SEQUENTIAL PRODUCT OF QUANTUM EFFECTS

AURELIAN GHEONDEA AND STANLEY GUDDER

ABSTRACT. Unsharp quantum measurements can be modelled by means of the class $\mathcal{E}(\mathcal{H})$ of positive contractions on a Hilbert space $\mathcal{H}$, in brief, quantum effects. For $A, B \in \mathcal{E}(\mathcal{H})$ the operation of sequential product $A \circ B = A^{1/2}BA^{1/2}$ was proposed as a model for sequential quantum measurements, cf. [5] and the literature quoted there. We continue these investigations on sequential product and answer positively the following question: the assumption $A \circ B \geq B$ implies $AB = BA = B$.

Then we propose a geometric approach of quantum effects and their sequential product by means of contractively contained Hilbert spaces and operator ranges. This framework leads us naturally to consider lattice properties of quantum effects, sums and intersections, and to prove that the sequential product is left distributive with respect to the intersection.

1. Introduction

Unsharp quantum measurements experiments can be modelled by means of the class $\mathcal{E}(\mathcal{H})$ of positive contractions on a Hilbert space $\mathcal{H}$. For $A, B \in \mathcal{E}(\mathcal{H})$ the operation of sequential product $A \circ B = A^{1/2}BA^{1/2}$ was proposed as a model for sequential measurements. A careful investigation of properties of the sequential product has been carried over in [5] (see also [6]). In that paper it was proved that, if the underlying Hilbert space $\mathcal{H}$ is finite dimensional, then, by preceding the effect B with another effect A, we cannot amplify the effect B. In terms of the operator model, this means that if for some $A, B \in \mathcal{E}(\mathcal{H})$ we have $A \circ B \geq B$, then $AB = BA = B$. The question, raised in [5], is whether this is true for infinite dimensional Hilbert spaces as well. We answer positively this question in Theorem 2.6. The idea of our proof is to iterate the inequality $A \circ B \geq B$ and to show that, at the limit, we obtain exactly the condition $AB = BA = B$. Here we use the borelian functional calculus for selfadjoint operators, that is, we pass to the von Neumann algebra generated by the operator A to get at the limit the orthogonal projection onto $\operatorname{Ker}(I - A)$, a common procedure in ergodic theory. The second section contains also a few other results on the sequential product, especially concerning comparison.

In the third section we propose a geometric approach of the sequential product based on the interpretation of quantum effects as contractively contained Hilbert spaces. This is a particular situation of the theory of continuously contained Hilbert spaces in quasi-complete locally convex spaces of L. Schwartz [8]. Similar ideas have been used in the case of contractively contained Hilbert spaces in the de Branges-Rovnyak model in quantum scattering [2], and operator ranges, cf. [4] and [7]. This allows us to consider lattice properties, as sums and intersections of the set of quantum effects,

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that lead naturally to the parallel sum of quantum effects. The sequential product is a morphism, in the second variable, with respect to the binary operation of parallel sum, cf. Theorem 3.10. Rephrased in terms of contractively contained Hilbert spaces, this means that the sequential product is left distributive with respect to the intersection.

2. Sequential Product

Let $\mathcal{H}$ be a Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the C^*-algebra of linear and bounded operators on $\mathcal{H}$, $\mathcal{B}(\mathcal{H})^+ = \{A \in \mathcal{B}(\mathcal{H}) \mid A \geq 0\}$ its positive cone, and by $\mathcal{E}(\mathcal{H})$ we denote the set of *quantum effects*, or simply, *effects*, that is, operators $A \in \mathcal{B}(\mathcal{H})$ such that $0 \leq A \leq I$. For two effects $A, B \in \mathcal{E}(\mathcal{H})$ we denote by $A \circ B = A^{1/2}BA^{1/2}$ the *sequential product* of the effects A and B, that is, the effect obtained by doing the mesurements in the order, first A and second B. Clearly, $A \circ B \leq A$ and a first question is when $A \circ B \leq B$. A sufficient condition to have this is the case when A and B are *compatible*, that is, $AB = BA$, but we would like to have a complete characterization. The answer is given by a particular case of the general majorization theorem (e.g. see [3]).

Theorem 2.1. *Let $T_i \in \mathcal{B}(\mathcal{H}_i, \mathcal{G})$ for Hilbert spaces $\mathcal{H}_i$ and $\mathcal{G}$, $i = 1, 2$. Then for $0 \leq \gamma < \infty$ the following conditions are equivalent:*

(a) $\|T_1^* g\|_{\mathcal{H}_1} \leq \gamma \|T_2^* g\|_{\mathcal{H}_2}$ *for all $g \in \mathcal{G}$;*
(b) $T_1 T_1^* \leq \gamma T_2 T_2^*$*;*
(c) $T_1 = T_2 S$ *for some operator $S \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ with $\|S\| \leq \gamma$.*

Each of these equivalent conditions implies

(d) $\operatorname{Ran}(T_1) \subseteq \operatorname{Ran}(T_2)$.

Conversely, (d) *implies all of* (a) *through* (c)*, for some $\gamma < \infty$, the operator S can be chosen such that* $\operatorname{Ker}(T_1) \subseteq \operatorname{Ker}(S)$ *and* $\operatorname{Ker}(T_2) \subseteq \operatorname{Ker}(S^*)$*, in which case, it is uniquely determined.*

As a consequence, given two operators $X, Y \in \mathcal{B}(\mathcal{H})$, we have $XX^* \leq YY^*$ if and only if $X = YT$ for some $T \in \mathcal{B}(\mathcal{H})$, $\|T\| \leq 1$. In the following, for $A \in \mathcal{E}(\mathcal{H})$ we denote by P_A the selfadjoint projection onto the closure of $\operatorname{Ran}(A)$ (which is the same with the closure of $\operatorname{Ran}(A^{1/2})$).

Theorem 2.2. *Let $A, B \in \mathcal{E}(\mathcal{H})$ be effects. Then $A \leq B$ if and only if there exists another effect $C \in \mathcal{E}(\mathcal{H})$ such that $A = B \circ C$. In addition, C can be chosen such that $C \leq P_A$ and in this case it is uniquely determined.*

Proof. Clearly, if $A = B \circ C$ for some effect $C \in \mathcal{E}(\mathcal{H})$, then $A = B \circ C \leq B$.

Conversely, let us assume that $A \leq B$. Then $A^{1/2}A^{1/2} \leq B^{1/2}B^{1/2}$ and hence, by Theorem 2.1, there exists $T \in \mathcal{B}(\mathcal{H})$, such that $\|T\| \leq 1$ and $A^{1/2} = B^{1/2}T$. Then $C = TT^* \in \mathcal{E}(\mathcal{H})$ and hence $A = B^{1/2}TT^*B^{1/2} = B \circ C$. The uniqueness part follows from the uniqueness part of Theorem 2.1. ☐

Corollary 2.3. *Let $A, B \in \mathcal{E}(\mathcal{H})$. Then $A \circ B \leq B$ if and only if there exists $C \in \mathcal{E}(\mathcal{H})$ such that $A \circ B = B \circ C$.*

Corollary 2.3 says that the effect $A \circ B$ diminishes the effect B if and only if there is another effect C such that, performing first the measurement B and then C, the obtained sequential measurement $B \circ C$ is the same as $A \circ B$.

In Theorem 3.2 in [5] it is proved that, given $A, B \in \mathcal{E}(\mathcal{H})$ we have the associativity property $(A \circ B) \circ C = A \circ (B \circ C)$, for all $C \in \mathcal{E}(\mathcal{H})$, if and only if A and B are compatible, that is, $AB = BA$. As a consequence of Theorem 2.2, we have:

Corollary 2.4. *If $A, B, C \in \mathcal{E}(\mathcal{H})$ are effects then there exists an effect $D \in \mathcal{E}(\mathcal{H})$ such that $A \circ (B \circ C) = (A \circ B) \circ D$.*

Proof. Since $B \circ C \leq B$, we have $A \circ (B \circ C) \leq A \circ B$ and then by Theorem 2.2, there exists $D \in \mathcal{E}(\mathcal{H})$ such that $A \circ (B \circ C) = (A \circ B) \circ D$. □

For a fixed $A \in \mathcal{E}(\mathcal{H})$, the mapping $\mathcal{E}(\mathcal{H}) \ni B \mapsto s_A(B) = A \circ B \in \mathcal{E}(\mathcal{H})$ is strongly continuous and affine. We are interested in sections of the sequential mapping s_A. Since $A \circ B \leq A$, it follows that such a section should be defined on $[0, A] = \{C \in \mathcal{E}(\mathcal{H}) \mid C \leq A\}$. Recall that P_A denotes the selfadjoint projection onto the closure of $\operatorname{Ran}(A)$. By Theorem 2.2, for any $B \in [0, A]$ there exists a unique $C \in [0, P_A]$ such that $B = A \circ C$. Denote $B/A = C$ and call it the *sequential quotient* of B by A. In this way, we can define a mapping

$$f_A \colon [0, A] \to [0, P_A], \quad f_A(B) = B/A, \ B \in [0, A]. \tag{2.1}$$

Clearly, f_A is a section of s_A.

Let us also note that the "segment" $[0, A]$ has a natural sequential product, that we denote by $\times$,

$$C \times D = A \circ \big((C/A) \circ (D/A)\big), \quad C, D \in [0, A]. \tag{2.2}$$

Theorem 2.5. *For any $A \in \mathcal{E}(\mathcal{H})$, the mapping f_A defined as in* (2.1) *is an affine strongly continous homeomorphism, such that*

$$f_A(C \times D) = f_A(C) \circ f_A(D), \quad C, D \in [0, A]. \tag{2.3}$$

Proof. Note first that f_A is bijective: its inverse is $s_A|[0, P_A] \colon [0, P_A] \to [0, A]$. Thus, since s_A is affine and $[0, A]$ is convex, the same is $s_A|[0, P_A]$, hence f_A is affine. To see that f_A is strongly continuous, note that

$$\langle Bx, y \rangle = \langle (B/A) A^{1/2} x, A^{1/2} y \rangle, \quad B \in \mathcal{E}(\mathcal{H}), \ x, y \in \mathcal{H},$$

and recall that the strong operator topology coincides with the weak operator topology on bounded sets of positive operators. Thus, f_A is an affine strongly continuous homeomorphism. The property expressed at (2.3) is just another way of writing (2.2). □

We are now interested in the question when $A \circ B \geq B$. In Theorem 2.6 in [5] it is proved that, if $\mathcal{H}$ is finite dimensional, by preceding the effect B with another effect A, we cannot amplify the effect B, more precisely, if $A \circ B \geq B$ then $AB = BA = B$, and it is asked whether this holds for infinite dimensional spaces $\mathcal{H}$. In the following theorem we answer positively this question.

Theorem 2.6. *Let $A, B \in \mathcal{E}(\mathcal{H})$, and such that $A \circ B \geq B$. Then $AB = BA = B$ and, consequently, $A \circ B = B$.*

Proof. In the following we will repeatedly use the following elementary fact: if $C, D \in \mathcal{B}(\mathcal{H})$ are selfadjoint and such that $C \leq D$ then for any $X \in \mathcal{B}(\mathcal{H})$ we have $X^*CX \leq X^*DX$.

Since $B \leq I$ it follows that $A^{1/2}BA^{1/2} \leq A^{1/2}A^{1/2} = A$. Therefore, by hypothesis we have

$$0 \leq B \leq A^{1/2}BA^{1/2} \leq A \leq I. \tag{2.4}$$

By applying $A^{1/2} \cdot A^{1/2}$ to (2.4), it follows that

$$0 \leq A^{1/2}BA^{1/2} \leq A^{1/2}A^{1/2}BA^{1/2}A^{1/2} = ABA \leq A^{1/2}AA^{1/2} = A^2 \leq A,$$

and hence, using again (2.4), we have

$$0 \leq B \leq A^{1/2}BA^{1/2} \leq A^{2/2}BA^{2/2} \leq A^2 \leq A \leq I.$$

Performing a similar procedure, we obtain inductively that for all $n \geq 1$ we have

$$0 \leq B \leq A^{1/2}BA^{1/2} \leq \cdots \leq A^{n/2}BA^{n/2} \leq A^n \leq \cdots \leq A \leq I. \tag{2.5}$$

From (2.5) we keep only

$$0 \leq B \leq A^n \leq I, \quad \forall n \geq 1. \tag{2.6}$$

Let us now consider the sequence of functions $f_n(t) = t^n$, $f_n \colon [0,1] \to \mathbb{R}_+$. Note that $|f_n(t)| \leq 1$ for all $n \geq 1$ and that

$$\lim_{n\to\infty} f_n(t) = \chi_{\{1\}}(t) = \begin{cases} 1, & t = 1, \\ 0, & 0 \leq t < 1, \end{cases} \qquad \forall t \in [0,1],$$

that is, the sequence of functions t^n converges boundedly pointwise, on the compact interval $[0,1] \supseteq \sigma(A)$, to the characteristic function of the set $\{1\}$. Thus, by borelian functional calculus for selfadjoint operators we have

$$\text{so-}\lim_{n\to\infty} A^n = \text{so-}\lim_{n\to\infty} f_n(A) = \chi_{\{1\}}(A) = P_{\operatorname{Ker}(I-A)}. \tag{2.7}$$

Consequently, letting $n \to \infty$ in (2.6), by (2.7) we obtain

$$0 \leq B \leq P_{\operatorname{Ker}(I-A)}. \tag{2.8}$$

If we represent the selfadjoint operators A and B with respect to the decomposition

$$\mathcal{H} = \operatorname{Ker}(I-A) \oplus (\mathcal{H} \ominus \operatorname{Ker}(I-A)),$$

by (2.8) we get

$$A = \begin{bmatrix} I & 0 \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

and now it is easy to see that $AB = BA = B$. $\square$

3. Contractively Contained Hilbert Spaces

Let $\mathcal{K}$ and $\mathcal{H}$ be Hilbert spaces. We say that $\mathcal{K}$ is *continuously contained* in $\mathcal{H}$, and we write $\mathcal{K} \hookrightarrow \mathcal{H}$, if $\mathcal{K} \subseteq \mathcal{H}$ and the inclusion mapping $\iota_{\mathcal{K}} \colon \mathcal{K} \hookrightarrow \mathcal{H}$ is continuous, that is, there exists $\gamma \geq 0$ such that $\|k\|_{\mathcal{H}} \leq \gamma \|k\|_{\mathcal{K}}$, for all $k \in \mathcal{K}$. $\mathcal{K}$ is *contractively contained* in $\mathcal{H}$ if $\mathcal{K} \subseteq \mathcal{H}$ and the inclusion mapping $\iota_{\mathcal{K}} \colon \mathcal{K} \hookrightarrow \mathcal{H}$ is contractive, that is, $\|k\|_{\mathcal{H}} \leq \|k\|_{\mathcal{K}}$, for all $k \in \mathcal{K}$.

Let $\mathcal{K} \hookrightarrow \mathcal{H}$ and denote $A = \iota_{\mathcal{K}} \iota_{\mathcal{K}}^* \in \mathcal{B}(\mathcal{H})$. Then $A \geq 0$ and, following L. Schwartz [8], we call it the *kernel* of $\mathcal{K}$ relative to $\mathcal{H}$.

In the following we propose a geometric approach of quantum effects and their sequential product. Let $A \in \mathcal{B}(\mathcal{H})$, $A \geq 0$. Define $\mathcal{R}_A = \operatorname{Ran}(A^{1/2})$ and the inner product

$$\langle A^{1/2}x, A^{1/2}y \rangle_{\mathcal{R}_A} = \langle P_{\mathcal{H} \ominus \operatorname{Ker}(A)} x, y \rangle_{\mathcal{H}}, \quad x, y \in \mathcal{H}. \tag{3.1}$$

More precisely, we define a Hilbert space structure on $\operatorname{Ran}(A^{1/2})$ in such a way that the operator $A^{1/2} \colon \mathcal{H} \to \operatorname{Ran}(A^{1/2})$ is a coisometry. Thus, $(\mathcal{R}_A, \langle \cdot, \cdot \rangle_{\mathcal{R}_A})$ is a Hilbert space. We note that

$$\|A^{1/2}x\|_{\mathcal{R}_A} = \|P_{\mathcal{H} \ominus \operatorname{Ker}(A)} x\|, \quad x \in \mathcal{H}.$$

Then

$$\begin{aligned} \|A^{1/2}x\|_{\mathcal{H}} &= \|A^{1/2} P_{\mathcal{H} \ominus \operatorname{Ker}(A)} x\|_{\mathcal{H}} \leq \|A^{1/2}\| \cdot \|P_{\mathcal{H} \ominus \operatorname{Ker}(A)} x\| \\ &= \|A^{1/2}\| \cdot \|A^{1/2}x\|_{\mathcal{R}_A}, \quad x \in \mathcal{H}, \end{aligned}$$

Thus, $\mathcal{R}_A \hookrightarrow \mathcal{H}$ and $\|\iota_{\mathcal{R}_A}\| \leq \|A^{1/2}\|$, in particular, if $A \leq I$ then $\mathcal{R}_A \hookrightarrow \mathcal{H}$ contractively.

Let $x, y \in \mathcal{H}$ be arbitrary. Then

$$\begin{aligned} \langle \iota_{\mathcal{R}_A} y, A^{1/2}x \rangle_{\mathcal{R}_A} &= \langle y, A^{1/2}x \rangle_{\mathcal{H}} = \langle A^{1/2}y, x \rangle_{\mathcal{H}} \\ &= \langle P_{\mathcal{H} \ominus \operatorname{Ker}(A)} A^{1/2} y, x \rangle_{\mathcal{H}} = \langle A^{1/2} A^{1/2} y, A^{1/2} x \rangle_{\mathcal{R}_A}, \end{aligned}$$

that is,

$$\iota_{\mathcal{R}_A} y = A^{1/2} A^{1/2} y = P_{\mathcal{H} \ominus \operatorname{Ker}(A)} A y, \quad y \in \mathcal{H},$$

and hence $\iota_{\mathcal{R}_A} \iota_{\mathcal{R}_A}^* = A$. Thus A is the kernel of $\mathcal{R}_A$ relative to $\mathcal{H}$. In particular, if $\mathcal{R}_A \hookrightarrow \mathcal{H}$ contractively, then $A \leq I$.

Let $\mathcal{K}$ be another Hilbert space continuously contained in $\mathcal{H}$ and such that $\iota_{\mathcal{K}} \iota_{\mathcal{K}}^* = A$. Then $\operatorname{Ran}(A) = \operatorname{Ran}(\iota_{\mathcal{K}}^*)$ is dense in $\mathcal{K}$. For arbitray $x, y \in \mathcal{H}$ we have

$$\begin{aligned} \langle Ax, Ay \rangle_{\mathcal{R}_A} &= \langle A^{1/2}x, A^{1/2}y \rangle_{\mathcal{H}} = \langle Ax, y \rangle_{\mathcal{H}} = \langle \iota_{\mathcal{K}} \iota_{\mathcal{K}}^* x, y \rangle_{\mathcal{H}} \\ &= \langle \iota_{\mathcal{K}}^* x, \iota_{\mathcal{K}}^* y \rangle_{\mathcal{K}} = \langle Ax, Ay \rangle_{\mathcal{K}}, \end{aligned}$$

that is, $\langle \cdot, \cdot \rangle_{\mathcal{R}_A}$ coincide with $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ on a dense linear manifold, hence $\mathcal{R}_A = \mathcal{K}$ as Hilbert spaces.

We thus have proved the following theorem, that is a particular case of results in [8] (see also [2] and [4]).

Theorem 3.1. *Let $A \in \mathcal{B}(\mathcal{H})^+$. Then there exists uniquely a Hilbert space $\mathcal{R}_A \hookrightarrow \mathcal{H}$ such that A is the kernel of $\mathcal{R}_A$ relative to $\mathcal{H}$. In addition, $\mathcal{R}_A \hookrightarrow \mathcal{H}$ contractively if and only if $A \in \mathcal{E}(\mathcal{H})$.*

Recall that for a Hilbert space $\mathcal{H}$ we denote by $\mathcal{E}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \mid 0 \leq A \leq I\}$ identified with the set of quantum effects on $\mathcal{H}$, and by $\mathcal{P}(\mathcal{H}) = \{P \in \mathcal{B}(\mathcal{H}) \mid P = P^* = P^2\}$ the set of orthogonal projections, identified with the set of sharp effects. As a consequence of Theorem 3.1, it follows that the mapping $A \mapsto \mathcal{R}_A$ induces a bijection between $\mathcal{E}(\mathcal{H})$ and the class of Hilbert spaces contractively contained in $\mathcal{H}$. This extends the bijective correspondence of $\mathcal{P}(\mathcal{H})$ with the class of subspaces of $\mathcal{H}$; a Hilbert space $\mathcal{K}$ contractively contained in $\mathcal{H}$ is a subspace of $\mathcal{H}$ if and only if the inclusion is isometric, and in this case $\iota_\mathcal{K} \iota_\mathcal{K}^*$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$. Note that, in this correspondence, subspace means that $\langle \cdot, \cdot \rangle_\mathcal{K}$ coincides with the restriction of $\langle \cdot, \cdot \rangle_\mathcal{H}$; if we impose only the condition that the strong topology of $\mathcal{K}$ is inherited from $\mathcal{H}$ we get the class of kernels with closed range. We make precise this in our context.

Corollary 3.2. *Let $A, B \in \mathcal{B}(\mathcal{H})^+$. The following assertions are equivalent:*

(i) *$\mathcal{R}_A \subseteq \mathcal{R}_B$ (set inclusion);*
(ii) *$\mathcal{R}_A \hookrightarrow \mathcal{R}_B$;*
(iii) *There exists $\gamma \geq 0$ such that $A \leq \gamma B$.*

In addition, denoting $\iota \colon \mathcal{R}_A \hookrightarrow \mathcal{R}_B$, the optimal constant γ in (iii) *is $\|\iota\|$. In particular, $\mathcal{R}_A \hookrightarrow \mathcal{R}_B$ contractively if and only if $A \leq B$.*

Proof. Consequence of Theorem 2.1 and Theorem 3.1. □

Corollary 3.3. *Let $A, B \in \mathcal{B}(\mathcal{H})^+$. The following assertions are equivalent:*

(i) *$\mathcal{R}_A = \mathcal{R}_B$ (equality of sets);*
(ii) *$\mathcal{R}_A = \mathcal{R}_B$ (equality of sets) and the norms are equivalent;*
(iii) *There exists $\gamma > 0$ such that $\frac{1}{\gamma} B \leq A \leq \gamma B$.*

As a consequence of Corollary 3.3, two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, continuously contained in the Hilbert space $\mathcal{H}$, coincide as sets, if and only they coincide topologically, that is, their strong topologies coincide.

On the ground of Theorem 3.1 we can define an operation of *sequential product* for contractively contained Hilbert spaces. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces contractively contained in $\mathcal{H}$, and let $A, B \in \mathcal{E}(\mathcal{H})$ be their kernels, that is, $\mathcal{H}_1 = \mathcal{R}_A$ and $\mathcal{H}_2 = \mathcal{R}_B$. Then $A \circ B = A^{1/2} B A^{1/2} \in \mathcal{E}(\mathcal{H})$ and we define $\mathcal{H}_1 \circ \mathcal{H}_2 = \mathcal{R}_{A \circ B} \hookrightarrow \mathcal{H}$ contractively. Of course, this definition can be extended for continuously contained Hilbert spaces, but in this more general case we have only that $\mathcal{H}_1 \circ \mathcal{H}_2$ is continuosly contained in $\mathcal{H}$, unless $\mathcal{H}_1$ is contractively contained in $\mathcal{H}$. The next corollary says that the sequential product is natural for the study of contractively contained Hilbert spaces (compare with Theorem 2.2).

Corollary 3.4. *Let $\mathcal{H}_1 \hookrightarrow \mathcal{H}$ contractively. Then, for any Hilbert space $\mathcal{H}_2 \hookrightarrow \mathcal{H}_1$ contractively, there exists $\mathcal{H}_3 \hookrightarrow \mathcal{H}$ contractively, such that $\mathcal{H}_2 = \mathcal{H}_1 \circ \mathcal{H}_3$.*

Proof. Consequence of Theorem 2.2 and Corollary 3.2. □

Another consequence of Theorem 3.1 is that we can define a 'sequential quotient' for contractively contained Hilbert spaces as the 'inverse' of the operation in Corollary 3.4: if the Hilbert space $\mathcal{H}_2 \hookrightarrow \mathcal{H}_1$ contractively, then $\mathcal{H}_2/\mathcal{H}_1$ is the unique Hilbert space $\mathcal{H}_3 \hookrightarrow \mathcal{H}$ contractively, such that $\mathcal{H}_2 = \mathcal{H}_1 \circ \mathcal{H}_3$.

Let now $\mathcal{K}$ be a Hilbert space contractively contained in $\mathcal{H}$. For any Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ contractively contained in $\mathcal{K}$, a natural 'sequential product' denoted by $\times$ can be defined

$$\mathcal{H}_1 \times \mathcal{H}_2 = \mathcal{K} \circ ((\mathcal{H}_1/\mathcal{K}) \circ (\mathcal{H}_2/\mathcal{K})), \tag{3.2}$$

by analogy with (2.2). Then the sequential quotient is left distributive with respect to these sequential products, which is just another way of writing (3.2), by analogy with (2.3).

Corollary 3.5. *Let $\mathcal{K}$ be a Hilbert space contractively contained in $\mathcal{H}$. For any Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ contractively contained in $\mathcal{K}$, we have*

$$(\mathcal{H}_1 \times \mathcal{H}_2)/\mathcal{K} = (\mathcal{H}_1/\mathcal{K}) \circ (\mathcal{H}_2/\mathcal{K}).$$

The geometric approach of quantum effects allows us to consider the lattice properties of $\mathcal{E}(\mathcal{H})$, especially those in connection with 'addition' and 'intersection' of continuously contained Hilbert spaces. The addition of continously contained Hilbert spaces should correspond to addition of the kernels, cf. [8]. Here is a direct argument, cf. [4].

Let $A, B \in \mathcal{B}(\mathcal{H})^+$ be such that $\mathcal{H}_1 = \mathcal{R}_A$ and $\mathcal{H}_2 = \mathcal{R}_B$. We consider the bounded operator $T = \begin{bmatrix} A^{1/2} & -B^{1/2} \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}, \mathcal{H})$. Then

$$\begin{aligned} \mathcal{R}_A + \mathcal{R}_B &= \operatorname{Ran}(A^{1/2}) + \operatorname{Ran}(B^{1/2}) = \operatorname{Ran}(T) = \operatorname{Ran}((TT^*)^{1/2}) \\ &= \operatorname{Ran}((A+B)^{1/2}) = \mathcal{R}_{A+B}, \end{aligned}$$

where we took into account that, by polar decomposition, $T = |T^*|V$, where $|T^*| = (TT^*)^{1/2}$ and V is a partial isometry. Consequently, we have

Theorem 3.6. (e.g. see [8] and [4]) *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces continuously contained in $\mathcal{H}$. Then the algebraic sum $\mathcal{H}_1 + \mathcal{H}_2$ can be naturally organized as a Hilbert space continuously contained in $\mathcal{H}$, more precisely, if $A, B \in \mathcal{B}(\mathcal{H})^+$ are the kernels of $\mathcal{H}_1$ and respectively $\mathcal{H}_2$, that is, $\mathcal{H}_1 = \mathcal{R}_A$ and $\mathcal{H}_2 = \mathcal{R}_B$, then $\mathcal{R}_A + \mathcal{R}_B = \mathcal{R}_{A+B}$, as sets.*

The intersection of continuously contained Hilbert spaces is a much more subtle operation that requires the definition of *parallel sum* of kernels, as introduced for the finite dimenensional case in [1], and in general in [4] (cf. [7] for a further study). Let $A, B \in \mathcal{B}(\mathcal{H})^+$. Since $A \leq A + B$, by Theorem 2.1 there exists a unique contraction $X \in \mathcal{B}(\mathcal{H})$ such that

$$A^{1/2} = (A+B)^{1/2} X, \quad \operatorname{Ker}(X^*) \supseteq \operatorname{Ker}(A+B). \tag{3.3}$$

Similarly, since $B \leq A + B$, there exists a unique contraction $Y \in \mathcal{B}(\mathcal{H})$ such that

$$B^{1/2} = (A+B)^{1/2} Y, \quad \operatorname{Ker}(Y^*) \supseteq \operatorname{Ker}(A+B). \tag{3.4}$$

The parallel sum of A and B is defined as

$$A\colon B = A^{1/2}X^*YB^{1/2}. \tag{3.5}$$

Theorem 3.7. ([4] and [7]) *Let $A, B \in \mathcal{B}(\mathcal{H})^+$. Then:*

(i) $0 \leq A\colon B \leq A, B$;
(ii) $A\colon B = B\colon A$;
(iii) $\mathrm{Ran}((A\colon B)^{1/2}) = \mathrm{Ran}(A^{1/2}) \cap \mathrm{Ran}(B^{1/2})$;
(iv) *If $A_1, B_1 \in \mathcal{B}(\mathcal{H})^+$ are such that $A \leq A_1$ and $B \leq B_1$, then $A\colon B \leq A_1\colon B_1$;*
(v) *If $A, B \neq 0$ then $\|A\colon B\| \leq (\|A\|^{-1} + \|B\|^{-1})^{-1}$;*
(vi) $\langle (A : B)h, h\rangle = \inf\{\langle Aa, a\rangle + \langle Bb, b\rangle \mid h = a + b\}$, *for all $h \in \mathcal{H}$;*
(vii) *If $A_n \searrow A$ and $B_n \searrow B$ strongly, then $A_n : B_n \searrow A : B$ strongly.*

According to (iii) we can give an interpretation of the binary operation of 'intersection' for continuously contained Hilbert spaces as follows:

Theorem 3.8. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces continuously contained in the Hilbert space $\mathcal{H}$. Then the vector space $\mathcal{H}_1 \cap \mathcal{H}_2$ can be naturally organized as a Hilbert space continuously contained in $\mathcal{H}$, more precisely, letting $\mathcal{H}_1 = \mathcal{R}_A$ and $\mathcal{H}_2 = \mathcal{R}_2$ for $A, B \in \mathcal{B}(\mathcal{H})^+$, identify $\mathcal{H}_1 \cap \mathcal{H}_2$ with $\mathcal{R}_{A:B}$. If either of $\mathcal{H}_i$, $i = 1, 2$ is contractively contained in $\mathcal{H}$, that is, its kernel is a quantum effect, then $\mathcal{H}_1 \cap \mathcal{H}_2$ is contractively contained in $\mathcal{H}$.*

Another simple consequence of Theorem 3.7 is the stability of the set of quantum effects is under the operation of parallel sum, even with the factor 2.

Corollary 3.9. *Let $A, B \in \mathcal{E}(\mathcal{H})$. Then $2(A\colon B) \in \mathcal{E}(\mathcal{H})$.*

Proof. By Theorem 3.7.(i), we have $A\colon B \geq 0$, so it remains to prove that $2(A\colon B) \leq I$. To see this we use Theorem 3.7.(v) and the inequality between the harmonic mean and the geometric mean to see that

$$\|2(A\colon B)\| \leq \frac{2\|A\|\,\|B\|}{\|A\| + \|B\|} \leq \sqrt{\|A\|\,\|B\|} \leq 1,$$

and hence $2(A\colon B) \leq I$. □

We end by proving that the sequential product is a morphism, in the second variable, with respect to the parallel sum. Recall that for any $A \in \mathcal{E}(\mathcal{H})$ we denote $[0, A] = \{B \mid 0 \leq B \leq A\}$ and P_A is the orthogonal projection onto the closure of $\mathrm{Ran}(A)$.

Theorem 3.10. *Let $A \in \mathcal{E}(\mathcal{H})$ and consider the mapping $s_A\colon [0, P_A] \to [0, A]$, $s_A(B) = A \circ B = A^{1/2}BA^{1/2}$, for all $B \in [0, P_A]$. Then*

$$s_A(C : D) = s_A(C) : s_A(D), \quad C, D \in [0, P_A].$$

Proof. Let $C, D \in [0, P_A]$ and consider the operators X, Y uniquely determined such that

$$C^{1/2} = (C + D)^{1/2}X, \quad \mathrm{Ker}(C^*) \supseteq \mathrm{Ker}(C + D),$$

$$D^{1/2} = (C + D)^{1/2}Y, \quad \mathrm{Ker}(D^*) \supseteq \mathrm{Ker}(C + D).$$

Then, by definition,

$$C : D = C^{1/2} X^* Y D^{1/2}, \quad s_A(C : D) = A^{1/2} C^{1/2} X^* Y D^{1/2} A^{1/2}.$$

Since $s_A(C) + s_A(D) = A^{1/2}(C + D)A^{1/2}$, it follows that there exists a uniquely determined partial isometry V, with appropriate supports, such that

$$(s_A(C) + s_A(D))^{1/2} V = A^{1/2}(C + D)^{1/2}.$$

Similarly, there exist uniquely determined partial isometries U and W, with appropriate supports, such that

$$s_A(C)^{1/2} = A^{1/2} C^{1/2} U, \quad s_A(D)^{1/2} = A^{1/2} D^{1/2} W.$$

Therefore,

$$s_A(C)^{1/2} = A^{1/2} C^{1/2} U = A^{1/2}(C + D)^{1/2} X U = (s_A(C) + s_A(D))^{1/2} V X U,$$

$$s_A(D)^{1/2} = A^{1/2} D^{1/2} U = A^{1/2}(C + D)^{1/2} X W = (s_A(C) + s_A(D))^{1/2} V X W,$$

and hence, by the definition of the parallel sum and taking into account that, since C and D are in $[0, P_A]$ it follows that multiplication on the left with $A^{1/2}$ does not affect the kernels, we have

$$\begin{aligned} s_A(C) : s_A(D) &= s_A(C)^{1/2} U^* X^* V^* V Y W^* s_A(D)^{1/2} \\ &= A^{1/2} C^{1/2} X^* Y D^{1/2} A^{1/2} = s_A(C : D). \end{aligned}$$

$\square$

With the definition of 'intersection' of continuously contained Hilbert spaces, as in Theorem 3.8, and the definition of 'sequential product' for contractively contained Hilbert spaces as a consequence of Theorem 3.1, Theorem 3.10 can be rephrased to say that the sequential product is left distributive with respect to the intersection.

Theorem 3.11. *Let the Hilbert space $\mathcal{H}_0$ be contractively contained into the Hilbert space $\mathcal{H}$. Then, for any Hilbert spaces $\mathcal{K}$ and $\mathcal{G}$, contractively contained into $\overline{\mathcal{H}_0}$, the closure of $\mathcal{H}_0$ in $\mathcal{H}$, we have*

$$\mathcal{H}_0 \circ (\mathcal{K} \cap \mathcal{G}) = (\mathcal{H}_0 \circ \mathcal{K}) \cap (\mathcal{H}_0 \circ \mathcal{G}).$$

Remark 3.12. Alternatively, Theorem 3.10 ca be proved by Theorem 3.7.(vi). Thus, for any quantum effects $0 \leq C, D \leq P_A$ and $h \in \mathcal{H}$ we have

$$\langle s_A(C : D)h, h\rangle = \inf\{\langle Cx, x\rangle + \langle Dy, y\rangle \mid A^{1/2} h = x + y\}.$$

Both C and D have supports in $P_A \mathcal{H} = \overline{\operatorname{Ran}(A^{1/2})}$ and hence

$$\begin{aligned} \langle s_A(C : D))h, h\rangle &= \inf\{\langle C A^{1/2} c, A^{1/2} c\rangle + \langle D A^{1/2} d, A^{1/2} d\rangle \mid A^{1/2} h = A^{1/2} c + A^{1/2} d\} \\ &= \inf\{\langle A^{1/2} C A^{1/2} c, c\rangle + \langle A^{1/2} D A^{1/2} d, d\rangle \mid h = c + d\} \\ &= \langle s_A(C) : s_A(D)h, h\rangle, \end{aligned}$$

where we took into account that, without restricting the generality, we can take all the vectors h, c, and d in $P_A \mathcal{H}$ and that on this subspace $A^{1/2}$ is one-to-one.

Corollary 3.13. *Let $A \in \mathcal{E}(\mathcal{H})$ and consider the function f_A as defined in* (2.1). *Then,*

$$f_A(E : F) = f_A(E) : f_A(F), \quad E, F \in [0, A].$$

Proof. The function f_A is the inverse of the function s_A as in Theorem 3.10. □

This corollary can be also rephrased, in terms of the 'sequential quotient' as defined in (3.2), saying that the 'sequential product' of contractively contained Hilbert spaces is right distributive with respect to 'intersection'.

Corollary 3.14. *Let $\mathcal{K}$ be a Hilbert space contractively contained in $\mathcal{H}$. For any Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ contractively contained in $\mathcal{K}$, we have*

$$(\mathcal{H}_1 \cap \mathcal{H}_2)/\mathcal{K} = (\mathcal{H}_1/\mathcal{K}) \cap (\mathcal{H}_2/\mathcal{K}).$$

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Institutul de Matematică al Academiei Române, C.P. 1-764, 70700 Bucureşti, România
E-mail address: gheondea@theta.ro

Department of Mathematics, University of Denver, Denver, Colorado 80208, USA
E-mail address: sgudder@math.du.edu