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PhD THESIS SUMMARY

ASYMPTOTIC PHENOMENA OF DIFFUSION EQUATIONS

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To Micu

"I don't know anything! That's the greatest spiritual ethic of all."
Sir Anthony Hopkins

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CONTENTS

OUTLINE OF THE MAIN RESULTS

Chapter 1: INTRODUCTION

This thesis explores a variety of scenarios in which asymptotic methods play a crucial role in the analysis of partial differential equations. The motivations for using mathematical approximations are diverse: on one hand, they allow the replacement of complex models with simpler, approximate ones, whose properties can be then transferred to the former more intricate objects. A relevant example in this context is the use of numerical schemes to effectively compute solutions to differential equations. On the other hand, approximations allow us to predict the behaviour of certain models when their parameters reach extreme values. For instance, large-time asymptotic profiles for evolution equations provide valuable insight into their solutions when the temporal variable becomes large. This work provides multiple asymptotic results for both local and non-local evolution equations, defined on Euclidean spaces as well as in the non-Euclidean context of hyperbolic spaces.

The primary chapters of this thesis (3-6) are grounded in four of my co-authored papers that explore asymptotic phenomena of parabolic equations. Two of these articles have been published [13, 24] while the others are under review in reputable journals [9, 19].

In the first part (Chapter 3), we study the large time asymptotic behaviour of the heat equation with Hardy inverse-square potential on corner spaces $\mathbb{R}^{N-k} \times (0,\infty)^k$, $k \geq 0$. We first show a new improved Hardy-Poincaré inequality for the quantum harmonic oscillator with Hardy potential. In view of that, we construct the appropriate functional setting in order to pose the Cauchy problem. Then we obtain optimal polynomial large time decay rates and subsequently the first term in the asymptotic expansion of the solutions in $L^2(\mathbb{R}^{N-k} \times (0,\infty)^k)$. Particularly, we extend and improve the results obtained by Vázquez and Zuazua (J. Funct. Anal. 2000), which correspond to the case $k = 0$, to any $k \geq 0$. We emphasise that the higher the value of k the better time decay rates are. We employ a different and simplified approach than Vázquez and Zuazua, managing to remove the usage of spherical harmonics decomposition in our analysis.

The second part (Chapter 4) of this work is dedicated to the study of a non-local evolution equation on the hyperbolic space $\mathbb{H}^N$. We first consider a model for particle transport governed by a non-local interaction kernel defined on the tangent bundle and invariant under the geodesic flow. We study the relaxation limit of this model to a local transport problem, as the kernel gets concentrated near the origin of each tangent space. Under some regularity and integrability conditions on the kernel, we prove that the solution of the rescaled non-local problem converges to that of the local transport equation. Then, we construct a large class of interaction kernels that satisfy those conditions.

We also consider a non-local, non-linear convection-diffusion equation on $\mathbb{H}^N$ governed by two kernels, one for each of the diffusion and convection parts, and we prove that the solutions converge to the solution of a local problem as the kernels get concentrated. We prove and then use in this sense a compactness tool on manifolds inspired by the work of Bourgain-Brezis-Mironescu.

The main motivation behind the third part (Chapter 5) of this work stems from a notable gap in the existing literature: the absence of a discrete counterpart to the Laplace-Beltrami operator on Riemannian manifolds, which can be effectively used to solve PDEs. We consider that the natural approach to pioneer this field is to first explore one of the simplest non-trivial (i.e., non-Euclidean) scenario, specifically focusing on the 2-dimensional hyperbolic space $\mathbb{H}^2$. To this end, we present two variants of discrete finite-difference operator tailored to this constant negatively curved space, both serving as approximations to the (continuous) Laplace-Beltrami operator within the L^2 framework. Moreover, we prove that the discrete heat equation associated to both aforesaid operators exhibits stability and converges towards the continuous heat-Beltrami Cauchy problem on $\mathbb{H}^2$. Eventually, we illustrate that a discrete Laplacian specifically designed for the geometry of the hyperbolic space yields a more precise approximation and offers advantages from both theoretical and computational perspectives.

In the last part (Chapter 6), we analyze the preservation of asymptotic properties of partially dissipative hyperbolic systems when switching to a discrete setting. We prove that one of the simplest consistent and unconditionally stable numerical methods – the central finite difference scheme – preserves both the asymptotic behaviour and the parabolic relaxation limit of one-dimensional partially dissipative hyperbolic systems which satisfy the Kalman rank condition.

The large time asymptotic-preserving property is achieved by conceiving time-weighted perturbed energy functionals in the spirit of the hypocoercivity theory. For the relaxation-preserving property, drawing inspiration from the observation that solutions in the continuous case exhibit distinct behaviours in low and high frequencies, we introduce a novel discrete Littlewood-Paley theory tailored to the central finite difference scheme. This allows us to prove Bernstein-type estimates for discrete differential operators and leads to a new relaxation result: the strong convergence of the discrete linearized compressible Euler system with damping towards the discrete heat equation, uniformly with respect to the mesh parameter.

The thesis concludes by presenting (in Chapter 7) a collection of smaller, less refined asymptotic results, intended for further development and publication, along with identifying and proposing new research directions inspired by the findings in this thesis. The extension of these asymptotic results to other types of differential equations and the exploration of the potential of the involved methods in various mathematical contexts are considered. Together, these currently unfinished projects and novel research directions have the potential to open up new horizons in mathematical analysis and to make significant contributions towards the theoretical and applied development of the field.

Chapter 2: PRELIMINARIES

This chapter of the thesis is dedicated to a brief presentation of rather classical results that are useful along the thesis.

First, we present some elementary notions of Semigroup Theory, a field of Analysis that forms the basis for the rigorous study of evolution equations. For a detailed introduction in this theory, as well as the rigorous construction of Lebesgue and Sobolev spaces on real intervals with values in a Banach space, the interested reader could consult [14].

Secondly, we provide a brief introduction into Riemannian geometry, differential operators and function spaces on Riemannian manifolds (refer to [25]) and then we recall some classical aspects about the hyperbolic space $\mathbb{H}^N$ and its Riemannian geodesic flow. We begin with a presentation of two models of $\mathbb{H}^N$, each of them to be used when most convenient in specific computations. We refer to [32] for more details about hyperbolic geometry and models.

Eventually, we introduce the semi-discrete one-dimensional Fourier transform and revisit some fundamental properties such as invertibility and Parseval's equality. Subsequently, we use these properties to study the solutions of discrete hyperbolic systems. We refer to [38, Section 2.2] and [36, Chapter 2] for more details concerning this topic.

Chapter 3: LONG TIME ASYMPTOTICS FOR THE HEAT EQUATION WITH HARDY POTENTIAL ON CORNER SPACES

The first original result in this thesis delves into the study of the heat equation with a singular potential on corner spaces $\mathbb{R}^{N,k}_+ := \mathbb{R}^{N-k} \times (0,\infty)^k$. The problem is inspired by the paper of Vazquez and Zuazua [39] which pertains to the case of the entire space $\mathbb{R}^N$. More precisely, we aim to find the asymptotic profile of an equation that describes a joint behaviour between a diffusion process of particles and an accumulation near the corner of the space (i.e., the origin $0_{\mathbb{R}^N}$). The diffusion is simply characterised by the Laplace operator, whereas the accumulation is described in terms on an inverse-square potential $\frac{\lambda}{|x|^2}$, where the parameter λ quantifies the strength of the gain of substance near the origin. The stability of the model requires that this rate of accumulation is less than a certain critical value $\lambda_{N,k} := \left(\frac{N-2}{2}+k\right)^2$, which in turn is the optimal constant of the Hardy inequality in corner spaces – see [37]:

$$\int_{\mathbb{R}^{N,k}_+} |\nabla u|^2 \mathrm{d}x \geq \lambda_{N,k} \int_{\mathbb{R}^{N,k}_+} \frac{u^2}{|x|^2}\mathrm{d}x, \quad \forall u \in C^\infty_c(\mathbb{R}^{N,k}_+). \tag{3.1}$$

This inequality leads to the well-posedness of the Cauchy problem that quantifies the phenomenon described in the preceding paragraph – see [5]:

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \frac{\lambda}{|x|^2} u(t,x), & t>0, x\in \mathbb{R}^{N,k}_+, \\ u(t,x) = 0, & t>0,\ x\in \partial\mathbb{R}^{N,k}_+, \\ u(0,x) = u_0(x), & x\in \mathbb{R}^{N,k}_+, \end{cases} \tag{3.2}$$

where $u_0 \in L^2(\mathbb{R}^{N,k}_+)$ and $\lambda \in (-\infty, \lambda_{N,k}]$. Although the problem is well-posed for any initial datum $u_0 \in L^2(\mathbb{R}^{N,k}_+)$ and, additionally, the $L^2(\mathbb{R}^{N,k}_+)$ norm of the solution does not increase in time, this is all we can say about the long-time behaviour of $\|u(t)\|_{L^2(\mathbb{R}^{N,k}_+)}$. Therefore, decay estimates can be obtained only if we restrict the initial datum to a subspace of $L^2(\mathbb{R}^{N,k}_+)$. Thus, inspired by [22, 39], we consider the weighted L^2 space:

$$L^2(\mathbb{R}^{N,k}_+; K) := \left\{ u\in L^2(\mathbb{R}^{N,k}_+) \ \Big| \ \int_{\mathbb{R}^{N,k}_+} |u(x)|^2 K(x)\mathrm{d}x < \infty \right\}, \quad K(x) := e^{\frac{|x|^2}{4}},$$

for which we obtain the sharp polynomial decay for the solutions of (3.2), together with the asymptotic profile which, in particular, describes how the information accumulates near the origin, for a very large time. These results are summarised in the following:

Theorem 3.1. *For any initial data $u_0 \in L^2(\mathbb{R}^{N,k}_+; K)$, the solution u of problem* (3.2) *satisfies:*

$$\|u(t)\|_{L^2(\mathbb{R}^{N,k}_+)} \le (1+t)^{-\frac{1+m_\lambda}{2}} \|u_0\|_{L^2(\mathbb{R}^{N,k}_+;K)}, \quad \forall t\ge 0, \tag{3.3}$$

where $m_\lambda := \sqrt{\lambda_{N,k} - \lambda}$.

Moreover, this polynomial decay rate is sharp and the asymptotic profile is described by the following convergence:

$$\lim_{t\to\infty} t^{\frac{1+m_\lambda}{2}} \left\| u(t) - \beta\, t^{-(1+m_\lambda)} e^{-\frac{|x|^2}{4t}} e^{\frac{|x|^2}{8}} \alpha_{k,\lambda} \right\|_{L^2(\mathbb{R}^{N,k}_+)} = 0, \tag{3.4}$$

where the function $\alpha_{k,\lambda} \in C^\infty(\mathbb{R}^{N,k}_+)$ defined as

$$\alpha_{k,\lambda}(x) := e^{-\frac{|x|^2}{8}} |x|^{m_\lambda - \frac{N-2}{2}} \frac{x_{N-k+1}x_{N-k+2}\cdots x_N}{|x|^k}, \tag{3.5}$$

is the first eigenfunction of the following elliptic eigenvalue problem:

$$\Delta\phi(x) + \left(\frac{\lambda}{|x|^2} - \frac{|x|^2}{16} \right) \phi(x) = \mu\, \phi(x), \quad \forall x \in \mathbb{R}^{N,k}_+, \tag{3.6}$$

with the corresponding eigenvalue $\mu_1 = \frac{1+m_\lambda}{2}$.

Also, β is a normalisation constant which depends on the initial datum u_0:

$$\beta := \left(\|\alpha_{k,\lambda}\|_{L^2(\mathbb{R}_+^{N,k})}\right)^{-1} \int_{\mathbb{R}_+^{N,k}} u_0(x) e^{\frac{|x|^2}{8}} \alpha_{k,\lambda}(x) \mathrm{d}x.$$

The construction of an appropriate functional framework for the study of the problem (3.2) with $L^2(\mathbb{R}_+^{N,K};K)$ initial data, the proof of Theorem 3.1 and other connected results are covered in Chapter 3 of this thesis, which is, in turn, based on my co-authored published article [13].

Chapter 4: NON-LOCAL TO LOCAL ASYMPTOTICS IN NON-EUCLIDEAN SETTING

In this chapter we study a type of asymptotics between operators which encode different behaviours in terms of the process they describe: a local motion versus a non-local movement of particles. While local PDEs such as the heat equation describe particles that instantaneously move only in a neighbourhood of their initial location, the non-local model that we escribe quantifies instant jumps of the particles between any points in the space; see [2]. This non-local motion is characterised by several (possibly non-symmetric) kernels $G_i(x,y)$ which account for the probability of a jump between the two positions x and y in space.

The second novel result in this thesis is a non-Euclidean counterpart of the convergence results in [30, 29]. The major change is that the space we work on is the hyperbolic space, which is not flat anymore, having constant sectional curvature -1. Although the N-dimensional hyperbolic space $\mathbb{H}^N$ is one of the simplest examples of non-Euclidean geometries, it already poses some important challenges even in terms of the formulation of the non-local problem. For instance, a task as simple as defining a kernel $G(x,y)$ depending only on the difference $y-x$ in the Euclidean setting becomes non-trivial when the curvature is not zero. In order to overcome this issue, we make use of the notion of *geodesic flow* on a Riemannian manifold and later transfer the computations from the manifold to tangent spaces via the exponential mapping.

To state the main result of this part, we need to briefly describe the hyperbolic space $\mathbb{H}^N$ and its geodesic flow. For simplicity, we will only consider a model of this space (i.e., the half-space model), referring to [32] for a more detailed presentation of the notions involved. The supporting set for this model is $\mathbb{H}^N \simeq \mathbb{R}^N_+ := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times (0, \infty)\}$, with the Riemannian metric defined by:

$$g_{ij}(x) = \frac{1}{x_N^2}\delta_{ij}, \quad i, j = \overline{1, N}.$$

In this setting, the expressions of the Riemannian gradient, divergence and Laplacian are:

$$\nabla_g f = x_N^2 \nabla_e f, \qquad \operatorname{div}_g(Y) = x_N^N \operatorname{div}_e\left(\tfrac{1}{x_N^N} Y\right), \qquad \Delta_g f = x_N^N \operatorname{div}_e\left(\tfrac{1}{x_N^{N-2}} \nabla_e f\right),$$

where ∇_e and div_e are the usual Euclidean gradient and divergence operators.

The volume form on $\mathbb{H}^N$ and on its tangent space at a point x become $\mathrm{d}\mu(x) = x_N^{-N}\,\mathrm{d}x$, respectively $\mathrm{d}\mu(V) = x_N^{-N}\,\mathrm{d}V$, meaning that the integration of functions on $\mathbb{H}^N$ and $T_x\mathbb{H}^N$ is given by the following formulas:

$$\int_{\mathbb{H}^N} f(x)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}^N_+} f(x)\frac{1}{x_N^N}\,\mathrm{d}x \quad \text{and} \quad \int_{T_x\mathbb{H}^N} f(V)\,\mathrm{d}\mu(V) = \frac{1}{x_N^N}\int_{\mathbb{R}^N} f(V)\,\mathrm{d}V.$$

The geodesics (i.e., the shortest-length curves) in this models are the straight Euclidean vertical half-lines (i.e., x' is constant) and the semicircles centred on the base $\{x_N = 0\}$ which are perpendicular on the base. For every two distinct points $x, y \in \mathbb{H}^N$, there exists exactly one (unparametrised) geodesic passing through them – see Figure 3.1. Moreover, given any vector V in the tangent space $T_x\mathbb{H}^N$, there is a unique parametrised geodesic $\gamma_{x,V} : \mathbb{R} \to \mathbb{H}^N$ such that $\gamma_{x,V}(0) = x$ and $\gamma'_{x,V}(0) = V$. If $\gamma_{x,V}(1) = y$, then we call $V_{x,y} := V$, meaning that the vector V transports through geodesics the point x to the point y. This vector $V_{x,y} \in T_x\mathbb{H}^N$ is the natural equivalent of the Euclidean vector $y - x$. We refer again to Figure 3.1.

Further, we define the geodesic flow of, $\mathbb{H}^N$ $(\Phi_t)_{t\in\mathbb{R}}$ which acts on the tangent bundle $T\mathbb{H}^N$ as follows:

$$\Phi_t(x, V) = (\gamma_{x,V}(t), \gamma'_{x,V}(t)) \in T\mathbb{H}^N,\ \forall (x, V) \in T\mathbb{H}^N.$$

Now, we state the first main theorem in this section, concerning the non-local transport problem on $\mathbb{H}^N$.

Theorem 3.2. *Let $\varepsilon_0 > 0$, $u_0 \in L^2(\mathbb{H}^N)$ and the family of scaled kernels $G_\varepsilon : \mathbb{H}^N \times \mathbb{H}^N \to [0, \infty)$, $\varepsilon \in (0, \varepsilon_0)$ satisfying:*

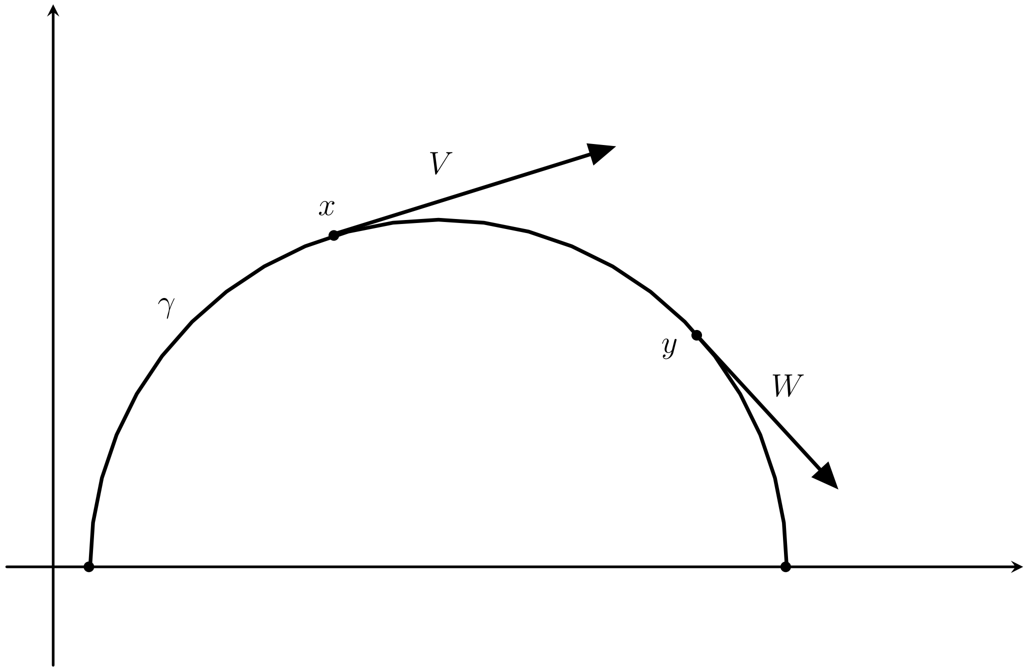

Figure 3.1: A geodesic γ through the points x and y in the half-space model, tangent to the vectors V and W.

- $G_\varepsilon(x,y) = \varepsilon^{-N-1}\widetilde{G}\left(x, \frac{V_{x,y}}{\varepsilon}\right)$, *where* $\widetilde{G} : T\mathbb{H}^N \to [0,\infty)$ *is invariant to the geodesic flow* $(\Phi_t)_{t\in\mathbb{R}}$*, meaning that:*

$$\widetilde{G}(\Phi_t(x,V)) = \widetilde{G}(x,V), \quad \forall t \in \mathbb{R} \text{ and } (x,V) \in T\mathbb{H}^N.$$

- *The following integrability condition holds true:*

$$\int_0^\infty \sup_{x\in\mathbb{H}^N, |W|=r} \widetilde{G}(x,W)(1+r)\left(e^r \sinh(r)\right)^{N-1} \mathrm{d}r < \infty,$$

where $|W|$ *stands for the hyperbolic norm of the tangent vector* $W \in T_x\mathbb{H}^N$.

- *The* first moment vector field X_G *on* $\mathbb{H}^N$*:*

$$X_G(x) \coloneqq -\int_{T_x\mathbb{H}^N} \widetilde{G}(x,W)W\mathrm{d}W, \forall x \in \mathbb{H}^N \tag{3.7}$$

is of class C^1.

Then, the family of solutions $(u^\varepsilon)_{\varepsilon>0}$ *of the non-local problems:*

$$\begin{cases} \partial_t u^\varepsilon(t,x) = \displaystyle\int_{\mathbb{H}^N} G_\varepsilon(x,y)(u^\varepsilon(t,y) - u^\varepsilon(t,x))\mathrm{d}\mu(y), & x \in \mathbb{H}^N, t \geq 0; \\ u^\varepsilon(0,x) = u_0(x), & x \in \mathbb{H}^N. \end{cases} \tag{3.8}$$

converges weakly in $L^2([0,T], L^2(\mathbb{H}^N))$, *as* $\varepsilon \to 0$, *to the unique distributional solution of the local transport problem:*

$$\begin{cases} \partial_t u(t,x) = -\mathrm{div}_g(u(t)X_G)(x), & x \in \mathbb{H}^N, t \geq 0; \\ u(0,x) = u_0(x), & x \in \mathbb{H}^N. \end{cases}$$

In the following, we will provide a similar asymptotic result for a more general non-local non-linear problem driven by two kernels, one of them being the transport kernel G_ε, whereas the other one (denoted by $\widetilde{J}_\varepsilon$) is responsible for the spread of particles evenly in all directions (non-local diffusion). For this reason, the kernel $\widetilde{J}_\varepsilon$ depends only on the geodesic distance $d_g(x,y)$ between the points $x, y \in \mathbb{H}^N$. We refer to [4] for the study of a similar problem involving only the non-local diffusion kernel $\widetilde{J}_\varepsilon$.

Theorem 3.3. *Let all the assumptions of Theorem 3.2 in place. Given the continuous function* $J : [0,\infty) \to [0,\infty)$ *that satisfies* $J(0) > 0$ *and*

$$\int_0^\infty J(r)\,(1+r^2)\,(e^r \sinh(r))^{N-1}\,\mathrm{d}r < \infty,$$

we consider the isotropic scaled kernel $\widetilde{J}_\varepsilon : \mathbb{H}^N \times \mathbb{H}^N \to [0,\infty)$,

$$\widetilde{J}_\varepsilon(x,y) = \varepsilon^{-N-2} J\left(\frac{d_g(x,y)}{\varepsilon}\right).$$

Further, for $q \geq 1$, *we set* $f(r) := |r|^{q-1}r$ *and let* u^ε *be the unique solution of the following non-local non-linear problem:*

$$\begin{cases} \partial_t u^\varepsilon(t,x) = \displaystyle\int_{\mathbb{H}^N} \widetilde{J}_\varepsilon(x,y)(u^\varepsilon(t,y) - u^\varepsilon(t,x))\mathrm{d}\mu(y) & \\ \qquad\qquad + \displaystyle\int_{\mathbb{H}^N} G_\varepsilon(x,y)(f(u^\varepsilon(t,y)) - f(u^\varepsilon(t,x)))\mathrm{d}\mu(y), & x \in \mathbb{H}^N, t \geq 0; \\ u(0,x) = u_0(x), & x \in \mathbb{H}^N. \end{cases}$$

Then, the family $(u^\varepsilon)_{\varepsilon>0}$ *converges weakly in* $L^2([0,T], L^2(\mathbb{H}^N))$ *and strongly in* $L^2([0,T], L^2_{loc}(\mathbb{H}^N))$ *to the unique weak solution of the local convection-diffusion problem:*

$$\begin{cases} \partial_t u(t,x) = A_J \Delta_g u(t,x) - \mathrm{div}_g(f(u(t))X_G)(x), & x \in \mathbb{H}^N, t \geq 0; \\ u(0,x) = u_0(x), & x \in \mathbb{H}^N, \end{cases}$$

where the diffusivity constant A_J *is:*

$$A_J := \frac{1}{2N}\mathrm{Area}(\mathbb{S}^{N-1}) \int_0^\infty J(r) r^{N+1}\,\mathrm{d}r,$$

$\mathrm{Area}(\mathbb{S}^{N-1})$ *stands for the Euclidean surface area of the Euclidean unit sphere and the first moment vector field* X_G *is given in* (3.7).

The detailed proof of the above theorems, together with the construction of a large class of kernels G_ε with the required properties is the subject of Chapter 4 of this thesis, which is based on the published paper [24].

Chapter 5: DISCRETE TO CONTINUOUS ASYMPTOTICS IN NON-EUCLIDEAN SETTING

The next original result in this thesis is a finite difference numerical approximation in the framework of a Riemannian manifold. Since the scientific literature lacks references about this type of numerical method in Riemannian context (the only existing schemes for numerical solutions of equations on manifolds are finite element [27], finite volume [1, 23] and Monte Carlo [20] approximations), we have started with one of the simplest examples, namely the 2-dimensional hyperbolic space $\mathbb{H}^2$. We present two variants of discrete finite difference Laplace operator, then we perform a comparison between them, in terms of the accuracy of approximation and computational efficiency. It will be shown that using a finite difference grid which is better adapted to the geometry of the hyperbolic space leads to a more accurate approximation, together with an improvement in the usage of computational resources.

Each of the two discrete Laplacians is constructed, similar to the case of the Euclidean plane, as a linear combination of the values of the function in five adjacent points of a numerical grid, their weights accounting for the even diffusion of heat across the curved space. We will prove the effectiveness of these discrete operators in building numerical schemes that can be implemented on a computer in order to approximate the solutions of the heat equation with source on the hyperbolic space $\mathbb{H}^2$:

$$\begin{cases} \partial_t u(t,\mathbf{x}) = \Delta_g u(t,\mathbf{x}) + f(t,\mathbf{x})\,; & t \in (0,T],\ \mathbf{x} \in \mathbb{H}^2\,, \\ u(0,\mathbf{x}) = u_0(\mathbf{x})\,, & \mathbf{x} \in \mathbb{H}^2, \end{cases} \tag{5.9}$$

where $T > 0$ is any fixed time, Δ_g is the Laplace-Betrami operator on $\mathbb{H}^2$, $u_0 \in L^2(\mathbb{H}^2)$ is the initial data and $f \in C([0,T], \mathrm{L}^2(\mathbb{H}^2))$ is the source term.

Seeking to construct a numerical scheme to approximate the heat-Beltrami equation, we focus ourselves on a particular model of the 2-dimensional hyperbolic space, i.e. the half-plane model, which is suitable for embedding a finite difference grid. The metric and differential operators in this model, together with the integration formula for functions were presented in the previous chapter. We only recall the formula of the Laplace-Beltrami operator:

$$\Delta_g v(\mathbf{x}) = x_2^2(\partial_{x_1}^2 v + \partial_{x_2}^2 v)(\mathbf{x}), \text{ where } v \in C^2(\mathbb{H}^2) \text{ and } \mathbf{x} = (x_1, x_2) \in \mathbb{H}^2.$$

The effectiveness of the finite difference approximation of differential operators – which uses only a discrete set of values of the functions involved – requires a certain regularity of these functions. In order to obtain the convergence of the numerical scheme to the solution of the continuous problem (5.9), we impose the following restrictions on the initial datum u_0 and on the source term f:

$$u_0 \in \mathcal{M} := \left\{ v \in C^6(\mathbb{H}^2) : \|v\|_{\mathcal{M}} := \sum_{\substack{\alpha,\beta\in\mathbb{N} \\ \alpha+\beta\leq 6}} \left\| e^{18 d_g(\mathrm{O},\mathbf{x})} \partial_{x_1}^{\alpha} \partial_{x_2}^{\beta} v(\mathbf{x}) \right\|_{\mathrm{L}^2(\mathbb{H}^2)} < \infty \right\} \quad (5.10)$$

and $f \in C([0,T], \mathcal{M})$, where O is an arbitrary point in $\mathbb{H}^2$. These conditions are necessary to obtain the convergence result and can also be made invariant to the particular model of $\mathbb{H}^2$. We note that, essentially, the weighted Sobolev norm in (5.10) characterises that the functions we are interested in decay sufficiently well for large space variable $\mathbf{x}$. This allows us to use the problem on a bounded domain as an effective approximation of the problem (5.9) on the whole hyperbolic space, as seen below.

As in the Euclidean case, four main ingredients are involved in the construction of the discrete counterpart of Δ_g: a finite difference grid, the function space associated to this mesh, the projection of continuous functions onto the grid and the discrete Laplacian itself. In what follows, we will construct two variants of discrete Laplace operator and describe the aforementioned elements in each of the two cases.

The first discrete Laplacian

The associated grid is the traditional Euclidean one, whose points are depicted in Figure 5.2 and consist of pairs of integral multiples of the mesh parameter $h > 0$:

$$(ih, jh), \text{ for } i \in \mathbb{Z}, j \in \mathbb{N}^*.$$

Around each of those points, we construct the finite difference cell:

$$\mathcal{C}_h^{i,j} := \left[ih - \frac{h}{2}, ih + \frac{h}{2}\right] \times \left[jh - \frac{h}{2}, jh + \frac{h}{2}\right]$$

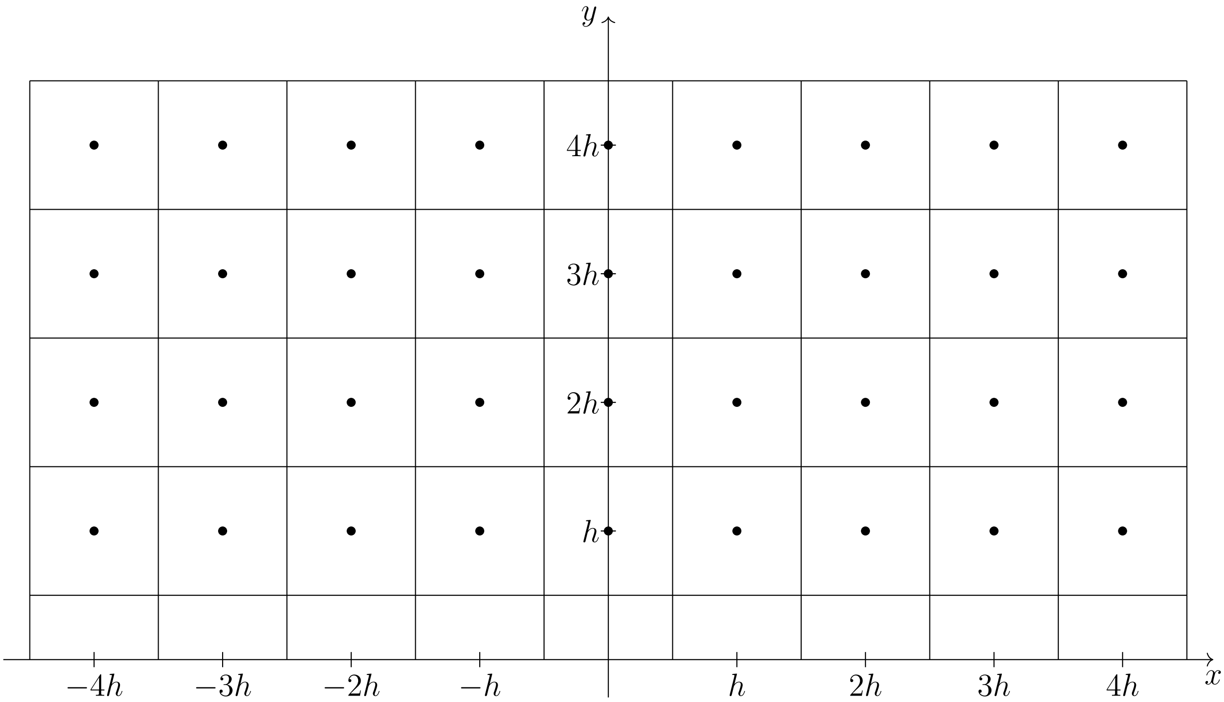

Figure 5.2: The grid corresponding to the first discrete Laplacian on $\mathbb{H}^2$.

with a hyperbolic area equal to $\frac{1}{j^2-\frac{1}{4}}$ and we introduce the grid functions space

$$\ell_h^2 := \left\{ \left(v_{i,j}^h\right)_{i\in\mathbb{Z},\, j\in\mathbb{N}^*} : \|v^h\|_{\ell_h^2}^2 := \sum_{i\in\mathbb{Z},\, j\in\mathbb{N}^*} \frac{1}{j^2-\frac{1}{4}} |v_{i,j}^h|^2 < +\infty \right\}$$

and the projection operator $\Pi_h : \mathrm{L}^2(\mathbb{H}^2) \to \ell_h^2$,

$$(\Pi_h v)_{i,j} = \left(j^2 - \frac{1}{4}\right) \int_{\mathcal{C}_h^{i,j}} v(\mathbf{x}) \frac{1}{x_2^2} \mathrm{d}\mathbf{x}.$$

Eventually, the discrete Laplace operator corresponding to this setup is obtained via Taylor expansions and has the following form for $v^h \in \ell_h^2$:

$$(\Delta_h^{(1)} v^h)_{i,j} := \left(j^2 - \frac{1}{4}\right) \left(v_{i+1,j}^h + v_{i-1,j}^h + v_{i,j+1}^h + v_{i,j-1}^h - 4v_{i,j}^h\right) \quad \forall\, i \in \mathbb{Z},\ j \in \mathbb{N}^*, \tag{5.11}$$

where we employ the convention $v_{i,0}^h = 0,\ \forall\, i \in \mathbb{Z}$.

The second discrete Laplacian

For the second variant of discrete Laplace operator, the associated grid is the one in Figure 5.3 and is tailored to the geometry of the hyperbolic space. More precisely, the hyperbolic distance between any adjacent nodes in the vertical direction is the

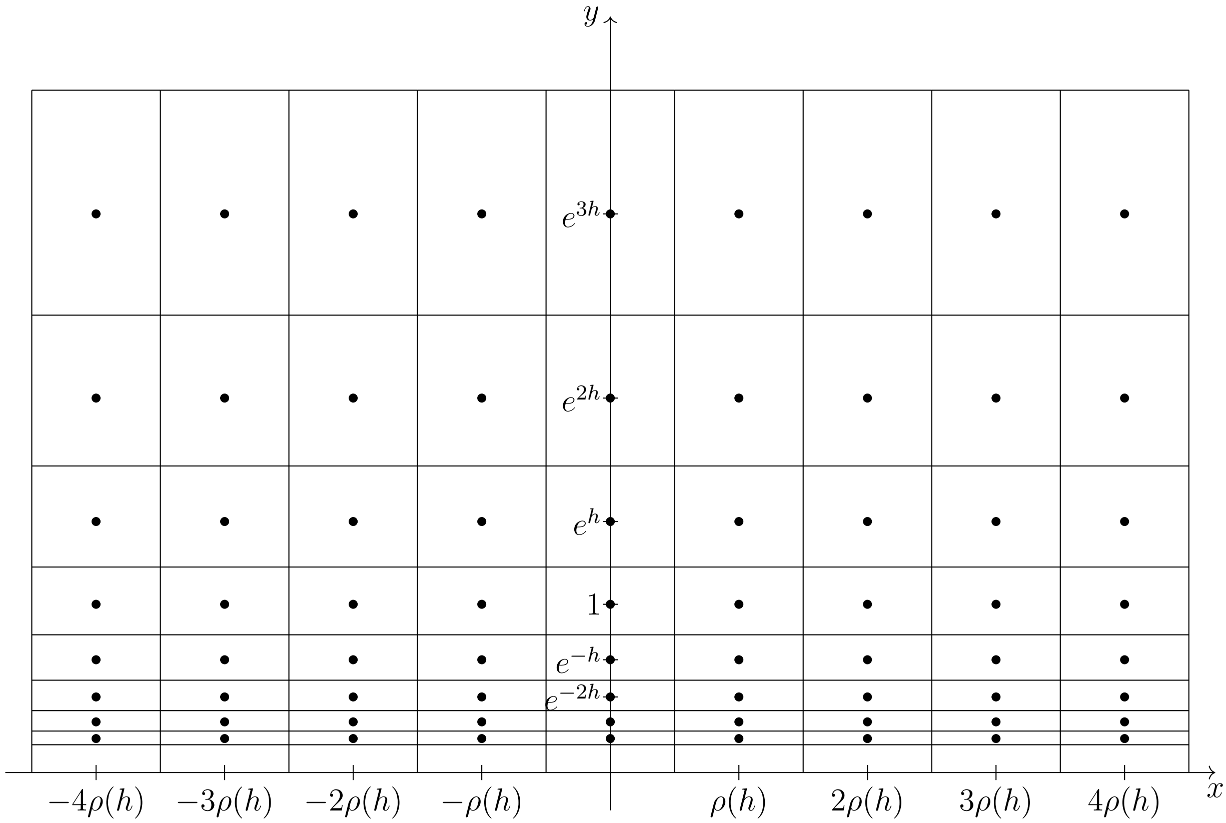

Figure 5.3: The grid corresponding to the second discrete Laplacian on $\mathbb{H}^2$, where we denote $\rho(h) := 2\sinh\left(\frac{h}{2}\right)$.

same, namely equal to the mesh parameter h. However, on the horizontal direction, the distances between consecutive nodes must increase as we approach the baseline $\{x_2 = 0\}$, because of the geometry of the hyperbolic space.

We have chosen the grid such that the hyperbolic distance between adjacent nodes in the horizontal direction on the level $\{x_2 = 1\}$ is equal to one; therefore the nodes are of the form $(i\rho(h), e^{jh}) \in \mathbb{H}^2$, $\forall i, j \in \mathbb{Z}$, where $\rho(h) := 2\sinh\left(\frac{h}{2}\right)$. Around each node of the grid, we define the finite difference cell:

$$\mathcal{C}_h^{i,j} := \left[\left(i - \frac{1}{2}\right)\rho(h), \left(i + \frac{1}{2}\right)\rho(h)\right] \times \left[e^{jh-\frac{h}{2}}, e^{jh+\frac{h}{2}}\right].$$

with the hyperbolic area equal to $\frac{(\rho(h))^2}{e^{jh}}$. The space of grid function is then defined as

$$\ell_h^2 := \left\{ \left(v_{i,j}^h\right)_{i,j\in\mathbb{Z}} : \|v^h\|_{\ell_h^2}^2 := \sum_{i,j\in\mathbb{Z}} (\rho(h))^2 e^{-jh} |v_{i,j}^h|^2 < +\infty \right\},$$

and the projection operator used to transfer information from the space $\mathrm{L}^2(\mathbb{H}^2)$ to the grid has the form:

$$(\Pi_h v)_{i,j} = \frac{e^{jh}}{(\rho(h))^2} \int_{\mathcal{C}_h^{i,j}} v(\mathbf{x}) \frac{1}{x_2^2} \mathrm{d}\mathbf{x}.$$

Finally, the second discrete Laplace operator defined on the space ℓ_h^2, obtained via Taylor expansions around the nodes $(i\rho(h), e^{jh})$, has the formula:

$$(\Delta_h^{(2)} v^h)_{i,j} := \frac{1}{(\rho(h))^2} \left[e^{2jh}(v_{i+1,j}^h + v_{i-1,j}^h - 2v_{i,j}^h) + \frac{2}{e^h+1} v_{i,j+1}^h + \frac{2e^h}{e^h+1} v_{i,j-1}^h - 2v_{i,j}^h \right]. \tag{5.12}$$

Reduction to a bounded domain

As we want to implement our numerical scheme on a computer (which, of course, has a limited amount of memory and processing speed), we need to reduce the infinite grids corresponding to $\Delta_h^{(1)}$ and $\Delta_h^{(2)}$ to bounded counterparts. The bounded domain $\mathbb{H}_D^2$ is constructed in order to grow as the mesh parameter h decreases to zero, in the following way:

$$\mathbb{H}_D^2 := [-D, D] \times [1/D, D] \subset \mathbb{H}^2 \tag{5.13}$$

where the size of the domain is chosen as $D = D_h := \frac{1}{h}$. In this way, as h approaches zero, the resulting bounded grid will not only refine, but also cover a larger portion of $\mathbb{H}^2$.

Next, since the heat equation on $\mathbb{H}^2$ has sufficient tail decay, we can use zero Dirichlet boundary conditions at the nodes lying on the boundary of $\mathbb{H}_D^2$ and obtain the discrete Laplace operators $\Delta_{h,D}^{(1)}$ and $\Delta_{h,D}^{(2)}$ with finite-dimensional domains. We note that the grid function space ℓ_h^2 and the projection operator Π_h are restricted accordingly in each case.

Semi-discrete finite difference scheme and convergence result

To build the semi-discrete numerical scheme, we consider $((\mathcal{C}_h^{i,j})_{(i,j)\in Z_1\times Z_2}, \ell_h^2, \Pi_h, \Delta_h)$ to be either of the two discrete Laplace operators, restricted to $\mathbb{H}_D^2$, with zero Dirichlet boundary conditions, where $D = D_h := \frac{1}{h}$, as discussed above. We note that the finite sets of integers Z_1 and Z_2, by which we index the grid cells, increase as h approaches zero.

Then, for u_0 and f satisfying Hypothesis (5.10), we consider the semi-discrete approximation of the heat-Beltrami equation with source (5.9):

$$\begin{cases} \partial_t u^h_{i,j}(t) = (\Delta_h u^h(t))_{i,j} + (\Pi_h(f(t)))_{i,j}\,, & t \in (0,T],\ (i,j) \in Z_1 \times Z_2; \\ u^h_{i,j}(0) = (\Pi_h(u_0))_{i,j}, & (i,j) \in Z_1 \times Z_2. \end{cases} \tag{5.14}$$

The order of convergence of this numerical scheme is given by the following theorem:

Theorem 5.4. *For every initial datum u_0 and source f satisfying Hypothesis* (5.10)*, there exists a constant $C_T > 0$ such that, for every $h \in (0, 1/2)$ and $t \in [0,T]$,*

$$\|u^h(t) - \Pi_h u(t)\|_{\ell^2_h} \leq h^2 C_T \left(\|u_0\|_{\mathcal{M}} + \|f\|_{C([0,T],\mathcal{M})}\right),$$

where u^h is the solution of (5.14) *and u is the solution of* (5.9)*.*

The proof of this theorem can be found in Chapter 5 of the thesis, in a more general setting where $D = D_{h,\gamma,\gamma} \coloneqq \zeta h^{-\gamma}$, with $\zeta > 2$ and $\gamma > 0$. We outline that the same order of convergence $\mathcal{O}(h^2)$ is valid when we discretise also the time, leading to a θ-scheme with the parameter $\theta \in [1/2, 1]$. This quadratic convergence order is experimentally proven to be sharp.

Moreover, the exponential long-time L^2 decay rate of the solutions of the homogeneous heat-Beltrami equation on $\mathbb{H}^2$ (i.e., equation (5.9) with $f \equiv 0$ and $T = \infty$) is preserved asymptotically by our finite difference scheme. More precisely, the optimal constant of the Poincaré inequality associated to both discrete Laplace operators $\Delta_h^{(1)}$ and $\Delta_h^{(2)}$ approaches, as h tends to zero, the value $\frac{1}{4}$, which is the sharp constant of the Poincaré inequality in the whole space $\mathbb{H}^2$ [35].

The full development of the ideas presented succinctly here is the subject of Chapter 5 of this thesis. The results have been submitted to be considered for publication [9].

Chapter 6: PUSHING ALL THE LIMITS: LONG-TIME, DISCRETE TO CONTINUOUS AND HYPERBOLIC TO PARABOLIC ASYMPTOTICS FOR PARTIALLY DISSIPATIVE HYPERBOLIC SYSTEMS

The last main original result in this thesis contains all classes of asymptotics described in this thesis, proving that, at least in some particular frameworks, there is a strong bond between all these types of approximation results. We will consider partially dissipative hyperbolic systems (systems of hyperbolic equations that have only some of the terms damped [15, 26, 8]) which satisfy a non-degeneracy property called *the Kalman rank condition* [7] and we emphasise that one of the most classical numerical schemes (based on the central finite difference operator [36]) not only approximates the solution of the continuous system, but also preserves both the long-time and hyperbolic-to-parabolic asymptotics inherent to the continuous model.

To describe the model, we refer to [7] and consider a positive integer $N \geq 2$, together with two symmetric matrices A and B such that B is *partially dissipative.* This means that B has the form

$$B = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{B} \end{pmatrix}, \tag{6.15}$$

where $\widetilde{B}$ is a positive definite symmetric $N_2 \times N_2$ matrix ($1 \leq N_2 < N$).

Let also $U^0 : \mathbb{R} \to \mathbb{R}^N$ be an L^2 function and the vector-valued unknown $U : [0, \infty) \times \mathbb{R} \to \mathbb{R}^N$ satisfy the following hyperbolic system:

$$\begin{cases} \partial_t U(t,x) + A\partial_x U(t,x) = -BU(t,x), & (t,x) \in (0,\infty) \times \mathbb{R}; \\ U(0,x) = U^0(x), & x \in \mathbb{R}. \end{cases} \tag{6.16}$$

We remark that, since the matrix B has the particular form (6.15), the damping term only acts on the last N_2 components U_2 of the solution vector $U = (U_1, U_2)$. However, if the matrices A and B satisfy the following non-degeneracy condition (the *Kalman* *rank condition*):

$$\text{the matrix } \mathcal{K}(A,B) := (B|AB|\dots|A^{N-1}B) \quad \text{has full rank } N, \tag{K}$$

then Crin-Barat, Shou and Zuazua [18] have obtained a decay result for the entire solution U of (6.16):

Proposition 6.5 ([18, Theorem 2.1])**.** *If $U^0 \in (H^1(\mathbb{R}))^N$, then the solution U of* (6.16) *satisfies:*

$$\|U_2(t)\|_{L^2(\mathbb{R})} + \|\partial_x U(t)\|_{L^2(\mathbb{R})} \leq C(1+t)^{-\frac{1}{2}}\|U^0\|_{H^1(\mathbb{R})}, \tag{6.17}$$

where $C > 0$ is a constant independent of time and U_0.

Our first main result in this chapter of the thesis is a discrete counterpart of Proposition 6.5. We consider the semi-discrete central finite difference scheme for the problem (6.16):

$$\begin{cases} \partial_t(U(t))_n + A(\mathcal{D}_h U(t))_n = -B(U(t))_n, & (t,n) \in (0,\infty)\times\mathbb{Z}; \\ (U(0))_n = (U^0)_n, & n \in \mathbb{Z} \end{cases} \tag{6.18}$$

where now the initial datum and the solution are defined on an infinite grid of width $h > 0$. More precisely, $U^0 = (U^0_n)_{n\in Z} : \mathbb{Z} \to \mathbb{R}^N$, $U = (U_n)_{n\in Z} : (0,\infty)\times\mathbb{Z} \to \mathbb{R}^N$ and the operator D_h acts on a bilateral sequence $v : \mathbb{Z} \to \mathbb{R}$ as follows:

$$(\mathcal{D}_h v)_n = \frac{v_{n+1} - v_{n-1}}{2h}.$$

We state the first main result in this chapter, namely the conservation of the long-time decay (6.17) in this discrete context:

Theorem 6.6. *There exists a constant $C > 0$ depending only on the matrices A and B such that, for every $h > 0$, every $U^0 \in (h^1_h)^N$ and $t > 0$, the following decay result holds:*

$$\|U_2(t)\|_{\ell^2_h} + \|\mathcal{D}_h U(t)\|_{\ell^2_h} \leq C(1+t)^{-\frac{1}{2}}\|U^0\|_{h^1_h},$$

where $\|\cdot\|_{\ell^2_h}$ and $\|\cdot\|_{h^1_h}$ are discrete Lebesgue and Sobolev norms.

The second type of asymptotics preserved by the central finite difference scheme in the context of partially dissipative systems refers to the relaxation of such a system towards a parabolic equation. A simpler, yet illustrative example in this sense is the following relaxed system:

$$\begin{cases} \partial_t \rho^\varepsilon(t,x) + \partial_x u^\varepsilon(t,x) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}; \\ \varepsilon^2 \partial_t u^\varepsilon(t,x) + \partial_x \rho^\varepsilon(t,x) + u^\varepsilon(t,x) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}; \\ (\rho^\varepsilon, u^\varepsilon)(0,x) = (\rho_0, u_0)(x), & x \in \mathbb{R}, \end{cases} \tag{6.19}$$

which can be seen as a linearised version of the compressible Euler system with damping (refer to [17, Introduction]). We note that the functions ρ and u are real-valued functions and resemble the density and speed of a flow on the line. When the relaxation parameter ε is equal to one, the system (6.19) fits into the framework of (6.16)-(K).

Following the argumentation of Crin-Barat and Danchin [17] one obtains easily the strong convergence of the solution of the system (6.19) towards the solution of the heat equation:

Proposition 6.7. *Assume the initial data $\rho_0, u_0 \in H^{s'}(\mathbb{R})$ for some $s' > 2$. Then, for every $s \in (2, s')$, the first component ρ^ε of the solution of (6.19) converges strongly as $\varepsilon \to 0$ in the spaces $L^\infty([0,\infty), \dot{H}^{s-2}(\mathbb{R}))$ and $L^1([0,\infty), \dot{H}^s(\mathbb{R}))$ to the solution ρ of the heat equation on $\mathbb{R}$:*

$$\begin{cases} \partial_t \rho(t,x) - \partial_{xx} \rho(t,x) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}; \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}. \end{cases}$$

Moreover, for small ε, the second component u^ε approximates the derivative of the first component ρ^ε, in the sense that the sum $(\partial_x \rho^\varepsilon + u^\varepsilon)$ converges to zero in $L^1([0,T], \dot{H}^{s-1}(\mathbb{R}))$. We note that the homogeneous Sobolev norm [3, Section 1.3] is defined as $\|v\|_{\dot{H}^s(\mathbb{R})} := \int_{\mathbb{R}} |\hat{v}(\xi)|^2 |\xi|^{2s} \mathrm{d}\xi$.

Our second main result in this chapter of the thesis establishes that this relaxation limit remains valid – uniformly with respect to the grid width h – even when transitioning to the discrete formulations of both the linearised Euler system and the heat equation. In order to achieve the uniformity, we need a method to essentially use the same initial data regardless of the parameters ε and h. In this sense we introduce, as in [36, Chapter 10], the truncation operator $\mathcal{T}_h : L^2(\mathbb{R}) \to \ell^2_h$,

$$(\mathcal{T}_h v)_n = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{i\xi nh} \hat{v}(\xi) \mathrm{d}\xi.$$

We note that the operator $\mathcal{T}_h$ is defined such that the continuous Fourier transform of v and the discrete Fourier transform of $\mathcal{T}_h v$ coincide on the common domain $\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$, thus the name *truncation operator*. In the statement of the second main result below, the initial data of the discrete equations are obtained by truncating functions defined on the real line.

Theorem 6.8. *For $h > 0$, $s' > 2$ and $\widetilde{\rho_0}, \widetilde{u_0} \in H^{s'}(\mathbb{R})$, we consider the truncations $\rho_0 = \mathcal{T}_h\widetilde{\rho_0}$ and $u_0 = \mathcal{T}_h\widetilde{u_0}$ as initial data for the discrete counterpart of the linearised Euler system* (6.19)*:*

$$\begin{cases} \partial_t \rho^\varepsilon + \mathcal{D}_h u^\varepsilon = 0, & (t,n) \in (0,\infty) \times \mathbb{Z} \\ \varepsilon^2 \partial_t u^\varepsilon + \mathcal{D}_h \rho^\varepsilon + u^\varepsilon = 0, & (t,n) \in (0,\infty) \times \mathbb{Z} \\ (\rho^\varepsilon, u^\varepsilon)(0) = (\rho_0, u_0), & n \in \mathbb{Z} \end{cases} \tag{6.20}$$

where $\rho^\varepsilon, u^\varepsilon : (0,\infty) \times \mathbb{Z} \to \mathbb{R}$. Then, for every $s \in (2, s')$ the first component ρ^ε of the solution converges in $L^\infty([0,\infty), \dot{h}_h^{s-2})$ and $L^1([0,\infty), \dot{h}_h^s)$, uniformly with respect to $h > 0$, as ε tends to zero, towards the solution ρ of the discrete heat equation:

$$\begin{cases} \partial_t \rho - \mathcal{D}_h^2 \rho = 0, & (t,n) \in (0,\infty) \times \mathbb{Z} \\ \rho(0) = \rho_0, & n \in \mathbb{Z}. \end{cases}$$

Note*: the discrete $\dot{h}_h^s$ Sobolev norm is given by $\|v\|^2_{\dot{h}_h^s} := \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |\hat{v}(\xi)|^2 \left|\frac{\sin(\xi h)}{h}\right|^{2s} \mathrm{d}\xi$.*

Moreover, the quantity $(\mathcal{D}_h\rho^\varepsilon + u^\varepsilon)$ converges to zero in $L^1([0,\infty), \dot{h}_h^{s-1})$, uniformly with respect to the grid width h.

For a sharper version of this relaxation limit – valid in a discrete counterpart of Besov norms – together with error estimates, numerical simulations and other extensions, we refer to Chapter 6 of the thesis and my co-authored paper [19] which has been submitted for publication.

Chapter 7: FURTHER ASYMPTOTIC RESULTS AND ONGOING PROJECTS

In this chapter, we introduce four additional studies that either expand on the previously discussed topics or are closely connected to them.

Asymptotic behaviour of coupled linear convection-diffusion on the real line

This topic fits within the class of long-time asymptotics, similar to Chapter 3. However, unlike the model analysed there, in this section the coefficients governing the evolution process are discontinuous. Specifically, we examine a convection-diffusion problem on the real line, where the particle velocity takes two distinct values, one on each half-line:

$$\begin{cases} u : [0,\infty) \times \mathbb{R} \to \mathbb{R}; & \\ \partial_t u(t,x) = \partial_{xx} u(t,x) - a\partial_x u(t,x), & t > 0, x < 0; \\ \partial_t u(t,x) = \partial_{xx} u(t,x) - b\partial_x u(t,x), & t > 0, x > 0; \\ u(t,0-) = u(t,0+), & t > 0; \\ \partial_x u(t,0+) - \partial_x u(t,0-) = (b-a)u(t,0), & t > 0; \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{7.21}$$

The initial datum u_0 is smooth with compact support and, without losing generality, we assume that the speed parameters satisfy $a < b$ and $b > 0$. Then, the asymptotic profile of (7.21) will either be a shifted Gaussian $\exp\left(\frac{-|x-bt|^2}{4t}\right)$ or a linear combination of two shifted Gaussians $\exp\left(\frac{-|x-bt|^2}{4t}\right)$ and $\exp\left(\frac{-|x-at|^2}{4t}\right)$, depending on the sign of a.

Non-local non-linear convection-diffusion on metric trees

The next study explores a similar non-local to local limit as discussed Chapter 4, but this time the ambient space is a metric graph Γ.

A metric graph is a pair consisting of a set of vertices and one of (oriented) edges, with an additional metric structure given by parametrisations of the edges with intervals of the real line. This parametrisation allows the application of differential operators to functions defined on these graphs, which, together with suitable coupling conditions at vertices, characterise dynamical processes on the network structure.

The description of the following transport-diffusion problem is taken from my GitHub repository [33]: we pose the convection-diffusion system on the metric tree Γ (i.e., the graph has no cycles), with the set of vertices V and the set of edges E:

$$\begin{cases} \partial_t u_e(t,x) - \delta \partial_{xx} u_e(t,x) + \alpha_e \partial_x u_e(t,x) = 0, & x \in e \in E, t \geq 0; \\ u_{e_1}(t,v) = u_{e_2}(t,v), & v \in V_{\text{int}}, e_1, e_2 \in E_v, t \geq 0; \\ \sum_{e \in E_v^{\text{in}}} \partial_x u_e(t,v) = \sum_{e \in E_v^{\text{out}}} \partial_x u_e(t,x), & v \in V_{\text{int}}, t \geq 0; \\ u(0,x) = u_0(x), & x \in \Gamma; \\ u(t,v) = u_v(t), & v \in V_\partial, t \geq 0. \end{cases} \tag{7.22}$$

Here, V_{int} stands for the set of interior vertices, and the set of boundary vertices is denoted by V_∂. The set E_v of the edges adjacent to a vertex v is divided into the set of incoming edges E_v^{in} and the set of outgoing edges E_v^{out}. A function u defined on the graph is a family $(u_e)_{e \in E}$, where $u_e : e \to \mathbb{R}$, for every edge $e \in E$. The diffusivity coefficient δ is a positive constant and the non-negative speeds α_e corresponding to each edge satisfy, for each interior vertex v:

$$\sum_{e \in E_v^{\text{in}}} \alpha_e = \sum_{e \in E_v^{\text{out}}} \alpha_e.$$

We refer to the same repository [33] for an animation of the dynamics given by the above equation. For the study of the well-posedness of the problem, we refer to [21] and for the analysis of its boundary controllability, see [6].

In this section, we consider a metric tree without boundary vertices and construct a suitable non-local approximation of the solutions of (7.22), similar to Chapter 4. The diffusion-only model (i.e., $\alpha_e = 0$, $\forall e \in E$) was analysed in [28], where the authors introduced a non-local diffusion kernel depending on the distance inherent to metric graphs. The main challenge that we faced when introducing transport terms is to define an appropriate sequence of kernels $(G_\varepsilon(x,y))_{\varepsilon>0}$ which asymptotically describes the convection behaviour.

We also prove the non-local to local convergence in the case of non-linear convection-diffusion, namely we replaced the term $\alpha_e \partial_x u_e(t,x)$ with $\alpha_e \partial_x (f(u_e(t,x)))$, for $f(r) = |r|^{q-1} r$, with $q \geq 1$.

L^p decay estimates for the heat equation with Hardy potential

The next result complements the decay estimates Chapter 3 which deal with L^2 spaces, to more general L^p context, $p \geq 2$.

This time, we start with the initial datum u_0 of the heat equation with Hardy potential (3.2) satisfying:

$$\|u_0\|_{L^p(\mathbb{R}_+^{N,k};K^{\frac{p}{2}})} := \left(\int_{\mathbb{R}_+^{N,k}} |u_0(x)|^p K^{\frac{p}{2}}(x)\mathrm{d}x\right)^{\frac{1}{p}} < \infty,$$

where we recall that $K(x) := e^{\frac{|x|^2}{4}}$. Then, if we restrict to values of the parameter λ in (3.2) to the range $\left(-\infty, \frac{4(p-1)}{p^2}\lambda_{N,k}\right]$, where $\lambda_{N,k} = \left(\frac{N-2}{2}+k\right)^2$, then the solution $u(t,x)$ possesses the following long-time decay:

$$\|u(t)\|_{L^p(\mathbb{R}_+^{N,k})} \leq (t+1)^{\frac{N}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{\sqrt{p-1}}{p}(m_{N,k}(\lambda,p)+1)}\|u_0\|_{L^p(\mathbb{R}_+^{N,k};K^{\frac{p}{2}})}, \tag{7.23}$$

where $m_{N,k}(\lambda,p) = \sqrt{\lambda_{N,k} - \frac{p^2}{4(p-1)}\lambda}$.

Branching optimal constants in the Caffarelli Caffarelli-Kohn-Nirenberg inequality

The Caffarelli-Kohn-Nirenberg (CKN) inequality [10] is a general integral inequality that includes, as particular cases, several famous inequalities such as Poincaré and Hardy, as well as the Heisenberg and Hydrogen uncertainty principles. As it could be observed throughout this thesis (Chapters 3 and 5), the study of optimal constants for integral inequalities is an indispensable tool for accurately determining the asymptotic behaviour of partial differential equations. Therefore, a unified study of the sharp constant for the Caffarelli-Kohn-Nirenberg inequality is an important step forward in establishing the asymptotic properties of as many classes of equations as possible. The CKN inequality has the following form:

$$\int_{\mathbb{R}^N} \frac{|u|^r}{|x|^{\gamma r}}\mathrm{d}x \leq C\left(\int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|x|^{\alpha p}}\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{\mathrm{supp}(u)} \frac{|u|^{\frac{p(r-1)}{p-1}}}{|x|^{\beta}}\mathrm{d}x\right)^{\frac{p-1}{p}}, \tag{7.24}$$

where the parameters $p > 1, r > 0$ and $\alpha, \beta, \gamma \in \mathbb{R}$ satisfy certain compatibility conditions (see [34, Introduction]).

As far as we know, one of the most up-to-date studies related to the optimal constant $C > 0$ in the Caffarelli-Kohn-Nirenberg inequality is that of Nguyen [34], which presents a wide variety of parameter combinations for which the optimal constant is equal to $\frac{r}{N-\gamma r}$. However, Cazacu et. al. [12] have shown that in certain situations (specifically, when $p = r = 2$), the optimal constant bifurcates depending on the values of the other parameters involved.

In this section of the thesis, we analyse the bifurcation behaviour of the sharp constant for the Caffarelli-Kohn-Nirenberg inequality for as many parameter combinations as possible, thereby attempting to fill the gap in the literature between the two aforementioned papers. More precisely, we obtain that in the case $p = 2$, if the parameters of (7.24) satisfy $r > p$ and

$$N - \beta - \left(1 + \alpha - \frac{\beta}{p}\right) \frac{p(r-1)}{r-p} > 0,$$

then the optimal constant is $\frac{r}{r(N-2-2\alpha)+\left(1+\alpha+\frac{\beta}{2}-N\right)}$.

CONCLUSION

This work delves into various asymptotic methods within partial differential equations, underscoring the essential role of approximation in PDEs and across Mathematical Analysis as a whole. From fundamental concepts like continuity to advanced numerical schemes, the process of passage to the limit and estimation of approximation error forms the foundation of Analysis.

Moreover, in our view, asymptotics represents a defining characteristic of Analysis, setting it apart from other mathematical branches and establishing it as a vital bridge between Mathematics and the real world. But why is this so?

First, the real world enters the domain of Mathematics through the construction approximate models of experimental data. Such models, like equations representing simplified particle dynamics, are abstract mathematical structures, enabling pure mathematical techniques to be applied to physical phenomena. For example, even though the heat equation only serves as an idealised model for heat propagation [11, 31, 16],it still offers valuable insights into temperature distribution over time.

Often, models inspired by physical phenomena are further simplified, allowing for a deeper and more comprehensive analysis, as in the long-time approximations of differential equations in Chapter 3.

Second, approximations enable the use of practical tools like computers, which, despite their inherent limitations, can solve mathematical problems with a quantifiably small error. This is precisely the main objective of Numerical Analysis: to gain insights into the quantitative and qualitative properties of objects intrinsic to pure mathematics by constructing approximate models that rely on a finite set of parameters. These models can be evaluated using computational methods, allowing their properties to be projected back onto the abstract, original objects. Chapters 5 and 6 illustrate this concept. Therefore, through asymptotics, Mathematics rigorously addresses the inherent lack of rigour in the world.

Moreover, describing asymptotic phenomena effectively requires tools from nearly all mathematical branches. Integral Inequalities (Chapters 3 and 5), Topology (Chapter 4), Spectral Theory (Chapter 3), Differential Geometry (Chapters 4 and 5), and Linear Algebra (Chapter 6) all play roles here. Constructing meaningful asymptotic results thus demands a broad mathematical understanding and the ability to integrate diverse concepts in order to approximately model and interpret real-world processes, with the purpose of studying their properties in a rigorous way.

In conclusion, we believe this work establishes a foundation for new research directions, with the potential to contribute significantly to both theoretical and applied Mathematics and further unify diverse mathematical fields.

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