ROMANIAN ACADEMY

Advanced Studies School of the Romanian Academy
"Simion Stoilow" Institute of Mathematics

DOCTORAL THESIS ABSTRACT

INVERSE PROBLEMS WITH APPLICATIONS TO DIFFUSION AND TRANSPORT IN POROUS MEDIA

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Contents

1 INTRODUCTION

In this thesis we deal with some inverse problems related to the phenomenon of mud filtrate diffusion in an oil reservoir (the invasion phenomenon). The starting point in the elaboration of this thesis consists of two articles we published in 2017 in Journal of Petroleum Science and Engineering (now Geoenergy Science and Engineering) [6], [7].

In order to study the invasion phenomenon we present the first nonlinear single phase mud filtrate invasion model in Chapter 2. This model consists of a mixed problem for an one dimensional parabolic equation (that models the mud filtrate diffusion in the oil reservoir) and a Cauchy problem for a first-order differential equation (that models the mud-cake growth). The two equations of the state system are coupled by the so-called the "filtration rate" [16], denoted in thesis by q. We present two identification problems related to this model (see Chapters 2 and 3). In Chapter 4 we introduce the second nonlinear single phase mud filtrate invasion model [5]. In this model we take into account the variation of the oil reservoir permeability due to the invasion phenomenon. In this context we introduce the so-called damaged permeability. We present two identification problems related to this model in Chapter 4 of this thesis.

1.1 Structure of the thesis and main results

In this subsection we briefly present the inverse problems we deal with and the results we obtained. Subsection 1.2 contains the definitions, notations and standard theorems of functional analysis we used in this thesis.

Chapter 2 is structured on the paper [3]

In this chapter we study an inverse problem (identification problem) related to the first nonlinear model of the invasion phenomenon. In the inverse problem we deal with the identification of the filtration rate u starting from the experimental values of mud filtrate concentration at the final time T of the invasion phenomenon.

The main results of this chapter are:

1) we prove the existence and uniqueness of the solution of the state system;

2) we prove that the minimization problem has at least one solution and we obtain the first order optimality condition;

3) we obtain the second order sufficient optimality condition.

Chapter 3 is based on the papers [6] and [7].

The inverse problem that we deal with here aims at identifying the oil reservoir permeability $k \in [k_m, k_M]$ which is the solution of the following minimization problem:

$$\inf_{k \in [k_m, k_M]} \frac{1}{2} \left\| y^k(T, \cdot) - y_{ref}(\cdot) \right\|^2 ,$$

subject to the first nonlinear single phase mud filtrate invasion model (the state system).

The main results of this chapter are:

1) we prove the existence and uniqueness of the solution of the state system;

2) we prove that the minimization problem has at least one solution and we obtain the first order optimality condition;

3) we obtain the second order sufficient optimality condition.

Chapter 4 is structured on the papers [5] and [4].

Here we deal with two identification problems related to the second nonlinear single phase mud filtrate invasion model. In the first problem we want to identify the damaged permeability $\chi \in U$ by solving the following minimization problem

$$\inf_{\chi \in U} \frac{1}{2} \left\| c^{\chi}(T, \cdot) - c_{ref}(\cdot) \right\|^{2}_{L^{2}(1,L)}$$

subject to the second nonlinear mud filtrate invasion model (the state system). In the second identification problem we impose the following damaged permeability

$$\chi(\tau) = \frac{1}{\alpha + (1-\alpha)(1-\tau)^{2}}$$

where we determine the scalar value $\alpha \in [k_c, 1]$ by solving the following minimization problem:

$$\inf_{\alpha \in [k_c,1]} \frac{1}{2} \left\| c^{\alpha}(T, \cdot) - c_{ref}(\cdot) \right\|^{2}_{L^{2}(1,L)}.$$

Here the state system is the same as in the previous problem.
The main results of this chapter are:
1) we propose a nonlinear model of invasion phenomenon (called the second nonlinear single phase mud filtrate invasion model);
2) we prove the existence and uniqueness of the solution of the state system;
3) we prove that each one of the minimization problems has at least one solution and we obtain the first order optimality conditions.

1.2 Some results of functional analysis

In this subsection we present the definitions, notations and standard mathematical results that we used in this thesis.

1.2.1 Banach spaces

Here we present Schauder's fixed point theorem, the definition of Gâteaux and Fréchet differentiable functions and we define the variational triple (Gelfand triple).

1.2.2 Hölder spaces

Here we present the definition of the Hölder continuous function with exponent γ, we introduce the Hölder space $C^{m,\gamma}(I)$, and the parabolic Hölder spaces $C^{\frac{\gamma}{2},\gamma}(Q)$ and $C^{1+\frac{\gamma}{2},2+\gamma}(Q)$.

1.2.3 Lebesgue and Sobolev spaces

For $\Phi \subset \mathbb{R}^n$ open and bounded with smooth enough boundary $\partial\Omega$, and $1 \leq p \leq \infty$ in this subsection we introduce the Lebesgue spaces $L^p(\Omega)$, the Sobolev spaces $W^{k,p}(I)$, $I = (1, L)$, $L > 1$, the Hilbert spaces $H^k(I) = W^{k,2}(I)$, and we state Stampacchia's lemma ([13], p. 289) and Arzelà's theorem ([8], p. 111, [17], p. 85).

1.2.4 Vector valued functions

Let X be a real vector space with norm $\|\cdot\|_X$. We call $f : [0, T] \to X$ a vector valued function. In this subsection we define the space of continuous vector valued functions $C([0, T]; X)$, the Lebesque-Bochner space $L^p(0, T; X)$, $1 \leq p \leq \infty$, and the weak derivative f' of $f \in L^1(0, T; X)$.

Let V be a reflexive Banach space with V' its dual space, H a Hilbert space and $V \subset H$ with dense and continuous injection. We identify H with its dual H' and we consider the variational triple $V \subset H \subset V'$. At the end of this subsection we state Lions-Aubin's theorem [12] and Ascoli-Arzelà's theorem [11], p. 55.

1.2.5 The Cauchy problem within the variational approach

In this section we state Lion's theorem. We use this theorem several times throughout the thesis.

1.3 Notations

We use the following notations: μ_f is the mud filtrate viscosity $(Pa \cdot s)$; ρ_c is the mudcake density (kg/m^3); Φ is the oil reservoir porosity (dimensionless, $0 < \Phi < 1$); Φ_c is the mudcake porosity (dimensionless, $0 < \Phi_c < 1$); ξ is the mudcake thickness (m); c (or C) is the mud filtrate concentration (kg/m^3); C_f is the mud filtrate concentration in the drilling fluid (kg/m^3); c_{ref} (or C_{ref}) is a reference concentration value (kg/m^3); c_s is the solid particle concentration in the drilling fluid (kg/m^3); c_0 (or C_0) is the initial mud filtrate concentration (kg/m^3); D is the diffusion coefficient (m^2/s); h is the oil reservoir thickness (m); k is the oil reservoir permeability (m^2); k_0 is the oil reservoir permeability in the uncontaminated zone (m^2); k_c is the mudcake permeability (m^2); m_c is the mass of the particles deposited on the well wall (kg); p_e is the oil reservoir pressure (Pa); S_{wir} is the irreducible water saturation (dimensionless); S_{or} is the residual oil saturation (dimensionless); q is the invasion rate (m^3/s); x is the radial distance from the wellbore axis (m); x_w is the wellbore radius (m); x_e is the drainage radius (m).

2 IDENTIFICATION OF THE INVASION RATE IN THE FIRST NONLINEAR SINGLE PHASE MUD FILTRATE INVASION MODEL

In this chapter we study an identification problem related to the so-called *first nonlinear single phase mud filtrate invasion model.* We briefly present this model in Section 2.1, using the works [16], [9].

2.1 The first nonlinear single phase mud filtrate invasion model

The model of the invasion phenomenon that we use in this chapter contains two parts: the first part describes the mudcake growth and the second part describes the mud filtrate dispersion (invasion) through the oil reservoir (see [16], [9]).

a. The mudcake thickness growth

Using some simplifying assumptions, the authors show in [16] that ξ (the mudcake thickness) satisfies the following problem:

$$\frac{d\xi}{dt} = c_q \frac{q(\xi)}{x_w - \xi}, \tag{2.1}$$

$$\xi(0) = 0, \tag{2.2}$$

where

$$q(\xi) = q_0 \frac{\frac{k_c}{k} \ln \frac{x_e}{x_w}}{\frac{k_c}{k} \ln \frac{x_e}{x_w} - \ln(1 - \frac{\xi}{x_w})}, \tag{2.3}$$

$$q_0 = \frac{2\pi kh\Delta p}{\mu_f \ln \frac{x_e}{x_w}}, \tag{2.4}$$

$$c_q = \frac{(1-\Phi)\, c_s}{2\pi h\,(1-\Phi_c)\,\rho_c} \tag{2.5}$$

For a physical reason we impose the following condition:

$$0 \leq \xi \leq \xi_M < x_w - x_d. \tag{2.6}$$

b. The mud filtrate dispersion through the oil reservoir

In [16] is shown that the mud filtrate dispersion C in the oil reservoir is the solution of the following mixed problem :

$$\frac{\partial C}{\partial t} = \frac{1}{x}\frac{\partial}{\partial x}\left(xD(\xi)\frac{\partial C}{\partial x}\right) - \frac{q\,(\xi)}{2\pi xh\Phi\,(1-S_{wir}-S_{or})}\frac{\partial C}{\partial x}, \tag{2.7}$$

for $(t,x) \in (0,T) \times (x_w, x_e)\,,\, T > 0,\, x_w < x_e,$

$$C\,(0,x) = C_0(x),\;\; x \in (x_w, x_e)\,, \tag{2.8}$$

$$C\,(t,x_w) = C_f,\;\; C\,(t,x_e) = 0,\;\; t \in (0,T)\,. \tag{2.9}$$

We consider that the initial mud filtrate concentration C_0 verifies the following condition:

$$0 \leq C_0 \leq C_f, \tag{2.10}$$

and we impose the condition

$$0 < S_{wir} + S_{or} < 1. \tag{2.11}$$

The most used formula in order to calculate the diffusion coefficient is:

$$D(\xi) = \alpha \left(\frac{q(\xi)}{2\pi x h} \right)^{\beta}, \tag{2.12}$$

where $\alpha > 0$ and $\beta > 1$ are empirical parameters. In dimensionless form, the model (2.1)-(2.2), (2.7)-(2.9) becomes

$$\frac{d\xi}{dt} = c_1 \frac{q(\xi)}{1-\xi}, \tag{2.13}$$

$$\xi(0) = 0, \tag{2.14}$$

$$q(\xi) = \frac{\frac{k_c}{k} \ln \frac{x_e}{x_w}}{\frac{k_c}{k} \ln \frac{x_e}{x_w} - \ln(1-\xi)}, \tag{2.15}$$

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x} \left(Pe^{-1} x^{-\beta} q^{\beta}(\xi) \frac{\partial C}{\partial x} \right) + \left(Pe^{-1} x^{-\beta-1} q^{\beta}(\xi) - x^{-1} q(\xi) \right) \frac{\partial C}{\partial x}, \tag{2.16}$$

$$C(0,x) = C_0, \ x \in (1, L), \tag{2.17}$$

$$C(t,1) = 1, \ C(t,L) = 0, \ t \in (0,T). \tag{2.18}$$

where

$$c_1 = \frac{\Phi(1-\Phi)(1 - S_{wir} - S_{or}) c_s}{(1-\Phi_c)\rho_c}, \tag{2.19}$$

$$Pe = \frac{x_w^2}{\alpha t_0 \left(\frac{q_0}{2\pi x_w h} \right)^{\beta}} \tag{2.20}$$

is the Péclet number. Recall that in (2.20) we have $\beta > 1$. Conditions (2.10) and (2.6) in dimensionless form become:

$$0 \le C_0 \le 1, \tag{2.21}$$

$$0 \leq \xi \leq \xi_M < 1. \tag{2.22}$$

We call the system (2.13)-(2.18) the *first nonlinear single phase mud filtrate invasion model.*

2.2 The inverse problem

As we saw in Section 2.1 the invasion phenomenon is described by the pair (ξ, C) (the mudcake thickness growth ξ and the mud filtrate concentration C). Recall that the mudcake thickness growth is described by the Cauchy problem (2.13)-(2.14). If we know all the physical quantities that appear in (2.15) and (2.19), we can solve the Cauchy problem (2.13)-(2.14) and we can determine the function $\xi = \xi(t)$. We consider the function

$$u : [0, T] \to \mathbb{R}_+, \; u(t) := q(\xi(t)). \tag{2.23}$$

If we use this function in (2.16), the mixed problem (2.16)-(2.18) becomes:

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(Pe^{-1}x^{-\beta}u^{\beta}(t)\frac{\partial C}{\partial x}\right) + \left(Pe^{-1}x^{-\beta-1}u^{\beta}(t) - x^{-1}u(t)\right)\frac{\partial C}{\partial x}, \tag{2.24}$$

$$C(0, x) = C_0, \; x \in (1, L), \tag{2.25}$$

$$C(t, 1) = 1, \; C(t, L) = 0, \; t \in (0, T). \tag{2.26}$$

We obtain the mud filtrate concentration $C(t, x)$ by solving the mixed problem (2.24)-(2.26).

In this chapter we suppose that we do not know the values of all physical quantities that appear in (2.15) and (2.19) (for example we do not know the value of the mudcake permeability k_c), so we cannot determine the invasion rate q by using formula (2.15). As a consequence we cannot determine the solution ξ of the Cauchy problem (2.13)-(2.14) and the volumetric flow rate u given by (2.23). Thus, in the inverse problem that we study in this chapter we deal with the determination of the volumetric flow rate u given by (2.23) starting from the observed values (reference values) of the mud filtrate concentration $C(T, x)$ available at a final time T. Let C_{ref} the observed values (reference values) of the mud filtrate concentration $C(T, x)$ available at a final time T. In what follows we identify the volumetric flow rate u by solving the following minimization problem:

$$Minimize\ \frac{1}{2}\left\|C\left(T, \cdot\right) - C_{ref}(\cdot)\right\|^2_{L^2(1,L)},$$

where $C\left(T, \cdot\right)$ is the solution of the mixed problem (2.24)-(2.26), for all u in the set

$$U = \{u \in W^{1,\infty}\left(0, T\right);\ 0 < u_m \leq u\left(t\right) \leq 1,\ \text{a.e. } t \in (0, T),$$

$$u(0) = 1,\ u(T) = u_m,\ |u'(t)| \leq u_\infty\}, \tag{2.27}$$

where $0 < u_m < 1$ and $u_\infty > 0$ are given constants. We reduce this inverse problem to an optimal control one, in which the control appears in all the coefficients of the parabolic equation. For $u \in U$, let $C(t, x)$ be the solution of the system (2.24)-(2.26). We call this system the state system. Let $C_{ref} \in L^2(0, 1)$ a known function. We define the cost functional

$$J_1\left(u\right) = \frac{1}{2}\left\|C\left(T, \cdot\right) - C_{ref}\left(\cdot\right)\right\|^2_{L^2(1,L)}, \tag{2.28}$$

and we introduce the following minimization problem:

$$\inf_{u \in U} J_1\left(u\right). \tag{2.29}$$

If we make the transformation

$$C = y + \frac{L - x}{L - 1}. \tag{2.30}$$

then the state system (2.24)-(2.26) takes the following form:

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left(a\left(u\right)\frac{\partial y}{\partial x}\right) + b\left(u\right)\frac{\partial y}{\partial x} + f\left(u\right), \ \ (t, x) \in Q, \tag{2.31}$$

$$y\left(0, x\right) = y_0\left(x\right), \ \ x \in \left(1, L\right), \tag{2.32}$$

$$y\left(t, 1\right) = y\left(t, L\right) = 0, t \in \left(0, T\right). \tag{2.33}$$

Here, we must specify that the state system coefficients a, b and f from (2.31) depend on x and u. In what follows, it is very important for us how these coefficients depend on the control u. Therefore, for simplicity, we write:

$$a\left(u\right) = Pe^{-1}x^{-\beta}u^{\beta}, \tag{2.34}$$

$$b\left(u\right) = Pe^{-1}x^{-\beta-1}u^{\beta} - x^{-1}u, \tag{2.35}$$

$$f\left(u\right) = Pe^{-1}\left(\beta - 1\right)\left(L - 1\right)^{-1}x^{-\beta-1}u^{\beta} - \left(L - 1\right)^{-1}x^{-1}u, \tag{2.36}$$

$$y_0\left(x\right) = C_0(x) + \left(L - 1\right)^{-1}\left(x - L\right). \tag{2.37}$$

With the transformation (2.30) the cost functional (2.28) becomes:

$$J\left(u\right) = \frac{1}{2}\left\|y^u\left(T, \cdot\right) - y_{ref}\left(\cdot\right)\right\|^2_{L^2(1,L)}, \tag{2.38}$$

where

$$y_{ref}\left(x\right) = C_{ref}\left(x\right) - \frac{L - x}{L - 1}. \tag{2.39}$$

In (2.38) we denoted by y^u the solution of the state system (2.31)-(2.33) which corresponds to $u \in U$. Under these circumstances the minimization problem (2.29) becomes:

$$\inf_{u\in U} J(u). \tag{P}$$

By a prime mark we denote a derivative with respect to the control u. Therefore we write:

$$a'(u) := \beta \cdot Pe^{-1} \cdot x^{-\beta} \cdot u^{\beta-1}, \tag{2.40}$$

$$b'(u) := \beta \cdot Pe^{-1} \cdot x^{-\beta-1} \cdot u^{\beta-1} - x^{-1}, \tag{2.41}$$

$$f'(u) := \frac{\beta(\beta-1)}{Pe\cdot(L-1)} \cdot \frac{u^{\beta-1}}{x^{\beta+1}} - \frac{1}{(L-1)\cdot x}. \tag{2.42}$$

2.3 Existence theorem for the state system

Let us denote $V = H_0^1(1,L)$, $H = L^2(1,L)$ and $V' = H^{-1}(1,L)$ the dual of V. We identify H with its own dual and we have $V \hookrightarrow H \equiv H' \hookrightarrow V'$, with continuous and dense embeddings. Let $y_0 \in H$ (see (2.37)) and $u \in U$.

Theorem 2.1 *Let $y_0 \in H$ and $u \in U$. The state system (2.31)-(2.33) has an unique solution y^u,*

$$y^u \in C([0,T];H) \cap L^2(0,T;V) \cap W^{1,2}(0,T;V'). \tag{2.43}$$

We have the estimates

$$\sup_{t\in[0,T]} \|y^u(t)\|_H^2 + \int_0^T \|y^u(t)\|_V^2\, dt \le C, \tag{2.44}$$

$$\int_0^T \left\| \frac{dy^u}{dt}(t) \right\|_{V'}^2 dt \le C_1, \tag{2.45}$$

where the constants C and C_1 do not depend on u. If $u_i \in U$, $i = 1,2$ and $y^{u_i}, i = 1,2$ are the corresponding solutions of the state system then the estimate

$$\|y^{u_1} - y^{u_2}\|^2_{L^\infty(0,T;H)} + \|y^{u_1} - y^{u_2}\|^2_{L^2(0,T;V)} \le C_2 \|u_1 - u_2\|^2_\infty, \tag{2.46}$$

holds, where C_2 does not depend on u_1 and u_2.

Theorem 2.2 *Let $y \in L^2(0,T;V)$ with $y' \in L^2(0,T;V')$ the weak solution of the problem (2.31)-(2.33).*
i) If $y_0 \in V$ then

$$y \in L^2(0,T;H^2(1,L)) \cap L^\infty(0,T;V), y' \in L^2(0,T;H) \tag{2.47}$$

and

$$\operatorname*{ess\,sup}_{1\le t\le T} \|y(t)\|_V^2 + \|y\|^2_{L^2(0,T;H^2(1,L))} + \|y'\|^2_{L^2(0,T;H)}$$

$$\le C(\|f(u(\cdot))\|^2_{L^2(0,T;H)} + \|y_0\|_V^2). \tag{2.48}$$

ii) If $y_0 \in V \cap H^2(1,L)$ then

$$y \in L^\infty(0,T;H^2(1,L)),\ y' \in L^\infty(0,T;H) \cap L^2(0,T;V),\ y'' \in L^2(0,T;V') \tag{2.49}$$

and

$$\operatorname*{ess\,sup}_{1\le t\le T}(\|y(t)\|^2_{H^2(1,L)} + \|y'\|^2_H) + \|y'\|^2_{L^2(0,T;V)} + \|y''\|^2_{L^2(0,T;V')}$$

$$\le C(\|f(u(\cdot))\|^2_{H^1(0,T;H)} + \|y_0\|^2_{H^2(1,L)}). \tag{2.50}$$

2.4 Existence theorem for the problem (P)

We show that the minimization problem (P) has at least one solution.

Theorem 2.3 *Let $y_{ref} \in H$ (see 2.39). The problem (P) has at least one solution u^* with the corresponding state*

$$y^* \in C\left([0,T];H\right) \cap L^2\left(0,T;V\right) \cap H^1\left(0,T;V'\right). \tag{2.51}$$

2.5 The first order variations system

Theorem 2.1 asserts that for each $u \in U$ there exists only one solution y^u of the state system (2.31)-(2.33). This fact allows us to define the control-to-state mapping S. Let:

$$X := L^\infty\left(0,T\right),\ Y := C\left([0,T];H\right) \cap L^2\left(0,T;V\right). \tag{2.52}$$

We define the control-to-state mapping

$$S : U \subset X \to Y,\ S\left(u\right) := y^u. \tag{2.53}$$

In what follows we show that S is Gâteaux differentiable in u^* and we obtain its Gâteaux derivative. In this context we consider *the first order variations system*:

$$\frac{\partial z}{\partial t} - \frac{\partial}{\partial x}\left(a\left(u^*\right)\frac{\partial z}{\partial x}\right) - b\left(u^*\right)\frac{\partial z}{\partial x}$$
$$= \frac{\partial}{\partial x}\left(a'\left(u^*\right)\frac{\partial y^*}{\partial x}\right)w + b'\left(u^*\right)\frac{\partial y^*}{\partial x}w + f'\left(u^*\right)w,\ \ (t,x) \in Q,$$

$$z\left(0,x\right)=0,\ \ x\in\left(1,L\right), \tag{2.54}$$

$$z\left(t,1\right)=z\left(t,L\right)=0,\ \text{a.e. } t\in\left(0,T\right),$$

where $w \in W^{1,\infty}(0,T)$.

Theorem 2.4 *Let $w\in W^{1,\infty}\left(0,T\right)$. The system (2.54) has an unique solution*

$$z\in C\left(\left[0,T\right];H\right)\cap L^{2}\left(0,T;V\right)\cap H^{1}\left(0,T;V'\right). \tag{2.55}$$

The solution z satisfies the estimate

$$\sup_{t\in[0,T]}\left\|z\left(t\right)\right\|_{H}^{2}+\int_{0}^{T}\left\|z\left(t\right)\right\|_{V}^{2}dt\leq C\left\|w\right\|_{X}^{2}, \tag{2.56}$$

where the constant C does not depend on w and z (2.54). In addition

$$z\in W^{1,2}(0,T;H)\cap L^{\infty}(0,T;V)\cap L^{2}(0,T;H^{2}) \tag{2.57}$$

2.6 The dual system and the necessary optimality condition

Let us consider the dual system:

$$\frac{\partial p}{\partial t}=-\frac{\partial}{\partial x}\left(a\left(u^{*}\right)\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial x}\left(b\left(u^{*}\right)p\right)\ \ in\ Q, \tag{2.58}$$

$$p\left(T,x\right)=y^{*}\left(T,x\right)-y_{ref}\left(x\right)\ \ in\ \ \left(1,L\right), \tag{2.59}$$

$$p\left(t,1\right)=0,p\left(t,L\right)=0,\ \ a.e.\ t\in\left(0,T\right). \tag{2.60}$$

Theorem 2.5 *The dual system (2.58)-(2.60) has an unique solution*

$$p\in C\left(\left[0,T\right],H\right)\cap L^{2}\left(0,T;V\right)\cap H^{1}\left(0,T;V^{'}\right). \tag{2.61}$$

If $y_0 \in H^2(1,L) \cap V$, $y_{ref} \in V$, *then*

$$p \in W^{1,2}(0,T;H) \cap L^\infty(0,t;V) \cap L^2(0,T;H^2). \tag{2.62}$$

Theorem 2.6 *If* u^* *is a solution of the problem (P),* y^* *is the solution of the state system corresponding to* u^* *and* p *is the solution of the dual system (2.58)-(2.60), then* u^* *satisfies the necessary condition*

$$\int_0^T (u^*(t) - u(t))\, \Phi(t)\, dt \geq 0,\ \ (\forall) u \in U, \tag{2.63}$$

where

$$\Phi(t) := \int_1^L \left(a'(u^*) \frac{\partial p}{\partial x} \frac{\partial y^*}{\partial x} - b'(u^*)\, p \frac{\partial y^*}{\partial x} - f'(u^*)\, p \right) dx,\ \ a.e.\ t \in (0,T). \tag{2.64}$$

2.7 Second order sufficient optimality condition

In this subsection we obtain the second order sufficient optimality condition. For this purpose we prove that the cost functional J is twice continuously Fréchet differentiable from $\overset{\circ}{U}$ into $\mathbb{R}$. Firstly, we prove that the control-to-state mapping (2.53) is twice continuously Fréchet differentiable.

3 IDENTIFICATION OF OIL RESERVOIR PERMEABILITY IN THE FIRST NON-LINEAR SINGLE PHASE MUD FILTRATE INVASION MODEL

In this chapter we study an identification problem related to the first nonlinear single phase mud filtrate invasion model presented in Chapter 2, Section 2.1. We want to identify the oil reservoir permeability k starting from the experimental observations of the mud filtrate concentration C.

3.1 Introduction

We consider the first nonlinear single phase mud filtrate invasion model under the following dimensionless form:

$$\frac{d\xi}{dt} = g(\xi, k), \tag{3.1}$$

$$\xi(0) = 0, \tag{3.2}$$

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}(a(x, \xi, k)\frac{\partial y}{\partial x}) + b(x, \xi, k)\frac{\partial y}{\partial x} + f(x, \xi, k), \ (t, x) \in Q, \tag{3.3}$$

$$y(0, x) = y_0(x), \ x \in (1, L), \tag{3.4}$$

$$y(t, 1) = 0, \ y(t, L) = 0, \ t \in (0, T). \tag{3.5}$$

where $Q = (0, T) \times (1, L)$, and

$$a(\xi, k) = \frac{1}{Pe}(\frac{q(\xi, k)}{x})^{\beta}, \tag{3.6}$$

$$b(\xi, k) = \frac{1}{x}a(x, \xi, k) - \frac{q(\xi, k)}{xS(k)}, \tag{3.7}$$

$$f(\xi, k) = \frac{\beta}{(L-1)x}a(x, \xi, k) - \frac{1}{L-1}b(x, \xi, k), \tag{3.8}$$

$$g(\xi, k) = a_1 \frac{k}{(1-\xi)(a_2 - k\ln(1-\xi))}, \tag{3.9}$$

$$q(\xi, k) = \frac{k}{a_2 - k\ln(1-\xi)}, \tag{3.10}$$

$$y_0(x) = C_0(x) - \frac{L-x}{L-1}. \tag{3.11}$$

We impose the condition (see (2.22)):

$$0 \leq \xi \leq \xi_M < 1.$$

We want to determine $k \in [k_m,\ k_M]$, starting from the experimental observations (reference values) of the solution of equation (3.3) at a final time T. Let $y_{ref}(x)$, $x \in [1, L]$ be these reference values. In order to determine k we study the problem

$$\text{Minimize } \frac{1}{2}\int_1^L (y^k(T, x) - y_{ref}(x))^2 dx \tag{3.12}$$

subject to (3.1)-(3.5) for all $k \in U$, where the control set is $U = [k_m, k_M]$. In (3.12) the notation y^k indicates the solution of (3.3)-(3.5) which corresponds to k.

3.2 Existence theorem for the state system

We call the system (3.1)-(3.5) the *state system.*

Theorem 3.1 *Let $k \in [k_m, k_M]$ and $y_0 \in V \cap H^2(1, L)$. i) The system (3.1), (3.2), (3.3)-(3.5) has an unique solution (ξ^k, y^k), with the following properties:*

$$y^k \in L^\infty(0, T; H^2(1, L)),\ \frac{dy^k}{dt} \in L^\infty(0, T; H) \cap L^2(0, T; V), \tag{3.13}$$

$$\underset{t\in[0,T]}{sup} (\left\|y^k(t)\right\|^2_{H^2(1,L)} + \| \frac{dy^k}{dt}(t) \|^2_{L^2(1,L)}) + \| \frac{dy^k}{dt}(t) \|^2_{L^2(0,T;V)} \leq C_1 \tag{3.14}$$

ii) If $k_1,\ k_2 \in [k_m, k_M]$ and (ξ^{k_1}, y^{k_1}), (ξ^{k_2}, y^{k_2}) are the unique solutions of system (3.1)-(3.5) corresponding to k_1 and k_2, respectively, then we have:

$$\left\|\xi^{k_1} - \xi^{k_2}\right\|_{C([0,T])} \leq L \left|k_1 - k_2\right|, \tag{3.15}$$

$$\sup_{t\in[0,T]} \left\|y^{k_1} - y^{k_2}\right\|^2_H + \left\|y^{k_1} - y^{k_2}\right\|^2_{L^2(0,T;V)} \leq C_2 \left|k_1 - k_2\right|^2. \tag{3.16}$$

3.3 Existence theorem for problem (P)

In this section we show that the minimization problem (3.12) has at least one solution. Let $y_{ref} \in V$ a given function and let the cost functional

$$J(k) = \frac{1}{2} \left\|y^k(T, x) - y_{ref}(x)\right\|^2_H. \tag{3.17}$$

We consider the following minimization problem

$$\inf_{k\in[k_m,k_M]} J(k), \tag{P}$$

subject to (3.1)-(3.5).

Theorem 3.2 *The problem (P) has at least one solution* $k^* \in [k_m, k_M]$ *with the corresponding state* (ξ^*, y^*) *, with the following regularities*

$$
\begin{aligned}
y^* &\in C([0,T];V) \cap L^2(0,T;H^2(1,L) \cap V), \ \frac{dy^*}{dt} \in L^2(0,T;H), \\
\xi^* &\in C^1([0,T]).
\end{aligned}
$$

3.4 The first order variations system

In this section we show that the control-to-state mapping S is Gâteaux differentiable and we obtain the expression of its first derivative.

Let k^* an optimal control (a solution of problem (P)), $h_1 \in \mathbb{R}$, and let *the first order variations system*:

$$
\begin{aligned}
\frac{dz_1}{dt} &= \frac{\partial g}{\partial \xi}(\xi^*, k^*) z_1 + \frac{\partial g}{\partial k}(\xi^*, k^*) h_1, \ t \in (0,T), && (3.18) \\
z_1(0) &= 0, && (3.19)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial z_2}{\partial t} &= \frac{\partial}{\partial x}(a(\xi^*, k^*)\frac{\partial z_2}{\partial x}) + b(\xi^*, k^*)\frac{\partial z_2}{\partial x} \\
&+ [\frac{\partial}{\partial x}(\frac{\partial a}{\partial \xi}(\xi^*, k^*)\frac{\partial y^*}{\partial x}) + \frac{\partial b}{\partial \xi}(\xi^*, k^*)\frac{\partial y^*}{\partial x} + \frac{\partial f}{\partial \xi}(\xi^*, k^*)] z_1 \\
&+ [\frac{\partial}{\partial x}(\frac{\partial a}{\partial k}(\xi^*, k^*)\frac{\partial y^*}{\partial x}) + \frac{\partial b}{\partial k}(\xi^*, k^*)\frac{\partial y^*}{\partial x} + \frac{\partial f}{\partial k}(\xi^*, k^*)] h_1, \quad (3.20)
\end{aligned}
$$

for $(t,x) \in Q$,

$$z_2(0,x) = 0, \ x \in (1,L), \quad (3.21)$$

$$z_2(t,1) = 0, \ z_2(t,L) = 0, \ t \in (0,T). \quad (3.22)$$

Theorem 3.3 *i) The system (3.18)-(3.22) has an unique solution (z_1, z_2) so that*

$$z_2 \in C([0,T];V) \cap L^2(0,T;H^2(1,L) \cap V), \ \frac{dz_2}{dt} \in L^2(0,T;H), \tag{3.23}$$

ii) The solution (z_1, z_2) satisfies the estimates

$$\|z_2\|^2_{C([0,T];H)} + \|z_2\|^2_{L^2(0,T;V)} \, dt \le C_6 \, |h_1|^2, \tag{3.24}$$

$$\|z_1\|_{C([0,T])} \le C_7 \, |h_1| . \tag{3.25}$$

3.5 The dual system and the necessary optimality condition

In this section we obtain the first order necessary condition of optimality (see Theorem 3.5). Let $k^* \in [k_m, k_M]$ be a solution of problem (P) and (ξ^*, y^*) the solution of the state system (3.1)-(3.5) corresponding to k^*. The cost functional J given by (3.17) is Gâteaux differentiable in k^* and its Gâteaux derivative $J'(k^*) : \mathbb{R} \to \mathbb{R}$ is

$$J'(k^*)(h_1) = \int_1^L (y^*(T,x) - y_{ref}(x)) z_2(T,x) dx, \tag{3.26}$$

where z_2 is the solution of (3.20)-(3.22) corresponding to h_1. Due to the fact that J is Gâteaux differentiable in k^* and $[k_m, k_M]$ is convex and closed the necessary condition of optimality is ([14], Theorem 1-2.1, p. 1180):

$$J'(k^*)(k - k^*) \ge 0, \ \text{for all } k \in [k_m, \ k_M]. \tag{3.27}$$

From (3.26) and (3.27) we obtain that the condition

$$(k - k^*) \int_1^L (y^*(T, x) - y_{ref}(x))\psi_2(T, x)dx \geq 0, \tag{3.28}$$

holds for all $k \in [k_m, k_M]$. We introduce the *dual system*:

$$\begin{aligned} -\frac{dp_1}{dt} &= \frac{\partial g}{\partial \xi}(\xi^*, k^*)p_1 - \int_1^L \frac{\partial a}{\partial \xi}(\xi^*, k^*)\frac{\partial y^*}{\partial x}\frac{\partial p_2}{\partial x}dx \\ &\quad + \int_1^L \frac{\partial b}{\partial \xi}(\xi^*, k^*)\frac{\partial y^*}{\partial x}p_2 dx + \int_1^L \frac{\partial f}{\partial \xi}(\xi^*, k^*)p_2 dx, \qquad (3.29) \\ p_1(T) &= 0, \qquad (3.30) \end{aligned}$$

$$\begin{aligned} -\frac{\partial p_2}{\partial t} &= \frac{\partial}{\partial x}(a(\xi^*, k^*)\frac{\partial p_2}{\partial x}) - \frac{\partial}{\partial x}(b(\xi^*, k^*)p_2), \ (t, x) \in Q \qquad (3.31) \\ p_2(T, x) &= y^*(T, x) - y_{ref}(x), \ x \in (1, L) \qquad (3.32) \\ p_2(t, 1) &= 0, \ p_2(t, L) = 0, \ t \in (0, T). \qquad (3.33) \end{aligned}$$

Theorem 3.4 *The system (3.29)-(3.33) has an unique solution* (p_1, p_2) *with the following properties:*

$$p_2 \in C([0, T]; V) \cap L^2(0, T; H^2(1, L) \cap V), \ \frac{dp_2}{dt} \in L^2(0, T; H), \tag{3.34}$$

$$\operatorname*{ess\,sup}_{t \in [0,T]} \|p_2(t)\|_V^2 + \| \, p_2 \, \|^2_{L^2(0,T;H^2(1,L))} + \| \, \frac{dp_2}{dt}(t) \, \|^2_{L^2(0,T;H)} \leq C_{11}. \tag{3.35}$$

Theorem 3.5 *Let* $k^* \in [k_m, k_M]$ *be a solution of problem (P),* (ξ^*, y^*) *the solution of the state system (3.1)-(3.5) and* (p_1, p_2) *the solution of dual system (3.29)-(3.33). Then* k^* *satisfies the necessary optimality condition*

$$(k - k^*)\Phi_1(k^*) \geq 0 \textit{ for all } k \in [k_m, k_M], \tag{3.36}$$

where

$$\begin{aligned}\Phi_1(k^*) &= \int_0^T \int_1^L \{[-\frac{\partial a}{\partial \xi}(\xi^*, k^*)\psi_1 - \frac{\partial a}{\partial k}(\xi^*, k^*)]\frac{\partial y^*}{\partial x}\frac{\partial p_2}{\partial x} \\ &+[\frac{\partial b}{\partial \xi}(\xi^*, k^*)\psi_1 + \frac{\partial b}{\partial k}(\xi^*, k^*)]\frac{\partial y^*}{\partial x}p_2 \\ &+[\frac{\partial f}{\partial \xi}(\xi^*, k^*)\psi_1 + \frac{\partial f}{\partial k}(\xi^*, k^*)]p_2\}dxdt, \qquad (3.37)\end{aligned}$$

and ψ_1 *is the solution of the following problem*

$$\frac{d\psi_1}{dt} = \frac{\partial g}{\partial \xi}(\xi^*, k^*)\psi_1 + \frac{\partial g}{\partial k}(\xi^*, k^*), \qquad (3.38)$$

$$\psi_1(0) = 0. \qquad (3.39)$$

3.6 The second order variations system

In this section we prove that the control to state mapping S is twice Gâteaux differentiable in k^*. In this context we introduce *the second order variations system* and we prove that this system has an unique solution (w_1, w_2) so that

$$w_2 \in C([0,T];V) \cap L^2(0,T;H^2(1,L) \cap V),\ \frac{dw_2}{dt} \in L^2(0,T;H). \qquad (3.40)$$

3.7 The second dual and sufficient optimality condition

In this section we obtain the sufficient local optimality condition. Let $k^* \in [k_m, k_M]$ a solution of problem (P). We prove that the cost functional J given by (3.17) is twice Gâteaux differentiable in k^*. We introduce the second dual system and we prove that this system has an unique solution(q_1, q_2) so that

$$q_2 \in C([0,T];V) \cap L^2(0,T;H^2(1,L) \cap V),\ \frac{dq_2}{dt} \in L^2(0,T;H).$$

3.8 Numerical results

In this section we present a numerical implementation of the inverse method presented in this chapter. We want to determine the oil reservoir permeability value k^* starting from the reference values $C_{ref}(x)$ of the mud filtrate concentration $C(T, x)$.

4 IDENTIFICATION PROBLEMS RELATED TO THE SECOND NONLINEAR SINGLE PHASE MUD FILTRATE INVASION MODEL

In this chapter we study two identification problems related to the second nonlinear single phase mud filtrate invasion model proposed by us in [5].

4.1 The second nonlinear single phase mud filtrate invasion model

In the following we use the simplified assumptions as in [16], but regarding the oil reservoir permeability we use the following assumption (see [5]): the reservoir permeability has a constant value k_0 in the uncontaminated zone, but in the invaded zone the oil reservoir permeability k_s is affected by the invasion phenomenon as follows from the formula

$$k_s = k_0 f_d \left(c/c_f \right), \tag{4.1}$$

In (4.1) c_f is the mud filtrate concentration in the drilling fluid (which is equal to the mud filtrate concentration at the well wall) and $f_d : [0,1] \to \left[\frac{k_c}{k_0}, 1 \right]$ is a known positive decreasing function so that $\frac{1}{x f_d \left(C/C_f \right)}$ is integrable on $[x_w, x_e]$ and $f_d(0) = 1$. The mud-cake permeability is lower than the reservoir permeability so we have $\frac{k_c}{k_0} \leq f_d \left(c/c_f \right) \leq 1$. We call the function given by (4.1) the *damaged permeability* [16].

We impose the following dimensionless condition as in Chapters 2 and 3 (see (2.6)):

$$0 \leq \xi \leq \xi_M < 1. \tag{4.2}$$

We use the following notations:

$$\chi(c) = \frac{1}{f_d(c)}. \tag{4.3}$$

$$I(c) = \int_1^L \frac{1}{x}\chi(c)dx, \tag{4.4}$$

$$q(c,\xi) = \frac{k_c \ln L}{k_c I(c) - \ln(1-\xi)}, \tag{4.5}$$

$$a(c,\xi) = Pe^{-1}x^{-\beta}q^{\beta}(c,\xi), \tag{4.6}$$

$$b(c,\xi) = Pe^{-1}x^{-\beta-1}q^{\beta}(c,\xi) - x^{-1}q(c,\xi), \tag{4.7}$$

$$f(c,\xi) = \frac{\alpha_1}{(1-\xi)\left[k_c I(c) - \ln(1-\xi)\right]}. \tag{4.8}$$

Thus, the *second nonlinear single phase mud filtrate invasion model* may be written as:

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(a(c,\xi)\frac{\partial c}{\partial x}\right) + b(c,\xi)\frac{\partial c}{\partial x}, \tag{4.9}$$

$$c(0,x) = c_0(x),\ x \in (1,L), \tag{4.10}$$

$$c(t,1) = 1,\ c(t,L) = 0,\ t \in (0,T), \tag{4.11}$$

$$\frac{d\xi}{dt} = f(c,\xi), \tag{4.12}$$

$$\xi(0) = 0. \tag{4.13}$$

In what follows, we consider that the initial datum c_0 verifies the following conditions:

$$c_0(1) = 1,\ c_0(L) = 0,\ 0 \le c_0 \le 1. \tag{4.14}$$

4.2 Identification of the damaged permeability

In this section, our purpose is to identify (based on certain observations) the damaged permeability in the system (4.9)-(4.13). We consider the following control set

$$U = \{\chi \in W^{2,\infty}(0,1)\,;\ 1 \leq \chi(\tau) \leq \frac{1}{k_c}, \|\chi\|_{W^{2,\infty}(0,1)} \leq \chi_\infty\}, \tag{4.15}$$

where $\chi_\infty > 0$ is given. In (4.15) k_c represents the dimensionless mud-cake permeability. We consider the following minimization problem:

$$\text{Minimize } \frac{1}{2}\left\|c^\chi(T,\cdot) - c_{ref}(\cdot)\right\|^2_{L^2(1,L)}, \tag{4.16}$$

for all $\chi \in U$ where c_{ref} is a reference (observed) mud filtrate concentration and (c^χ, ξ^χ) is the solution of (4.9)-(4.13) corresponding to χ.

4.2.1 Existence theorem for the state system

We use the following notations: $H = L^2(1,L)$, $V = H_0^1(1,L)$, $Q = (0,T) \times (1,L)$.

Theorem 4.1 *Let*

$$c_0 \in W^{1,\infty}(1,L), c_0(1) = 1,\ c_0(L) = 0, 0 \leq c_0 \leq 1. \tag{4.17}$$

The state system (4.9)-(4.13) has an unique solution (c^χ, ξ^χ)*, with the following properties:*
i) $c^\chi \in C^{\frac{\gamma}{2},\gamma}(\overline{Q})$, $\frac{\partial c^\chi}{\partial t} \in C^{\frac{\gamma}{2},\gamma}(Q)$, $\frac{\partial^2 c^\chi}{\partial x^2} \in C^{\frac{\gamma}{2},\gamma}(Q)$; *ii)* $0 \leq c^\chi \leq 1$; *iii)* $\frac{\partial c^\chi}{\partial x}$ *is bounded on* $\overline{Q}$: *there exists* $M > 0$, *which does not depend on* χ, z *and* c^z,

so that

$$\sup_{\overline{Q}} \left| \frac{\partial c^{\chi}}{\partial x} \right| \leq M; \tag{4.18}$$

iv)

$$\xi^{\chi} \in C^1([0,T]),\ 0 \leq \xi^{\chi}(t) \leq \xi_M,\ \ (\forall)\ \ t \in [0,T]\,.$$

4.2.2 Existence theorem for the minimization problem (4.16)

The purpose of this section is to show that the minimization problem (4.16) has at least one solution.

Theorem 4.2 *The minimization problem (4.16) has at least one solution* χ^* *with the corresponding state* (c^*, ξ^*)*.*

4.2.3 The first order variations system

In this section we show that the control-to-state mapping S is Gâteaux differentiable and we construct its Gâteaux derivative. For this purpose we introduce the first order variations system (4.22)-(4.26). For $\chi \in U$ we denote (c^{χ}, ξ^{χ}) the unique solution of the state system (4.9)-(4.13). Let the control-to-state mapping:

$$S : U \to \mathcal{Y},\ S(\chi) = (c^{\chi}, \xi^{\chi})\,, \tag{4.19}$$

where $\mathcal{Y}$ is the Banach space

$$\mathcal{Y} = C\left([0,T]\,; H\right) \times C\left([0,T]\right). \tag{4.20}$$

endowed with the norm

$$\|(\theta,\xi)\|_{\mathcal{Y}}^2 = \sup_{t\in[0,T]} (\|\theta(t)\|_H^2 + |\xi(t)|^2). \tag{4.21}$$

Let χ^* be a solution of the problem (4.16) and (c^*,ξ^*) the solution of the state system (4.9)-(4.13) corresponding to χ^* and $w \in W^{2,\infty}(0,1)$. Let the *first order variations system*:

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial x}\left(a(c^*,\xi^*)\frac{\partial z}{\partial x}\right) + b(c^*,\xi^*)\frac{\partial z}{\partial x} + (\int_1^L \frac{1}{x}\chi^{*\prime}(c^*)\, z dx)\frac{\partial}{\partial x}(a_I(c^*,\xi^*)\frac{\partial c^*}{\partial x})$$

$$+(\int_1^L \frac{1}{x}\chi^{*\prime}(c^*)\, z dx) b_I(c^*,\xi^*)\frac{\partial c^*}{\partial x} + \frac{\partial}{\partial x}(a_\xi(c^*,\xi^*)\frac{\partial c^*}{\partial x}\psi) + b_\xi(c^*,\xi^*)\frac{\partial c^*}{\partial x}\psi$$

$$+(\int_1^L \frac{1}{x} w(c^*)\, dx)\frac{\partial}{\partial x}(a_I(c^*,\xi^*)\frac{\partial c^*}{\partial x}) + (\int_1^L \frac{1}{x} w(c^*)\, dx) b_I(c^*,\xi^*)\frac{\partial c^*}{\partial x},\ \ (t,x)\in Q, \tag{4.22}$$

$$z(0,x) = 0,\ \ x \in (1,L), \tag{4.23}$$

$$z(t,1) = z(t,L) = 0,\ \ \text{a.e. } t \in (0,T), \tag{4.24}$$

$$\frac{d\psi}{dt} = f_I(c^*,\xi^*)\int_1^L \frac{1}{x}\chi^{*\prime}(c^*)\, z dx + f_\xi(c^*,\xi^*)\psi + f_I(c^*,\xi^*)\int_1^L \frac{1}{x} w(c^*)\, dx, \tag{4.25}$$

$$\psi(0) = 0. \tag{4.26}$$

Theorem 4.3 *i) For every $w \in W^{2,\infty}(0,1)$ the system (4.22)-(4.26) has an unique solution (z,ψ) so that*

$$z \in C([0,T];V) \cap L^2(0,T;H^2(1,L)\cap V),\ \ \frac{dz}{dt} \in L^2(0,T;H). \tag{4.27}$$

ii) We have

$$\|(z,\psi)\|_{\mathcal{Y}}^2 + \|z\|_{L^2(0,T;V)}^2 \le \mathcal{K}_5 \|w\|_{W^{1,\infty}(0,1)}^2, \tag{4.28}$$

where

$$\|(z,\psi)\|_{\mathcal{Y}}^2 = \|z\|_{C([0,T];H)}^2 + \|\psi\|_{C([0,T])}^2$$

and $\mathcal{K}_5 > 0$ *does not depend of* $\chi^* \in U$, z^*, ξ^*, z *and* ξ.

4.2.4 The dual system and the necessary optimality condition

In this section we obtain the necessary optimality condition (see Theorem 4.5). Let the *dual system:*

$$\frac{\partial p_1}{\partial t} = -\frac{\partial}{\partial x}\left(a(c^*,\xi^*)\frac{\partial p_1}{\partial x}\right) + \frac{\partial}{\partial x}(b(c^*,\xi^*)p_1) + \frac{1}{x}\chi^{*\prime}(c^*)\int_1^L a_I(c^*,\xi^*)\frac{\partial c^*}{\partial x}\frac{\partial p_1}{\partial x}dx$$

$$-\frac{1}{x}\chi^{*\prime}(c^*)\int_1^L b_I(c^*,\xi^*)\frac{\partial c^*}{\partial x}p_1 dx - \frac{1}{x}f_I(c^*,\xi^*)\chi^{*\prime}(c^*)\,p_2, \tag{4.29}$$

$$p_1(T,x) = c^*(T,x) - c_{ref}(x), \text{ in } (1,L),$$

$$p_1(t,1) = p_1(t,L) = 0, \text{ a.e. } t \in (0,T),$$

$$\frac{dp_2}{dt} + f_\xi(c^*,\xi^*)p_2 = \int_1^L a_\xi(c^*,\xi^*)\frac{\partial c^*}{\partial x}\frac{\partial p_1}{\partial x}dx - \int_1^L b_\xi(c^*,\xi^*)\frac{\partial c^*}{\partial x}p_1 dx, \ t \in [0,T], \tag{4.30}$$

$$p_2(T) = 0. \tag{4.31}$$

Theorem 4.4 *Let*

$$c_{ref} \in H^1(1,L), \ c_{ref}(1) = 1, \ c_{ref}(L) = 0, \ 0 \le c_{ref} \le 1. \tag{4.32}$$

The dual system (4.29)-(4.31) has an unique solution (ζ,η) *with the following regularity*

$$\zeta \in C([0,T];V) \cap L^2(0,T;H^2(1,L)\cap V), \ \frac{d\zeta}{dt} \in L^2(0,T;H). \tag{4.33}$$

Theorem 4.5 contains the necessary optimality condition.

Theorem 4.5 *Let χ^* be a solution of the problem (4.16), (c^*, ξ^*) the solution of the state system (4.9)-(4.13) corresponding to χ^* and (p_1, p_2) the solution of the dual system (4.29)-(4.31). Under these circumstances χ^* satisfies*

$$\int_0^T \left(\int_1^L \frac{1}{x}\chi(c^*)dx\right)\Phi(\chi^*, t)dt \geq \int_0^T \left(\int_1^L \frac{1}{x}\chi^*(c^*)dx\right)\Phi(\chi^*, t)dt, \ (\forall)\chi \in U, \tag{4.34}$$

where

$$\Phi(\chi^*, t) := \int_1^L \left(-a_I(c^*, \xi^*)\frac{\partial c^*}{\partial x}\frac{\partial p_1}{\partial x} + b_I(c^*, \xi^*)\frac{\partial c^*}{\partial x}p_1\right)dx + f_I(c^*, \xi^*)p_2, \tag{4.35}$$

a.e. $t \in (0, T)$.

4.3 Identification of the minimum value of oil reservoir permeability

Here we search the function χ (the damaged permeability) under the following form:

$$\chi(\tau) = \frac{1}{\alpha + (1-\alpha)(1-\tau)^2}, \ \tau \in [0, 1], \tag{4.36}$$

where α is a positive number so that $0 < k_c \leq \alpha \leq 1$.

We want to identify α that is a solution of the following minimization problem:

$$\text{Minimize } \frac{1}{2}\left\|c^\alpha(T, \cdot) - c_{ref}(\cdot)\right\|^2_{L^2(1,L)}, \tag{4.37}$$

for all $\alpha \in [k_c, 1]$, where c_{ref} is a know (observed) mud filtrate concentration.

4.3.1 Existence theorem for the state system

The theorem below asserts that the state system (4.9)-(4.13) has an unique solution for any control $\alpha \in [k_c, 1]$.

Theorem 4.6 *Let $\alpha \in [k_c, 1]$. Then the state system (4.9)-(4.13), where χ is given by (4.36), has an unique solution (c^α, ξ^α) with the following properties: i) $c^\alpha \in C^{\frac{\gamma}{2},\gamma}(\overline{Q})$, $\frac{\partial c^\alpha}{\partial t} \in C^{\frac{\gamma}{2},\gamma}(Q)$, $\frac{\partial^2 c^\alpha}{\partial x^2} \in C^{\frac{\gamma}{2},\gamma}(Q)$;ii) $0 \leq c^\alpha(t,x) \leq 1$, for all $(t,x) \in \overline{Q}$;iii) $\frac{\partial c^\alpha}{\partial x}$ is bounded on $\overline{Q}$: there exists $M > 0$, that does not depend on χ, α and c^α, so that $\sup_{\overline{Q}} \left|\frac{\partial c^\alpha}{\partial x}\right| \leq M$; iv) $\xi^\alpha \in C^1([0,T]), 0 \leq \xi(t) \leq \xi_M$, for all $t \in [0,T]$.*

4.3.2 Existence theorem for the minimization problem (4.37)

In this section we prove that the problem (4.37) has at least one solution (see Theorem 4.7). We consider c_{ref} a given function so that

$$c_{ref} \in H^1(1,L), c_{ref}(1) = 1, c_{ref}(L) = 0, \ 0 \leq c_{ref} \leq 1. \tag{4.38}$$

Theorem 4.7 *The problem (4.37) has at least one solution $\alpha^* \in [k_c.1]$ with the corresponding state $z^* = (c^*, \xi^*)$.*

4.3.3 The first order variations system

In this section we show that the control to state mapping S is Gâteaux differentiable. In order to determine its Gâteaux derivative we use the first order variations system (4.39)-(4.40). Let us consider the *first order variations system*:

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial x}(a(c^*,\xi^*)\frac{\partial z}{\partial x}) + b(c^*,\xi^*)\frac{\partial z}{\partial x}$$

$$+2(1-\alpha^*)(\int_1^L \frac{1}{x}\chi^{*2}(1-c^*)zdx)\frac{\partial}{\partial x}(a_I(c^*,\xi^*)\frac{\partial c^*}{\partial x})$$

$$+2(1-\alpha^*)(\int_1^L \frac{1}{x}\chi^{*2}(1-c^*)zdx)b_I(c^*,\xi^*)\frac{\partial c^*}{\partial x}$$

$$+\frac{\partial}{\partial x}(a_\xi(c^*,\xi^*)\frac{\partial c^*}{\partial x}\psi)+b_\xi(c^*,\xi^*)\frac{\partial c^*}{\partial x}\psi$$
$$+w\sigma\frac{\partial}{\partial x}(a_I(c^*,\xi^*)\frac{\partial c^*}{\partial x})+w\sigma b_I(c^*,\xi^*)\frac{\partial c^*}{\partial x},\ \ (t,x)\in Q, \tag{4.39}$$

$z\,(0,x)=0,\ \ x\in(1,L)\,,$

$z\,(t,1)=z\,(t,L)=0,\ \text{a.e. } t\in(0,T)\,,$

$$\frac{d\psi}{dt}=2\,(1-\alpha^*)\,f_I(c^*,\xi^*)\int_1^L\frac{1}{x}\chi^{*2}(1-c^*)zdx+f_\xi(c^*,\xi^*)\psi+wf_I(c^*,\xi^*)\sigma,$$
$$\psi\,(0)=0, \tag{4.40}$$

where

$$\sigma(t)=-\int_1^L\frac{1}{x}\chi^{*2}(2c^*-c^{*2})dx. \tag{4.41}$$

Theorem 4.8 *i) The system (4.39)-(4.40) has an unique solution* (z,ψ) *with the following regularities*

$$z\in C\,([0,T];V)\cap L^2(0,T;H^2(1,L)\cap V),\ \frac{dz}{dt}\in L^2\,(0,T;H)\,. \tag{4.42}$$

ii) We have

$$\|(z,\psi)\|^2_{\mathcal{Y}}+\|z\|^2_{L^2(0,T;V)}\,dt\leq\mathcal{K}_4\,|w|^2\,, \tag{4.43}$$

where $\mathcal{K}_4$ *does not depend of* z, ψ *and* w

4.3.4 The dual system and the necessary optimality condition

In this section we obtain the necessary optimality condition (4.48) (see Theorem 4.10). Let the *dual system*

$$\begin{aligned}\frac{\partial p_1}{\partial t} &= -\frac{\partial}{\partial x}\left(a(c^*,\xi^*)\frac{\partial p_1}{\partial x}\right)+\frac{\partial}{\partial x}\left(b(c^*,\xi^*)p_1\right)\\ &\quad+2(1-\alpha^*)(1-c^*)\frac{1}{x}\chi^{*2}\int_1^L a_I(c^*,\xi^*)\frac{\partial c^*}{\partial x}\frac{\partial p_1}{\partial x}dx\end{aligned}$$

$$-2(1-\alpha^*)(1-c^*)\frac{1}{x}\chi^{*2}\int_1^L b_I(c^*,\xi^*)\frac{\partial c^*}{\partial x}p_1dx-2(1-\alpha^*)(1-c^*)\frac{1}{x}f_I(c^*,\xi^*)\chi^{*2}p_2, \tag{4.44}$$

$$p_1(T,x)=c^*(T,x)-c_{ref}(x), \text{ in } (1,L),$$

$$p_1(t,1)=p_1(t,L)=0, \text{ a.e. } t\in(0,T),$$

$$\frac{dp_2}{dt}+f_\xi(c^*,\xi^*)p_2=\int_1^L a_\xi(c^*,\xi^*)\frac{\partial c^*}{\partial x}\frac{\partial p_1}{\partial x}dx-\int_1^L b_\xi(c^*,\xi^*)\frac{\partial c^*}{\partial x}p_1dx, \ t\in[0,T], \tag{4.45}$$

$$p_2(T)=0. \tag{4.46}$$

Theorem 4.9 *The system (4.44)-(4.46) has an unique solution* (ζ,η) *with the following regularity*

$$\zeta\in C([0,T];V)\cap L^2(0,T;H^2(1,L)\cap V), \ \zeta'\in L^2(0,T;H). \tag{4.47}$$

Theorem 4.10 *Let* (c^*,ξ^*) *the solution of the state system (4.9)-(4.13) which corresponds to* α^* *and* (p_1,p_2) *be the solution of the dual system (4.44)-(4.46). Then* α^* *satisfies the necessary condition*

$$(\alpha-\alpha^*)\int_0^T\Phi(\alpha^*,t)dt\geq 0, \ (\forall)\alpha\in[k_c,1], \tag{4.48}$$

where

$$\Phi(\alpha^*,t):=(\int_1^L(-a_I(c^*,\xi^*)\frac{\partial c^*}{\partial x}\frac{\partial p_1}{\partial x}+b_I(c^*,\xi^*)\frac{\partial c^*}{\partial x}p_1)dx+f_I(c^*,\xi^*)p_2)\cdot\sigma(t), \tag{4.49}$$

a.e. $t\in(0,T)$, *and* σ *is given by (4.41).*

4.3.5 Numerical results

This subsection is dedicated to the presentation of some numerical results.

Further directions of research

We present here five direction of research starting from the results obtained in this thesis .

Published articles and conference participation during the doctoral internship

A. Articles

The published articles during the doctoral internship are [1], [2], [3], [4] and [5].

B. Conferences

During the doctoral internship I participated at 12 conferences.

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