then the h-transform of the subordination of X by means of $\xi = (\xi_t)_{t\geqslant 0}$, denoted by $X^{\xi,\mu} = (X_t^{\xi,\mu})_{t\geqslant 0}$, is a right (Markov) process with state space E_h, which is equivalent with the subordinate process by means of ξ of the h-transform X^h of X.

The assertion follows observing that for every $t > 0$ we have the equality of kernels $(P_t^\mu)^h = (P_t^h)^\mu$. Further, we write $P_t^{\mu,h}$ instead of $(P_t^\mu)^h$ and $P_t^{h,\mu}$ instead of $(P_t^h)^\mu$.

3.5 h-transform for Bochner subordinate L^p-semigroups

If m is a σ-finite measure on E such that $m \circ P_t \leqslant m$ for all $t > 0$ (such a measure is called *subinvariant*), then for every $p \in [1, \infty)$:

(3.2)

The transition function of a right (Markov) process X induces a C_0-semigroup of sub-Markovian contractions on $L^p(E, m)$.

A converse of this result was provided in [11] and it was completed in [46] for L^p-semigroups subordinate in the sense of Bochner.

Our aim in this section is to obtain a converse of (3.2) in the frame of $\mathbf{h}$-transform of Bochner subordinate L^p-semigroups; it will be given in Proposition 3.6 below.

Let $p \in [1, \infty)$ and $(P_t)_{t\geqslant 0}$ be a strongly continuous semigroup of sub-Markovian contractions on $L^p(E, m)$, where m is a σ-finite measure on the Lusin measurable space $(E, \mathcal{B})$.

Let $\mathbf{h} \in L^p_+(E, m)$, such that $P_t\mathbf{h} \leqslant \mathbf{h}$ for all $t > 0$. Assume that $\mathbf{h} \neq 0$ and consider the

set $E_{\mathbf{h}} \in \mathcal{B}(E)$,

$$E_{\mathbf{h}} := [\mathbf{h} > 0].$$

Let further $m_{\mathbf{h}}$ be the measure on $(E_{\mathbf{h}}, \mathcal{B}|_{E_{\mathbf{h}}})$, having the density $\mathbf{h}^p$ with respect to m, $m_{\mathbf{h}} = \mathbf{h}^p \cdot m$. For each $t \geqslant 0$ let $P_t^{\mathbf{h}}$ be the operator on $L^p(E_{\mathbf{h}}, m_{\mathbf{h}})$ defined as:

$$P_t^{\mathbf{h}} u := \frac{1}{\mathbf{h}} P_t(\mathbf{h}u), \quad u \in L^p(E_{\mathbf{h}}, m_{\mathbf{h}}). \tag{3.3}$$

Clearly, $P_t^{\mathbf{h}}$ is a sub-Markovian contraction semigroup on $L^p(E_{\mathbf{h}}, m_{\mathbf{h}})$ and it is a straightforward calculation to see that the family $(P_t^{\mathbf{h}})_{t\geqslant 0}$ becomes a strongly continuous semigroup of sub-Markovian contractions on $L^p(E_{\mathbf{h}}, m_{\mathbf{h}})$, called the **h**-*transform of the* L^p-*semigroup* $(P_t)_{t\geqslant 0}$.

We can state now a version of (3.1) for the Bochner subordination and **h**-transform on L^p-spaces: *The Bochner subordination and the* **h**-*transform commute. More precisely, if* $\mu = (\mu_t)_{t\geqslant 0}$ *is a convolution semigroup on* $\mathbb{R}_+$, *then* $P_t^{\mathbf{h},\mu} = P_t^{\mu,\mathbf{h}}$ *for all* $t > 0$, *hence the* C_0-*semigroups* $(P_t^{\mathbf{h},\mu})_{t\geqslant 0}$ *and* $(P_t^{\mu,\mathbf{h}})_{t\geqslant 0}$ *on* $L^p(E_{\mathbf{h}}, m_{\mathbf{h}})$ *coincide.*

(3.4)

Let $(\mathcal{L}^{\mu},\ \mathcal{D}(\mathcal{L}^{\mu}))$, respectively $(\mathcal{L}^{\mu,\mathbf{h}}, \mathcal{D}(\mathcal{L}^{\mu,\mathbf{h}}))$, be the infinitesimal generator of the semigroup $(P_t^{\mu})_{t\geqslant 0}$ on $L^p(E, m)$, respectively of the semigroup $(P_t^{\mu,\mathbf{h}})_{t\geqslant 0}$ on $L^p(E_{\mathbf{h}}, m_{\mathbf{h}})$. Then:

$$\mathbf{h} \cdot \mathcal{D}(\mathcal{L}^{\mu,\mathbf{h}}) \subseteq \mathcal{D}(\mathcal{L}^{\mu})|_{E_{\mathbf{h}}} \text{ and } \mathcal{L}^{\mu,\mathbf{h}} u = \frac{1}{h} \mathcal{L}^{\mu}(\mathbf{h}u)|_{E_{\mathbf{h}}} \text{ for all } u \in \mathcal{D}(\mathcal{L}^{\mu,\mathbf{h}}).$$

The main result of this section is the following proposition.

Proposition 3.6. *Let* $p \in [1, \infty)$ *and* $(P_t)_{t\geq 0}$ *be a* C_0-*semigroup of sub-Markovian contractions on* $L^p(E, m)$, *where* $(E, \mathcal{B})$ *is a Lusin measurable space and* m *is a* σ-*fine measure on* $(E, \mathcal{B})$, *such that* $\int P_t f dm \leq \int f dm$ *for every* $f \in L^p(E, m)$, $f \geq 0$ *and* $t > 0$. *Assume that there exists* $f \in L^p(E, m)$, $f > 0$, *such that* $V_\alpha f \leq 1$ *for all* $\alpha > 0$, *where* $(V_\alpha)_{\alpha>0}$ *is the* C_0-*continuous resolvent of contractions on* $L^p(E, m)$ *induced by* $(P_t)_{t\geq 0}$, $V_\alpha = \int_0^\infty e^{-\alpha t} P_t\, dt$, $\alpha > 0$.

Let further $\mu = (\mu_t)_{t\geq 0}$ *be a convolution semigroup on* $\mathbb{R}_+$ *and* ξ *the positive real-valued*

stationary stochastic process with independent nonnegative increments induced by μ.

Then there exist a Lusin topological space E_1, $E \subset E_1$, $E \in \mathcal{B}_1$ (the σ-algebra of all Borel subsets of E_1), $\mathcal{B} = \mathcal{B}|_{E_1}$, and a right (Markov) process X with state space E_1, such that:
(i) The transition function of X and $(P_t)_{t\geqslant 0}$ coincide, regarded as families of operators on $L^p(E_1, m_1)$, where m_1 is the measure on $(E_1, \mathcal{B}_1)$ extending m with zero on $E_1 \setminus E$.
(ii) The C_0-semigroup $(P^{\mu,\mathbf{h}}_t)_{t\geq 0}$ ($\mathbf{h}$-transform of subordinate in the sense of Bochner of $(P_t)_{t\geqslant 0}$ w.r.t. μ) coincides as a family of operators on $L^p(E_h, m_h)$ with the transition function of $X^{\xi,h}$ (the h-transform of the subordination of X by means of ξ), where h is a function on E_1 which is excessive w.r.t. X, equals in $L^p(E_1, m_1)-$ classes with $\mathbf{h}$ and $E_h := [h > 0]$. So, we have

$$P^{\mu,\mathbf{h}}_t u = \mathbb{E}^{\cdot} u(X^{\xi,h}_t),\ t > 0, u \in L^p(E_h, m_h).$$

(iii) Let ν be a probability measure on E_h, having density with respect to m_h, $\frac{d\nu}{dm_h} \in L^{p'}_+(E_h, m_h)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, and consider the probability $\mathbb{P}^{\xi,\nu/h}$ of $X^{\xi,h}$ with initial distribution ν. Then for every $u \in \mathcal{D}(\mathcal{L}^{\mu,\mathbf{h}})$ the process

$$u(X^{\xi,h}_t) - \int_0^t \left(\frac{1}{h}\mathcal{L}^{\mu}(\mathbf{h}u)\right)(X^{\xi,h}_s)\, ds,\ \ t \geq 0,$$

is a martingale under $\mathbb{P}^{\xi,\nu/h}$ with respect to the filtration of $X^{\xi,h}$.

4 $\widehat{h}$-TRANSFORM OF A NONLOCAL BRANCHING PROCESS

.

The space of finite configurations. The state space for a discrete branching process is the set $\widehat{E}$ of finite sums of Dirac measures on E, defined as follows:

$$\widehat{E} := \left\{ \sum_{n\leqslant n_0} \delta_{x_n} : n_0 \in \mathbb{N}^*, x_n \in E \text{ for all } \ 1 \leqslant n \leqslant n_0 \right\} \cup \{\mathbf{0}\},$$

where $\mathbf{0}$ denotes the zero measure. The set $\widehat{E}$ is identified with the union of all symmetric m-th powers $E^{(m)}$ of E (that is, if $m \geqslant 1$ then $E^{(m)}$ is the factorization of the Cartesian product

E^m by the equivalence relation induced by the permutation group σ^m; see, [21] and [18] for details), hence $\widehat{E} = \bigcup_{m\geqslant 0} E^{(m)}$, where $E^{(0)} := \{\mathbf{0}\}$ (see, [40]).

The set $\widehat{E}$ is called the *space of all finite configurations of* E and it is endowed with the disjoint union topology of topological spaces and the corresponding Borel σ-algebra $\mathcal{B}(\widehat{E})$.

Remark 4.1. *Let N be a sub-Markovian kernel on $(\widehat{E}, \mathcal{B}(\widehat{E}))$. Then N is a branching kernel on $\widehat{E}$ if and only if there exists a sub-Markovian kernel $\mathbf{N}$ from $\widehat{E}$ to E, such that $N\widehat{\varphi} = \widehat{\mathbf{N}\varphi}$, for every $\varphi \in b\mathcal{B}_+(E)$.*

A (right) Markov process with state space $\widehat{E}$ is called *a branching process* provided that its transition function is formed by branching kernels. For the probabilistic interpretation of the analytic branching property, see [35], page 337.

4.1 Multiplicative functions and branching kernels on $\widehat{E}$

For every real-valued φ, $\mathcal{B}(E)$-measurable we define the *multiplicative function* $\widehat{\varphi} : \widehat{E} \to \mathbb{R}_+$, as:

$$\widehat{\varphi}(\mathbf{x}) := \begin{cases} \prod_n \varphi(x_n), \text{ if } \mathbf{x} = (x_n)_{n\geq 1} \in \widehat{E},\ \mathbf{x} \neq \mathbf{0}, \\ \\ 1, \quad \text{if} \quad \mathbf{x} = \mathbf{0}, \end{cases}$$

A multiplicative function is called *coherent state* in the analysis on configuration spaces, (see, [37]).

We consider a sequence of Markovian kernels $(K_n)_{n\geq 1}$, where K_n is a kernel from $E^{(n)}$ to E, for each $n \geq 1$ and $(q_n)_{n\geq 1} \subseteq b\mathcal{B}(E)$, with $\sum_{n\geq 1} q_n \leq 1$. Assume that $\sum_{n\geq 1} nq_n$ is a bounded function on E.

The branching mechanism will be the sub-Markovian kernel K from $\widehat{E}$ to E, defined as

$$Kg = \sum_{n\geq 1} q_n K_n(g|_{E^{(n)}}) \text{ for all } g \in b\mathcal{B}_+(\widehat{E}).$$

We identify K with the pair of sequence $(K_n)_{n\geq 1}$ and $(q_n)_{n\geq 1}$.

By (3.2) (see and (3.1) from [18]) there exists a branching kernel $\widehat{K}$ on $\widehat{E}$, such that for all

$\varphi \in \mathcal{B}_+(E)$, $\varphi \leqslant 1$, we have

$$\widehat{K}\widehat{\varphi}|_E = \sum_{n\geq 1} q_n K_n \varphi^{(n)},$$

Let $h \in \mathcal{B}_+(E)$, $h \leq 1$, such that $K\widehat{h} \leqslant h$, and consider the kernel K^h from $\widehat{E_h}$ to E_h, given by

$$K^h g := \frac{1}{h} K(\widehat{h}g) \text{ for all } g \in \mathcal{B}_+(\widehat{E_h}). \tag{4.1}$$

The notation K^h is compatible with the previous one N^h from Section 3, when the kernel N is from E to E. Indeed, because N induces a kernel $\overline{N}$ from $\widehat{E}$ to E, defined as $\overline{N}g := N(g|_E)$, for $g \in \mathcal{B}_+(\widehat{E})$, we have $\overline{N}^h = \overline{N^h}$. Assume that

$$K\widehat{h} \leq h \text{ and } \frac{1}{h}\sum_{n\geq 1} n q_n \text{ is a bounded function on } E_h. \tag{4.2}$$

The first condition implies that K^h is a sub-Markovian kernel from $\widehat{E}_h$ to E_h, i.e., $K^h 1 \leq 1$.

Fie $\widehat{X} = (\widehat{X}_t)_{t\geqslant 0}$ be the branching process with state space $\widehat{E}$ and transition function $(\widehat{H}_t)_{t\geqslant 0}$, induced by a Markov process X with state space E, by a branching mechanism K described before, and by a killing constant $c > 0$. For the existence of the process $\widehat{X}$, see [18]. The process $\widehat{X}$ has nonlocal branching in the sense that is not required that the direct descendants start at the terminal position of the parent particle. More precisely, in the description of this branching process we have: *$q_n(x)$ is the probability that a particle destroyed at the point $x \in E$ has precisely n descendants , while $K_{n,x}$ (= the probability induced by the Markovian kernel K_n at x) is the distribution of the n descendants.*

We can state now our main result.

Teorema 4.2. *The following assertions hold.*

(i) Assume that $\varphi \in \mathcal{E}(\mathcal{U}_c)$, $\varphi \leq 1$, is such that $K\widehat{\varphi} \leq \varphi$. Then the function $\widehat{\varphi}$ belongs to $\mathcal{E}(\widehat{\mathcal{U}})$, i.e., it is excessive with respect to the branching process $\widehat{X}$ on $\widehat{E}$.

(ii) Let $h \in \mathcal{E}(\mathcal{U})$, $h \leq 1$. Then the $\widehat{h}$-transform of the branching process $\widehat{X}$ is a branching process on $\widehat{E_h}$. More precisely, consider the branching semigroup on $\widehat{E_h}$ induced by the h-transform process X^h of X and the sub-Markovian kernel K^h from $\widehat{E_h}$ to E_h, given in (4.1). Then this semigroup is the transition function of the $\widehat{h}$-transform of $\widehat{X}$.

(iii) Let A be an absorbing subset of E (w.r.t. X) and suppose that for every n and $x \in A$ we have $K_{n,x}(\widehat{A^c} \cap E^{(n)}) = 0$, that is, the probability measure on $E^{(n)}$ induced by the Markovian

kernel K_n at x is carried by $(\widehat{A^c})^c \cap E^{(n)}$. Then $\widehat{A^c}$ is the complement of an absorbing subset of $\widehat{E}$ (w.r.t. $\widehat{X}$) and the restriction of the branching process $\widehat{X}$ to $\widehat{A^c}$ is a branching process, it is induced by the restrictions of X and K to A^c.

4.2 The nonlinear evolution equation associated with the $\widehat{h}$-transform

We emphasize the probabilistic representation of the solution to the nonlinear evolution equation associated with the $\widehat{h}$-transform, using essentially the branching processes.

For $r > 0$ we consider the following condition:

$$\sum_{n\geqslant 1} r^{n-1} q_n \leqslant h \text{ and } \frac{1}{h}\sum_{n\geqslant 1} r^{n-1} n q_n \text{ is a bounded function on } E_h. \tag{4.3}$$

Corollary 4.3. *Let $\varphi \in b\mathcal{B}_+(E_h)$ be such that condition (4.3) is verified for some $r \geqslant \varphi$. Then the integral equation*

$$k_t = e^{-ct} P_t^h \varphi + c\int_0^t e^{-cs} P_s^h K^h \widehat{k_{t-s}}\, ds \text{ for all } t \geqslant 0. \tag{4.4}$$

has a solution k_t, $t \geqslant 0$, with the probabilistic representation

$$k_t(x) = \widehat{\mathbb{E}}^{\delta_x/\widehat{h}} \widehat{\varphi}(\widehat{X}_t^{\widehat{h}}),\ t \geqslant 0,\ x \in E_h.$$

Remark 4.4. (i) *For the meaning of the right hand side of the above probabilistic representation of k_t, see assertion (i) of Remark 4.4 in [20].*
(ii) *Recall that the nonlinear operation associated with the branching process $\widehat{X} = (\widehat{X}_t)_{t\geqslant 0}$ is defined as*

$$\Lambda u := (L - c)u + cK\widehat{u}. \tag{4.5}$$

It is in general an integral-differential operator. It was proved in [21] and [47] that in the case when L is the Laplace operator on a bounded smooth Euclidean domain, then the Dirichlet problem associated with the nonlinear operator Λ has a solution, using the associated branching process, if the boundary date is sub-unitary. Notice that in în [22] it was proved an extension of the solvability from sub-unitary to bounded boundary data, analogue to the procedure from Corollary 4.3, and this was the answer to a question of P. Hsu.

(iii) *Let Λ^h be the operator associated with the branching process $\widehat{X}^{\widehat{h}}$, obtained by applying Theorem 4.2,*

$$\Lambda^h u := (L^h - c)u + cK^h\widehat{u},\ u \in \mathcal{D}(L^h),$$

where L^h is the weak generator of X^h.

5 h-TRANSFORM OF A NONLINEAR RESOLVENT

Let E be a Lusin measurable space, i.e., it is measurable isomorphic to a Borel subset of a metrizable compact space endowed with its Borel σ-algebra.

We consider $b\mathcal{B}$ the set of all bounded $\mathcal{B}$-measurable functions on $(E, \mathcal{B}_+)$, $\mathcal{B}_+$ the set of all positive $\mathcal{B}$-measurable function on E and let $C := \frac{1}{h}b\mathcal{B}$, where h is strictly positive $\mathcal{B}$-measurable functions on E. The set C is a vector lattice, its generates the σ-algebra $\mathcal{B}$, and it is solid in order, that is, for all $v \in C$ and $|u| \leq v$, we have that $u \in C$.

If $f \in C$, then we define $\|f\|_0 := \|hf\|_\infty$, with $\|f\|_\infty := \sup\{|f(x)| : x \in E\}$. Then, $(C, \|\|_0)$ is a Banach space, and the map $b\mathcal{B} \ni v \longmapsto \frac{1}{h}v$ is an isometry between the spaces $(b\mathcal{B}, \|\|_\infty)$ and $(C, \|\|_0)$.

Let $j : (b\mathcal{B}, \|\|_\infty) \longrightarrow (C, \|\|_0)$ be the bijection mapping defined as $j(u) := \frac{1}{h}u$, hence $\|j(u)\|_0 = \|u\|_\infty$, for all $u \in b\mathcal{B}$. To any operator $V : C \longrightarrow C$ we associate the operator $\overline{V} : b\mathcal{B} \longrightarrow b\mathcal{B}$, defined as $\overline{V}(u) = V(j(u))$.

Lemma 5.1. *If V satisfies the complete maximum principle on C, then $\overline{V}$ satisfies the complete maximum principle on $b\mathcal{B}$.*

Let $V : C \longrightarrow C$, $\alpha \in (0, 1)$ and $c \in \mathbb{R}_+^*$. V is called hölderian (of constants (α, c)) if for every $f, g \in C$, we have: $\|Vf - Vg\|_0 \leq c\,\|f - g\|_0^\alpha$.

5.1 The V^h operator

We introduce the operator V^h associated to the h-transform and we emphasize its properties that we shall use in the next sections in studying the real Monge-Ampère operator and its h-transform.

Proposition 5.2. *If V is a n-homogeneous operator, u is a V-dominant function and $\alpha > 0$, then αu is a V-dominant function.*

Assume further that V is n-homogeneous. We consider now the h-transform of the nonlinear operator V, analogously with the linear case treated in [10].

Define the operator $V^h : C \longrightarrow C$, as:

$$V^h u := \frac{1}{h} V(h^n u), \text{for any function } \ u \in C.$$

Teorema 5.3. (i) *Let $V : b\mathcal{B} \longrightarrow b\mathcal{B}$ be a hölderian compact operator on $b\mathcal{B}$. Then V^h is a hölderian compact operator on C.*
(ii) *If the operator V satisfies the complete maximum principle on $b\mathcal{B}$, then $V^h : C \longrightarrow C$ satisfies the complete maximum principle on C.*
(iii) *For the hölderian operator V^h on C satisfying the complete maximum principle on C, there exists and it is unique a sub-Markovian resolvent $\mathcal{V}^h = (V_p^h)_{p>0}$ on C associated with V^h.*

5.2 h-transform and exact functions

Linear case.

Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$. Assume that $\mathcal{U}$ is the resolvent associated to a right process X with state space E (a Lusin topological space).

Let $(L, \mathcal{D}(L))$ be the weak generator of X. We fix a bounded function $k \in \mathcal{B}_+(E)$ and let $(P_t)_{t\geq 0}$ be the transition function of the process X. We consider the semigroup $({}^kP_t)_{t\geq 0}$ given by the Feynman-Kac formula, as:

$${}^kP_t f(x) = \mathbb{E}^x \left(e^{-\int_0^t k(X_s)ds} f(X_t) \right), \ \ x \in E, \ t \geq 0.$$

We fix a function $h \in \mathcal{E}(\mathcal{U})$.

Proposition 5.4. *Let $({}^kP_t^h)_{t\geq 0}$ be the transition function of the process $({}^kX)^h$ on E_h,*

$${}^kP_t^h f := \frac{1}{h} {}^kP_t(h\bar{f}).$$

Let $k' := k|_{E_h}$. The processes $({}^kX)^h$ and ${}^{k'}(X^h)$ are equivalent, more precisely we have ${}^kP_t^h = {}^{k'}(P_t^h), \ t \geq 0$.

Let A be an absorbing (finely closed) set for X and $(P'_t)_{t\geq 0}$, the transition function on A^c, such that

$$P'_t f = P_t \bar{f}|_{A^c},\ f \in \mathcal{B}_+(A^c). \tag{5.1}$$

According to Proposition 1.4 in [10]), we have that $(P'_t)_{t\geq 0}$ is the transition function of the right process $X' = X^h$ on A^c, where $h := 1_{A^c}$.

Corollary 5.5. *The processes ${}^{k'}(X')$ and $({}^kX)'$ are equivalent.*

Suppose that $h > 0$ is a bounded function and $h \in \mathcal{S}(\mathcal{U})$. Let $\mathcal{U}^h = (U^h_\alpha)_{\alpha>0}$ be the resolvent of kernels with initial kernel $U^h = M_{\frac{1}{h}} U M_h$.

Teorema 5.6. *Suppose that $l \geq 0$ is a bounded function (hence exact) and $\mathcal{U}'$ is the resolvent of kernels with initial kernel U_l, then $(\mathcal{U}')^h = (\mathcal{U}^h)'$.*

Nonlinear case.

In the following, we consider the nonlinear case for exact functions and the associated $h-$transform. We fix a bounded function h and assume that there exists $\alpha_0 > 0$, such that $h > \alpha_0$.

Let $\mathcal{L}$ be an algebra of real-valued, bounded functions on a non-empty set E. Recall that $\mathcal{L}_h = \frac{1}{h}\mathcal{L}$.

Let $\mathcal{L}_0 := \{v \in \mathcal{L} : \exists\ c > 0,\ \text{ such that } |v| \leq c \cdot h\} \subset \mathcal{L}$. We define $\mathcal{L}'_h := \frac{1}{h}\mathcal{L}_0$.

Further, let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be the sub-Markovian resolvent of nonlinear operators on $\mathcal{L}$ associated with U.

Proposition 5.7. *The operator $T := M_{\frac{1}{h}} U M_h$ is a Lipschitz operator on $\mathcal{L}'_h$, provided that U is a Lipschitz operator on $\mathcal{L}$.*

Teorema 5.8. *Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a (nonlinear) sub-Markovian resolvent. Assume further that the function $h > 0$ satisfies $h \in \mathcal{S}(\mathcal{U})$. Then $\mathcal{U}^h = (U^h_\alpha)_{\alpha>0}$, $U^h_\alpha := M_{\frac{1}{h}} U_\alpha M_h$, $\alpha > 0$ is a (nonlinear) sub-Markovian resolvent on $\mathcal{L}'_h$ and $M_{\frac{1}{h}} U M_h$ is the initial operator of the resolvent $\mathcal{U}^h$.*

Teorema 5.9. *Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a nonlinear resolvent with initial operator U. If we define $V := UM_h$, $h \in \mathcal{L}^+$, then V satisfies the complete maximum principle and there exists a resolvent $\mathcal{V} = (V_\alpha)_{\alpha>0}$ having V as initial operator.*

Proposition 5.10. *Let h be a U-exact function of the corresponding application U_h. Let $\mathcal{V} = (V_\alpha)_{\alpha>0}$ be a nonlinear resolvent on $\mathcal{L}$, $V_\alpha : \mathcal{L} \longrightarrow \mathcal{L}$ having initial operator $V = UM_h$, given by the Theorem 5.9. Then, we have $V_1 = U_h M_h$.*

5.3 h-transform of Doob of the real Monge-Ampère operator

We introduce the Monge-Ampère resolvent, and then we consider the h-transform of the real Monge-Ampère operator.

Let $U \subset \mathbb{R}^d$, $d \geq 2$, be a bounded, open, and strictly convex non-empty subset and let λ be the Lebesgue measure on U.

Consider the following sets: $\mathcal{A}(U) := \{f : U \longrightarrow \overline{\mathbb{R}} : f \ \lambda - \text{measurable}\}$,

$$\mathcal{A}^+(U) := \{f \in \mathcal{A}(U) : f \geq 0\},$$

$$b\mathcal{A}(U) := \{f \in \mathcal{A}(U) : f \text{ bounded}\},$$

$$\mathcal{U}_0(U) := \{f : U \longrightarrow \mathbb{R} \ \text{ convex and continuous on } \overline{U}, f = 0 \text{ on } \partial U\},$$

$$-\mathcal{U}_0(U) := \{f : U \longrightarrow \mathbb{R} \text{ concave and continuous on } \overline{U}, f|_{\partial U} = 0\}.$$

Let $f \in b\mathcal{A}^+(U)$. We define $T : b\mathcal{A}^+(U) \longrightarrow -\mathcal{U}_0(U)$, as: $Tf := -H(f \cdot \lambda, 0), f \in b\mathcal{A}^+(U)$, with $H(f \cdot \lambda, 0) = H(f \cdot \lambda, 0, U)$ being convex and continuous on $\overline{U}$, such that $f|_{\partial U} = 0$ and $f \cdot \lambda$ denotes the curvature measure.

From [26], the Monge-Ampère operator on $\mathbb{R}^d$ can be expressed as an infimum of a family of linear operators, as follows:

$$d(\det D^2u(x))^{1/d} = \inf_{\det A=1} tr(A^t D^2u(x)A), \tag{5.2}$$

if in addition u is a strictly convex and bounded function. Here $A = (a_{ij})_{i,j=\overline{1,d}}$ is a symmetric matrix, with $\det A = 1$.

The relation (5.2) can be written in terms of operators as follows:

$$Mu = \frac{1}{d^d}\left(\inf_{\det A=1} tr(AD^2uA)\right)^d. \tag{5.3}$$

Let h be a strictly positive, bounded strictly convex function. Then, for every function u which is bounded and strictly convex, if $hu \in C^2(U)$ we define the operator M^h, by $M^h u := \frac{1}{h^d} M(hu)$.

Proposition 5.11. *If M is the Monge-Ampère operator satisfying relation* (5.3)*, then the h-transformed operator M^h has the representation*

$$M^h u := \frac{1}{d^d} \left(\inf_{\det A=1} tr(AD_h^2 uA) \right)^d, \tag{5.4}$$

where u is a bounded strictly convex function and $D_h^2(u) := \frac{1}{h} D^2(hu)$.

Remark 5.12. (i) *The approximation* (5.4) *for the operator M^h is analogue of the formula* (5.3) *for the Monge-Ampère operator. We can say that in this sense M^h is a Monge-Ampère, operator having the approximation with linear operators* (5.4)*.*

(ii) *The operator M^h has an approximation with linear operators analogue to the one valid for M, replacing the Hessian matrix D with D_h.*

Let Φ be a positive function such that $\lim_{\epsilon \to 0} \Phi(\epsilon) = \infty$ and $\lim_{\epsilon \to 0} \epsilon \Phi(\epsilon) = 0$. For every $x \in \Omega$ (fixed), the matrix

$$A^* := \det D^2 u(x)^{\frac{1}{2d}} D^2 u(x)^{-\frac{1}{2}}$$

has determinant equal to 1 and satisfies the relation $A^\star \leq \Phi(\epsilon) I$ for ϵ small enough.

The following result is a complement showing that the approximation with linear operators is preserved under the h-transform of the operator D^2.

Proposition 5.13. *Assume that $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^d$ and that all the eigenvalues of $D^2 u$ are positive (notice that in this case u is strictly convex) and let h be a strictly convex function, such that $\frac{u}{h}$ and, respectively $\frac{1}{h} D^2 u$ are bounded functions.*

Then, for ϵ small enough, we have

$$d \det \left[D_h^2 \left(\frac{u}{h} \right)(x) \right]^{\frac{1}{d}} = \inf_{\det A=1} tr \left[A^t D_h^2 \left(\frac{u}{h} \right)(x) A \right] \tag{5.5}$$

$$= \inf_{\det A=1, A \leq \Phi(\epsilon) I} tr \left[A^t D_h^2 \left(\frac{u}{h} \right)(x) A \right].$$

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