ROMANIAN ACADEMY
Advanced Studies School of the Romanian Academy
"Simion Stoilow" Institute of Mathematics

PhD THESIS SUMMARY

MATHEMATICAL AND NUMERICAL MODELLING IN FLUID-STRUCTURE INTERACTION PROBLEMS

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Chapter 1

INTRODUCTION

1.1 Fluid-structure interaction (FSI) problems - literature review

Fluid–structure interaction (FSI) is the interaction of some movable or deformable structure with an internal or surrounding fluid flow. In the case of fluid flow and elastic material, each of the subproblems acts on the other. The coupling is acting at the interface which is the surface between the fluid and solid problems, these being the surface-coupled multiphysics problems. The approach for computational fluid-structure interaction in real-life applications involves designing efficient coupling techniques. A mathematical analysis of the coupled FSI of equations that portrays the full FSI problem is still a novelty. Because the two subproblems, meaning the Navier-Stokes incompressible equations for the fluid and an equation for the elastic solid are mathematical obstacles, it is no wonder that results for the coupled problem are rarely spread. The interaction between a fluid and a solid (elastic or rigid) structure is a phenomenon that appears in many fields of real life. The distortion of aircraft wings caused by the aerodynamical forces, the influence of a vessel wall characteristics on the blood flow parameters, lubricant fluxes in ball-bearings are only a few important examples of (FSI) problems. Due to the practical applications and to the mathematical challenges as well, FSI problems have been studied extensively in the last years.

For the Navier-Stokes equations in fluid-dynamics and conservation equations for nonlinear hyperelastic materials, many theoretical questions are still not answered. Moreover, the motion of the fluid domain may lead to lose smoothness of the interface, where the coupling occurs. Having a fully coupled model for the whole problem, we can also use strongly coupled solution schemes like Newton linearization, multigrid or Krylov subspace methods for the complete problem without subiterating between fluid and solid domains. In the last 40 or 50 years, the Arbitrary Lagrangian Eulerian coordinate framework has been introduced to work with flow problems on moving domains [23], [39], [40] where numerical methods are used. Fluid-structure interaction is called a surface coupled multiphysics problem, as opposed to volume coupled multiphysics problems, where two (or more) subproblems all live in the same domain. A typical example for such volume coupled problems would be given by chemically reactive flows, where the chemical reaction interacts with a flow problem [11], [66]. As an example of FSI problem with moving domains, we can cite [57] where there are two rigid bodies moving. Existence and regularity for the coupled problem has been investigated in [2] (analyzes a parabolic-hyperbolic model of a coupled system which appears when an elastic structure is immersed in the fluid), [3] - [5] study a coupled parabolic-hyperbolic with the elastic structure immersed in the fluid by the semigroup theory, in [24] a time-dependent system that is modelling the interaction between a Stokes fluid and an elastic structure is described, [25] studies a linear fluid-structure interaction problem. For an in-depth introduction to optimization and parameter identification with partial differential equations, we refer to the literature [38], [49], [63], [64]. Optimization results for fluid-structure interaction problems can be found, e.g., in [14], [15], [27], [56], [59], [60]. Other works that study different aspects of interest from mathematical and numerical points of view regarding FSI problems are presented in what follows. Examples of venous insufficiency studies are in [20], [42], [46] and [50]. In [9] the authors prove the existence of the weak solutions in the interaction problem between a Stokes fluid and a multilayered poroelastic structure, [19] presents new algorithms for predicting the hemodynamics in large arteries, [21] proposes investigating from a numerical point of view of a fluid-structure interaction problem that is based on the decomposition theory, in [34] existence results are obtained in a fluid-structure interaction problem when in the elastic structure biological motors are considered, [52] represents a variational analysis in a fluid-structure interaction problem when Young's modulus and the density of the plate are big or small parameters, [53] presents an asymptotic analysis in a fluid-structure interaction depending on two small parameters, the thickness of the elastic medium and the fluid one, while [55] gives a mathematical introduction to modelling, analysis and simulation techniques for FSI problems.

1.2 Motivation and starting point of the study

This work is associated with the complex real-life phenomenon of blood flow through the circulatory system. Due to its obvious practical importance, numerous mathematical models have been elaborated in the specialized literature to describe as well as possible the characteristics of this phenomenon. Articles that take into account the complicated structure of the blood system ([8] describes a procedure of reconstruction of the pressure for thin domains, [41] proves the existence and uniqueness of the weak solution for steady-state Navier-Stokes equations in a thin tube structure), the interaction between the blood and the elastic vessel wall (see e.g., [10], [53]), or the role of the viscosity on the fluid flow (in [7] an incompressible micropolar fluid flowing through a pipe system, [54] studies a nonsteady viscous flow in a thin channel with elastic wall, [59] studies viscous fluid-elastic structure interaction problems) are only a few examples of recent works dedicated to this topic. In all the previous cited works it was supposed that the fluid and the elastic medium temperatures are constant; so, they do not affect the characteristics of the FSI. This assumption is a simplification, as most elastic materials and fluids have a strong dependency on the temperature. In real life there are many situations in which the interaction between a fluid and an elastic medium is strongly influenced by temperature variations. As an example of the important role played by the temperature we mention the connection between the ambient temperature and the variation of the blood pressure (see e.g., [44], [65]).

Before starting the study of the research topic presented in this thesis, in 2020, multiple searches have been made to see if there are other approaches for similar problems. We found only one article, [28], where it is presented a simplified model of the coupled FSI problem when the temperature variations of the two media are taken into account. Considering the stationary case, the authors study a weak coupling between the two media, as follows: they solve firstly the problem in the fluid domain; then, they solve the convection-difusion temperature equation in the entire domain, meaning the reunion of the fluid and solid subdomains; lastly, they solve the linear thermoelasticity system that describes the elastic medium deformation. This weak coupling diminishes the computation cost because it is not needed to solve a coupled monolithic system.

In addition to [28], we can mention two more articles ([47, 48]) published after 2020 dealing with thermal FSI problems. The characteristic of the models studied in these two articles is that the elastic structure has a lower dimension than that of the fluid domain (this means that the elastic structure represents a part of the fluid domain boundary).

1.3 Novelty

In this thesis we propose a double coupled mathematical model for describing a thermal (FSI) problem when the fluid and the solid domains have the same dimension. By means of this model, we obtain qualitative results such as existence, regularity, uniqueness of the unknown functions; moreover, we perform numerical simulations that highlight some aspects characterizing phenomena of real life. The novelty of our approach is that, unlike the problems studied in [28, 47, 48], we succeed in analyzing, both from mathematical and numerical viewpoints, a double coupled system with nonlinear equations and nonhomogeneous boundary and initial conditions, in the case when the fluid and the elastic structure occupy domains with same dimension. In our model the fluid motion is described by the incompressible Navier–Stokes system in the Boussinesq approximation; the behavior of the elastic medium is modeled by the linear thermoelasticity equation and, in addition, the variation of the temperature is given by a convection-diffusion equation corresponding to each medium (nonlinear in the fluid domain). The coupling conditions on the interface between the two media are the continuity of velocities and temperatures, and, also, the continuity of normal stresses and of heat fluxes.

Even though the specialty literature from the last years contains a big number of works that capture different aspects of the interaction between a viscous fluid and an elastic structure (for example, [13] studies a three-dimensional fluid-structure interaction problem between a viscous fluid and a thin elastic plate where strong convergence results and quantitative error estimates are obtained, [29] a stable semi-implicit scheme and a Chorin-Temam projection are studied in an incompressible fluid-elastic structure coupling, in [37] an unsteady nonlinear fluid–structure interaction problem without decoupling the fluid from the structure is studied, [51] describes a nonlinear, moving boundary fluid FSI problem between an incompressible, viscous Newtonian fluid with a two-dimensional Navier-Stokes equation and an elastic structure with Navier slip boundary conditions where the method used is the time discretization via operator splitting, [52] studies a two-dimensional time dependent model between a thin elastic plate and a Newtonian viscous fluid with an asymptotic expansion, in [54] a nonsteady viscous flow with viscosity constant almost everywhere and an asymptotic expansion of the solution is constructed, [59] determines a viscosity function that realizes an optimal blood pressure configuration), the thermal coupling between the two media is still a novelty.

1.4 Thesis structure and main results

The thesis is divided into six chapters and a bibliography containing 67 titles. In *Chapter 2* we present known functional analysis results from the literature that we use in the thesis. In *Chapter 3* we introduce and study the fully nonlinear coupled system modelling the thermal FSI problem with boundary, coupling and initial conditions. We define the FSI problem corresponding to a Navier-Stokes incompressible fluid and an elastic solid, both are coupled with temperature equations, the two media being coupled on the interface by the continuity of velocity, normal tensions, temperature and heat fluxes, presented in *Main Problem*. By linearizing the system and introducing dimensionless expressions, we obtain the partial differential coupled system analyzed in what follows. Then, we introduce two new unknown functions in order to reduce the number of unknowns. In this way, we transform the differential system into an integro-differential one. For obtaining the variational analysis of the problem, we present the functional spaces, the regularity of the data and we define two new unknowns satisfying homogeneous boundary and initial conditions. We introduce the notion of weak solution to the integro-differential system and we obtain the first main result of this chapter: the weak solution existence and uniqueness. The existence result is proven by means of the Galerkin's method applied for the variational problem associated with the new unknown functions. We obtain two sets of estimates for the unknowns and the derivatives of the unknowns. As the variational problem does not provide enough regularity in the elastic domain, we approximate it with a family of viscoelastic variational problems, depending on a small parameter ε. The difference between the initial variational problem and a viscoelastic problem from this family is represented by an additional small term contained by the last one. This term corresponds to the regularity in the elastic domain "uncovered" by the initial variational problem and it is necessary in what follows for obtaining the convergence of the numerical scheme. We prove existence, uniqueness and regularity results for the viscoelastic problems and some estimates. This approximation is justified by an error estimate theorem that gives an error of order $O(\varepsilon^{1/4})$ between the exact solution and the viscoelastic one with respect to suitable norms. As a consequence we obtain, as $\varepsilon \to 0$, the strong convergence of the family of viscoelastic solutions to the solution of the initial variational problem with respect to suitable norms. The last part of this chapter, is devoted to the numerical analysis of the problem for the case $n = 2$. For any fixed value of ε, we associate to the corresponding viscoelastic problem a numerical scheme, using a finite difference method in time and a finite element method for the space approximation. The additional viscoelastic term allows us to obtain suitable estimates for the unique solution of the numerical scheme in the elastic domain, followed by a stability theorem. The second main result of this chapter is the convergence theorem We show first that the sequence of solutions to the numerical scheme is weakly convergent (with respect to suitable spaces) to a triplet that satisfies the same variational equalities as the viscoelastic solution, but that has less regularity than this solution. Then, we improve step by step the regularity of this triplet until we obtain a regularity (still inferior to that of the viscoelastic solution), but which allows us to obtain a uniqueness result that has as consequence the fact that the previously obtained weak limit of the sequence of numerical solutions is unique and it represents the unique solution of the viscoelastic problem.

Chapter 4 presents theoretical results concerning the fluid pressure. The theoretical results of this chapter include the equivalence between the variational problem without pressure (VP) and the variational problem with pressure, the existence and uniqueness of the pressure by use of the Stokes formula. We prove the approximation of the pressure by a sequence of viscoelastic pressure functions and then, the weak convergence of this sequence to the pressure. We mention that the pressure uniqueness is not a trivial result, as it is known that Navier-Stokes problems do not give the uniqueness of the fluid pressure. We present a numerical approximation scheme with stability and convergence results. Last part of the chapter is dedicated to the Uzawa algorithm whose role is to approximate a function subject to restrictions by a sequence of functions without restrictions. Uzawa's algorithm for Stokes and Navier-Stokes equations is used, for example, in [62]. The role of this algorithm is to approximate a function that needs to verify the condition $\int_{\mathcal{S}} \operatorname{div} \vec{\alpha}_h \mathrm{d}x = 0, \forall \mathcal{S} \in \mathcal{S}_h^f$ with a sequence of functions that are not subjected anymore to these restrictions. This is necessary because it is difficult to build a precise basis of space $\mathcal{W}_h = \left\{ \vec{\alpha}_h \in W_h \,\middle|\, \int_{\mathcal{S}} \operatorname{div} \vec{\alpha}_h \mathrm{d}x = 0, \forall \mathcal{S} \in \mathcal{S}_h^f \right\}$ because of the restriction of this space. We generalise this algorithm for the studied doubled coupled problem and, in addition from what it is proven in [62], we also prove that the sequence of functions which approximates the pressure has an unique limit point.

Chapter 5 deals with some numerical simulations chosen in order to emphasize physical phenomena related to the considered problem. These simulations rely on the numerical schemes presented before and highlight the following aspects:

1. the way in which the temperature variation influences the variation of the fluid longitudinal velocity;

2. the way in which the changes of the exterior temperature influence the blood pressure variations;

3. the influence of the forces acting in the elastic domain on the fluid longitudinal velocity;

Chapter 6 presents the conclusions of our theoretical and numerical approach.

1.5 Interdisciplinarity of the study

This characteristic is highlighted by the fact that, starting from a physical problem (the influence of temperature on the blood flow through an elastic blood vessel), a mathematical model is constructed in order to describe this phenomenon and then, by means of this model, numerical simulations are performed, which validate the proposed model since these simulations lead to the expected physical results.

1.6 Limits of the research and perspectives for further research

Because of the complexity of the mathematical model (double coupled, nonlinear, nonhomogeneous system) introduced in order to describe as faithfully as possible the temperature influence on a viscous fluid-elastic structure interaction when the solid domain has the same dimension as the fluid one, it was necessary for its study to make a small simplification consisting of linearization of the system equations.

Since our work represents a mathematical and numerical study associated with a simplified model of blood flow through the circulatory system, some directions of future research could be to consider a more realistic geometry of the problem or a model closer to reality for modelling blood flow. A next stage would be domains in $\mathbb{R}^3$ and then a tube structure. The domain called tube structure is a mathematical model aimed to describe structures that contain several tubes connected at their points called nodes. More precisely, it is a special structure of thin domains. From the physical point of view, these structures can be used as geometrical models of the blood vessel systems, hydraulic installations, and so on.

1.7 Dissemination

Results of this thesis were presented at the following conferences:

1. Alexandra-Roxana Ciorogar, Ruxandra Stavre, *Variational and numerical analysis of a thermal fluid-structure interaction problem*, 39th Caius Iacob Conference on Fluid Mechanics and its Technical Applications, Bucharest, Romania, October 28-29, 2021

2. Alexandra-Roxana Ciorogar, Ruxandra Stavre, *Mathematical and numerical modeling of a given field of temperature influence on the pressure variation in a fluid-structure interaction problem*, International Symposium on Applied Mathematics and Engineering, Istanbul, Turkey, January 21-23, 2022

3. Alexandra-Roxana Ciorogar, Ruxandra Stavre, *A thermal fluid-structure interaction model; approximation methods and numerical algorithms*, International Applied Mathematics, Modelling and Simulation Conference, Paris, France, June 17-19, 2022

4. Alexandra-Roxana Ciorogar, Ruxandra Stavre, *Variational and numerical analysis for a thermal fluid-structure interaction problem*, International Workshop "Multiscale Modeling & Methods", Vilnius, Lithuania, October 24-26, 2022

1.8 Publications

Results of this thesis were published in:

1. Ciorogar, A., Stavre, R., 2023. *A Thermal Fluid–Structure Interaction Problem: Modeling, Variational and Numerical Analysis*, J. Math. Fluid Mech. 25, 37, DOI: `https://doi.org/10.1007/s00021-023-00783-x`

2. Stavre, R., Ciorogar, A., 2025. *Influence of a Given Field of Temperature on the Blood Pressure Variation: Variational Analysis, Numerical Algorithms and Simulations*, Axioms, 14, 88, DOI: `https://doi.org/10.3390/axioms14020088`.

Chapter 2

A THERMAL FLUID-STRUCTURE INTERACTION PROBLEM: MODELLING, VARIATIONAL AND NUMERICAL ANALYSIS

The results presented in this chapter were published in *Ciorogar, A., Stavre, R., 2023. A Thermal Fluid–Structure Interaction Problem: Modeling, Variational and Numerical Analysis, J. Math. Fluid Mech. 25, 37.*

2.1 Description of the mathematical model

We consider the nonstationary interaction between a viscous, incompressible fluid and an elastic solid. Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ be an open, bounded, Lipschitz set with a C^1-piecewise boundary and $\Omega_f \subset \Omega$ with the same properties. Denote $\Omega_s = \Omega \setminus \overline{\Omega_f}$. Ω_f represents the domain occupied by the viscous fluid and Ω_s is the elastic domain. The fluid flow is described by the Navier-Stokes equations in Boussinesq approximation, where we consider the temperature variation. We also consider the influence of body forces, corresponding to the weight, $\rho\vec{g}$ where ρ is the density and $\vec{g}$ is the gravitational acceleration. The unknowns are *the velocity* $\vec{v}$, *the fluid temperature* T_f and *the pressure* p. In this approximation, the density variation is neglected in the inertial terms but the gravity is sufficiently important to highlight the density variation. The density depends linearly on the temperature with the expression $\rho = \rho_0 - \rho_0\alpha_f(T - T_0)$, ρ_0 being a positive constant representing the fluid density at temperature T_0 and α_f the thermal expansion coefficient.

The elastic medium behavior is modeled by the linear thermoelasticity equation, with the small strain tensor components given in terms of the displacement vector $\vec{u}$, considered to be decomposed by $\varepsilon_{ij} = \varepsilon^M_{ij} + \varepsilon^T_{ij}$, ε^M_{ij} representing the components of mechanical strain and ε^T_{ij} the components of thermal strain (see e.g., [45]), the unknown variables being *the displacement vector* $\vec{u}$ and *the solid temperature* T_s. Using the theory from [58], we have $\sigma_{ij} = \underbrace{\lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}}_{\varepsilon^M_{ij}} \underbrace{-k\alpha_s(T - T_0)\delta_{ij}}_{\varepsilon^T_{ij}}$.

In addition to the previous equations, we consider two non-stationary convection-diffusion equations corresponding to the heat transfer in the fluid and in the elastic solid.

We suppose that the elastic medium deformation is small, so the equations for the two media can be written with a good approximation in the initial corresponding domains. The boundaries of the two phases have the following properties: $\partial\Omega_f = \Gamma_f \cup \Gamma$, $\partial\Omega_s = \Gamma_s \cup \Gamma$, with $\text{meas}(\Gamma_f \cap \Gamma) = 0$, $\text{meas}(\Gamma_s \cap \Gamma) = 0$, $\Gamma_f \cup \Gamma_s = \partial\Omega$, $\text{meas}(\Gamma_f \cap \Gamma_s) = 0$ and $\Gamma = \partial\Omega_f \cap \partial\Omega_s$ represents the two phases interface. In addition, corresponding to different types of boundary conditions, we consider $\Gamma_s = \Gamma^1_s \cup \Gamma^2_s$, with $\text{meas}(\Gamma^1_s \cap \Gamma^2_s) = 0$. The coupling conditions (represented by the velocity, the stress vector, temperature and heat flux continuity) are imposed on the fixed interface separating these domains (for details concerning the coupling conditions in FSI see e.g., [55]).

As we previously said, the blood system has a complicated geometric structure. However, for the purpose of our approach, this structure is not one of the main aspects, so we consider a simplified geometry of the problem.

For a better identification of the boundaries we present below particular domains in $\mathbb{R}^2$ and $\mathbb{R}^3$.

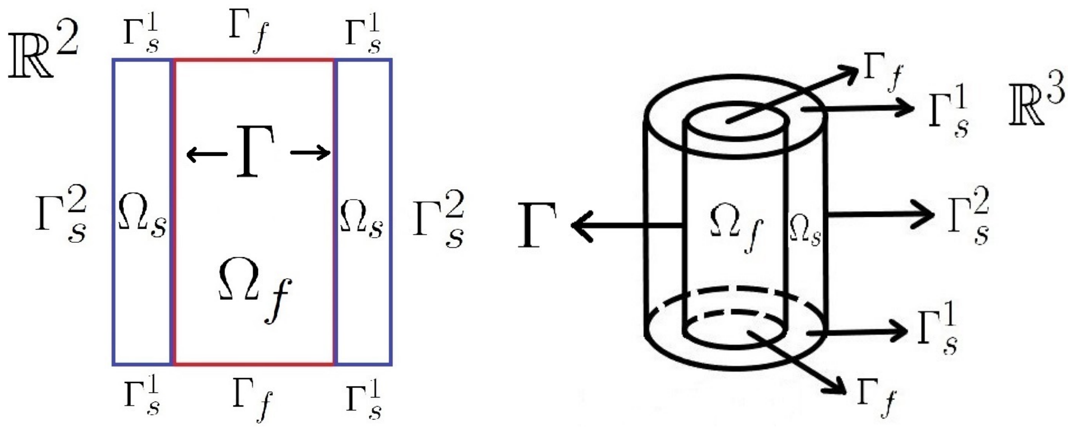

Figure 2.1: *Particular domains in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$

We formulate now the **Main Problem**, the nonstationary thermal FSI modeled by the nonlinear system of equations.

Find $\vec{v}$, $\vec{u}$, p, T_f, T_s solutions of the nonlinear partial differential equation system:

$$\begin{cases} \rho_0\vec{v}\,'+\rho_0(\vec{v}\cdot\nabla)\vec{v}-2\nu\,\mathrm{div}(D(\vec{v}))+\rho_0\alpha_f(T_f-T_{f0})\vec{g}+\nabla p=\rho_0\vec{g} & \\ & \text{in } \Omega_f\times(0,\tau), \\ \mathrm{div}\,\vec{v}=0 & \\ \rho_s\vec{u}\,''-\displaystyle\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial\vec{u}}{\partial x_j}\right)+k\alpha_s\nabla(T_s-T_{s0})=\vec{f}_s & \text{in } \Omega_s\times(0,\tau), \\ \rho_0 c_f T_f'+\rho_0 c_f\vec{v}\cdot\nabla T_f-k_f\Delta T_f=Q_f & \text{in } \Omega_f\times(0,\tau), \\ \rho_s c_s T_s'+k\alpha_s T_{s0}(\mathrm{div}\,\vec{u}\,')-k_s\Delta T_s=Q_s & \text{in } \Omega_s\times(0,\tau), \end{cases} \tag{2.1}$$

with boundary conditions

$$\begin{cases} \vec{v}=\vec{0} & \text{on } \Gamma_f\times(0,\tau), \\ \vec{u}=\vec{0} & \text{on } \Gamma_s\times(0,\tau), \\ \dfrac{\partial T_f}{\partial n}=0 & \text{on } \Gamma_f\times(0,\tau), \\ \dfrac{\partial T_s}{\partial n}=0 & \text{on } \Gamma_s^1\times(0,\tau), \\ T_s=T_g & \text{on } \Gamma_s^2\times(0,\tau), \end{cases} \tag{2.2}$$

junction conditions

$$\begin{cases} \vec{v} = \vec{u}' \\ -p\vec{n} + 2\nu D(\vec{v})\vec{n} = \sum_{i,j=1}^{n} A_{ij} \frac{\partial \vec{u}}{\partial x_j} n_i - k\alpha_s (T_s - T_{s_0})\vec{n} \\ T_f = T_s \\ k_f \frac{\partial T_f}{\partial n} = k_s \frac{\partial T_s}{\partial n} \end{cases} \quad \text{on } \Gamma \times (0,\tau), \tag{2.3}$$

and initial conditions

$$\begin{cases} \vec{v}(0) = \vec{v}_0 & \text{in } \Omega_f, \\ \vec{u}(0) = \vec{u}'(0) = \vec{0} & \text{in } \Omega_s, \\ T(0) = T_0 & \text{in } \Omega. \end{cases} \tag{2.4}$$

2.2 The variational problem

Variational Problem

Find $(\vec{\omega}, S) \in H_{\mathcal{W}} \times H_{\mathcal{T}}$ solution of:

$$\begin{aligned} &\int_\Omega \chi_1 \vec{\omega}'(t) \cdot \vec{\varphi} + \int_{\Omega_f} (\vec{\omega}(t) \cdot \nabla)\vec{w}_0 \cdot \vec{\varphi} + \int_{\Omega_f} (\vec{w}_0 \cdot \nabla)\vec{\omega}(t) \cdot \vec{\varphi} \\ &+ \frac{2}{\mathrm{Re}} \int_{\Omega_f} D(\vec{\omega}(t)) : D(\vec{\varphi}) + \alpha_f \int_{\Omega_f} S(t)\vec{g} \cdot \vec{\varphi} + k\alpha_s \frac{\rho_s}{\rho_0} \int_\Omega \nabla S(t) \cdot \vec{\varphi} \\ &+ \frac{\rho_s}{\rho_0} \sum_{i,j=1}^{n} \int_{\Omega_s} A_{ij} \frac{\partial}{\partial x_j} \left(\int_0^t \vec{\omega}(s)\mathrm{d}s \right) \cdot \frac{\partial \vec{\varphi}}{\partial x_i} = \int_\Omega \vec{K}(t) \cdot \vec{\varphi}, \\ &\qquad\qquad\qquad\qquad \forall\, \vec{\varphi} \in \mathcal{W} \text{ a.e. in } (0,\tau), \\ &\int_\Omega \chi_2 S'(t)\eta + \int_{\Omega_f} \eta\vec{\omega}(t) \cdot \nabla T_0 + \int_{\Omega_f} \eta\vec{w}_0 \cdot \nabla S(t) + \int_\Omega \chi_3 \nabla S(t) \cdot \nabla\eta \\ &- \frac{c_s}{c_f}\frac{\rho_s}{\rho_0} \mathrm{Ec} k\alpha_s \int_\Omega \nabla(T_0\eta) \cdot \vec{\omega}(t) = \int_\Omega G(t)\eta - \int_\Omega \vec{F}(t) \cdot \nabla\eta, \\ &\qquad\qquad\qquad\qquad \forall\, \eta \in \mathcal{T} \text{ a.e. in } (0,\tau), \\ &\vec{\omega}(0) = \vec{0} \text{ in } \mathcal{H}; S(0) = 0 \text{ in } L^2(\Omega). \end{aligned} \tag{VP}$$

2.3 Approximation of the variational problem with a family of viscoelastic problems

Proposition 2.1. *The pair $(\vec{\omega}, S)$ is solution for (VP) if and only if the triplet $(\vec{\omega}, \vec{u}, S)$ is solution for the problem:*

$$
\begin{aligned}
&\text{Find } (\vec{\omega}, \vec{u}, S) \in H_{\mathcal{W}} \times H_{\mathcal{W}^s} \times H_{\mathcal{T}} \text{ such that:}\\
&\int_\Omega \chi_1 \vec{\omega}'(t) \cdot \vec{\varphi} + \int_{\Omega_f} (\vec{\omega}(t) \cdot \nabla) \vec{w}_0 \cdot \vec{\varphi} + \int_{\Omega_f} (\vec{w}_0 \cdot \nabla) \vec{\omega}(t) \cdot \vec{\varphi}\\
&+ \frac{2}{Re} \int_{\Omega_f} D(\vec{\omega}(t)) : D(\vec{\varphi}) + \alpha_f \int_{\Omega_f} S(t) \vec{g} \cdot \vec{\varphi} + \frac{\rho_s}{\rho_0} \sum_{i,j=1}^{n} \int_{\Omega_s} A_{ij} \frac{\partial \vec{u}(t)}{\partial x_j} \cdot \frac{\partial \vec{\varphi}}{\partial x_i}\\
&+ k \alpha_s \frac{\rho_s}{\rho_0} \int_\Omega \nabla S(t) \cdot \vec{\varphi} = \int_\Omega \vec{K}(t) \cdot \vec{\varphi}, \ \forall \vec{\varphi} \in \mathcal{W} \text{ a.e. in } (0, \tau),\\
&\sum_{i,j=1}^{n} \int_{\Omega_s} A_{ij} \frac{\partial \vec{u}'(t)}{\partial x_j} \cdot \frac{\partial \vec{\psi}}{\partial x_i} = \sum_{i,j=1}^{n} \int_{\Omega_s} A_{ij} \frac{\partial \vec{\omega}(t)}{\partial x_j} \cdot \frac{\partial \vec{\psi}}{\partial x_i}, \ \forall \vec{\psi} \in \mathcal{W}^s \text{ a.e. in } (0, \tau),\\
&\int_\Omega \chi_2 S'(t) \eta + \int_{\Omega_f} \eta \vec{\omega}(t) \cdot \nabla T_0 + \int_{\Omega_f} \eta \vec{w}_0 \cdot \nabla S(t)\\
&+ \int_\Omega \chi_3 \nabla S(t) \cdot \nabla \eta - \frac{c_s}{c_f} \frac{\rho_s}{\rho_0} Eck \alpha_s \int_\Omega \nabla (T_0 \eta) \cdot \vec{\omega}(t)\\
&= \int_\Omega G(t) \eta - \int_\Omega \vec{F}(t) \cdot \nabla \eta, \ \forall \eta \in \mathcal{T} \text{ a.e. in } (0, \tau),\\
&\vec{\omega}(0) = \vec{0} \text{ in } \mathcal{H};\ \vec{u}(0) = \vec{0} \text{ in } \mathcal{W}^s;\ S(0) = 0 \text{ in } L^2(\Omega).
\end{aligned}
\tag{2.5}
$$

2.3.1 The family of viscoelastic problems

For proving the convergence of the numerical scheme to the initial variational problem, the H^1-regularity of $\vec{\omega}$ in Ω_s should be provided by a corresponding term, since here it is not possible anymore to use the data regularity in time, the numerical scheme being independent of t.

Let $\varepsilon > 0$ be a small parameter. Consider the problem:

Viscoelastic Variational Problem

$$
\begin{aligned}
&\text{Find } (\vec{\omega}_\varepsilon, \vec{u}_\varepsilon, S_\varepsilon) \in H_{\mathcal{W}} \times H_{\mathcal{W}^s} \times H_{\mathcal{T}} \text{ such that:}\\
&\int_\Omega \chi_1 \vec{\omega}_\varepsilon'(t)\cdot\vec{\varphi} + \int_{\Omega_f} (\vec{\omega}_\varepsilon(t)\cdot\nabla)\vec{w}_0\cdot\vec{\varphi} + \int_{\Omega_f} (\vec{w}_0\cdot\nabla)\vec{\omega}_\varepsilon(t)\cdot\vec{\varphi}\\
&+\frac{2}{\mathrm{Re}}\int_{\Omega_f} D(\vec{\omega}_\varepsilon(t)) : D(\vec{\varphi}) + \alpha_f\int_{\Omega_f} S_\varepsilon(t)\vec{g}\cdot\vec{\varphi} + \frac{\rho_s}{\rho_0}\sum_{i,j=1}^{n}\int_{\Omega_s} A_{ij}\frac{\partial \vec{u}_\varepsilon(t)}{\partial x_j}\cdot\frac{\partial\vec{\varphi}}{\partial x_i}\\
&+\varepsilon\frac{\rho_s}{\rho_0}\sum_{i,j=1}^{n}\int_{\Omega_s} B_{ij}\frac{\partial\vec{\omega}_\varepsilon(t)}{\partial x_j}\cdot\frac{\partial\vec{\varphi}}{\partial x_i} + k\alpha_s\frac{\rho_s}{\rho_0}\int_\Omega \nabla S_\varepsilon(t)\cdot\vec{\varphi} = \int_\Omega \vec{K}(t)\cdot\vec{\varphi},\\
&\hspace{20em}\forall\,\vec{\varphi}\in\mathcal{W} \text{ a.e. in } (0,\tau),\\
&\sum_{i,j=1}^{n}\int_{\Omega_s} A_{ij}\frac{\partial\vec{u}_\varepsilon'(t)}{\partial x_j}\cdot\frac{\partial\vec{\psi}}{\partial x_i} = \sum_{i,j=1}^{n}\int_{\Omega_s} A_{ij}\frac{\partial\vec{\omega}_\varepsilon(t)}{\partial x_j}\cdot\frac{\partial\vec{\psi}}{\partial x_i},\ \forall\,\vec{\psi}\in\mathcal{W}^s \text{ a.e. in } (0,\tau),\\
&\int_\Omega \chi_2 S_\varepsilon'(t)\eta + \int_{\Omega_f}\eta\vec{\omega}_\varepsilon(t)\cdot\nabla T_0 + \int_{\Omega_f}\eta\vec{w}_0\cdot\nabla S_\varepsilon(t)\\
&+\int_\Omega \chi_3\nabla S_\varepsilon(t)\cdot\nabla\eta - \frac{c_s}{c_f}\frac{\rho_s}{\rho_0}\mathrm{Ec}k\alpha_s\int_\Omega\nabla(T_0\eta)\cdot\vec{\omega}_\varepsilon(t)\\
&=\int_\Omega G(t)\eta - \int_\Omega \vec{F}(t)\cdot\nabla\eta,\ \forall\,\eta\in\mathcal{T} \text{ a.e. in } (0,\tau),\\
&\vec{\omega}_\varepsilon(0) = \vec{0} \text{ in } \mathcal{H};\ \vec{u}_\varepsilon(0) = \vec{0} \text{ in } \mathcal{W}^s;\ S_\varepsilon(0) = 0 \text{ in } L^2(\Omega).
\end{aligned}
\tag{VVP}
$$

Corollary 2.1. *Let $(\vec{\omega}, \vec{u}, S)$ be the unique solution of the problem (2.5) and $\{(\vec{\omega}_\varepsilon, \vec{u}_\varepsilon, S_\varepsilon)\}_\varepsilon$ the family of viscoelastic solutions. Then, the following convergences hold, when $\varepsilon \to 0$:*

$$
\begin{cases}
\vec{\omega}_\varepsilon \to \vec{\omega} \text{ strongly in } L^\infty(0,\tau;(L^2(\Omega))^n),\\
S_\varepsilon \to S \text{ strongly in } L^\infty(0,\tau;L^2(\Omega)),\\
\vec{\omega}_\varepsilon \to \vec{\omega} \text{ strongly in } L^2(0,\tau;(H^1(\Omega_f))^n),\\
S_\varepsilon \to S \text{ strongly in } L^2(0,\tau;H^1(\Omega)),\\
\vec{u}_\varepsilon \to \vec{u} \text{ strongly in } L^\infty(0,\tau;(H^1(\Omega_s))^n).
\end{cases}
$$

2.4 Numerical approximation schemes, estimates, stability, convergence for the case $n = 2$

We consider a viscoelastic problem corresponding to a fixed ε and we associate it a numerical scheme using a finite element approximation in space and a finite difference approximation in time. All the unknowns appearing in what follows depend on ε, but, for simplifying the writing, we omit it.

2.4.1 The numerical scheme independent of time

For a fixed value of ε, we associate to the viscoelastic problem (VVP) the following numerical scheme

$$
\left\{
\begin{aligned}
&\text{For } \vec{\omega}^0_{h,N}, .., \vec{\omega}^{m-1}_{h,N} \in \mathcal{W}_h; \vec{u}^0_{h,N}, .., \vec{u}^{m-1}_{h,N} \in W^s_h; S^0_{h,N}, .., S^{m-1}_{h,N} \in \mathcal{T}_h \text{ given,}\\
&(\vec{\omega}^0_{h,N}, \vec{u}^0_{h,N}, S^0_{h,N}) = (\vec{0}, \vec{0}, 0), \text{ find } (\vec{\omega}^m_{h,N}, \vec{u}^m_{h,N}, S^m_{h,N}) \in \mathcal{W}_h \times W^s_h \times \mathcal{T}_h \text{ s.t.}\\
&\frac{1}{\tau/N}\int_\Omega \chi_1 \vec{\omega}^m_{h,N}\cdot\vec{\varphi}_h + \int_{\Omega_f}(\vec{\omega}^m_{h,N}\cdot\nabla)\vec{w}_0\cdot\vec{\varphi}_h + \int_{\Omega_f}(\vec{w}_0\cdot\nabla)\vec{\omega}^m_{h,N}\cdot\vec{\varphi}_h\\
&+\frac{2}{\mathrm{Re}}\int_{\Omega_f} D(\vec{\omega}^m_{h,N}) : D(\vec{\varphi}_h) + \alpha_f \int_{\Omega_f} S^m_{h,N}\vec{g}\cdot\vec{\varphi}_h\\
&+\frac{\rho_s}{\rho_0}\sum_{i,j=1}^{2}\int_{\Omega_s} A_{ij}\frac{\partial \vec{u}^m_{h,N}}{\partial x_j}\cdot\frac{\partial\vec{\varphi}_h}{\partial x_i} + \varepsilon\frac{\rho_s}{\rho_0}\sum_{i,j=1}^{2}\int_{\Omega_s} B_{ij}\frac{\partial \vec{\omega}^m_{h,N}}{\partial x_j}\cdot\frac{\partial\vec{\varphi}_h}{\partial x_i}\\
&+k\alpha_s\frac{\rho_s}{\rho_0}\int_\Omega \nabla S^m_{h,N}\cdot\vec{\varphi}_h = \frac{1}{\tau/N}\int_\Omega \chi_1\vec{\omega}^{m-1}_{h,N}\cdot\vec{\varphi}_h + \int_\Omega \vec{K}^m_N\cdot\vec{\varphi}_h,\ \forall\,\vec{\varphi}_h\in\mathcal{W}_h,\\
&\frac{1}{\tau/N}\sum_{i,j=1}^{2}\int_{\Omega_s} A_{ij}\frac{\partial \vec{u}^m_{h,N}}{\partial x_j}\cdot\frac{\partial\vec{\psi}_h}{\partial x_i} - \sum_{i,j=1}^{2}\int_{\Omega_s} A_{ij}\frac{\partial \vec{\omega}^m_{h,N}}{\partial x_j}\cdot\frac{\partial\vec{\psi}_h}{\partial x_i}\\
&=\frac{1}{\tau/N}\sum_{i,j=1}^{2}\int_{\Omega_s} A_{ij}\frac{\partial \vec{u}^{m-1}_{h,N}}{\partial x_j}\cdot\frac{\partial\vec{\psi}_h}{\partial x_i},\ \forall\,\vec{\psi}_h\in W^s_h,\\
&\frac{1}{\tau/N}\int_\Omega \chi_2 S^m_{h,N}\eta_h + \int_{\Omega_f}(\vec{\omega}^m_{h,N}\cdot\nabla T_0)\eta_h + \int_{\Omega_f}(\vec{w}_0\cdot\nabla S^m_{h,N})\eta_h\\
&+\int_\Omega \chi_3\nabla S^m_{h,N}\cdot\nabla\eta_h - \frac{c_s}{c_f}\frac{\rho_s}{\rho_0}\mathrm{Ec}k\alpha_s\int_\Omega \nabla(T_0\eta_h)\cdot\vec{\omega}^m_{h,N}\\
&=\frac{1}{\tau/N}\int_\Omega \chi_2 S^{m-1}_{h,N}\eta_h + \int_\Omega G^m_N\eta_h - \int_\Omega \vec{F}^m_N\cdot\nabla\eta_h,\ \forall\,\eta_h\in\mathcal{T}_h.
\end{aligned}
\right.
\tag{NS}
$$

Theorem 2.1. *There exists $N_0 \in \mathbb{N}^*$ such that for any $N \geq N_0$, $m \in \{1, \ldots, N\}$ and $h > 0$ the problem (NS) has a unique solution.*

2.4.2 Stability

Let $(\vec{\omega}^m_{h,N}, \vec{u}^m_{h,N}, S^m_{h,N})$ be the unique solution to the system (NS); define the following functions depending on x and t:

$$
\left\{
\begin{aligned}
&\vec{\omega}_{h,N} \in L^2(0,\tau;\mathcal{W}_h)\\
&\vec{\omega}_{h,N}(t) = \vec{\omega}^m_{h,N} \text{ if } t \in \left[(m-1)\frac{\tau}{N}, m\frac{\tau}{N}\right),\ m = \overline{1,N},
\end{aligned}
\right.
\tag{2.6}
$$

$$\begin{cases} \vec{u}_{h,N} \in L^2(0,\tau;W_h^s) \\ \vec{u}_{h,N}(t) = \vec{u}^m_{h,N} \text{ if } t \in \left[(m-1)\frac{\tau}{N}, m\frac{\tau}{N}\right), \ m = \overline{1,N}, \end{cases} \tag{2.7}$$

$$\begin{cases} S_{h,N} \in L^2(0,\tau;\mathcal{T}_h) \\ S_{h,N}(t) = S^m_{h,N} \text{ if } t \in \left[(m-1)\frac{\tau}{N}, m\frac{\tau}{N}\right), \ m = \overline{1,N}, \end{cases} \tag{2.8}$$

and

$$\begin{cases} \vec{w}_{h,N} \in C^0([0,\tau];\mathcal{W}_h) \\ \vec{w}_{h,N}(t) = (\vec{\omega}^m_{h,N} - \vec{\omega}^{m-1}_{h,N})\left(\frac{t}{\tau/N} - m\right) + \vec{\omega}^m_{h,N}, \ t \in \left[(m-1)\frac{\tau}{N}, m\frac{\tau}{N}\right), m = \overline{1,N}, \end{cases} \tag{2.9}$$

$$\begin{cases} \vec{\xi}_{h,N} \in C^0([0,\tau];W_h^s) \\ \vec{\xi}_{h,N}(t) = (\vec{u}^m_{h,N} - \vec{u}^{m-1}_{h,N})\left(\frac{t}{\tau/N} - m\right) + \vec{u}^m_{h,N}, \ t \in \left[(m-1)\frac{\tau}{N}, m\frac{\tau}{N}\right), m = \overline{1,N}, \end{cases} \tag{2.10}$$

$$\begin{cases} \sigma_{h,N} \in C^0([0,\tau];\mathcal{T}_h) \\ \sigma_{h,N}(t) = (S^m_{h,N} - S^{m-1}_{h,N})\left(\frac{t}{\tau/N} - m\right) + S^m_{h,N}, \ t \in \left[(m-1)\frac{\tau}{N}, m\frac{\tau}{N}\right), m = \overline{1,N}. \end{cases} \tag{2.11}$$

Let us define the sets:

$$\begin{cases} \mathcal{E}_\omega = \{\vec{\omega}_{h,N}, \ h > 0, \ N \in \mathbb{N}^*, \ N \geq \hat{N}\}, \\ \mathcal{E}_u = \{\vec{u}_{h,N}, \ h > 0, \ N \in \mathbb{N}^*, \ N \geq \hat{N}\}, \\ \mathcal{E}_S = \{S_{h,N}, \ h > 0, \ N \in \mathbb{N}^*, \ N \geq \hat{N}\}. \end{cases}$$

Theorem 2.2. *Let $h > 0$ and $N \in \mathbb{N}^*$, $N \geq \hat{N}$ be the two parameters characterizing the space and the time discretization, respectively. Then:*

$$\begin{aligned} &(i) \text{ the set } \mathcal{E}_\omega \text{ is } L^\infty(0,\tau;(L^2(\Omega))^2) \cap L^2(0,\tau;(H_0^1(\Omega))^2) \text{ stable}; \\ &(ii) \text{ the set } \mathcal{E}_u \text{ is } L^\infty(0;\tau;\mathcal{W}^s) \text{ stable}; \\ &(iii) \text{ the set } \mathcal{E}_S \text{ is } L^\infty(0,\tau;L^2(\Omega)) \cap L^2(0,\tau;\mathcal{T}) \text{ stable}. \end{aligned} \tag{2.12}$$

2.4.3 The convergence of the numerical scheme

Define the functions:

$$\begin{cases} \vec{\tilde{K}}_N \in L^2(0,\tau;(L^2(\Omega))^2) \\ \vec{\tilde{K}}_N(t) = \vec{K}_N^m \text{ for } t \in \left[(m-1)\frac{\tau}{N}, m\frac{\tau}{N}\right), \; m = \overline{1,N}, \\ \tilde{G}_N \in L^2(0,\tau;L^2(\Omega)) \\ \tilde{G}_N(t) = G_N^m \text{ for } t \in \left[(m-1)\frac{\tau}{N}, m\frac{\tau}{N}\right), \; m = \overline{1,N}, \\ \vec{\tilde{F}}_N \in L^2(0,\tau;(L^2(\Omega))^2) \\ \vec{\tilde{F}}_N(t) = \vec{F}_N^m \text{ for } t \in \left[(m-1)\frac{\tau}{N}, m\frac{\tau}{N}\right), \; m = \overline{1,N}. \end{cases} \tag{2.13}$$

Proposition 2.2. *If $(\vec{\omega}^m_{h,N}, \vec{u}^m_{h,N}, S^m_{h,N})$ represents the unique solution of the problem (NS), then, $\vec{\omega}_{h,N}$, $\vec{u}_{h,N}$, $S_{h,N}$, $\vec{w}_{h,N}$, $\vec{\xi}_{h,N}$, $\sigma_{h,N}$ defined by (2.6) - (2.11) verify the time dependent numerical scheme:*

$$\begin{cases} \int_\Omega \chi_1 \vec{w}'_{h,N}(t)\cdot\vec{\varphi}_h + \int_{\Omega_f}(\vec{\omega}_{h,N}(t)\cdot\nabla)\vec{w}_0\cdot\vec{\varphi}_h + \int_{\Omega_f}(\vec{w}_0\cdot\nabla)\vec{\omega}_{h,N}(t)\cdot\vec{\varphi}_h \\ +\frac{2}{Re}\int_{\Omega_f} D(\vec{\omega}_{h,N}(t)) : D(\vec{\varphi}_h) + \alpha_f\int_{\Omega_f} S_{h,N}(t)\vec{g}\cdot\vec{\varphi}_h \\ +\frac{\rho_s}{\rho_0}\sum_{i,j=1}^2\int_{\Omega_s} A_{ij}\frac{\partial\vec{u}_{h,N}(t)}{\partial x_j}\cdot\frac{\partial\vec{\varphi}_h}{\partial x_i} + \varepsilon\frac{\rho_s}{\rho_0}\sum_{i,j=1}^2\int_{\Omega_s} B_{ij}\frac{\partial\vec{\omega}_{h,N}(t)}{\partial x_j}\cdot\frac{\partial\vec{\varphi}_h}{\partial x_i} \\ + k\alpha_s\frac{\rho_s}{\rho_0}\int_\Omega\nabla S_{h,N}(t)\cdot\vec{\varphi}_h = \int_\Omega \vec{\tilde{K}}_N(t)\cdot\vec{\varphi}_h, \; \forall\,\vec{\varphi}_h\in\mathcal{W}_h \text{ a.e. in } (0,\tau), \\ \sum_{i,j=1}^2\int_{\Omega_s} A_{ij}\frac{\partial\vec{\xi}'_{h,N}(t)}{\partial x_j}\cdot\frac{\partial\vec{\psi}_h}{\partial x_i} = \sum_{i,j=1}^2\int_{\Omega_s} A_{ij}\frac{\partial\vec{\omega}_{h,N}(t)}{\partial x_j}\cdot\frac{\partial\vec{\psi}_h}{\partial x_i}, \\ \qquad\qquad \forall\,\vec{\psi}_h\in W_h^s \text{ a.e. in } (0,\tau), \\ \int_\Omega \chi_2\sigma'_{h,N}(t)\eta_h + \int_{\Omega_f}(\vec{\omega}_{h,N}(t)\cdot\nabla T_0)\eta_h + \int_{\Omega_f}(\vec{w}_0\cdot\nabla S_{h,N}(t))\eta_h \\ +\int_\Omega \chi_3\nabla S_{h,N}(t)\cdot\nabla\eta_h - \frac{c_s}{c_f}\frac{\rho_s}{\rho_0}Eck\alpha_s\int_\Omega\nabla(T_0\eta_h)\cdot\vec{\omega}_{h,N}(t) \\ = \int_\Omega \tilde{G}_N(t)\eta_h - \int_\Omega \vec{\tilde{F}}_N(t)\cdot\nabla\eta_h, \; \forall\,\eta_h\in\mathcal{T}_h \text{ a.e. in } (0,\tau). \end{cases} \tag{NST}$$

Theorem 2.3. *For any $h > 0$, $N \in \mathbb{N}^*$, $N \geq \hat{N}$ and $m = \overline{1,N}$, let $(\vec{\omega}^m_{h,N}, \vec{u}^m_{h,N}, S^m_{h,N})$ be the unique solution of the problem (NS) and $\vec{\omega}_{h,N}$, $\vec{u}_{h,N}$, $S_{h,N}$, $\vec{w}_{h,N}$, $\vec{\xi}_{h,N}$, $\sigma_{h,N}$ the functions defined by (2.6) - (2.11). Let $(\vec{\omega}_\varepsilon, \vec{u}_\varepsilon, S_\varepsilon)$ be the unique solution of the viscoelastic problem (VVP). Then, $(\vec{\omega}_\varepsilon, \vec{u}_\varepsilon, S_\varepsilon)$ is the only weak limit point of the sequence $\{(\vec{\omega}_{h,N}, \vec{u}_{h,N}, S_{h,N})\}_{h,N}$ when $h \to 0$, $N \to \infty$ in the sense of the weak convergences in the sets where we have stability given by Theorem 2.2.*

Chapter 3

ANALYSIS OF THE FLUID PRESSURE: VARIATIONAL STUDY, APPROXIMATION, NUMERICAL ALGORITHMS, UZAWA ALGORITHM

The results presented in this chapter were published in *Stavre, R., Ciorogar, A., 2025. Influence of a Given Field of Temperature on the Blood Pressure Variation: Variational Analysis, Numerical Algorithms and Simulations, Axioms, 14, 88.*

3.1 The viscoelastic variational problem with pressure

Viscoelastic Variational Problem with Pressure

$$
\begin{aligned}
&(\vec{\omega}_\varepsilon, \vec{u}_\varepsilon, p_\varepsilon, S_\varepsilon) \in \tilde{H} \times H_{\mathcal{W}^s} \times L^2(\Omega_f \times (0,\tau)) \times H_{\mathcal{T}} \\
&\int_\Omega \chi_1 \vec{\omega}'_\varepsilon \cdot \vec{\varphi} + \int_{\Omega_f} (\vec{\omega}_\varepsilon \cdot \nabla)\vec{w}_0 \cdot \vec{\varphi} + \int_{\Omega_f} (\vec{w}_0 \cdot \nabla)\vec{\omega}_\varepsilon \cdot \vec{\varphi} \\
&+ \frac{2}{\mathrm{Re}} \int_{\Omega_f} D(\vec{\omega}_\varepsilon) : D(\vec{\varphi}) - \int_{\Omega_f} p_\varepsilon \mathrm{div}\, \vec{\varphi} + \alpha_f \int_{\Omega_f} S_\varepsilon \vec{g} \cdot \vec{\varphi} \\
&+ \frac{\rho_s}{\rho_0} \sum_{i,j=1}^n \int_{\Omega_s} A_{ij} \frac{\partial \vec{u}_\varepsilon}{\partial x_j} \cdot \frac{\partial \vec{\varphi}}{\partial x_i} + \varepsilon \frac{\rho_s}{\rho_0} \sum_{i,j=1}^n \int_{\Omega_s} B_{ij} \frac{\partial \vec{\omega}_\varepsilon}{\partial x_j} \cdot \frac{\partial \vec{\varphi}}{\partial x_i} \\
&+ k\alpha_s \frac{\rho_s}{\rho_0} \int_\Omega \nabla S_\varepsilon(t) \cdot \vec{\varphi} + k\alpha_s \frac{\rho_s}{\rho_0} \int_{\Omega_f} S_\varepsilon \mathrm{div}\, \vec{\varphi} \\
&= \int_\Omega \vec{K} \cdot \vec{\varphi} - k\alpha_s \frac{\rho_s}{\rho_0} \int_{\Omega_f} (\tilde{T}_g - T_0) \mathrm{div}\, \vec{\varphi}, \quad \forall\, \vec{\varphi} \in (H^1_0(\Omega))^n \text{ in } L^2(0,\tau), \\
&\sum_{i,j=1}^n \int_{\Omega_s} A_{ij} \frac{\partial \vec{u}'_\varepsilon}{\partial x_j} \cdot \frac{\partial \vec{\psi}}{\partial x_i} = \sum_{i,j=1}^n \int_{\Omega_s} A_{ij} \frac{\partial \vec{\omega}_\varepsilon}{\partial x_j} \cdot \frac{\partial \vec{\psi}}{\partial x_i}, \ \ \forall\, \vec{\psi} \in \mathcal{W}^s \text{ in } L^2(0,\tau), \\
&\int_\Omega \chi_2 S'_\varepsilon \eta + \int_{\Omega_f} \eta \vec{\omega}_\varepsilon(t) \cdot \nabla T_0 + \int_{\Omega_f} \eta \vec{w}_0 \cdot \nabla S_\varepsilon + \int_\Omega \chi_3 \nabla S_\varepsilon \cdot \nabla \eta \\
&- \frac{c_s}{c_f} \frac{\rho_s}{\rho_0} \mathrm{Ec} k \alpha_s \int_\Omega \nabla(T_0 \eta) \cdot \vec{\omega}_\varepsilon = \int_\Omega G\eta - \int_\Omega \vec{F} \cdot \nabla \eta, \ \ \forall\, \eta \in \mathcal{T} \text{ in } L^2(0,\tau), \\
&\int_{\Omega_f} q \mathrm{div}\, \vec{\omega}_\varepsilon = 0, \ \forall\, q \in L^2(\Omega_f) \text{ in } L^2(0,\tau), \\
&\vec{\omega}_\varepsilon(0) = \vec{0} \text{ in } \mathcal{H},\ \vec{u}_\varepsilon(0) = \vec{0} \text{ in } \mathcal{W}^s,\ S_\varepsilon(0) = 0 \text{ in } L^2(\Omega).
\end{aligned}
\tag{VVPP}
$$

3.2 The numerical approximation scheme with pressure

$$
\left\{
\begin{aligned}
&\frac{1}{\tau/N}\int_{\Omega}\chi_1\vec{\omega}^m_{h,N}\cdot\vec{\varphi}_h+\int_{\Omega_f}(\vec{\omega}^m_{h,N}\cdot\nabla)\vec{w}_0\cdot\vec{\varphi}_h+\int_{\Omega_f}(\vec{w}_0\cdot\nabla)\vec{\omega}^m_{h,N}\cdot\vec{\varphi}_h\\
&+\frac{2}{\mathrm{Re}}\int_{\Omega_f}D(\vec{\omega}^m_{h,N}):D(\vec{\varphi}_h)+\alpha_f\int_{\Omega_f}S^m_{h,N}\vec{g}\cdot\vec{\varphi}_h-\int_{\Omega_f}\pi^m_{h,N}D_h\vec{\varphi}_h\\
&+k\alpha_s\frac{\rho_s}{\rho_0}\int_{\Omega_f}S^m_{h,N}\mathrm{div}\,\vec{\varphi}_h+\frac{\rho_s}{\rho_0}\sum_{i,j=1}^{2}\int_{\Omega_s}A_{ij}\frac{\partial\vec{u}^m_{h,N}}{\partial x_j}\cdot\frac{\partial\vec{\varphi}_h}{\partial x_i}\\
&+\varepsilon\frac{\rho_s}{\rho_0}\sum_{i,j=1}^{2}\int_{\Omega_s}B_{ij}\frac{\partial\vec{\omega}^m_{h,N}}{\partial x_j}\cdot\frac{\partial\vec{\varphi}_h}{\partial x_i}+k\alpha_s\frac{\rho_s}{\rho_0}\int_{\Omega}\nabla S^m_{h,N}\cdot\vec{\varphi}_h=\frac{1}{\tau/N}\int_{\Omega}\chi_1\vec{\omega}^{m-1}_{h,N}\cdot\vec{\varphi}_h\\
&+\int_{\Omega}\vec{K}^m_N\cdot\vec{\varphi}_h-k\alpha_s\frac{\rho_s}{\rho_0}\int_{\Omega_f}((\tilde{T}_g)^m_N-T_0)\mathrm{div}\,\vec{\varphi}_h,\ \forall\,\vec{\varphi}_h\in W_h,\\[2ex]
&\frac{1}{\tau/N}\sum_{i,j=1}^{2}\int_{\Omega_s}A_{ij}\frac{\partial\vec{u}^m_{h,N}}{\partial x_j}\cdot\frac{\partial\vec{\psi}_h}{\partial x_i}-\sum_{i,j=1}^{2}\int_{\Omega_s}A_{ij}\frac{\partial\vec{\omega}^m_{h,N}}{\partial x_j}\cdot\frac{\partial\vec{\psi}_h}{\partial x_i}\\
&=\frac{1}{\tau/N}\sum_{i,j=1}^{2}\int_{\Omega_s}A_{ij}\frac{\partial\vec{u}^{m-1}_{h,N}}{\partial x_j}\cdot\frac{\partial\vec{\psi}_h}{\partial x_i},\ \forall\,\vec{\psi}_h\in W^s_h,\\[2ex]
&\frac{1}{\tau/N}\int_{\Omega}\chi_2S^m_{h,N}\eta_h+\int_{\Omega_f}\eta_h\vec{\omega}^m_{h,N}\cdot\nabla T_0+\int_{\Omega_f}\eta_h\vec{w}_0\cdot\nabla S^m_{h,N}\\
&+\int_{\Omega}\chi_3\nabla S^m_{h,N}\cdot\nabla\eta_h-\frac{c_s}{c_f}\frac{\rho_s}{\rho_0}\mathrm{Ec}k\alpha_s\int_{\Omega}\nabla(T_0\eta_h)\cdot\vec{\omega}^m_{h,N}\\
&=\frac{1}{\tau/N}\int_{\Omega}\chi_2S^{m-1}_{h,N}\eta_h+\int_{\Omega}G^m_N\eta_h-\int_{\Omega}\vec{F}^m_N\cdot\nabla\eta_h,\ \forall\,\eta_h\in\mathcal{T}_h.
\end{aligned}
\right. \tag{NSP}
$$

Time dependent numerical scheme

Proposition 3.1. *$(\vec{\omega}^m_{h,N}, \vec{u}^m_{h,N}, \pi^m_{h,N}, S^m_{h,N})$ is a solution for (NSP) if and only if $\vec{\omega}_{h,N}$, $\vec{u}_{h,N}$, $\pi_{h,N}$, $S_{h,N}$; $\vec{w}_{h,N}$, $\vec{\xi}_{h,N}$, $\sigma_{h,N}$ verify the problem*

$$\left\{\begin{array}{l} \displaystyle\int_\Omega \chi_1 \vec{w}'_{h,N}(t)\cdot\vec{\varphi}_h + \int_{\Omega_f}(\vec{\omega}_{h,N}(t)\cdot\nabla)\vec{w}_0\cdot\vec{\varphi}_h + \int_{\Omega_f}(\vec{w}_0\cdot\nabla)\vec{\omega}_{h,N}(t)\cdot\vec{\varphi}_h \\ \displaystyle+\frac{2}{Re}\int_{\Omega_f} D(\vec{\omega}_{h,N}(t)) : D(\vec{\varphi}_h) + \alpha_f\int_{\Omega_f} S_{h,N}(t)\vec{g}\cdot\vec{\varphi}_h - \int_{\Omega_f}\pi_{h,N}(t) D_h\vec{\varphi}_h \\ \displaystyle+ k\alpha_s\frac{\rho_s}{\rho_0}\int_{\Omega_f} S_{h,N}(t) div\,\vec{\varphi}_h + \frac{\rho_s}{\rho_0}\sum_{i,j=1}^{2}\int_{\Omega_s} A_{ij}\frac{\partial\vec{u}_{h,N}(t)}{\partial x_j}\cdot\frac{\partial\vec{\varphi}_h}{\partial x_i} \\ \displaystyle+\varepsilon\frac{\rho_s}{\rho_0}\sum_{i,j=1}^{2}\int_{\Omega_s} B_{ij}\frac{\partial\vec{\omega}_{h,N}(t)}{\partial x_j}\cdot\frac{\partial\vec{\varphi}_h}{\partial x_i} + k\alpha_s\frac{\rho_s}{\rho_0}\int_\Omega \nabla S_{h,N}(t)\cdot\vec{\varphi}_h \\ \displaystyle=\int_\Omega \tilde{\vec{K}}_N(t)\cdot\vec{\varphi}_h + k\alpha_s\frac{\rho_s}{\rho_0}\int_{\Omega_f}(T_0 - \widetilde{(\tilde{T}_g)_N})div\,\vec{\varphi}_h,\ \forall\,\vec{\varphi}_h\in W_h\ in\ L^2(0,\tau), \\ \\ \displaystyle\sum_{i,j=1}^{2}\int_{\Omega_s} A_{ij}\frac{\partial\vec{\xi}'_{h,N}(t)}{\partial x_j}\cdot\frac{\partial\vec{\psi}_h}{\partial x_i} = \sum_{i,j=1}^{2}\int_{\Omega_s} A_{ij}\frac{\partial\vec{\omega}_{h,N}(t)}{\partial x_j}\cdot\frac{\partial\vec{\psi}_h}{\partial x_i}, \\ \qquad\qquad\qquad\qquad\qquad \forall\,\vec{\psi}_h\in W^s_h\ in\ L^2(0,\tau), \\ \\ \displaystyle\int_\Omega \chi_2\sigma'_{h,N}(t)\eta_h + \int_{\Omega_f}\eta_h\vec{\omega}_{h,N}(t)\cdot\nabla T_0 + \int_{\Omega_f}\eta_h\vec{w}_0\cdot\nabla S_{h,N}(t) \\ \displaystyle+\int_\Omega \chi_3\nabla S_{h,N}(t)\cdot\nabla\eta_h - \frac{c_s}{c_f}\frac{\rho_s}{\rho_0}Eck\alpha_s\int_\Omega \nabla(T_0\eta_h)\cdot\vec{\omega}_{h,N}(t) \\ \displaystyle=\int_\Omega \tilde{G}_N(t)\eta_h - \int_\Omega \tilde{\vec{F}}_N(t)\cdot\nabla\eta_h,\ \ \forall\,\eta_h\in\mathcal{T}_h\ in\ L^2(0,\tau). \end{array}\right. \tag{NSPT}$$

The following theorem gives the convergence of the scheme (NSPT) to the problem (VVPP).

Theorem 3.1. *Let $h > 0$, $N \in \mathbb{N}^*$, $N \geq \hat{N}$. For all $m \in \{1,\ldots,N\}$ let $(\vec{\omega}^m_{h,N}, \vec{u}^m_{h,N}, \pi^m_{h,N}, S^m_{h,N})$ be the unique solution of problem (NSP) Let $(\vec{\omega}_\varepsilon, \vec{u}_\varepsilon, p_\varepsilon, S_\varepsilon)$ be the unique solution of the problem (VVPP). Then, this solution represents the only weak limit point of the sequence $\left\{(\vec{\omega}_{h,N}, \vec{u}_{h,N}, \pi_{h,N}, S_{h,N})\right\}_{h,N}$ when $h \to 0$, $N \to \infty$ with respect to the weak convergence in the sets for which the stability of the sets $\mathcal{E}_\omega$, $\mathcal{E}_u$, $\mathcal{E}_\pi$, $\mathcal{E}_S$ holds.*

3.3 Uzawa's algorithm

Uzawa's Algorithm

For $(\vec{\omega}^0_{h,N}, \vec{u}^0_{h,N}, \pi^{1,0}_{h,N}, S^0_{h,N}) = (\vec{0}, \vec{0}, 0, 0)$, find, for all $m \in \{1, \ldots, N\}$ and for all $r \in \mathbb{N}$, $(\vec{\omega}^{m,r+1}_{h,N}, \vec{u}^{m,r+1}_{h,N}, \pi^{m,r+1}_{h,N}, S^{m,r+1}_{h,N}) \in W_h \times W^s_h \times X_h \times \mathcal{T}_h$ such that

$$
\begin{aligned}
&\frac{1}{\tau/N}\int_\Omega \chi_1 \vec{\omega}^{m,r+1}_{h,N}\cdot\vec{\varphi}_h + \int_{\Omega_f}(\vec{\omega}^{m,r+1}_{h,N}\cdot\nabla)\vec{w}_0\cdot\vec{\varphi}_h + \int_{\Omega_f}(\vec{w}_0\cdot\nabla)\vec{\omega}^{m,r+1}_{h,N}\cdot\vec{\varphi}_h\\
&+\frac{2}{\mathrm{Re}}\int_{\Omega_f} D(\vec{\omega}^{m,r+1}_{h,N}) : D(\vec{\varphi}_h) + \alpha_f\int_{\Omega_f} S^{m,r+1}_{h,N}\vec{g}\cdot\vec{\varphi}_h - \int_{\Omega_f}\pi^{m,r}_{h,N} D_h\vec{\varphi}_h\\
&+k\alpha_s\frac{\rho_s}{\rho_0}\int_{\Omega_f} S^{m,r+1}_{h,N}\mathrm{div}\,\vec{\varphi}_h + \frac{\rho_s}{\rho_0}\sum_{i,j=1}^2\int_{\Omega_s} A_{ij}\frac{\partial\vec{u}^{m,r+1}_{h,N}}{\partial x_j}\cdot\frac{\partial\vec{\varphi}_h}{\partial x_i}\\
&+\varepsilon\frac{\rho_s}{\rho_0}\sum_{i,j=1}^2\int_{\Omega_s} B_{ij}\frac{\partial\vec{\omega}^{m,r+1}_{h,N}}{\partial x_j}\cdot\frac{\partial\vec{\varphi}_h}{\partial x_i} + k\alpha_s\frac{\rho_s}{\rho_0}\int_\Omega \nabla S^{m,r+1}_{h,N}\cdot\vec{\varphi}_h\\
&=\frac{1}{\tau/N}\int_\Omega\chi_1\vec{\omega}^{m-1}_{h,N}\cdot\vec{\varphi}_h + \int_\Omega \vec{K}^m_N\cdot\vec{\varphi}_h - k\alpha_s\frac{\rho_s}{\rho_0}\int_{\Omega_f}((\tilde{T}_g)^m_N - T_0)\mathrm{div}\,\vec{\varphi}_h,\\
&\hfill \forall\,\vec{\varphi}_h\in W_h,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\tau/N}\sum_{i,j=1}^2\int_{\Omega_s} A_{ij}\frac{\partial\vec{u}^{m,r+1}_{h,N}}{\partial x_j}\cdot\frac{\partial\vec{\psi}_h}{\partial x_i} - \sum_{i,j=1}^2\int_{\Omega_s} A_{ij}\frac{\partial\vec{\omega}^{m,r+1}_{h,N}}{\partial x_j}\cdot\frac{\partial\vec{\psi}_h}{\partial x_i}\\
&\qquad = \frac{1}{\tau/N}\sum_{i,j=1}^2\int_{\Omega_s} A_{ij}\frac{\partial\vec{u}^{m-1}_{h,N}}{\partial x_j}\cdot\frac{\partial\vec{\psi}_h}{\partial x_i},\ \ \forall\,\vec{\psi}_h\in\mathcal{W}^s_h,
\end{aligned}
\tag{UA}
$$

$$
\begin{aligned}
&\frac{1}{\tau/N}\int_\Omega\chi_2 S^{m,r+1}_{h,N}\eta_h + \int_{\Omega_f}\eta_h\vec{\omega}^{m,r+1}_{h,N}\cdot\nabla T_0 + \int_{\Omega_f}\eta_h\vec{w}_0\cdot\nabla S^{m,r+1}_{h,N}\\
&+\int_\Omega\chi_3\nabla S^{m,r+1}_{h,N}\cdot\nabla\eta_h - \frac{c_s}{c_f}\frac{\rho_s}{\rho_0}\mathrm{Ec}k\alpha_s\int_\Omega\nabla(T_0\eta_h)\cdot\vec{\omega}^{m,r+1}_{h,N}\\
&=\frac{1}{\tau/N}\int_\Omega\chi_2 S^{m-1}_{h,N}\eta_h + \int_\Omega G^m_N\eta_h - \int_\Omega\vec{F}^m_N\cdot\nabla\eta_h,\ \ \forall\,\eta_h\in\mathcal{T}_h,\\
&\int_{\Omega_f}\pi^{m,r+1}_{h,N}q_h + \rho\int_{\Omega_f} D_h\vec{\omega}^{m,r+1}_{h,N}q_h = \int_{\Omega_f}\pi^{m,r}_{h,N}q_h,\ \forall\, q_h\in X_h.
\end{aligned}
$$

The unknown of the previous problem is $(\vec{\omega}^{m,r+1}_{h,N}, \vec{u}^{m,r+1}_{h,N}, \pi^{m,r+1}_{h,N}, S^{m,r+1}_{h,N}) \in W_h \times W^s_h \times X_h \times \mathcal{T}_h$. We notice that, unlike $\vec{\omega}^m_{h,N}$, which is subject to the constraint of the space $\mathcal{W}_h$, the first component of the solution to (UA) belongs to the space without constraint W_h.

We prove next the convergence of the Uzawa's algorithm.

Theorem 3.2. *Let $h > 0$, $N \in \mathbb{N}^*$ with $N \geq \hat{N}$, $m \in \{1, \ldots, N\}$ and $r \in \mathbb{N}$. We suppose that ρ satisfies:*

$$0 < \rho < \frac{1}{2Re}. \tag{3.1}$$

Then, for fixed values of h, N, m, we have the following convergences when $r \to \infty$:

$$\begin{cases} \vec{\omega}^{m,r}_{h,N} \to \vec{\omega}^{m}_{h,N} \ \textit{strongly in} \ W_h, \\ \pi^{m,r}_{h,N} \to \pi^{m}_{h,N} \ \textit{strongly in} \ L^2(\Omega_f), \\ \vec{u}^{m,r}_{h,N} \to \vec{u}^{m}_{h,N} \ \textit{strongly in} \ W^s_h, \\ S^{m,r}_{h,N} \to S^{m}_{h,N} \ \textit{strongly in} \ \mathcal{T}_h. \end{cases} \tag{3.2}$$

The results of this section correspond to the case $n = 2$.

Chapter 4

NUMERICAL SIMULATIONS

The results presented in this chapter were published in *Stavre, R., Ciorogar, A., 2025. Influence of a Given Field of Temperature on the Blood Pressure Variation: Variational Analysis, Numerical Algorithms and Simulations, Axioms, 14, 88*.

The last part of this thesis is devoted to numerical simulations chosen in order to emphasize physical phenomena related to the considered problem. The software used for the following simulations was MATLAB.

In all examples presented below, we considered a simplified domain, namely $\Omega = [0, 4] \times [0, 12]$, representing the cross-section of a blood vessel, as in Figure 2.1, left.

The approximation schemes used for the simulations are associated with the viscoelastic problems. We use a finite difference method for time and a finite element method for space. Then, we use Uzawa's algorithm. The small parameter ε introduced with the family of viscoelastic problems, which is a fixed parameter in the numerical schemes, must be higher than $\max\left\{\frac{\tau}{N}, \mathrm{d}xf, \mathrm{d}xs, \mathrm{d}y\right\}$, where $\frac{\tau}{N}$ is the time step and $\mathrm{d}xf, \mathrm{d}xs$ are the space steps on Ox in the fluid, solid domain respectively and $\mathrm{d}y$ is the space step on the Oy axis. We have chosen different space and time steps in order to obtain optimum convergence.

The results of the following sections are obtained for the two-dimensional case and have been done starting from the (UA) system.

4.1 Temperature influence on the fluid–structure coupling

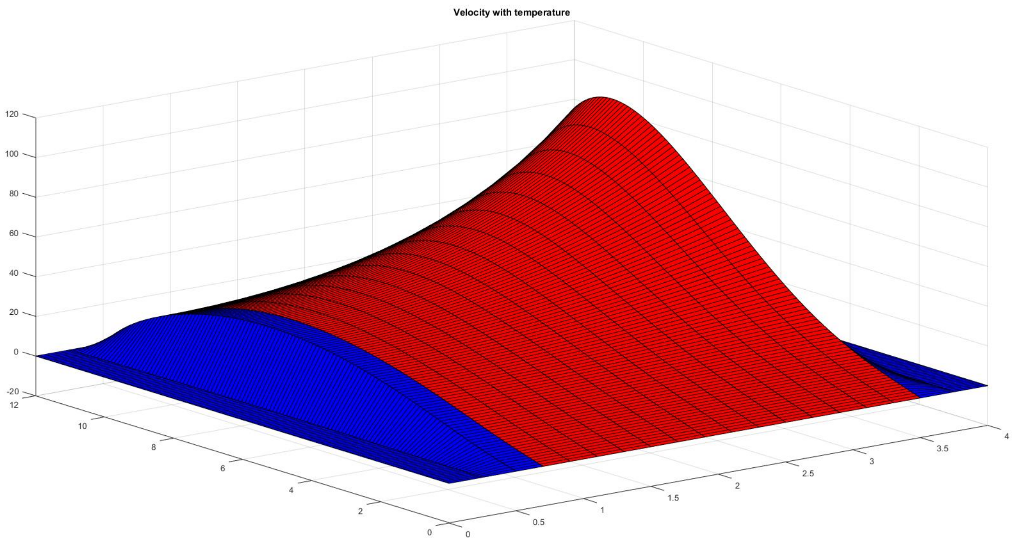

Figure 4.1: The velocity with temperature.

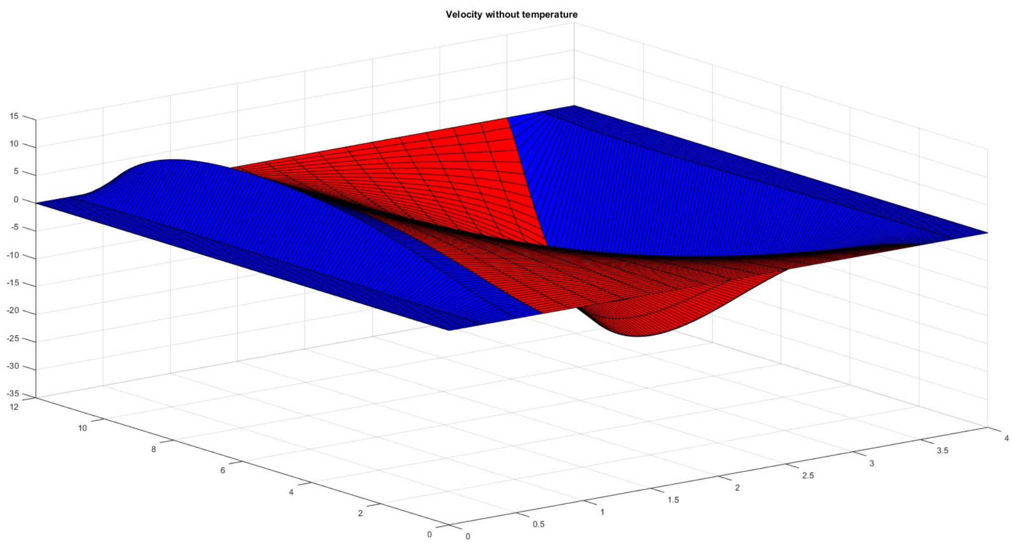

Figure 4.2: The velocity without temperature.

4.2 Influence of a given field of temperature on the blood pressure variation

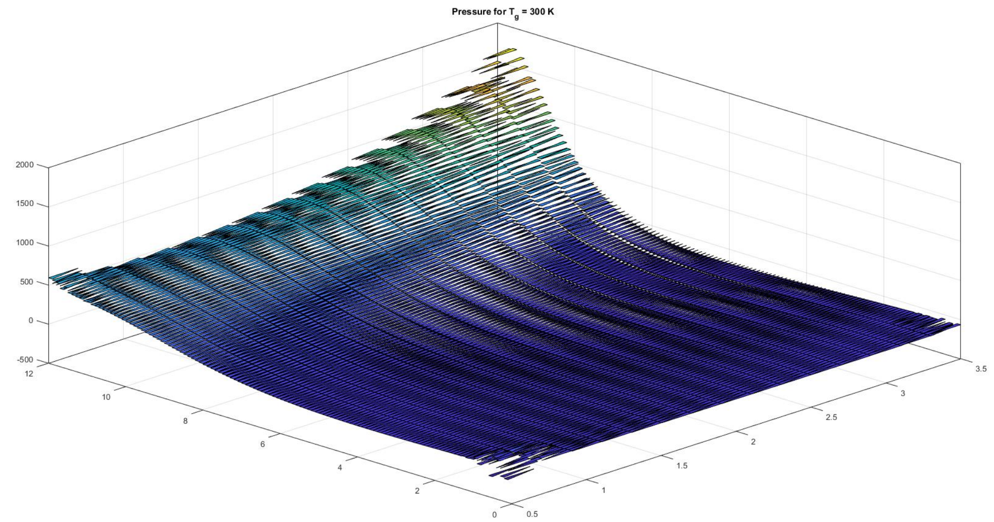

Figure 4.3: The pressure profile in fluid for T_g = 300 K.

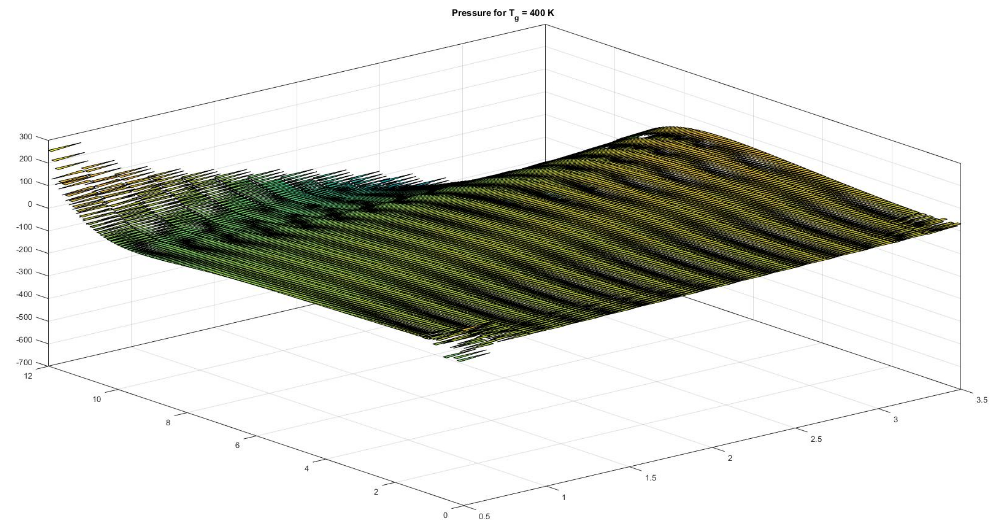

Figure 4.4: The pressure profile in fluid for T_g = 400 K.

4.3 Influence of the compression forces on the blood reflux phenomena in venous insufficiency

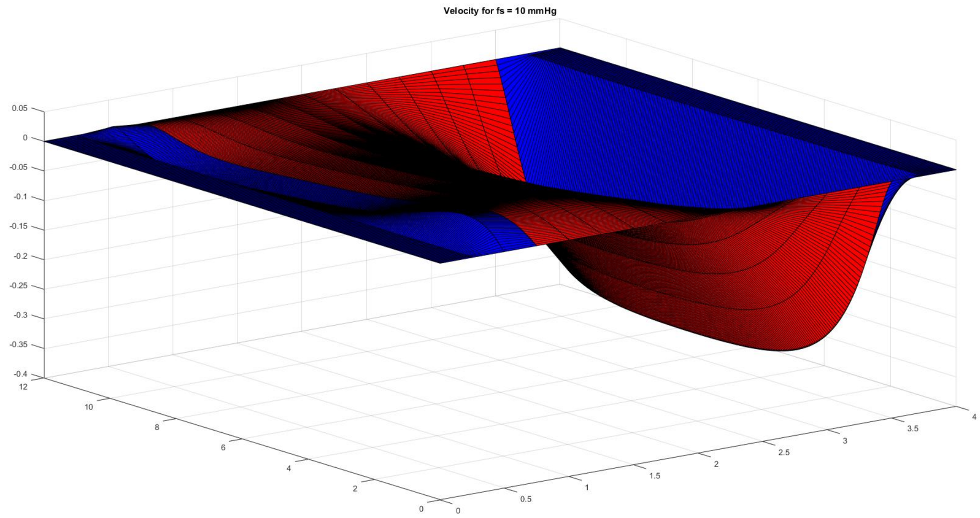

Figure 4.5: The velocity profile for f_s = 10 mm Hg.

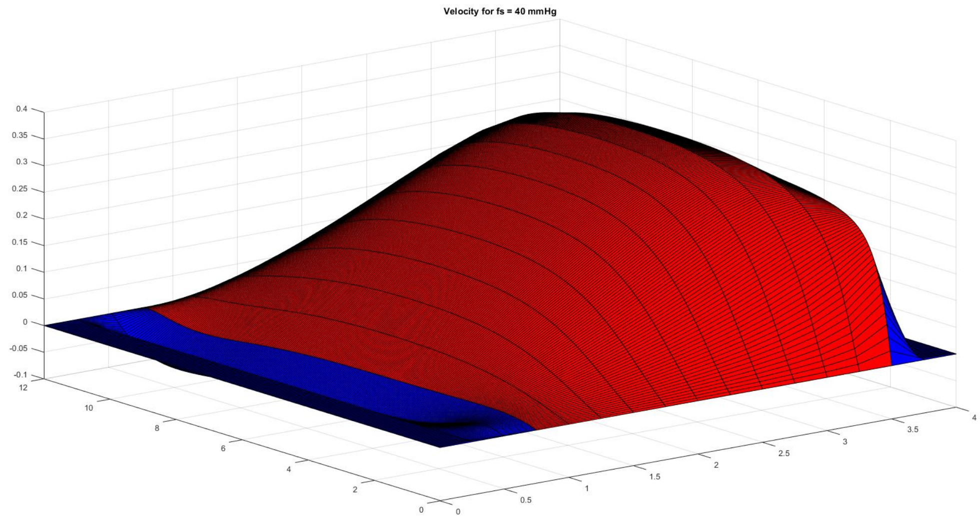

Figure 4.6: The velocity profile for f_s = 40 mm Hg.

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