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On a model for the maximal function of an n-hypercontraction

Ilie Valusescu *

Abstract. Starting from the characterization given by Agler [2] to an n-hypercontraction of the class $C_{0\cdot}$ as a part of the backward shift on a Bergman space, parallel with results about maximal functions attached to a contraction operator on a Hilbert space [20, 21], the particular case of an n-hypercontraction is analyzed. The functional model from H^2 given by the maximal function of a contraction in the $C_{\cdot 0}$ case and some connections between the maximal function and systems are recalled, and corresponding results about the generalized maximal function attached to an n-hypercontraction are given. Such a way, a functional model from a Bergman space A_n for an n-hypercontraction of the class $C_{0\cdot}$ is characterized and partial results about the connections between the generalized maximal function of an n-hypercontraction and systems are given.

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Keywords: n-hypercontraction, maximal function of a contraction, shift operator, Bergman spaces, linear systems, functional model.

1 n-hypercontractions

The class $\mathcal{C}_n$ of the n-hypercontraction operators on a Hilbert space was introduced by J. Agler [2], [1] where he proved that an n-hypercontraction from the class $C_{0.}$ is modeled as a restriction to an invariant subspace of the adjoint shift operator on a standard weighted Bergman space.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}(\mathcal{H})$ the C^*-algebra of all linear bounded operators on $\mathcal{H}$. For a fixed integer $n \geq 1$ an operator $T \in \mathcal{L}(\mathcal{H})$ is called an *n-hypercontraction*, if it verifies the inequalities

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k} T^{*k}T^k \geq 0 \tag{1.1}$$

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in $\mathcal{L}(\mathcal{H})$ for all $1 \leq m \leq n$.

Obviously $\mathcal{C}_n$ is a particular case of the class of contraction operators on $\mathcal{H}$. For $n = 1$ the contraction case is obtained.

Based on the inequalities (1.1), there exist n square roots operators in $\mathcal{L}(\mathcal{H})$ if we take

$$D_{m,T} = \Big[\sum_{k=0}^{m}(-1)^k \binom{m}{k} T^{*k}T^k\Big]^{1/2} \tag{1.2}$$

for $1 \leq m \leq n$, which generalize the notion of defect operator for a contraction $T \in \mathcal{L}(\mathcal{H})$, $D_T = (I - T^*T)^{1/2}$. Also the notion of defect space of a contraction $\mathcal{D}_T = \overline{D_T\mathcal{H}}$ is generalized to n closed subspaces of $\mathcal{H}$ taking $\mathcal{D}_{m,T} = \overline{D_{m,T}\mathcal{H}}$, for $1 \leq m \leq n$.

Actually Agler [2] introduced the notion of n-hypercontraction based on hereditary polynomials $p(x, y)$ in two noncommuting variables x and y of the form

$$p(x, y) = \sum c_{ij} y^j x^i$$

and using the fact that if p is a hereditary polynomial of this form and $b \in B$, where B is any C^*-algebra with identity, then $p(b)$ is defined by

$$p(b) = \sum c_{ij} b^{*j} b.$$

In the particular case when $T \in \mathcal{L}(\mathcal{H})$ - the C^*-algebra of all linear bounded operators in $\mathcal{H}$ we have

$$(1 - yx)^m(T) = \sum_{k=0}^{m} \binom{m}{k} T^{*k}T^k. \tag{1.3}$$

Then $T \in \mathcal{L}(\mathcal{H})$ as an n-hypercontraction was defined having the property that $(1 - yx)^m(T) \geq 0$ for $1 \leq m \leq n$, and the class of n-hypercontractions on $\mathcal{H}$ was denoted by $\mathcal{C}_n(\mathcal{H})$. Also, $T \in \mathcal{C}_n(\mathcal{H})$ was called a strong n-hypercontraction if $T^m \to 0$ strongly as $m \to \infty$.

The standard weighted Bergman space on the unit disc $\mathbb{D}$, denoted by $A_n(\mathbb{D})$ is the Hilbert space of analytic functions on $\mathbb{D}$

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

with the norm

$$\|f\|^2_{A_n} = \sum_{k=0}^{\infty} |a_k|^2 \mu_{n;k} \tag{1.4}$$

where a_k are the Taylor coefficients of the function f and the weights are given by

$$\mu_{n;k} = 1/\binom{k+n-1}{k}, \tag{1.5}$$

which actually are the Taylor coefficients from the decomposition of the function $(1-x)^{-n} = \sum_0^\infty \mu_{n;k} x^k$.

Equivalently, $A_n(\mathbb{D})$ can be defined as the Hilbert space of square integrable analytic functions on $\mathbb{D}$ corresponding to the weighted area measure

$$\mathrm{d}\mu_n(z) = (n-1)(1-|z|^2)^{n-2}\mathrm{d}A(z), \tag{1.6}$$

where $\mathrm{d}A(z) = \frac{1}{\pi}\mathrm{d}x\mathrm{d}y$, z=x+iy, is the usual planar normalized Lebesgue area measure. Therefore the norm (1.4) on $A_n(\mathbb{D})$ is equivalent with the norm

$$\|f\|^2_{A_n} = \lim_{r\to 1} \int_{\bar{\mathbb{D}}} |f(rz)|^2 \,\mathrm{d}\mu_n(z), \tag{1.7}$$

and $A_n(\mathbb{D})$ can be seen as $H^2(\mu_n)$.

Let us remark that in the particular case $n = 1$ the space $A_1(\mathbb{D})$ is the Hardy space $H^2(\mathbb{D})$, $\mathrm{d}\mu_1$ is the normalized arc measure on the unit circle $\mathbb{T}$ and $\mu_{n;k}$ are the moments of the measure $\mathrm{d}\mu_n$ defined by

$$\mu_{n;k} = \int_{\bar{\mathbb{D}}} |z|^{2k} \,\mathrm{d}\mu_n(z) = 1/\binom{k+n-1}{k}, \qquad k \geq 0. \tag{1.8}$$

There exist a lot of references for Bergman space, but the most cited is the book of Hedenmalm, Korenblum and Zhu [10].

For our purpose we consider the vectorial case, namely $A_n(\mathcal{E})$ is the standard weighted Bergman space of $\mathcal{E}$-valued analytic functions on $\mathbb{D}$,

$$f(z) = \sum_{k=0}^\infty a_k z^k, \qquad a_k \in \mathcal{E},$$

where $\mathcal{E}$ is a Hilbert space. Therefore $A_n(\mathcal{E})$ is the Hilbert space of all $\mathcal{E}$-valued analytic functions on $\mathbb{D}$ with the finite norm

$$\|f\|^2_{A_n} = \sum_{k=0}^\infty \|a_k\|^2 \mu_{n;k}, \tag{1.9}$$

where $\mu_{n;k}$ are given by (1.8), or equivalently,

$$\|f\|^2_{A_n} = \lim_{r\to 1} \int_{\bar{\mathbb{D}}} \|f(rz)\|^2 \,\mathrm{d}\mu_n(z), \tag{1.10}$$

where $\mathrm{d}\mu_n$ is the standard weighted area measure given by (1.6). Obviously $A_1(\mathcal{E})$ is the Hardy space $H^2(\mathcal{E})$.

As usually, the shift operator on $A_n(\mathcal{E})$ is defined by

$$(S_n f)(z) = zf(z) = \sum_{k=1}^{\infty} a_{k-1} z^k, \qquad z \in \mathbb{D} \tag{1.11}$$

and it is a bounded operator on $A_n(\mathcal{E})$. Since the weight sequence $\{\mu_{n;k}\}$ is decreasing and $\lim_{k\to\infty} \frac{\mu_{n;k+1}}{\mu_{n;k}} = 1$, it follows that the norm of S_n is 1.

Notice that S_1 is the unilateral shift on $H^2(\mathcal{E})$, S_2 is the Bergman shift on the Bergman space $A_2(\mathcal{E})$ and that, in general, S_n is a weighted shift.

The adjoint operator S_n^* of S_n has the form

$$(S_n^* f)(z) = \sum_{k=0}^{\infty} \frac{\mu_{n;k+1}}{\mu_{n;k}} a_{k+1} z^k, \qquad z \in \mathbb{D}. \tag{1.12}$$

Indeed, we have

$$\langle S_n^* f, f\rangle_{A_n} = \langle f, S_n f\rangle_{A_n} = \left\langle \sum_{k=0}^{\infty} a_k z^k, \sum_{k=1}^{\infty} a_{k-1} z^k \right\rangle_{A_n} =$$

$$= \left\langle \sum_{k=0}^{\infty} a_k z^k, \sum_{k=0}^{\infty} a_{k-1} z^k \right\rangle_{A_n} = \sum_{k=0}^{\infty} \left\langle a_k z^k, a_{k-1} z^k \right\rangle_{\mathcal{E}} \mu_{n;k} =$$

$$= \sum_{k=1}^{\infty} \left\langle a_k z^k, a_{k-1} z^k \right\rangle_{\mathcal{E}} \mu_{n;k} = \sum_{j=0}^{\infty} \left\langle a_{j+1} z^{j+1}, a_j z^{j+1} \right\rangle_{\mathcal{E}} \mu_{n;j+1} =$$

$$= \sum_{j=0}^{\infty} \left\langle \frac{\mu_{n;j+1}}{\mu_{n;j}} a_{j+1} z^j, a_j z^j \right\rangle_{\mathcal{E}} \mu_{n;j} = \left\langle \sum_{k=0}^{\infty} \frac{\mu_{n;k+1}}{\mu_{n;k}} a_{k+1} z^k, \sum_{k=0}^{\infty} a_k z^k \right\rangle_{A_n},$$

and it follows that

$$(S_n^* f)(z) = \sum_{k=0}^{\infty} \frac{\mu_{n;k+1}}{\mu_{n;k}} a_{k+1} z^k, \qquad z \in \mathbb{D}.$$

An operator A is *a part* of an operator $B \in \mathcal{L}(\mathcal{H})$ if A is a restriction to an invariant subspace $\mathcal{E}$ of B, $A = B|_{\mathcal{E}}$. Sometimes B is called an *extension* of the operator A. Due to [7] and [18] it is known the following

THEOREM 1.1. *If S is the unilateral shift of multiplicity one and $S^{*(\infty)}$ is the direct sum of a countably infinite number of copies of S^*, then an operator T has an extension to $S^{*(\infty)}$ if $\|T\| \leq 1$ and $T^m \to 0$ as $m \to \infty$.*

This result was applied by Agler [1] to a Bergman shift on $A_2(\mathcal{E})$, and generalized in [2] to the case of n-hypercontractions, the main result being the following theorem ([2], Theorem1.12).

THEOREM 1.2. *The operator T from $\mathcal{L}(\mathcal{H})$ has an extension to S_n^* if and only if T is an n-hypercontraction of $C_{0.}$ class.*

2 Maximal function

Let T be a contraction on a complex separable Hilbert space $\mathcal{H}$. As usually, by $D_T = (I - T^*T)^{1/2}$, $D_{T^*} = (I - TT^*)^{1/2}$ will be denoted the *defect operators* of T, and by $\mathcal{D}_T = \overline{D_T\mathcal{H}}$, $\mathcal{D}_{T^*} = \overline{D_{T^*}\mathcal{H}}$ will be denoted the *defect spaces* of the contraction T.

Using the factorization theorem of an operator valued semispectral measure, the maximal outer function $\{\mathcal{H}, \mathcal{D}_{T^*}, M_T(\lambda)\}$ in the particular case of the semispectral measure attached to a contraction T was obtained [20] into the form

$$M_T(\lambda) = D_{T^*}(I - \lambda T^*)^{-1}, \qquad (\lambda \in \mathbb{D}), \tag{2.1}$$

and was called the *maximal function* of the contraction T.

Analogously, the maximal function of T^* will be of the form

$$M_{T^*}(\lambda) = D_T(I - \lambda T)^{-1}, \qquad (\lambda \in \mathbb{D}). \tag{2.2}$$

Between the maximal function $M_T(\lambda)$, and the characteristic function of the contraction T,

$$\Theta_T(\lambda) = [-T + \lambda D_{T^*}(I - \lambda T^*)^{-1} D_T]|\mathcal{D}_T,$$

there exists the obvious relation

$$\Theta_T(\lambda) = [-T + \lambda M_T(\lambda) D_T]|\mathcal{D}_T.$$

To the maximal function $M_T(\lambda)$ the following operator

$$M_T : \mathcal{H} \to H^2(\mathcal{D}_{T^*})$$

is attached by

$$(M_T h)(\lambda) = M_T(\lambda)h. \tag{2.3}$$

Analogously, for the maximal function of T^* the operator

$$M_{T^*} : \mathcal{H} \to H^2(\mathcal{D}_T)$$

is defined by

$$(M_{T^*}h)(\lambda) = M_{T^*}(\lambda)h. \tag{2.4}$$

In some investigations , generalizing the $C_{0.}$ and $C_{.0}$ case, an operator T on $\mathcal{H}$ is called *stable* if $T^n \to 0$, and $*$-*stable* if $T^{*n} \to 0$, strongly as $n \to 0$. Actually, in the linear system theory, a system with the main operator T is called stable, if it is stable and $*$-stable.

Proposition 2.1. *If T is a contraction of the $C_{.0}$ class, then its maximal function $\{\mathcal{H}, \mathcal{D}_{T^*}, M_T(\lambda)\}$ is bounded, and the attached operator M_T defined by* (2.3) *is an isometry. Moreover, the Sz.-Nagy–Foias functional model [18] reduces to a functional representation given by the maximal function $M_T(\lambda)$. Namely, the imbedding of $\mathcal{H}$ into the functional model is given by*

$$\boldsymbol{H} = M_T\mathcal{H} \subset H^2(\mathcal{D}_{T^*}), \tag{2.5}$$

and the functional representation of the contraction T is given by the backward shift S^* on $H^2(\mathcal{D}_{T^*})$

$$\boldsymbol{T}^* u(\lambda) = \frac{1}{\lambda}[M_T(\lambda)h - M_T(0)h] = S^*u(\lambda). \tag{2.6}$$

Moreover, M_T intertwines T^* with S^*,

$$M_T T^* = S^* M_T. \tag{2.7}$$

Proof. Using the Sz.-Nagy–Foias notations from [18], the functional model of a contraction is obtained by a unitary imbedding Φ of the dilation space $\mathfrak{K}$ of T into a functional space, where

$$\mathfrak{K} = M(\mathfrak{L}_*) \oplus \mathfrak{R} \quad \text{and} \quad \mathfrak{L}_* = U\mathfrak{L}^*,$$

$$\mathfrak{L}^* = \overline{(U^* - T^*)\mathcal{H}},\ M(\mathfrak{L}_*) = \oplus_{-\infty}^{+\infty} U^n \mathfrak{L}_*.$$

Also it is known that the space of the isometric dilation $U_+ = U|\mathfrak{K}_+$ is

$$\mathfrak{K}_+ = M_+(\mathfrak{L}_*) \oplus \mathfrak{R} = \mathcal{H} \oplus M_+(\mathfrak{L}).$$

Of course $\mathfrak{R} = \mathfrak{K} \ominus M(\mathfrak{L}_*)$ reduces to $\{0\}$ if and only if $T^{*n} \longrightarrow O$, i.e. $T \in C_{.0}$ class. In this case $\Phi = \Phi^{\mathcal{D}_{T^*}}$ - the Fourier representation of $M_+(\mathcal{D}_{T^*})$, taking account by the unitary transformation of $\mathfrak{L}_*$ into $\mathcal{D}_{T^*}$ and the fact that $P^{\mathfrak{L}_*}M_+(\mathfrak{L}) \subset M_+(\mathfrak{L}_*)$ [cf. [18], Chap II, Theorem 2.1], where $P^{\mathfrak{L}_*}$ is the orthogonal projection of $\mathfrak{K}$ on $M(\mathfrak{L}_*)$. Therefore we have

$$\mathbf{H} = \Phi\mathcal{H} = \Phi^{\mathcal{D}_{T^*}}\mathcal{H}$$

and by [20], Prop.1, it follows that

$\mathbf{H} = V_{M_T}\mathcal{H}$, where $V_{M_T} = \Phi^{\mathcal{D}_{T^*}} P^{\mathcal{L}_*}|\mathcal{H}$.

The relation (2.6) is a consequence of the fact that $T^* = U_+^*|\mathcal{H}$.

For any contraction T and $h \in \mathcal{H}$ we have

$\sum\limits_{k=0}^{n} \left\|D_{T^*}T^{*k}h\right\|^2 = \sum\limits_{k=0}^{n} \left\langle D_{T^*}^2 T^{*k}h, T^{*k}h\right\rangle =$

$= \sum\limits_{k=0}^{n} (\left\|T^{*k}h\right\|^2 - \left\|T^{*k+1}h\right\|^2) =$

$= \sum\limits_{k=0}^{n} \left\|T^{*k}h\right\|^2 - \sum\limits_{k=1}^{n+1} \left\|T^{*k}h\right\|^2 =$

$= \|h\|^2 - \left\|T^{*k+1}h\right\|^2.$

Since T is $*$-stable, the previous relation becomes $\sum\limits_{n=0}^{\infty} \|D_{T^*}T^{*n}h\|^2 = \|h\|^2$, and taking into account that $M_T(\lambda) = \sum\limits_{n=0}^{\infty} D_{T^*}T^{*n}\lambda^n$, it follows that the attached operator $M_T : \mathcal{H} \to H^2(\mathcal{D}_{T^*})$ is an isometry.

The intertwining relation (2.7) is verified taking into account the form of the maximal function of T.

$$S^*M_T h(\lambda) = \frac{1}{\lambda}[M_T(\lambda)h - M_T(0)h] = \frac{1}{\lambda}[D_{T^*}(I-\lambda T^*)^{-1}h - M_T(0)h] =$$

$$= \frac{1}{\lambda}\Big[\sum_{k=0}^{\infty} D_{T^*}T^{*k}h\lambda^k - D_{T^*}\Big] = \frac{1}{\lambda}\Big[\sum_{k=1}^{\infty} D_{T^*}T^{*k}h\lambda^k\Big] =$$

$$= \sum_{k=1}^{\infty} D_{T^*}T^{*k}h\lambda^{k-1} = \sum_{k=0}^{\infty} D_{T^*}T^{*k+1}h\lambda^k = D_{T^*}(I-\lambda T^*)T^*h = M_T(\lambda)T^*h,$$

which implies the intertwining relation (2.7).

$\square$

A dual result can be found for $T \in C_{0\cdot}$, which implies T^* in $C_{\cdot 0}$ class, and we have

Proposition 2.2. *If T is a contraction of the $C_{0\cdot}$ class, then the maximal function $\{\mathcal{H}, \mathcal{D}_T, M_{T^*}(\lambda)\}$ is bounded, and the attached operator M_{T^*} defined by* (2.4) *is an isometry. Moreover, the Sz.-Nagy–Foias functional model [18] reduces to a functional representation given by the maximal function $M_{T^*}(\lambda)$. Namely, the imbedding of $\mathcal{H}$ into the functional model is given by*

$$\boldsymbol{H} = M_{T^*}\mathcal{H} \subset H^2(\mathcal{D}_T), \tag{2.8}$$

and the functional representation of the contraction T is given by the backward shift S^* on $H^2(\mathcal{D}_T)$

$$\boldsymbol{T}u(\lambda) = \frac{1}{\lambda}[M_{T^*}(\lambda)h - M_{T^*}(0)h], \tag{2.9}$$

Moreover, M_{T^*} intertwines T with S^*,

$$M_{T^*}T = S^* M_{T^*}. \tag{2.10}$$

In what follows, some applications of the maximal function of a contraction will be presented, especially based on the results from [21].

Let $\mathcal{H}, \mathcal{U}, \mathcal{Y}$ be separable Hilbert spaces and $A \in \mathcal{L}(\mathcal{H})$, $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$, $C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. A linear system $\sigma = (A, B, C, D; \mathcal{H}, \mathcal{U}, \mathcal{Y})$ of the form

$$\begin{cases} h_{n+1} = & Ah_n + Bu_n, \quad (n \geq 0) \\ \quad y_n = & Ch_n + Du_n, \end{cases} \tag{2.11}$$

where $\{h_n\} \subset \mathcal{H}$, $\{u_n\} \subset \mathcal{U}$, $\{y_n\} \subset \mathcal{Y}$, is called a *discrete-time system.*

Usually the spaces $\mathcal{H}$, $\mathcal{U}$, $\mathcal{Y}$ are called, respectively, the *state* space, the *input* space, and the *output* space, and the operators A, B, C and D are called, respectively, the *main* operator, the *control* operator, the *observation* operator, and the *feedthrough* operator of the system σ.

Let us consider the bloc operator matrix (*colligation*) $S : \mathcal{H} \oplus \mathcal{U} \to \mathcal{H} \oplus \mathcal{Y}$,

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{Y} \end{bmatrix}. \tag{2.12}$$

Then (2.11) can be written into the matrix form $\begin{bmatrix} h_{n+1} \\ y_n \end{bmatrix} = S \begin{bmatrix} h_n \\ u_n \end{bmatrix}$.

The system σ will be called: *passive*, *isometric*, *co-isometric*, *conservative*, if S is, respectively, a contraction, an isometry, a co-isometry, or unitary.

The operator valued function $\Theta_\sigma(\lambda) : \mathcal{U} \to \mathcal{Y}$, $(\lambda \in \mathbb{D})$, attached to a system σ by

$$\Theta_\sigma(\lambda) = D + \lambda C(I_{\mathcal{H}} - \lambda A)^{-1}B \qquad (\lambda \in \mathbb{D}), \tag{2.13}$$

is called the *transfer function* (or frequency response function) of the system.

The transfer function is the basic connection between state-space and frequency-domain in the linear systems theory.

The references for the linear systems are very large, I mention here only a few of them [7, 8, 11, 5, 15, 3]. The aim of this paper is not an exhaustive study on linear systems, but only to analyse some connections between the maximal function and disctete linear systems.

If σ is a passive system, then $\Theta_\sigma(\lambda)$ is a contractive holomorphic function on $\mathbb{D}$, i.e. $\Theta_\sigma(\lambda)$ belongs to the Schur class $S(\mathcal{U}, \mathcal{Y})$.

For a system σ, the following subspaces of $\mathcal{H}$ are considered:

$$\mathcal{C}_\sigma = \bigvee_{n\geq 0} A^n B\mathcal{U} \qquad \text{(the \textit{controllable} space)}, \tag{2.14}$$

and

$$\mathcal{O}_\sigma = \bigvee_{n\geq 0} A^{*n} C^*\mathcal{Y} \qquad \text{(the \textit{observable} space)}. \tag{2.15}$$

Generally we have $\mathcal{H} = \big(\mathcal{C}_\sigma \bigvee \mathcal{O}_\sigma\big) \oplus \big(\mathcal{C}_\sigma^\perp \cap \mathcal{O}_\sigma^\perp\big)$. The system σ is called *controllable* if $\mathcal{C}_\sigma = \mathcal{H}$, *observable* if $\mathcal{O}_\sigma = \mathcal{H}$, and *minimal* if σ is both observable and controllable. The system σ is *simple* if $\mathcal{C}_\sigma \bigvee \mathcal{O}_\sigma = \mathcal{H}$.

From(2.14) it follows that $\big(\mathcal{C}_\sigma\big)^\perp = \bigcap_{n=0}^{\infty} \ker(B^* A^{*n})$, and from (2.15) we have $\big(\mathcal{O}_\sigma\big)^\perp = \bigcap_{n=0}^{\infty} \ker(CA^n)$. Hence the following characterizations occur: the system σ is, respecively, controllable iff $\bigcap_{n=0}^{\infty} \ker(B^* A^{*n}) = \{0\}$, observable iff $\bigcap_{n=0}^{\infty} \ker(CA^n) = \{0\}$, and simple iff $\big(\bigcap_{n=0}^{\infty} \ker(B^* A^{*n})\big) \cap \big(\bigcap_{n=0}^{\infty} \ker(CA^n)\big) = \{0\}$.

In this paper we are mainly concerned on the system $\mathcal{J}$ given by the following unitary operator (the rotation operator of T, or Julia operator)

$$J(T) = R_T = \begin{bmatrix} T & D_{T^*} \\ D_T & -T^* \end{bmatrix}. \tag{2.16}$$

In this particular case, the controllable and the observable subspaces of $\mathcal{H}$ will be, respectively,

$$\mathcal{C} = \bigvee_{n=0}^{\infty} T^n D_{T^*}\mathcal{D}_{T^*}, \qquad \mathcal{O} = \bigvee_{n=0}^{\infty} T^{*n} D_T\mathcal{D}_T, \tag{2.17}$$

and the corresponding orthogonals in the state space $\mathcal{H}$ will be

$$\mathcal{C}^\perp = \bigcap_{n=0}^{\infty} \ker(D_{T^*}T^{*n}) = \bigcap_{n} \ker D_{T^{*n}} = \{h \in \mathcal{H};\ \|T^{*n}h\| = \|h\|\}, \tag{2.18}$$

$$\mathcal{O}^\perp = \bigcap_{n=0}^{\infty} \ker(D_T T^n) = \bigcap_{n} \ker D_{T^n} = \{h \in \mathcal{H};\ \|T^n h\| = \|h\|\}. \tag{2.19}$$

Thus $T|\mathcal{O}^\perp$ and $T^*|\mathcal{C}^\perp$ are isometric operators and

$$\mathcal{C}^\perp \cap \mathcal{O}^\perp = \{h \in \mathcal{H};\ \|T^n h\| = \|h\| = \|T^{*n} h\|\} = \mathcal{H}_0, \tag{2.20}$$

where $\mathcal{H}_0$ is the subspace of the unitary part from the canonical decomposition [18] of the contraction $T = T_0 \oplus T_1 = \begin{bmatrix} T_0 & 0 \\ 0 & T_1 \end{bmatrix}$ into its unitary part and the completely non-unitary (c.n.u.) part on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$.

Also, let us remark that the transfer function of $\mathcal{J}$ is just the characteristic function of T^*, namely

$$\Theta_{\mathcal{J}}(\lambda) = -T^* + \lambda D_T (I - \lambda T)^{-1} D_{T^*} = \Theta_{T^*}(\lambda).$$

If we consider the system $\mathcal{J}^*$ given by the unitary bloc matrix (the rotation of T^*), $J(T^*) = R_{T^*} = \begin{bmatrix} T^* & D_T \\ D_{T^*} & -T \end{bmatrix}$, then the transfer function of $\mathcal{J}^*$ will be given by $\{\mathcal{D}_T, \mathcal{D}_{T^*}, \Theta_T(\lambda)\}$, the characteristic function of T.

Obviously $R_{T^*} = R_T^*$, and the corresponding linear systems $\mathcal{J}$ and $\mathcal{J}^*$ are dual, namely, if $\mathcal{J}$ is observable, then $\mathcal{J}^*$ is controllable, and conversely.

A characterization for the controllability (observability) of the system $\mathcal{J}$ can be done with the maximal function of the main operator T as follows.

PROPOSITION 2.3. *The discrete linear system $\mathcal{J}$ is controllable if and only if the operator M_T defined by the maximal function of T is one to one, and $\mathcal{J}$ is observable if and only if the operator M_{T^*} defined by the maximal function of T^* is one to one.*

Proof. If the system $\mathcal{J}$ is controllable, then $\mathcal{C}_{\mathcal{J}} = \mathcal{H}$, where $\mathcal{C}_{\mathcal{J}}$ is given by (2.17), or equivalently, $\mathcal{C}_{\mathcal{J}}^\perp = \bigcap \ker(D_{T^*} T^{*n}) = \{0\}$. That is, $D_{T^*} T^{*n} h = 0$ for any $n \geq 0$ if and only if $h = 0$. Taking into account that

$$M_T(\lambda) h = D_{T^*} h + D_{T^*} T^* \lambda h + D_{T^*} T^{*2} \lambda^2 h + \cdots ,$$

it follows that $\ker M_T = 0$.

Conversely, if $\ker M_T = 0$, then $M_T h = 0$ if and only if $h = 0$, i.e., $D_{T^*} T^{*n} h = 0$ for any $n \geq 0$ if and only if $h = 0$, which implies that $\mathcal{C}_{\mathcal{J}}^\perp = \{0\}$, or equivalently $\mathcal{C}_{\mathcal{J}} = \mathcal{H}$, and the system is controllable.

Analogously can be proved the same facts for the system $\mathcal{J}^*$, and taking into account the duality of the systems $\mathcal{J}$ and $\mathcal{J}^*$, the proof is finished. □

Therefore the operators M_T and M_{T^*} corresponding to the maximal functions $M_T(\lambda)$ and $M_{T^*}(\lambda)$ of T and T^*, respectively, contain the information about the structure of the corresponding systems. Actually M_T is the controllability operator and M_{T^*} is the observability operator for the system$\mathcal{J}$.

Starting from Aronsjain reproducing kernel Hilbert spaces [4] a caracterization for Hilbert spaces with positive kernels was done. It is known that if $\mathcal{K}$ is a Hilbert space of $\mathcal{E}$-valued functions defined on a set $\mathcal{S}$, then $\mathcal{K}$ is called a reproducing kernel Hilbert space (RKHS) if the point evaluations $e(s) : \mathcal{K} \to \mathcal{E}$ defined by $e(s)f = f(s)$ are continuous for each s in $\mathcal{S}$. In this case there exists a function $K : \mathcal{S} \times \mathcal{S} \to \mathcal{L}(\mathcal{E})$ such that for each $s \in \mathcal{S}$ and $a \in \mathcal{E}$

(i) $K(\cdot, s)a \in \mathcal{K}$

and for each $f \in \mathcal{K}, a \in \mathcal{E}, s \in \mathcal{S}$ we have

(ii) $\langle f(s), a\rangle_{\mathcal{E}} = \langle f, K(\cdot, s)a\rangle_{\mathcal{K}}$.

If K is a reproducing kernel for a Hilbert space $\mathcal{K}$, then K is a positive definite function on $\mathcal{S}$, and conversely, any positive definite function K on $\mathcal{S}$ is the reproducing kernel for a RKHS $\mathcal{K}$.

As an example, $H^2(\mathcal{E})$ is a RKHS with the reproducing kernel

$$K(\lambda, \mu) = \frac{1}{1 - \lambda\overline{\mu}} I_{\mathcal{E}},$$

where $\lambda, \mu \in \mathbb{D}$ and $I_{\mathcal{E}}$ is the identity operator on $\mathcal{E}$.

Also, the standard weighted Bergman space $A_n(\mathcal{E})$ is a RKHS with the reproducing kernel

$$K(\lambda, \mu) = \frac{1}{(1 - \lambda\overline{\mu})^n} I_{\mathcal{E}}.$$

In what follows, some positive definite kernels expressed in terms of the maximal functions $M_T(\lambda)$ and $M_{T^*}(\lambda)$ are analyzed and some applications are given

Proposition 2.4. *Let $\{\mathcal{D}_T, \mathcal{D}_{T^*}, \Theta_T(\lambda)\}$ and $\{\mathcal{D}_{T^*}, \mathcal{D}_T, \Theta_{T^*}(\lambda)\}$ be the characteristic functions of the contractions T and T^*, respectively. There exist the following relations*

$$K_T(\lambda, \mu) = \frac{I - \Theta_T(\lambda)\Theta_T(\mu)^*}{1 - \lambda\overline{\mu}} = M_T(\lambda)M_T(\mu)^* \tag{2.21}$$

and

$$K_{T^*}(\lambda, \mu) = \frac{I - \Theta_T(\mu)^*\Theta_T(\lambda)}{1 - \lambda\overline{\mu}} = M_{T^*}(\overline{\mu})M_{T^*}(\overline{\lambda})^*, \tag{2.22}$$

where $M_T(\lambda)$ and $M_{T^*}(\lambda)$ are the maximal functions of T and T^*, respectively.

Proof. It is known ([18], Chap.VI, (1.4)) that the defect function of the characteristic function $\{\mathcal{D}_T, \mathcal{D}_{T^*}, \Theta_T(\lambda)\}$ is obtained by

$$\langle f, f\rangle - \langle \Theta_T(\lambda)f, \Theta_T(\mu)f\rangle = (1-\lambda\overline{\mu})\left\langle (I-\lambda T^*)^{-1}D_T f, (I-\mu T^*)^{-1}D_T f\right\rangle,$$

hence $\Delta^2_{\Theta_T}(\lambda,\mu) = I - \Theta_T(\mu)^*\Theta_T(\lambda) = (1-\lambda\overline{\mu})D_T(I-\overline{\mu}T)^{-1}(I-\lambda T^*)^{-1}D_T$, and taking into account by (2.2) it follows that

$$\frac{I-\Theta_T(\mu)^*\Theta_T(\lambda)}{1-\lambda\overline{\mu}} = M_{T^*}(\overline{\mu})M_{T^*}(\overline{\lambda})^*.$$

An analogous calculus leads to

$$\Delta^2_{\Theta_{T^*}}(\lambda,\mu) = I - \Theta_T(\lambda)\Theta_T(\mu)^* = (1-\lambda\overline{\mu})D_{T^*}(I-\lambda T^*)^{-1}(I-\overline{\mu}T)^{-1}D_{T^*},$$

and by (2.1) it follows that

$$\frac{I-\Theta_T(\lambda)\Theta_T(\mu)^*}{1-\lambda\overline{\mu}} = \Theta_1(\lambda)\Theta_1(\mu)^*.$$

$\square$

REMARK. Taking into account the dual property of the characteristic function $\Theta_T(\overline{\lambda})^* = \Theta_{T^*}(\lambda)$, the properties (2.21) and (2.22) become, respectively,

$$K_T(\overline{\lambda},\overline{\mu}) = \frac{I-\Theta_{T^*}(\lambda)^*\Theta_{T^*}(\mu)}{1-\overline{\lambda}\mu} = M_T(\overline{\lambda})M_T(\overline{\mu})^*$$

and

$$K_{T^*}(\overline{\lambda},\overline{\mu}) = \frac{I-\Theta_{T^*}(\mu)\Theta_{T^*}(\lambda)^*}{1-\overline{\lambda}\mu} = M_{T^*}(\mu)M_{T^*}(\lambda)^*.$$

Taking into account Proposition 2.3 we have the following

COROLLARY 2.5. *The discrete linear system $\mathcal{J}$ is controllable if and only if the positive kernel $K_T(\lambda,\lambda)$ given by* (2.21) *is strictly positive, and $\mathcal{J}$ is observable if and only if the kernel $K_{T^*}(\lambda,\lambda)$ given by* (2.22) *is strictly positive.*

Other applications of the previous positive definite kernels can be done in the analysis of the structure of some invariant subspaces, and characterizations in terms of the maximal functions can be obtained.

If T is a contraction of the class $C_{0\cdot}$, then T^* is *-stable, and by Proposition 2.1 the functional model of $\mathcal{H}$ is the space $\mathbf{H}$ from $H^2(\mathcal{D}_T)$ and T is a part of the the backward shift S^* on $H^2(\mathcal{D}_T)$. Obviously $\mathbf{H}^\perp$ is an invariant subspace for the shift operator S on $H^2(\mathcal{D}_T)$, and it is characterized by Beurling theorem like a product of $H^2(\mathcal{D}_T)$ with an inner function. For $\mathbf{H}$ we have the following characterization given with $\{\mathcal{H}, \mathcal{D}_T, M_{T^*}(\lambda)\}$, the maximal function of the contraction T^*.

Proposition 2.6. *If T is a contraction of the $C_{0\cdot}$ class, then the functional model $\boldsymbol{H} \subset H^2(\mathcal{D}_T)$ of $\mathcal{H}$ is a reproducing kernel Hilbert space with the kernel*

$$K(\lambda, \mu) = M_{T^*}(\lambda) M_{T^*}(\mu)^*,$$

where $\{\mathcal{H}, \mathcal{D}_T, M_{T^*}(\lambda)\}$ is the maximal function of T^*.

Proof. If $f \in \mathbf{H}$, then $f = M_{T^*}h$, $h \in \mathcal{H}$. For any $a \in \mathcal{D}_T$ we have

$$\langle f(\lambda), a \rangle = \langle M_{T^*}(\lambda)h, a \rangle = \langle h, M_{T^*}(\lambda)^* a \rangle .$$

By Proposition 2.1, since $T \in C_{0\cdot}$, i.e. $T^* \in C_{\cdot 0}$ and M_T^* is an isometry, it follows that

$$\langle f(\lambda), a \rangle = \langle M_{T^*}h, M_{T^*}M_{T^*}(\lambda)^* a \rangle = \langle f, M_{T^*}M_{T^*}(\lambda)^* a \rangle = \langle f, K(\cdot, \lambda)a \rangle_{H^2(\mathcal{D}_T)} .$$

Therefore $\mathbf{H}$ is a reproducing kernel Hilbert space with the kernel $K(\lambda, \mu) = M_{T^*}(\lambda) M_{T^*}(\mu)^*$. □

3 On a model for n-hypercontractions

As was mentioned before, an operator $T \in \mathcal{L}(\mathcal{H})$ is an n-hypercontraction if verifies the set of inequalities

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*k} T^k \geq 0 \tag{3.1}$$

in $\mathcal{L}(\mathcal{H})$ for all $1 \leq m \leq n$, where $n \geq 1$ is a positive integer.

It is obvious by (3.1) that for $1 \leq m \leq n$ there exist n square roots operators $D_{m,T}$ in $\mathcal{L}(\mathcal{H})$ given by (1.2), which generalize the notion of defect operator $D_T = (I - T^*T)^{1/2}$ for a contraction $T \in \mathcal{L}(\mathcal{H})$. Also the notion of defect space of a contraction $\mathcal{D}_T = \overline{D_T\mathcal{H}}$ was generalized to n closed subspaces of $\mathcal{H}$ taking $\mathcal{D}_{m,T} = \overline{D_{m,T}\mathcal{H}}$, for $1 \leq m \leq n$.

Parallel with the results obtained for characterization of the functional model $\mathbf{H}$ from $H^2(\mathcal{D}_T)$ with the maximal function $\{\mathcal{H}, \mathcal{D}_T, M_{T^*}(\lambda)\}$, given by (2.8) and (2.9) in the $C_{0\cdot}$ case, in the following an extension is made for the n-hypercontraction case. To do this, based on (3.1) and (1.2), let us consider generalized maximal functions attached to an n-hypercontraction T on $\mathcal{H}$, $\{\mathcal{H}, \mathcal{D}_{n,T^*}, M_{n,T}(\lambda)\}$ and $\{\mathcal{H}, \mathcal{D}_{n,T}, M_{n,T^*}(\lambda)\}$, where

$$M_{n,T}(\lambda) = D_{n,T^*}(I - \lambda T^*)^{-n} \tag{3.2}$$

and

$$M_{n,T^*}(\lambda) = D_{n,T}(I - \lambda T)^{-n}. \tag{3.3}$$

Also, the operators $M_{n,T} : \mathcal{H} \to A_n(\mathcal{D}_{T^*})$ and $M_{n,T^*} : \mathcal{H} \to A_n(\mathcal{D}_T)$ will be introduced by

$$(M_{n,T}h)(\lambda) = M_{n,T}(\lambda)h, \qquad h \in \mathcal{H},\ \lambda \in \mathbb{D} \tag{3.4}$$

and

$$(M_{n,T^*}h)(\lambda) = M_{n,T^*}(\lambda)h, \qquad h \in \mathcal{H},\ \lambda \in \mathbb{D}. \tag{3.5}$$

We are going to prove in the case of an n-hypercontraction from $C_{0\cdot}$ class, the case considered by Agler [2], that some results obtained for $M_T(\lambda)$ and $M_{T^*}(\lambda)$ can be extended for $M_{n,T}(\lambda)$ and $M_{n,T^*}(\lambda)$. Namely the Proposition 2.2 is generalized in terms of $M_{n,T^*}(\lambda)$.

PROPOSITION 3.1. *Let T be an n-hypercontraction of the $C_{0\cdot}$ class, then the operator M_{n,T^*} associated to $\{\mathcal{H}, \mathcal{D}_{n,T}, M_{T^*}(\lambda)$ by (3.5) is an isometry from $\mathcal{H}$ into $A_n(\mathcal{D}_{n,T})$, modeling $\mathcal{H}$ as a subspace $\boldsymbol{H}$ of the Bergman space $A_n(\mathcal{D}_{n,T})$, satisfying the intertwining relation*

$$M_{n,T^*}T = S_n^* M_{n,T^*}. \tag{3.6}$$

Proof. Taking account by the definition (1.2) of $D_{m,T}$ we have for $1 \le m \le n$ and $h \in \mathcal{H}$

$$\|D_{m,T}h\|^2 = \langle D_{m,T}h, D_{m,T}h\rangle = \left\langle D_{m,T}^2 h, h\right\rangle =$$

$$= \left\langle \sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*k}T^k h, h \right\rangle = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left\|T^k h\right\|^2.$$

Using the known binomial coefficients formula $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ it folows that for each $h \in \mathcal{H}$ we have

$$\|D_{m+1,T}h\|^2 = \|D_{m,T}h\|^2 - \|D_{m,T}Th\|^2. \tag{3.7}$$

Putting $T^j h$ instead of h in (3.7) and summing for $j = 0, 1, \cdots, k-1$ it follows that

$$\sum_{j=0}^{k-1} \left\|D_{m+1,T}T^j h\right\|^2 = \sum_{j=0}^{k-1} (\left\|D_{m,T}T^j h\right\|^2 - \left\|D_{m,T}T^{j+1}h\right\|^2) =$$

$$= \|D_{m,T}h\|^2 - \left\|D_{m,T}T^k h\right\|^2.$$

Since $\left\|D_{m,T}T^k h\right\| \to 0$ as $k \to \infty$ it follows that

$$\|D_{m,T}h\|^2 = \sum_{k=0}^{\infty} \left\|D_{m+1,T}T^k h\right\|^2 . \tag{3.8}$$

Giving values $1 \leq m < n$, since $\|D_{1,T}h\|^2 = \|h\|^2 - \|Th\|^2$ it follows, using again the binomial coefficients relation,

$$\|h\|^2 - \|Th\|^2 = \sum_{k_1,\cdots,k_{n-1}=0}^{\infty} \left\|D_{n,T}T^{k_1+\cdots+k_{n-1}}h\right\|^2 = \sum_{k=0}^{\infty} \frac{1}{\mu_{n-1;k}} \left\|D_{n,T}T^k h\right\|^2 .$$

Therefore

$$\|h\|^2 - \|Th\|^2 = \sum_{k=0}^{\infty} \frac{1}{\mu_{n-1;k}} \left\|D_{n,T}T^k h\right\|^2 . \tag{3.9}$$

If we put $T^j h$ instead of h in (3.9), then for each $h \in \mathcal{H}$ we have

$$\left\|T^j h\right\|^2 - \left\|T^{j+1}h\right\|^2 = \sum_{k=0}^{\infty} \frac{1}{\mu_{n-1;k}} \left\|D_{n,T}T^{k+j}h\right\|^2 .$$

Giving again values for $j = 0, 1, \cdots, p-1$ and summing up we obtain

$$\|h\|^2 - \|T^p h\|^2 = \sum_{j=0}^{p-1} \sum_{k=0}^{\infty} \frac{1}{\mu_{n-1;k}} \left\|D_{n,T}T^{k+j}h\right\|^2$$

and taking account that

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left\|D_{n,T}T^{k+j}h\right\|^2 = \sum_{k=0}^{\infty} \frac{1}{\mu_{n;k}} \left\|D_{n,T}T^k h\right\|^2 ,$$

for $p \to \infty$ it follows that

$$\|h\|^2 - \lim_{p\to\infty} \|T^p h\|^2 = \sum_{k=0}^{\infty} \frac{1}{\mu_{n;k}} \left\|D_{n,T}T^k h\right\|^2 = \left\|\sum_{k=0}^{\infty} \binom{k+n+1}{k} D_{n,T}T^k\right\|^2 =$$

$$= \left\|D_{n,T}(I - \lambda T)^{-n}h\right\|^2 = \|M_{n,T^*}h\|^2 .$$

Thus, since T is of $C_{0\cdot}$ class, we obtain that $\|h\| = \|M_{n,T^*}h\|$, and M_{n,T^*} is an isometry.

Now, by (1.12) we have for each $\lambda \in \mathbb{D}$

$$(S_n^* M_{n,T^*} h)(\lambda) = \sum_{k=0}^{\infty} \frac{\mu_{n;k+1}}{\mu_{n;k}} A_{k+1} h \lambda^k,$$

where A_k are the Taylor coefficients of the function

$$(M_{n,T^*} h)(\lambda) = M_{n,T^*}(\lambda) h = D_{n,T}(I - \lambda T)^{-n} h =$$

$$= \sum_{k=0}^{\infty} \binom{k+n-1}{k} D_{n,T} T^k h \lambda^k = \sum_{k=0}^{\infty} \Big(\frac{1}{\mu_{n;k}} D_{n,T} T^k h \Big) \lambda^k.$$

Therefore

$$(S_n^* M_{n,T^*} h)(\lambda) = \sum_{k=0}^{\infty} \Big(\frac{\mu_{n;k+1}}{\mu_{n;k}} \frac{1}{\mu_{n;k+1}} D_{n,T} T^{k+1} h \Big) \lambda^k =$$

$$= \sum_{k=0}^{\infty} \Big(\frac{1}{\mu_{n;k}} D_{n,T} T^{k+1} h \Big) \lambda^k = (M_{n,T*} T h)(\lambda),$$

and the intertwining condition (3.6) is verified. □

Similarly with Proposition 2.6 where the functional model $\mathbf{H}$ of $\mathcal{H}$ from $H^2(\mathcal{D}_T)$ is a RKHS with the kernel given by the maximal function $M_{T^*}(\lambda)$, the functional model of $\mathcal{H}$ from $A_n(\mathcal{D}_{n,T})$ will be characterized by $M_{n,T}(\lambda)$ as follows.

Proposition 3.2. *If $T \in \mathcal{L}(\mathcal{H})$ is an n-hypercontraction of the $C_{0\cdot}$ class, then the functional model of $\mathcal{H}$, $\boldsymbol{H} = M_{n,T^*}\mathcal{H}$ from $A_n(\mathcal{D}_{n,T})$, is a reproducing kernel Hilbert space with the kernel given by*

$$K(\lambda, \mu) = M_{n,T^*}(\lambda) M_{n,T^*}(\mu)^*, \qquad \lambda, \mu \in \mathbb{D}. \tag{3.10}$$

Proof. For each function $f \in \mathbf{H}$, $f = M_{n,T^*} g$, and each element $d \in \mathcal{D}_{n,T}$, taking account by the fact that M_{n,T^*} is an isometry from $\mathcal{H}$ into $A_n(\mathcal{D}_{n,T})$, we have

$$\langle f(\mu), d \rangle_{\mathcal{H}} = \langle M_{n,T^*}(\mu) g, d \rangle_{\mathcal{H}} = \langle g, M_{n,T^*}(\mu)^* d \rangle_{\mathcal{H}} =$$

$$= \langle M_{n,T^*}(\lambda) g, M_{n,T^*}(\lambda) M_{n,T^*}(\mu)^* d \rangle_{A_n} = \langle f, M_{n,T^*}(\lambda) M_{n,T^*}(\mu)^* d \rangle_{A_n}.$$

Therefore $\mathbf{H}$ is a RKHS with the kernel given by (3.10).

□

Other applications of the extended maximal functions $M_{n,T}(\lambda)$ and $M_{n,T^*}(\lambda)$ and the attached operators $M_{n,T}$ and M_{n,T^*} given by (3.4) and (3.5), can be done in linear system theory as well as in studying the structure of invariant subspaces for the shift operator S_n and S_n^* on the Bergman space $A_n(\mathcal{D}_{n,T})$. Such a way, similarly with the fact that the operators M_T and M_{T^*} attached to the maximal functions of a contraction are the controllability and observability operators for the linear sistem corresponding to the rotation operator R_T, the operators $M_{n,T}$ and M_{n,T^*} defined by (3.4) and (3.5) will be the controllability and observability operators of a specific linear system having as the main operator an n-hypercontraction of the $C_{0\cdot}$ class. Also the controllability and observability gramians given by $C_{n,T} = M_{n,T}^* M_{n,T}$ and $O_{n,T} = M_{n,T^*}^* M_{n,T^*}$ are of interest in the study of the structure of a system.

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