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Summary Ph. D. Thesis

Boundary value problems and branching processes

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CONTENTS

1. INTRODUCTION

In this thesis we study the existence of solutions of nonlinear Dirichlet problems with general boundary data, not necessarily continuous. We essentially use the controlled convergence initiated by A. Cornea which replaces the pointwise convergence of the solution to the given boundary data (cf. [25] and [26]). It turns out that this type of convergence provides a way to describe the boundary behaviour of the solution to the boundary value problems for general (not necessarily continuous) boundary data and it was already used in the linear case for the Dirichlet problem on an Euclidean domain in [5], for the Dirichlet problem associated with the Gross-Laplace operator on an abstract Wiener space (in [8]), and also for the Neumann problem on a Euclidean ball (see [17]).

In Chapter 3 an important result for the controlled convergence is Theorem 3.4, where it is proven the existence of the stochastic solution of the linear Dirichlet problem with discontinuous boundary data associated with the generator of a Markov Process killed with the multiplicative functional induced by a Borel measurable positive and bounded function c. Likewise, the case of the Laplace operator is illustrated by Corollary 3.9 which is an improvement of the main result in [5], Theorem 4.8. We offer a uniqueness result for the solution of the Dirichlet problem associated with the operator $\frac{1}{2}\Delta + q$.

In Chapter 4 we study a semilinear elliptic equation on D, a regular and bounded domain in $\mathbb{R}^d$, $d \geqslant 3$, with Dirichlet boundary condition, which is a special case of the one treated by Chen, Williams and Zhao in [21]. If the boundary data φ is a positive and bounded continuous function defined on the boundary of D then the authors proved the existence of a *weak classical solution* which converges pointwise to φ, using Schauder's fixed point theorem and a compactness criteria for continuous functions. If φ is discontinuous on the boundary of D then the problem doesn't have a weak classical solution, so we need a more general type of solution. The strategy is to modify the procedure in [21], since we have to work with spaces of discontinuous functions, in particular, the compactness criteria mentioned above are not more suitable. However, the imposed additional hypothesis on the nonlinear term permits us to use the Banach fixed point theorem. As a byproduct we prove an uniqueness result and a probabilistic representation of the solution, an approximation with stochastic terms, which might be considered an analogue of the stochastic solution to the linear Dirichlet problem. We essentially use the Corollary 3.9 from Chapter 3 for the boundary behaviour of the solution using controlled convergence instead of pointwise convergence for the case of the Laplace operator and $c = 0$.

În Chapter 5, we study nonlinear Dirichlet problems associated with nonlocal branching processes with the spatial motion given by a right Markov space X with the weak generator L. The existence of the solutions of the problems with continuous boundary data φ was studied in [41], [14], [15] (see also [38]), where the solution converges pointwise to φ on the boundary of the domain.

The aim of this paper is twofolds. First, we intend to solve the problem for functions φ which are discontinuous, replacing the pointwise convergence to the boundary data with the controlled convergence. We consider two cases:

(1) L is the Laplace operator, more precisely L is the weak generator of the d-dimensional Brownian motion which is an extension to the Laplace operator. In this case the control function is given by a (real valued) harmonic function over a bounded regular domain $D \subset \mathbb{R}^d$.

(2) L is a gradient type operator, the generator of a continuous flow $\phi = (\phi_t)_{t \geqslant 0}$ on a Lusin topological space F, leaving in finite time a bounded domain $D \subset F$ with compact closure. As in în [8], we must consider an exceptional set for the controlled convergence at the boundary.

The second aim is to show that the problem has a generalized solution which admits a probabilistic representation. As in [11] and [41], a key tool of our approach is a nonlocal branching Markov process $\widehat{X} = (\widehat{X}_t, \widehat{\mathbb{P}}^\mu, \mu \in \widehat{E})$ with state space $\widehat{E}$ of all finite configurations on $E := \overline{D}$ which describes the time evolution of a system of particles; $\widehat{X}$ has the spatial motion a Markov process X and the branching mechanism is given by a sequence of Markovian kernels (for details see [11] and [13]).

We mention here the essential contribution of E.B. Dynkin in using the measure-valued superprocesses as instruments for solving semilinear equations, the typical one being $\Delta u = u^{\alpha}$, cu $1 < \alpha \leqslant 2$; see the monographs [31], [32], also [28]. It turns out that in order to follow the above mentioned program initiated by E.B. Dynkin in the early 1990s, we need the non-local branching process $\widehat{X}$ instead of a superprocess.

In the case (2), we follow [19], and we consider the branching Markov process $\widehat{X^0} = (\widehat{X^0_t}, \widehat{\mathbb{P}^0}^{\mu}, \mu \in \widehat{E})$ which has the same branching mechanism as $\widehat{X}$ but no spatial motion; $\widehat{X^0}$ is called a pure branching process. The measure-valued process $\widehat{X}$ admits a representation through a flow induced by the spatial motion Φ (= the flow on E obtained from the continuous flow ϕ stopped at the boundary of D) and $\widehat{X^0}$.

The original results of this thesis are included in the following ISI and World of Science articles:

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2. PRELIMINARIES

The purpose of this chapter is to introduce preliminary concepts that appear in this thesis. We present fundamental concepts and results such as transition functions, resolvents of kernels, right Markov processes together with Kolmogorov's classical theorem for the construction of Markov processes, stopping times, martingales (supermartingales), the strong, weak, and extended generator of a Markov process, multiplicative functionals, and the process killed by a multiplicative functional.

3. LINEAR DIRICHLET PROBLEMS WITH DISCONTINUOUS BOUNDARY DATA

3.1. General framework

Let F be a Lusin topological space and D a domain from F such that $E := \overline{D}$ is a compact set. Let $\mathcal{B}(F)$ be the Borel σ-algebra of the Lusin topological space F. For $A \in \mathcal{B}(F)$, we denote by $\mathcal{B}(A)$ the Borel σ–algebra of A, $\mathcal{B}(A) = \mathcal{B}(F)|_A$. Let further $\mathcal{B}_+(A)$ denote the convex cone of all numerical, positive $\mathcal{B}(A)$-measurable functions on A and $b\mathcal{B}_+(A)$ be the set of bounded functions from $\mathcal{B}_+(A)$, $C(A)$ the space of continuous real-valued functions on A, $C_b(A)$ the space of continuous and bounded real-valued functions on A.

Let $Y = (Y_t, \mathbb{P}^x, x \in F)$ be a diffusion on F, that is, a path continuous right Markov process with state space F. Suppose that Y has infinite life time.

Let τ be the *first entry time* of ∂D, $\tau := \inf\{t \geqslant 0 : Y_t \in \partial D\}$, τ_o be the *first hitting time* of ∂D, $\tau_o := \inf\{t > 0 : Y_t \in \partial D\}$, and $\tau_D := \inf\{t > 0 : Y_t \notin D\}$ *the first exit time of* D. Recall that if $x \in D$ then $\tau = \tau_o$ $\mathbb{P}^x$-a.s. Because D is an open subset we have that $\tau_o = \tau_D$.

We consider the process Y stopped at the boundary of D, that is, the process $X = (X_t, \mathbb{P}^x, x \in E)$ with state space E, defined as $X_t := Y_{t\wedge\tau}$. Recall that we denoted by E the closure of D. Suppose that $\mathbb{P}^x(\tau < \infty) = 1$ for all $x \in D$. Let L be the weak generator associated with the process X and $\mathcal{D}(L)$ the domain of L.

Let $(T_t)_{t\geqslant 0}$ be the transition function of X, that is, for every $f \in b\mathcal{B}_+(E)$, $T_t f(x) := \mathbb{E}^x f(X_t)$, $t \geqslant 0$. Let further $(T_t^c)_{t\geqslant 0}$ be the transition function of the process obtained from X by killing with the multiplicative functional induced by c, expressed as the

Feynman-Kac semigroup,

$$T_t^c f\,(x) = \mathbb{E}^x\{e^{-\int_0^t c(X_s)ds} f(X_t)\},\ f \in b\mathcal{B}_+(E), x \in E.$$

We have denoted by $\mathbb{E}^x$ the expectation under $\mathbb{P}^x$. Define the kernel P_τ^c on F as

$$P_\tau^c f(x) := \mathbb{E}^x\{e^{-\int_0^\tau c(Y_s)ds} f\,(Y_\tau)\},\ f \in b\mathcal{B}_+(F), x \in F. \tag{3.1}$$

Defining analogously the kernel $P_{\tau_o}^c$, we have $P_\tau^c f(x) = P_{\tau_o}^c f(x) = P_\tau^c(f 1_{\partial D})(x)$ if $x \in D$. We have that

$$\lim_{t\to\infty} T_t^c(f|_E)\,(x) = P_\tau^c f(x),\ x \in E, f \in b\mathcal{B}_+(F). \tag{3.2}$$

3.1.1. Perturbed operator

Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be the resolvent of the process X. Denote by $L - c$ the weak generator of $(T_t^c)_{t\geqslant 0}$ and $\mathcal{D}(L-c)$ the domain of $L-c$. There exists a Markov process having $(T_t^c)_{t\geqslant 0}$ as its transition function (see [42]); we say that it is the Markov process obtained from killing X with $m = (m_t)_{t\geqslant 0}$, the multiplicative functional induced by c, that is, $m_t = e^{-\int_0^t c(X_s)ds}, t \geqslant 0$.

Theorem 3.1. *If L is the weak generator of the process X and $c \in b\mathcal{B}_+(E)$ is a finely continuous function with respect to $\mathcal{U}$, then $\mathcal{D}(L) = \mathcal{D}(L-c)$ şi $(L-c)u = Lu - cu$.*

3.2. Strong Feller functions

Strong Feller property. A transition function $(T_t)_{t\geqslant 0}$ on a Lusin topological space F is called *strong Feller* provided that $T_t f$ is a continuous function on F for every $f \in b\mathcal{B}_+(F)$ and $t > 0$.

The next result shows that the strong Feller property of a transition function is preserved by killing.

Lemma 3.2. *([12]) Let $(S_t)_{t\geqslant 0}$ be the transition function of a right Markov process Y on the Lusin topological space F, $c \in b\mathcal{B}_+(F)$, and consider $(S_t^c)_{t\geqslant 0}$, the transition function of the process obtained from Y by killing with the multiplicative functional induced by c. Then the following assertions hold.*

(i) If $(S_t)_{t\geqslant 0}$ *is strong Feller then so is* $(S_t^c)_{t\geqslant 0}$.

(ii) Assume in addition that F *is a locally compact space with a countable base. If* $(S_t)_{t\geqslant 0}$ *acts on the space of continuous functions vanishing at infinity then* $(S_t^c)_{t\geqslant 0}$ *has the same property.*

3.3. The stochastic solution of the Dirichlet problem with discontinuous boundary data, based on the controlled convergence

Recall that a point $x \in \partial D$ is said to be *regular boundary point* of D if $\mathbb{P}^x(\tau_o = 0) = 1$; see e.g. [30], vol. II, page 32, or [24], page 23. The domain D is called *regular* provided that every point of ∂D is a regular boundary point of D.

Denote by V the kernel on F defined as

$$Vf(x) = \mathbb{E}^x \int_0^\infty e^{-\int_0^t c(Y_s)ds} f(Y_t)dt, \ f \in b\mathcal{B}_+(F), x \in F,$$

it is the *potential kernel* of the process on F obtained from Y by killing with the multiplicative functional induced by c.

The stochastic solution to the Dirichlet problem. The following result is a version of Theorem 13.1 from [30], vol II, page 32.

Proposition 3.3. *([12]) Assume that* F *is a locally compact space with countable base. If* Y *is a right Markov process with state space* F*, having continuous paths, such that its transition function* $(S_t)_{t\geqslant 0}$ *is strong Feller and acts on continuous functions vanishing at infinity, and* D *is a regular domain with compact closure. Suppose in addition that the potential kernel* V *is* proper*, that is, there exists a function* $h \in \mathcal{B}_+(F)$*,* $h > 0$*, such that* Vh *is a real-valued function. Then*

$$\lim_{D\ni x\to y} P_\tau^c f(x) = f(y) \text{ for every } f \in b\mathcal{C}_+(\partial D) \text{ and } y \in \partial D. \tag{3.3}$$

3.3.1. Controlled convergence

Controlled convergence; cf. [25] and [26]. Let $f : \partial D \to \overline{\mathbb{R}}$, $D_o \subset D$ and

$h, k : D \longrightarrow \overline{\mathbb{R}}$, $k \geqslant 0$, such that $h|_{D_o}$, $k|_{D_o}$ are real-valued. We say that h converges to f controlled by k on D_o (we write $h \xrightarrow{k} f$ on D_o) if for every $A \subset D_o$ and $y \in \partial D \cap \overline{A}$ the following conditions hold:

($*$) If $\limsup\limits_{A \ni x \to y} k(x) < \infty$ then $f(y) \in \mathbb{R}$ and $f(y) = \lim\limits_{A \ni x \to y} h(x)$,

($**$) If $\lim\limits_{A \ni x \to y} k(x) = \infty$ then $\lim\limits_{A \ni x \to y} \frac{h(x)}{1+k(x)} = 0$.

If the set D_o is not specified, then $D_o = D$ and we write h converges to f controlled by k. This case was considered in [26]. The function k is called a *control function.*

3.3.2. Main result

We consider now D a bounded and regular domain of F with compact closure. We present an important result on the controlled convergence for the stochastic solution of the linear Dirichlet problem with possible discontinuous boundary data. We then use it essentially in solving nonlinear Dirichlet problems with general (discontinuous) boundary data.

Theorem 3.4. *([12]) Let λ be a finite measure on D and $\sigma := \lambda \circ P^c_\tau$. Assume that (3.3) holds, and let $\varphi \in L^1_+(\partial D, \sigma)$, then there exists a Borel measurable function $g : \partial D \longrightarrow \overline{\mathbb{R}}_+$ such that $P^c_\tau \varphi \xrightarrow{k} \varphi$ on $D_o = [k < \infty]$, where $k := P^c_\tau g \in L^1(D, \lambda)$.*

Remark 3.5. *The set $D \setminus D_o = [k = \infty]$ in Lemma 3.4 is finely closed, λ–polar and λ-negligible.*

3.4. The case of the Laplace operator

Harmonic functions. The classical linear Dirichlet problem . We consider $F = \mathbb{R}^d$, $d \geqslant 1$, D a bounded domain in $\mathbb{R}^d$ where ∂D is the boundary of D. A (real-valued) function $h \in C^2(D)$ is *harmonic* on D if

$$\Delta h := \sum_{i=1}^{d} \frac{\partial^2 h}{\partial x_i^2} = 0 \textit{ on } D.$$

Δ is called the Laplace operator.

A solution for the *classical linear Dirichlet problem* on D with boundary data $\varphi : \partial D \longrightarrow \mathbb{R}$ is a harmonic function h defined on D which satisfies the boundary condition:

$$\lim_{D \ni x \to y} h(x) = \varphi(y) \text{ for all } y \in \partial D.$$

In this section $Y = (Y_t)_{t \geqslant 0}$ is the d-dimensional Brownian motion on $\mathbb{R}^d$, $d \geqslant 1$. If $f : \partial D \longrightarrow \overline{\mathbb{R}}$ is bounded below Borel measurable function we denote

$$H_D f(x) := \mathbb{E}^x f(Y_\tau), \; x \in D.$$

Remark 3.6. *According to Theorem 3.7 in [46], page 106, we have that $H_D f$ is a (real-valued) harmonic function on D if $H_D f$ is not equal ∞ everywhere on D.*

Dirichlet problem based on controlled convergence. A function $f : \partial D \longrightarrow \overline{\mathbb{R}}$ is called *resolutive* provided that there exists a harmonic function h on D which converges to f controlled by a real-valued, non-negative superharmonic function k. If f is resolutive, then the unique function h (see Corollary 3.8 below) is called the *solution on D to the Dirichlet problem with boundary data f.*

Remark 3.7. *A harmonic function h on D is the solution to the classical Dirichlet problem with boundary data f if and only if h converges to f controlled by a bounded function k; see [25], and Remark 5.2 (ii) from [8].*

The next corollary is a version of the Corollary 4.3 din [5].

Corollary 3.8. *([18])If the Dirichlet problem has a solution then it is unique. In particular, if u is a harmonic function on D which converges controlled by k to zero then $u = 0$ on D.*

Stochastic solution to the linear Dirichlet problem with general boundary data, associated with the Laplace operator Δ

The next result is a consequence of Theorem (3.4) which shows that the stochastic solution solves the Dirichlet problem with general boundary data in the case of the Laplace operator. It is an improvement of the result in [5], Theorem 4.8; for the relation with the resolutivity for the Perron-Wiener-Brelot method see Corollary 2.13 in [26].

Corollary 3.9. *([18]) Let $D \subset \mathbb{R}^d$ be a bounded regular domain. Let $f : \partial D \longrightarrow \mathbb{R}$ be a bounded below Borel measurable function and assume that $H_D f$ is not equal $+\infty$ everywhere on D. Then $H_D f$ is the unique solution to the Dirichlet problem with boundary data f. More precisely, there exists $g \in \mathcal{B}_+(\partial D)$ such that the function $k := H_D g$ is real-valued and $H_D f$ converges to f controlled by k.*

3.5. Uniqueness of the solution of the Dirichlet problem associated with the operator $\frac{1}{2}\Delta + q$ in the sense of the controlled convergence

Let $D \subset \mathbb{R}^d$, $d \geqslant 3$ a regular bounded domain. We consider J the Kato class of the Green function (see [24]). Let $\varphi \in b\mathcal{B}_+(\partial D)$. A function $u \in C(D)$ is called a *weak solution* of the Dirichlet problem associated withe the operator $\frac{1}{2}\Delta + q$, with boundary data φ, if there exists a superharmonic control function $k : D \longrightarrow \mathbb{R}_+$ such that

$$\begin{cases} \frac{1}{2}\Delta u + qu = 0 \text{ in } D \text{ în the weak sense,} \\ u \text{ converges to } \varphi \text{ controlled by } k. \end{cases}$$

Theorem 3.10. *([18]) Let $q \in J$ and $\varphi \in b\mathcal{B}_+(\partial D)$. If the linear Dirichlet problem associated with the operator $\frac{1}{2}\Delta + q$, with boundary data φ has a weak solution $C_b(D)$ then it is unique.*

4. SEMILINEAR DIRICHLET PROBLEMS WITH DISCONTINUOUS BOUNDARY DATA

This chapter is based on the article [18] where we study the following Dirichlet problem:

$$\begin{cases} \frac{1}{2}\Delta u - \mathbb{F}(\,\cdot\,,u) = 0 \ \text{ in } D, \\ u \xrightarrow{k} \varphi, \end{cases} \tag{4.1}$$

where D is a bounded regular domain in $\mathbb{R}^d$, $d \geq 3$, the boundary condition φ is a non-negative, bounded and Borel measurable real-valued function defined on the boundary ∂D of D, and $\mathbb{F}$ is a real-valued Borel measurable function on $D \times (0,b)$ for some $b \in (0,\infty]$ such that for every $x \in D$ the function $\mathbb{F}(x,\cdot)$ is continuous on $(0,b)$ and we have

$$0 \leq \mathbb{F}(x,u) \leq U(x)u \tag{4.2}$$

where U is a fixed positive Green-tight function on D. Here the regularity of D is in the sense of the linear Dirichlet problem.

The equation in the problem is in the weak sense and the behavior at the boundary of the solution is illustrated with the help of the controlled convergence where $k : D \longrightarrow \overline{\mathbb{R}}_+$ is a control function.

If φ is a positive, real-valued and continuous function defined on ∂D, then the problem above is a special case of the one studied in [21] where the solution u is continuous on $E = \overline{D}$ and converges pointwise to φ at the boundary of D, that is $\lim_{D\ni x\to y} u(x) = \varphi(y)$, for every $y \in \partial D$. Recall that in this case u is a *classical weak solution*. If φ is discontinuous then one can show that the problem doesn't have a classical weak solution. So, we must replace the pointwise convergence at the boundary data with the controlled convergence obtaining a more general type of solution. A function $u \in C(D)$ that solves the problem 4.1 where the equation is in the weak sense is called a *weak solution* to the nonlinear Dirichlet problem with boundary data $\varphi \in b\mathcal{B}_+(\partial D)$, associated with the operator $\frac{1}{2}\Delta u - \mathbb{F}(\cdot, u)$, provided that there exists a control function k which is superharmonic on D.

4.1. The existence of the solution

For the proof of the existence of the solution we use a few arguments from [21]. Let $b \in (0, \infty]$ such that $\|\varphi\|_\infty < b$, where $\|\cdot\|_\infty$ is the supremum norm. As in [21], (3.8)-(3.10), let

$$\gamma_0 := \inf\{\varphi(x) : x \in \partial D\},\ \beta := c\,||U||_D\,,$$

and

$$\Lambda := \{u \in b\mathcal{B}_+(D) : m := e^{-\beta}\gamma_0 \leq u \leq ||\varphi||_\infty =: \widetilde{m} \text{ pe } D\}.$$

Suppose that $\gamma_0 > 0$, so $m = e^{-\beta}\gamma_0 > 0$. We endow Λ with the metric induced by the supremum norm, so obviously we obtain a complete metric space.

Consider B a ball in $\mathbb{R}^d$ of radius R_o centered at the origin, containing D. Recall that $Y = (Y_t)_{t \geqslant 0}$ is the d-dimensional Brownian motion on $\mathbb{R}^d$, $d \geqslant 3$.

Theorem 4.1. *([18]) Let φ be a bounded, Borel measurable function on ∂D such that $\gamma_0 > 0$. Assume that $\mathbb{F}$ is a real-valued Borel measurable function on $D \times (0, b)$ satisfying condition (4.2) and suppose that for every $x \in D$ the function $H_x : [m, \widetilde{m}] \longrightarrow [0, \infty)$ defined as $H_x(y) := \frac{\mathbb{F}(x,y)}{y}$ is Lipschitz continuous on $[m, \widetilde{m}]$ with the constant C that does not depend on x. Suppose that φ is such that*

$$||\varphi||_\infty < \frac{d}{R_o^2 C}. \tag{4.3}$$

Then the nonlinear Dirichlet problem with boundary data φ associated with the operator $u \longmapsto \frac{1}{2}\Delta u - \mathbb{F}(\cdot, u)$ has a weak solution $u \in C(D)$, that is,

$$\begin{cases} \frac{1}{2}\Delta u - \mathbb{F}(\cdot, u) = 0 \text{ in } D \text{ in the weak sense,} \\ u \text{ converges to } \varphi \text{ controlled by } k, \end{cases} \tag{4.4}$$

where the control function is $k := H_D g$ for some function $g \in \mathcal{B}_+(\partial D)$.

Remark 4.2. *(i) The proof of Theorem 4.1 allows to emphasize the following probabilistic representation of the solution to the nonlinear Dirichlet problem* (4.1). *Let $v_0 \in \Lambda$ and define recurrently*

$$v_{n+1} := \mathbb{E}^{\cdot}[e_{q_{v_n}}(\tau_D)\,\varphi(Y_{\tau_D})] \quad \textit{for} \quad n \geq 0.$$

Then the sequence $(v_n)_{n\geq 0}$ from Λ converges uniformly to the solution to the problem (4.1).

(ii) Condition (4.3) over the "size" of φ is similar to condition (b) from Theorem 1.1 in [21].

4.2. Uniqueness of the solution

Theorem 4.3. *([18]) If the nonlinear Dirichlet problem associated with the operator $v \longmapsto \frac{1}{2}\Delta v - \mathbb{F}(\cdot, v)$, with boundary data $\varphi \in b\mathcal{B}_+(\partial D)$, has a weak solution in Λ then it is unique.*

5. NONLINEAR DIRICHLET PROBLEMS ASSOCIATED WITH NONLOCAL BRANCHING PROCESSES

This chapter is based on the article [12].

In this chapter we consider F a Lusin topological space and D a bounded domain in F such that $E := \overline{D}$ is a compact set. Let X be a right Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}^x)$ on $(E, \mathcal{B}(E))$ with infinite life time ξ, the transition function $(T_t)_{t\geqslant 0}$, $T_t f(x) = \mathbb{E}^x\{f(X_t)\}$, $f \in \mathcal{B}_+(E)$, $x \in E$, $t \geqslant 0$ and its weak generator L.

We consider the following boundary value problem:

$$(5.1)\qquad \begin{cases} (L-c)u + c\sum\limits_{j\geqslant 1} b_j B_j u^{(j)} = 0 \text{ în } D \\ u \xrightarrow{k} \varphi \end{cases}$$

where c is a positive, bounded and real-valued Borel measurable function on D, extended with 0 on $F\setminus D$, $(b_j)_{j\geqslant 1}$ is a sequence of positive Borel measurable functions on E such that $\sum_{j\geqslant 1} b_j \leqslant 1$, for every $j \geqslant 1$, B_j is a Markovian kernel from $E^{(j)}$ (the j-th symmetric power of E) to E, $\varphi : \partial D \longrightarrow \mathbb{R}_+$ is a positive, bounded, real-valued Borel measurable function, not necessarily continuous, u is a positive real-valued Borel measurable function on E such that it belongs to the domain of $L-c$, and $u \leqslant 1$, and for every $j \in \mathbb{N}$, $j \geqslant 1$, $u^{(j)} : E^{(j)} \longrightarrow \mathbb{R}$ is the function defined as $u^{(j)}(\mathbf{x}) := u(x_1)\cdot\ldots\cdot u(x_j)$ for every $\mathbf{x} = (x_1, ..., x_j) \in E^{(j)}$.

The behaviour at the boundary of the solution u is given by the controlled convergence to φ and the control is expressed with $k : D \longrightarrow \overline{\mathbb{R}}_+$, a L-superharmonic function on D. A Borel real-valued positive measurable function on E that solves the problem (5.1) is called a *generalized solution* of the problem. We suppose that u belongs to the domain of the weak generator $L-c$ and that the equation is satisfied pointwise on D.

A key tool is a non-local branching processes $\widehat{X}$ with the state space the set $\widehat{E}$, having the Markov process X as its spatial motion (of the particles between the branching moments) and the sequence $(B_j)_{j\geqslant 1}$ as its branching mechanism (see [4], [11] and [13]).

We next give the probabilistic meaning of the branching mechanism for the branching process $\widehat{X}$: if $k > 1$ and $x \in E$, then $b_k(x)$ is the probability that a particle destroyed at the point x has k descendants, while the probability $B_{k,x}$ induced by the Markovian kernel B_k is the distribution of the k descendants; recall that $B_{k,x}$ is the probability measure on $E^{(k)}$ such that $\int_{E^{(k)}} f dB_{k,x} = B_k f(x)$ for every $g \in \mathcal{B}_+(E^{(k)})$. Consequently, such a branching process is named *non-local*, since the descendants are not born from the point where the parent died. Only in the particular case related to the problem (5.1) the branching process has the property that the descendants always start from the point where the parent died, because in this case we have $B_{k,x} = \delta_x$ for all $x \in E$.

5.1. Nonlocal branching processes

We denote by $M(E)$ the set of all positive measures on E, endowed with the weak topology, that is $\mu_n \to \mu$ if and only if $\int_E f d\mu_n \to \int_E f d\mu$ for all $f \in C(E)$ bounded functions. We define by $\mathcal{M}(E)$ its corresponding Borel σ-algebra.

The space of finite configurations of E.

We consider the following space $\widehat{E} \subseteq M(E)$ of finite sums of Dirac measures on E,

$$\widehat{E} := \{\sum_{i \leqslant i_0} \delta_{x_i} : i_0 \in \mathbb{N},\ i_0 \geqslant 1,\ x_i \in E \text{ for every } 1 \leqslant i \leqslant i_0\} \cup \{\mathbf{0}\},$$

where $\mathbf{0}$ is the measure zero (see [44]). Recall that $E^{(j)}$ is the j-th symmetric power of E, meaning the factorization of the Cartesian product E^j with respect to the equivalence relation induced by the group of permutations σ^j. Then $\widehat{E}$ is identified as $\widehat{E} = \bigcup_{j \geqslant 0} E^{(j)}$, where $E^{(0)} := \{\mathbf{0}\}$. The set $\widehat{E}$ is called *the space of finite configurations* of E and it is endowed with the weak topology of finite measures on E and its corresponding Borel σ-algebra $\mathcal{B}(\widehat{E})$.

Branching processes. A right Markov process with state space $\widehat{E}$ is called *branching process* provided that for any independent copies X_1 and X_2 of the given process on $\widehat{E}$, starting respectively from two measures μ_1 and μ_2 from $\widehat{E}$, $X_1 + X_2$ and the process starting from $\mu_1 + \mu_2$ are equal in distribution.

Let p_1 and p_2 two finite measures on $\widehat{E}$. Recall that their convolution $p_1 * p_2$ is the finite measure on $\widehat{E}$ defined for every $F \in b\mathcal{B}_+(\widehat{E})$ by $\int_{\widehat{E}} p_1 * p_2 \,(d\nu)\, F\,(\nu) := \int_{\widehat{E}} p_1\,(d\nu_1) \int_{\widehat{E}} p_2\,(d\nu_2)\, F\,(\nu_1 + \nu_2)$.

Recall that (see e.g. [48]) a kernel N on $(\widehat{E}, \mathcal{B}(\widehat{E}))$ which is sub-Markovian (i.e. $N1 \leqslant 1$) is called *branching kernel* provided that for all $\mu, \nu \in \widehat{E}$ we have $N_{\mu+\nu} = N_\mu * N_u$, where N_μ denotes the measure on $\widehat{E}$ such that $\int_{\widehat{E}} g dN_\mu = Ng\,(\mu)$ for all $g \in b\mathcal{B}_+(\widehat{E})$.

A Markov process with state space $\widehat{E}$ is a branching process if and only if its transition function is formed from branching kernels (see e.g. [39] and [11]; see also [13] and [9]).

For every $\mathcal{B}$-measurable real-valued function f we denote the multiplicative function

$\widehat{f}:\widehat{E}\to\mathbb{R}_+$ as $\widehat{f}(\mathbf{x})=\begin{cases}\prod\limits_{i\geqslant 1} f(x_i), \text{ if } \mathbf{x}=(x_i)_{i\geqslant 1}\in\widehat{E},\ \mathbf{x}\neq\mathbf{0}\\ 1, \text{ if } \mathbf{x}=\mathbf{0},\end{cases}$ cf. [48]. In other words $\widehat{f}|_{E^{(j)}}=f^{(j)}$ if $j\geqslant 1$ and $\widehat{f}(\mathbf{0})=1$.

5.2. The case of the Laplace operator

Let $F=\mathbb{R}^d$, $d\geqslant 1$, $D\subset\mathbb{R}^d$ a bounded regular domain, and $E=\overline{D}$. Let $Y=(Y_t,\mathbb{P}^x,x\in\mathbb{R}^d)$ be the d-dimensional Brownian motion on $\mathbb{R}^d$, and τ the first entry time of ∂D of the Brownian motion.

We consider X the stopped Brownian motion at the boundary of D, that is, the process $X=(X_t,\mathbb{P}^x,x\in\mathbb{R}^d)$ with state space E, defined as $X_t=Y_{t\wedge\tau}$. Recall that X is a path continuous Markov process.

Let $L=\Delta$, that is, L is the weak generator of the d-dimensional Brownian motion stopped at the boundary of D. Recall that $\Delta-c$ is the weak generator of the d-dimensional Brownian motion stopped at the boundary of D and killed with the multiplicative functional induced by c.

Using techniques from [14], [15], [41] and [11] (see also [13]) we prove the following result.

Theorem 5.1. *([12]) Let $\varphi:\partial D\longrightarrow\mathbb{R}_+$ be a bounded Borel measurable function, extended as a function on E with the value zero on D. Let $r>0$ be such that $\|\varphi\|_\infty\leqslant r$. Assume that*

$$\sup_{x\in E}\sum_{j\geqslant 1} r^{j-1}b_j(x)\leqslant 1 \text{ și } \sup_{x\in E}\sum_{j\geqslant 1} jr^{j-1}b_j(x)<\infty. \tag{5.2}$$

Then there exists a constant $c_0>0$ such that if $c<c_0$, then there exist a Borel measurable function $g:\partial D\longrightarrow\overline{\mathbb{R}}_+$ and a non-local branching process $\widehat{X}=(\widehat{X}_t,\widehat{\mathbb{P}}^\mu,\mu\in\widehat{E})$ with state space $\widehat{E}$ (the set of all finite configurations of E), and spatial motion X, such that there exists

$$\lim_{t\to\infty} r\widehat{\mathbb{E}}^{\delta_x}\{\widehat{(\frac{\varphi}{r})}(\widehat{X}_t)\}=:u(x) \text{ for all } x\in E, \tag{5.3}$$

The function u is a generalized solution to the nonlinear Dirichlet problem (5.1) *with* *boundary data* φ, *associated with the operator* $u \longmapsto (\Delta - c)\, u + c\sum_{j\geqslant 1} b_j B_j u^{(j)}$, *that is*

(5.4)
$$\begin{cases} (\Delta - c)\, u + c\sum_{j\geqslant 1} b_j B_j u^{(j)} = 0 \text{ în } D \\ u \stackrel{k}{\longrightarrow} \varphi, \end{cases}$$

where k *is a harmonic function on* D, *defined as* $k = H_D g$.

5.3. The case of the gradient type operator

Let F be a Lusin topological space, D a bounded domain of F, $E = \overline{D}$, and $\phi = (\phi_t)_{t\geqslant 0}$ be a *right continuous flow* on F. Recall that a right continuous flow on F is a family $\phi = (\phi_t)_{t\geqslant 0}$ of mappings on F (cf. [47], page 41; see also [10]) provided that:

(1) $\phi_{t+s}(x) = \phi_t(\phi_s(x))$ for all $s, t > 0$ and $x \in F$;

(2) $\phi_0(x) = x$ for all $x \in F$;

(3) For each $t > 0$ the function $F \ni x \longmapsto \phi_t(x)$ is $\mathcal{B}(F)/\mathcal{B}(F)$-measurable;

(4) For each $x \in F$ the function $t \longmapsto \phi_t(x)$ is right continuous on $[0, \infty)$.

The right continuous flow ϕ may be regarded as a deterministic right Markov process on F with infinite lifetime, $Y = (\Omega, \mathcal{F}, \mathcal{F}_t, Y_t, \theta_t, \mathbb{P}^x)$: $\Omega = F$, $\mathcal{F} = \mathcal{F}_t = \mathcal{B}(\mathbb{R}^d)$, $Y_t(x) := \phi_t(x) =: \theta_t(x)$ for all $x \in \Omega$, and $\mathbb{P}^x = \delta_x$. Let $(S_t)_{t\geqslant 0}$ be the transition function of Y (that is, of ϕ), $S_t f(x) = f(\phi_t(x))$ for all $t \geqslant 0$, $x \in F$, and $f \in \mathcal{B}_+(F)$. In particular, the transition function $(S_t)_{t\geqslant 0}$ on F is Markovian.

Let Λ be the weak generator of Y. We say also that Λ is the *weak generator of* ϕ. It is a first order "gradient type operator" in the sense that the domain $\mathcal{D}(\Lambda)$ of Λ is an algebra and if $u \in \mathcal{D}(\Lambda)$ then $\Lambda(u^2) = 2u\Lambda u$; cf. [6].

Let $\tau : F \to [0, +\infty]$ the first entry time of ∂D by ϕ, that is, $\tau(x) = \inf\{t \geqslant 0 : \phi_t(x) \in \partial D\}$. Assume that τ is bounded, thus there exists $M \in \mathbb{R}$ such that $0 \leqslant \tau(x) \leqslant M$ for every $x \in F$. Suppose that

(5.5)
$$\lim_{D\ni x\to y} \tau(x) = 0 \text{ for every } y \in \partial D.$$

We consider Φ, the flow ϕ stopped at the boundary of D, that is, $\Phi_t = \phi_{t\wedge\tau}$ for $t \geqslant 0$. It follows (cf. [6] and [10]) that Φ is a right continuous flow on E. Clearly, its deterministic right Markov process X on E is precisely the process Y stopped at the boundary of D, $X_t = Y_{t\wedge\tau}$, $t \geqslant 0$. Let $(T_t)_{t\geqslant 0}$ be the transition function of X (i.e., of Φ), $T_t f(x) = f(\Phi_t(x))$ for all $t \geqslant 0$, $x \in E$, and $f \in \mathcal{B}_+(E)$.

Denote by L the weak generator of Φ. One can see that L coincides on D with the restriction to D of Λ, the weak generator of ϕ, in the following sense. If $f \in \mathcal{D}(\Lambda)$ then $f|_E \in \mathcal{D}(L)$ and $L(f|_E) = \Lambda f$ on D and $Lg = 0$ on ∂D for all $g \in \mathcal{D}(L)$.

For every Borel measurable function $f : \partial D \longrightarrow \overline{\mathbb{R}}$ we define

$$H_D f(x) := \mathbb{E}^x f(\Phi_\tau) = f(\phi_{\tau(x)}(x)) \text{ for all } x \in D.$$

Let $(b_j)_{j\geqslant 1}$ be a sequence of positive numbers such that $\sum\limits_{j\geqslant 1} b_j \leqslant 1$ and assume that $1 < m_1 := \sum\limits_{j\geqslant 1} j b_j < \infty$. We also fix a function $c : F \to \mathbb{R}$ such that $c(x) = c_1$ for $x \in D$, where c_1 is a constant and $0 < c_1 \leqslant \frac{m_1}{m_1 - 1}$, and $c(x) = 0$ for $x \in F \setminus D$.

Let $(T_t^c)_{t\geqslant 0}$ be the transition function of X killed with the multiplicative functional induced by c, i.e., for $x \in E$

$$T_t^c f(x) := \mathbb{E}^x\{e^{-\int_0^t c(\Phi_s)ds} f(\Phi_t)\} = e^{-\int_0^t c(\Phi_s(x))ds} f(\Phi_t(x)),\ t \geqslant 0,\ f \in b\mathcal{B}_+(E).$$

Because c is 0 on $F \setminus D$, we have that for $x \in E$, $t \geqslant 0$, and $f \in \mathcal{B}_+(F)$

$$\lim_{t\to\infty} T_t^c (f|_E)(x) = \lim_{t\to\infty} e^{-\int_0^t c(\Phi_s(x))ds} f|_E(\Phi_t(x)) = e^{-c_1\tau(x)} f(\phi_{\tau(x)}(x)) =: P_\tau^c f(x).$$

From (5.5) and the properties of ϕ it follows that for every $f \in b\mathcal{C}_+(\partial D)$ we have

$$\text{(5.6)} \qquad \lim_{D\ni x\to y} P_\tau^c f(x) = \lim_{D\ni x\to y} H_D f(x) = f(\phi_0(y)) = f(y) \text{ for every } y \in \partial D.$$

Notice that the validity of (5.6) means that condition (5.5) implies that the stochastic solution to the Dirichlet problem for the weak generator L of ϕ is the classical solution, provided that the boundary data is continuous.

Let $X^0 = (X_t^0, \mathbb{P}^x, x \in E)$ be the trivial Markov process on E for which every point is a trap, that is, for each $x \in E$, $\mathbb{P}^x(X_t^0 = x) = 1$ for all $t \geqslant 0$, or equivalently, each kernel from its transition function $(T_t^0)_{t\geqslant 0}$ is the identity operator, $T_t^0 f = f$ for every $t \geqslant 0$ and $f \in \mathcal{B}_+(E)$. We consider the non-local branching process $\widehat{X^0}$ on $\widehat{E}$ for which the spatial motion is X^0 (see [19]); because actually $\widehat{X^0}$ has no spatial motion, it is called *pure branching process.*

The main assumption is the following "commutation property" of the flow Φ and the branching mechanism induced by the sequence $(B_j)_{j\geqslant 1}$: for all $f \in \mathcal{B}_+(E)$, $f \leqslant 1$, we have $B_j(f\circ\Phi_t)^{(j)} = B_k f^{(j)}\circ\Phi_t,\ j\geqslant 1,\ t\geqslant 0$; cf. condition (4.5) from [19].

The following result corresponds to Theorem 5.1 in the frame of this subsection.

Theorem 5.2. *([12]) Let $\varphi \in b\mathcal{B}_+(\partial D)$, $r>0$ be such that $||\varphi||_\infty < r$, and assume that*

$$\sum_{j\geqslant 1} r^{j-1} b_j \leqslant 1 \text{ şi } \sum_{j\geqslant 1} j r^{j-1} b_j < \infty.$$

We also fix a finite measure λ on D. Suppose that τ is bounded on D, satisfies (5.5), and the mapping $[0,\infty)\times E \ni (t,x) \longmapsto \Phi_t(x)$ is jointly continuous. Then there exist a Borel measurable function $g : \partial D \longrightarrow \overline{\mathbb{R}}_+$, a λ-polar λ-negligible set $M_o \subset D$, and a non-local branching process $\widehat{X} = (\widehat{X}_t, \widehat{\mathbb{P}}^\mu, \mu \in \widehat{E})$ with state space $\widehat{E}$ and spatial motion Φ, such that there exists

$$(5.7) \qquad \lim_{t\to\infty} r\widehat{\mathbb{E}}^{\delta_x}\{\widehat{\left(\frac{\varphi}{r}\right)}(\widehat{X}_t)\} = \lim_{t\to\infty} r\widehat{\mathbb{E}^0}^{\delta_x}\{\widehat{\left(\frac{\varphi}{r}\right)}(\Phi_t(\widehat{X_t^0}))\} =: u(x) \text{ for all } x\in E.$$

The function u is a generalized solution to the nonlinear Dirichlet problem (5.1) with boundary data φ and L, the weak generator of Φ, that is

$$(5.8) \qquad \begin{cases} (L-c)u + c\sum_{j\geqslant 1} b_j B_j u^{(j)} = 0 \text{ în } D \\ u \overset{k}{\longrightarrow} \varphi \text{ pe } D\setminus M_o, \end{cases}$$

where $k = H_D g$ is λ-integrable and the exceptional λ-polar λ-negligible set is $M_o = [k=+\infty]$.

6. APPENDIX

We present general notions about harmonic, hyperharmonic and superharmonic functions and classical results such as the Monotone Class Theorem and the Minimum

Principle. We also present sketches of proofs for results used throughout this thesis (see Remark 1.1 in [5], Corollary 4.3 in [5] şi Theorem 2.1 in [41]).

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