INCLUSION RESULTS INVOLVING GAUSSIAN HYPERGEOMETRIC FUNCTIONS FOR UNIVALENT FUNCTIONS HAVING UNIVALENT DERIVATIVES

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Classes of analytic functions for which both f and f' are univalent in the open unit disc $\mathbb{E} = \{z : |z| < 1\}$ was investigated earlier by Silverman in 1987. However, the application of Gaussian hypergeometric functions on the classes of analytic functions for which both f and f' are univalent in the open unit disc $\mathbb{E}$ is not being studied in the literature. By exploring this, we investigate the necessary and sufficient conditions and inclusion relations for certain function involving Gaussian hypergeometric functions to be in few subclasses of analytic functions for which both f and f' are univalent in the open unit disc $\mathbb{E}$ in this article. Further, we consider an integral operator related to Gaussian hypergeometric functions and several mapping properties are discussed. We also pointed out certain corollaries and consequences of the main results.

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1. INTRODUCTION

Let $\mathcal{A}$ be the class of all functions $f : \mathbb{E} \longrightarrow \mathbb{C}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ which are analytic in $\mathbb{E} = \{z : |z| < 1\}$ defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Furthermore, let us denote by $\mathcal{S} \subset \mathcal{A}$ where the functions in $\mathcal{S}$ are also univalent in $\mathbb{E}$.

Let $\mathcal{T}$ denote the class of all functions given by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \tag{2}$$

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normalized by the conditions $f(0) = f'(0) - 1 = 0$ which are analytic in $\mathbb{E}$. For any function $f \in \mathcal{A}$ given in the form (1) and if $g \in \mathcal{A}$ is given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product(or Convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{E}.$$

Let $\mathcal{S}^\star(\eta)$ and $\mathcal{C}(\eta)$ be the subclasses of $\mathcal{S}$ consisting of functions that are starlike of order η and convex of order η, $0 \leq \eta < 1$. The analytic descriptions of the above two classes are, respectively, given by

$$\mathcal{S}^\star(\eta) = \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \eta,\, 0 \leq \eta < 1 \right\}$$

and

$$\mathcal{C}(\eta) = \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \eta,\, 0 \leq \eta < 1 \right\}.$$

Further, $\mathcal{C} \equiv \mathcal{C}(0)$ and $\mathcal{S}^\star \equiv \mathcal{S}(0)$, the well-known standard class of convex functions. It is an established fact that

$$f \in \mathcal{C}(\eta) \Longleftrightarrow zf' \in \mathcal{S}^*(\eta).$$

We denote by $\mathcal{T}^*(\eta)$ and $\mathcal{K}(\eta)$ the subclasses of $\mathcal{T}$ that are, respectively, starlike of order η and convex of order η. Silverman [17] investigated functions of the classes $\mathcal{T}^*(\eta) = \mathcal{T} \cap \mathcal{S}^*(\eta)$ and $\mathcal{K}(\eta) = \mathcal{T} \cap \mathcal{C}(\eta)$.

Let $\mathcal{S}_1$ be the subfamily of $\mathcal{S}$ consisting of functions f for which both f and f' are univalent in $\mathbb{E}$. A function $f(z)$ given in the form (1) is said to be in $\mathcal{S}_m$ if f and its first m derivatives are univalent in $\mathbb{E}$. Furthermore, let $\mathcal{T}_1$ be the subfamily of $\mathcal{T}$ consisting of functions f for which f and f' are univalent in $\mathbb{E}$. It is clear that the second coefficient of a function in $\mathcal{T}_1$ cannot vanish. Therefore, class $\mathcal{T}_1$ is non-empty as the function $z - \frac{z^2}{2}$ belongs to the class $\mathcal{T}_1$. A function $f(z)$ given in the form (2) is said to be in $\mathcal{T}_m$ if f and its first m derivatives are univalent in $\mathbb{E}$. If $f \in \mathcal{T}_m$, then f is said to be in $\mathcal{T}_\infty$.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{UCV}$ of uniformly convex functions in $\mathbb{E}$ if f is a normalized convex function in $\mathbb{E}$ and has the property that, for every circular arc Γ contained in the $\mathbb{E}$, with center ξ also in $\mathbb{E}$, the image curve $f(\Gamma)$ is a convex arc. Goodman [7] introduced the class of uniformly convex functions which is denoted by $\mathcal{UCV}$ and uniformly starlike conditions $\mathcal{UST}$ with the following analytic descriptions, respectively, given by

$$\mathcal{UCV} := \left\{ f \in \mathcal{S} : \Re\left\{ 1 + (z - \zeta)\frac{f''(z)}{f'(z)} \right\} \geq 0, \quad (z, \zeta) \in \mathbb{E} \times \mathbb{E} \right\}$$

and

$$\mathcal{UST} := \left\{ f \in \mathcal{S} : \Re\left\{ \frac{f(z) - f(\zeta)}{(z - \zeta)f'(z)} \right\} \geq 0, \quad (z, \zeta) \in \mathbb{E} \times \mathbb{E} \right\}.$$

Furthermore, we denote by $k-\mathcal{UCV}$ and $k-\mathcal{UST}$ two interesting subclasses of S consisting, of functions which are k-uniformly convex and k-starlike in $\mathbb{E}$, respectively. The analytic descriptions of the above two classes are, respectively, given by

$$k-\mathcal{UCV} := \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| (z \in \mathbb{E}; 0 \le k < \infty)\right\}$$

and

$$k-\mathcal{ST} := \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| (z \in \mathbb{E}; 0 \le k < \infty)\right\}.$$

The class $k-\mathcal{UCV}$ was introduced by Kanas and Wiśniowska [10] , where its geometric definition and connections with the conic domains were considered. The class $k-\mathcal{ST}$ was investigated in [11]. In fact, it is related to the class $k-\mathcal{UCV}$ by means of the well-known Alexander equivalence between the usual classes of convex and starlike functions; see also the work of Kanas and Srivastava [9] for further developments involving each of the classes $k-\mathcal{UCV}$ and $k-\mathcal{ST}$. In particular, if $k=1$, we obtain

$$1-\mathcal{UCV} \equiv \mathcal{UCV}, \quad 1-\mathcal{ST} = \mathcal{SP}, \tag{3}$$

where $\mathcal{UCV}$ and $\mathcal{SP}$ are the familiar classes of uniformly convex functions and parabolic starlike functions in $\mathbb{E}$, respectively (see for details, [6, 7, 14]). In fact, by making use of a certain fractional calculus operator, Srivastava and Mishra [23] presented a systematic and unified study of the classes $\mathcal{UCV}$ and $\mathcal{SP}$.

In 1995, Dixit and Pal [3] defined the class $\mathcal{R}^{\tau}(A,B)$ as follows: For $\tau \in \mathbb{C}\setminus\{0\}$ and $-1 \le B < A \le 1$, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(A,B)$ if it satisfies the following condition

$$\left|\frac{f'(z)-1}{(A-B)\tau - B[f'(z)-1]}\right| < 1, \quad z \in \mathbb{E}.$$

1.1. **Gaussian hypergeometric function-GHF**

The Gaussian hypergeometric function is defined for $|z|<1$ by the hypergeometric series

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n.$$

It is a solution of the hypergeometric differential equation

$$z(1-z)w''(z) + [c-(a+b+1)z]\,w' - abw(z) = 0,$$

and has rich applications in various fields such as conformal mappings, quasiconformal theory, continued fractions and so on. By the Gauss summation theorem, we have

$$F(a,b;c;1)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{4}$$

Here, a,b,c are complex numbers such that $c\neq 0,-1,-2,-3,\ldots,(a)_0=1$ for $a\neq 0$, and for each positive integer n, $(a)_n=a(a+1)(a+2)\cdots(a+n-1)$ is the Pochhammer symbol. In the case of $c=-k, k=0,1,2,\ldots$, $F(a,b;c;z)$ is defined if $a=-j$ or $b=-j$, where $j\leq k$. In this situation, $F(a,b;c;z)$ becomes a polynomial of degree j in z. Results regarding $F(a,b;c;z)$ when $\Re(c-a-b)$ is positive, zero, or negative are abundant in the literature. In particular, when $\Re(c-a-b)>0$ the function is bounded. The hypergeometric function $F(a,b;c;z)$ has been studied extensively by various authors and plays an important role in geometric function theory (for more detail see [13,20–22]). It is useful in unifying various functions by giving approprite values to the parameters a,b and c. We refer to [1,5–16] and references therein for some important results.

Consider a function $F_1(a,b;c;z)$ [19], a normalized form of hypergeometric function $F(a,b;c;z)$ as follows:

$$F_1(a,b;c;z)=z\big(2-F(a,b;c;z)\big)=z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n$$

which clearly implies $F_1(a,b;c;z)\in\mathcal{T}$.

Hohlov [8] introduced a linear operator $H_{a,b,c}:\mathcal{A}\longrightarrow\mathcal{A}$ defined by

$$H_{a,b,c}(f)(z)=zF(a,b;c;z)*f(z)$$

and

$$J_{a,b,c}(f)(z)=z(2-F(a,b;c;z)*f(z),$$

where $F(a,b;c;z)$ is the Gaussian hypergeometric function. Therefore, any function $f(z)$ given in (1), we have

$$H_{a,b,c}(f)(z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_nz^n,$$

and

$$J_{a,b,c}(f)(z)=z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_nz^n.$$

The integral representation of Gaussian hypergeometric function is given by

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\frac{dt}{(1-tz)^a},\qquad \Re(c)>\Re(b)>0.$$

Therefore, we have

$$\left[I_{a,b,c}(f)\right](z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\frac{f(tz)}{t}dt * \frac{z}{(1-tz)^a}.$$

If $f(z) = \frac{z}{1-z} \in \mathcal{C}$, then the operator $I_{a,b,c}(f) \equiv zF(a,b;c;z)$. If we take $a = 1, b = 1+\delta, c = 2+\delta$ with $\Re(\delta) > -1$ then the Hohlov operator $I_{a,b,c}(f)$ turns into Bernardi operator which is defined by

$$B_{f(z)}\left[I_{a,b,c}(f)\right](z) = \frac{1+\delta}{z^{\delta}} \int_0^1 t^{\delta-1}f(t)dt.$$

Indeed, $I_{1,1,2}(f)$ and $I_{1,2,3}(f)$ are known as Alexander and Libera operators, respectively.

These functions and their numerous generalizations have a wide range of applications in physical scientific challenges, including generating functions, proving combinatorial identities, and calculating fields from astronomy to quantum mechanics. In addition, there are many uses for hypergeometric functions in the fields of differential equations, geometry, topology, probability, statistics, and geometric function theory.

While classes of analytic functions for which both f and f' are univalent in $\mathbb{E}$ was investigated earlier by Silverman in 1987, the application of Gaussian hypergeometric functions on the classes of analytic functions for which both f and f' are univalent in $\mathbb{E}$ is not being studied in the literature. Stimulated by prior results on relations between different subclasses of analytic and univalent functions by using hyper geometric functions (see, for example, [2, 4, 24]) we obtain sufficient condition for the function $zF(a,b;c;z)$ to be in the classes $\mathcal{S}_1, \mathcal{T}_1, \mathcal{T}_m$, and information regarding the images of functions belonging in $\mathcal{R}^{\tau}(A,B)$ by using the convolution operator. Finally, we determined conditions for an integral operator to belong to the above classes. Several corollaries and consequences of the main results are also considered.

2. PRELIMINARY LEMMAS

To prove our main results, we need the following lemmas.

Lemma 2.1 ([18, Corollary 2]). *Let a function $f(z)$ given in the form* (2) *be such that $a_2 \neq 0$, for $n \geq 3$. Then $f \in \mathcal{S}_1$ if*

$$\sum_{n=3}^{\infty} n(n-1)|a_n| \leq 2|a_2|.$$

Lemma 2.2 ([18, Theorem 2]). *Let a function $f(z)$ given in the form (2) be such that $a_2 > 0$. Then $f \in \mathcal{T}_1$ if*

$$\sum_{n=3}^{\infty} n(n-1)a_n \leq 2a_2.$$

Further, the condition given is also sufficient for $0 < a_2 \leq \frac{1}{3}$.

Lemma 2.3 ([18, Theorem 6]). *If a function $f(z)$ is given in the form (2), with $\prod_{n=2}^{m+1} a_n \neq 0$, then $f \in \mathcal{T}_m$ if*

$$\sum_{n=k+2}^{\infty} (n-k)(n-k+1)\cdots n a_n \leq (k+1)! a_{k+1},$$

for $k = 1, 2, \ldots, m$.

Lemma 2.4 ([3]). *If a function $f \in \mathcal{R}^{\tau}(A, B)$ is of the form (1), then*

$$|a_n| \leq \frac{(A-B)|\tau|}{n}, \quad n \geq 2.$$

The result is sharp.

Lemma 2.5 ([10]). *Let $f \in \mathcal{A}$ be of the form (1) where $a_n = A_n$. If $f \in k-UCV$, then the coefficient inequality*

$$|A_n| \leq \frac{(P_1)_{n-1}}{(1)_n}, \quad n \geq 2,$$

holds true, where $0 \leq k < \infty$ and $P_1 = P_1(k)$ is denoted by

(5)
$$P_1(k) = \begin{cases} \frac{8(\arccos k)^2}{\pi^2(1-k^2)} & \text{for } 0 \leq k < 1, \\ \frac{8}{\pi^2} & \text{for } k = 1, \\ \frac{\pi^2}{4\sqrt{u}(1+u)(k^2-1)h^2(u)} & \text{for } k > 1, \end{cases}$$

where $t \in (0, 1)$ is determined by $k = \cosh(\pi h'(u)/4h(u))$, h is the Legendre's complete Elliptic integral of the first kind

$$h(u) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-u^2x^2)}}.$$

and $h'(u) = h(\sqrt{1-u^2})$ is the complementary integral of $h(u)$.

Lemma 2.6 ([11]). *Let $f \in \mathcal{A}$ be of the form (1) where $a_n = A_n$. If $f \in k-ST$, then the coefficient inequality*

$$|A_n| \leq \frac{(P_1)_{n-1}}{(1)_{n-1}}, n \geq 2,$$

holds true where $0 \leq k < \infty$ and $P_1 = P_1(k)$ is defined in (5).

LEMMA 2.7. *If* $f \in \mathcal{S}$, $f(z) = z + \sum_2^\infty a_n z^n$ *for* $z \in \mathbb{E}$, *then*

$$|a_n| \leq n \quad \text{for } n = 1, 2, 3 \ldots \tag{6}$$

Equality $|a_n| = n$ *for a given* $n \geq 2$ *holds if and only if* f *is a rotation of the Koebe function.*

3. SUFFICIENT CONDITIONS FOR THE GHF TO BE IN $\mathcal{T}_1, \mathcal{T}_2$ AND $\mathcal{T}_m$

THEOREM 3.1. *If* $a, b > 0$ *and* $c > a + b + 2$, *then a necessary condition for* $zF(a, b; c; z)$ *to be in* $\mathcal{S}_1$ *is that*

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{(a+1)(b+1)}{(c-a-b-2)} + 2\right] \leq 4, \tag{7}$$

holds. If $0 \leq ab \leq \frac{c}{3}$, *then condition* (7) *is necessary and also sufficient for* $F_1(a, b; c; z)$ *to be in* $\mathcal{T}_1$.

Proof. To prove $zF(a, b; c; z) \in \mathcal{S}_1$, by Lemma 2.1, it is sufficient to prove that

$$\sum_{n=3}^{\infty} n(n-1)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| \leq 2\left|\frac{ab}{c}\right|. \tag{8}$$

Noting that $(a)_n = a(a+1)_{n-1}$, $|(a)_n| \leq (|a|)_{n-1}$ and then applying (4), we may express left-hand side of (8).

$$\begin{aligned}
&\sum_{n=3}^{\infty} n(n-1)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| \\
&\quad \leq \sum_{n=3}^{\infty} n\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}} \\
&\quad = \sum_{n=3}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-3}} + 2\sum_{n=3}^{\infty} \frac{a(a+1)_{n-2}\, b(b+1)_{n-2}}{c(c+1)_{n-2}(1)_{n-2}} \\
&\quad = \frac{a(a+1)b(b+1)}{c(c+1)}\sum_{n=3}^{\infty} \frac{(a+2)_{n-3}(b+2)_{n-3}}{(c+2)_{n-3}(1)_{n-3}} \\
&\qquad + 2\frac{ab}{c}\sum_{n=3}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-2}} \\
&\quad = \frac{a(a+1)b(b+1)}{c(c+1)}F(a+2, b+2; c+2; 1) \\
&\qquad + 2\frac{ab}{c}\left[F(a+1, b+1; c+1; 1) - 1\right]
\end{aligned}$$

$$= \frac{a(a+1)b(b+1)\Gamma(c+2)\Gamma(c-a-b-2)}{c(c+1)\Gamma(c-a)\Gamma(c-b)} + \frac{2ab}{c}\left[\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} - 1\right].$$

Hence, we have

$$\sum_{n=3}^{\infty} n(n-1)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \frac{a(a+1)b(b+1)\Gamma(c+2)\Gamma(c-a-b-2)}{c(c+1)\Gamma(c-a)\Gamma(c-b)} + \frac{2ab}{c}\left[\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} - 1\right]. \tag{9}$$

But the expression (9) is bounded above by $\frac{2ab}{c}$ if (7) holds. By Lemma 2.2, condition (7) is sufficient for $F_1(a,b;c;z) \in \mathcal{T}_1$. This essentially completes the proof of Theorem 3.1. □

By fixing $b = a$, we can improve the assertion of Theorem 3.1 as follows.

Corollary 3.2. *If $a > 0$ and $c > 2a+2$, then a necessary condition for $zF(a,b;c;z)$ to be in $\mathcal{S}_1$ is that*

$$\frac{\Gamma(c+1)\Gamma(c-2a-1)}{[\Gamma(c-a)]^2}\left[\frac{(a+1)^2}{(c-2a-2)} + 2\right] \le 4, \tag{10}$$

holds. If $0 \le a \le \sqrt{\frac{c}{3}}$, then condition (10) is necessary and also sufficient for $F_1(a,b;c;z)$ to be in $\mathcal{T}_1$.

Theorem 3.3. *If $a, b > 0$ and $c > a+b+3$, then a sufficient condition for $F_1(a,b;c;z)$ to be in $\mathcal{T}_2$ is that*

$$\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{(a+2)(b+2)}{(c-a-b-3)} + 3\right] \le 6. \tag{11}$$

Proof. To prove $F_1(a,b;c;z) \in \mathcal{T}_2$, by Lemma 2.3, it is sufficient to show that

$$\sum_{n=4}^{\infty} n(n-2)(n-1)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \le 6\frac{a(a+1)\,b(b+1)}{c(c+1)(1)_2}. \tag{12}$$

Noting that $(a)_n = a(a+1)_{n-1}$, and then applying (4), we may express the left-hand side of (12)

$$\sum_{n=4}^{\infty} n(n-2)(n-1)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

$$= \sum_{n=4}^{\infty} n\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-3}}$$

$$
\begin{aligned}
&= \sum_{n=4}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-4}} + 3\sum_{n=4}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-3}} \\
&= \frac{a(a+1)(a+2)\, b(b+1)(b+2)}{c(c+1)(c+2)} \sum_{n=0}^{\infty} \frac{(a+3)_n(b+3)_n}{(c+3)_n(1)_n} \\
&\quad + \frac{3a(a+1)\, b(b+1)}{c(c+1)} \sum_{n=1}^{\infty} \frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n} \\
&= \frac{a(a+1)(a+2)\, b(b+1)(b+2)}{c(c+1)(c+2)} F(a+3, b+3; c+3; 1) \\
&\quad + \frac{3a(a+1)\, b(b+1)}{c(c+1)} [F(a+2, b+2; c+2; 1) - 1].
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=4}^{\infty} n(n-2)(n-1) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&\qquad = \frac{a(a+1)(a+2)b(b+1)(b+2)\Gamma(c+3)\Gamma(c-a-b-3)}{c(c+1)(c+2)\Gamma(c-a)\Gamma(c-b)} \\
&\qquad\quad + \frac{3a(a+1)b(b+1)\Gamma(c+2)\Gamma(c-a-b-2)}{c(c+1)\Gamma(c-a)\Gamma(c-b)} \\
&\qquad\quad - \frac{3a(a+1)b(b+1)}{c(c+1)}.
\end{aligned} \tag{13}
$$

But the expression (13) is bounded above by $\frac{6a(a+1)b(b+1)}{c(c+1)(1)_2}$, if (11) holds, which completes the proof of Theorem 3.3. □

Assuming $b = a$, the assertion of Theorem 3.3 yields the following result.

COROLLARY 3.4. *If $a > 0$ and $c > 2a + 3$, then a sufficient condition for $F_1(a, b; c; z)$ to be in $\mathcal{T}_2$ is that*

$$
\frac{\Gamma(c+2)\Gamma(c-2a-2)}{[\Gamma(c-a)]^2} \left[\frac{(a+2)^2}{(c-2a-3)} + 3 \right] \le 6. \tag{14}
$$

THEOREM 3.5. *If $a, b > 0$ and $c > a + b + (k+1)$, then a sufficient condition for $F_1(a, b; c; z)$ to be in $\mathcal{T}_m$ is that*

$$
\frac{\Gamma(c+k)\Gamma(c-a-b-k)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{(a+k)(b+k)}{(c-a-b-(k+1))} + (k+1) \right] \le 2(k+1). \tag{15}
$$

Proof. To prove $F_1(a, b; c; z) \in \mathcal{T}_m$ by Lemma 2.3, it is sufficient to show that

$$
\sum_{n=k+2}^{\infty} (n-k)(n-k+1)\cdots(n-1)n \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \le (k+1)! \frac{(a)_k(b)_k}{(c)_k(1)_k}, \tag{16}
$$

where $k = 1, 2, \ldots, m$. Equation (16) can be written as

$$\sum_{n=k+2}^{\infty} (n-k)(n-k+1)\cdots(n-1)n\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq (k+1)k!\frac{a(a+1)\cdots(a+k-1)b(b+1)\cdots(b+k-1)}{c(c+1)\cdots(c+k-1)k!}. \tag{17}$$

Noting that $(a)_n = a(a+1)_{n-1}$ and then applying (4), we may express left-hand side of (17) as

$$\begin{aligned}
&\sum_{n=k+2}^{\infty} (n-k)(n-k+1)\cdots(n-1)n\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&\quad = \sum_{n=k+2}^{\infty} n\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-(k+1)}} \\
&\qquad + \sum_{n=k+2}^{\infty} (n-(k+1))\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-(k+1)}} \\
&\qquad + (k+1)\sum_{n=k+2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-(k+1)}} \\
&\quad = \frac{(a+k)(b+k)}{(c+k)}\sum_{n=0}^{\infty}\frac{(a+(k+1))_n(b+(k+1))_n}{(c+(k+1))_n(1)_n} \\
&\qquad + (k+1)\sum_{n=1}^{\infty}\frac{(a+k)_n(b+k)_n}{(c+k)_n(1)_n} \\
&\quad = \frac{(a+k)(b+k)}{(c+k)}F\big(a+(k+1), b+(k+1); c+(k+1); 1\big) \\
&\qquad + (k+1)\big[F(a+k, b+k; c+k; 1) - 1\big].
\end{aligned}$$

That is,

$$\begin{aligned}
&\sum_{n=k+2}^{\infty} (n-k)(n-k+1)\cdots(n-1)n\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&\quad = \frac{(a+k)(b+k)\Gamma(c+k+1)\Gamma(c-a-b-k-1)}{(c+k)\Gamma(c-a)\Gamma(c-b)} \\
&\qquad + (k+1)\left[\frac{\Gamma(c+k)\Gamma(c-a-b-k)}{\Gamma(c-a)\Gamma(c-b)} - 1\right].
\end{aligned} \tag{18}$$

But the expression (18) is bounded above by $(k+1)!\frac{(a)_k(b)_k}{(c)_k(1)_k}$, if (15) holds, which once and for all completes the proof of Theorem 3.5. □

By noting that if $f \in \mathcal{T}_m$ for every integer m, then $f \in \mathcal{T}_\infty$, we have the following corollary.

Corollary 3.6. *If*

$$F_1(a,b;c;z) = z - \sum_{n=2}^{\infty} \left(\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right) z^n \in \mathcal{T},$$

$\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \neq 0$ *and* (16) *holds for every* k *then* $F_1(a,b;c;z) \in \mathcal{T}_\infty$.

Taking $b = a$, we state the assertion of Theorem 3.5 as follows.

Corollary 3.7. *If* $a > 0$ *and* $c > 2a+(k+1)$, *then a sufficient condition for* $F_1(a,b;c;z)$ *to be in* $\mathcal{S}_m$ *is that*

$$\frac{\Gamma(c+k)\Gamma(c-2a-k)}{[\Gamma(c-a)]^2} \left[\frac{(a+k)^2}{(c-2a-(k+1))} + (k+1) \right] \leq 2(k+1). \tag{19}$$

4. SUFFICIENT CONDITIONS FOR THE INTEGRAL FORM OF GHF

Let us consider the Integral operator $G(a,b;c;z)$ and $G_1(a,b;c;z)$ as follows

$$G(a,b;c;z) = \int_0^z F(a,b;c;t)dt = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} z^n,$$

$$G_1(a,b;c;z) = \int_0^z \big(2 - F(a,b;c;t)\big)dt = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} z^n.$$

Theorem 4.1. *If* $a, b > 0$ *and* $c > a+b+1$, *then a necessary condition for* $G(a,b;c;z) = \int_0^z F(a,b;c;t)dt$ *to be in* $\mathcal{S}_1$ *is that*

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \leq 2, \tag{20}$$

holds. If $0 \leq ab \leq \dfrac{c}{3}$, *then condition* (20) *is necessary and also sufficient for* $G_1(a,b;c;z)$ *to be in* $\mathcal{T}_1$.

Proof. To prove $G(a,b;c;z) \in \mathcal{S}_1$, by Lemma 2.1, it is sufficient to show that

$$\sum_{n=3}^{\infty} n(n-1) \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} \right| \leq \frac{2ab}{c}. \tag{21}$$

Noting that $(a)_n = a(a+1)_{n-1}$, and then applying (4), we may express left-hand side of (21)

$$\sum_{n=3}^{\infty} n(n-1)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n}\right| \leq \sum_{n=3}^{\infty} n(n-1)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_n}$$
$$= \frac{ab}{c}\sum_{n=1}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}.$$

Hence,

$$\sum_{n=3}^{\infty} n(n-1)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} \leq \frac{ab}{c}\left[\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} - 1\right]. \tag{22}$$

But the expression (22) is bounded above by $\frac{2ab}{c}$, if (20) holds. This essentially completes the proof of Theorem 4.1. □

Fixing $b = a$, we state the following result.

COROLLARY 4.2. *Let $a > 0$ and $c > 2a+1$, then a necessary condition for $G(a, a; c; z)$ to be in $\mathcal{S}_1$ is that*

$$\frac{\Gamma(c+1)\Gamma(c-2a-1)}{[\Gamma(c-a)]^2} \leq 2, \tag{23}$$

holds. If $0 \leq a \leq \sqrt{\frac{c}{3}}$, then condition (23) is necessary and also sufficient for $G_1(a, b; c; z)$ to be in $\mathcal{T}_1$.

THEOREM 4.3. *If $a, b > 0$ and $c > a+b+1$, then a sufficient condition for $G_1(a, b; c; z)$ to be in $\mathcal{T}_2$ is that*

$$\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)} \leq 2. \tag{24}$$

Proof. To prove $G_1(a, b; c; z) \in \mathcal{T}_2$, by Lemma 2.3, it is sufficient to show that

$$\sum_{n=4}^{\infty} n(n-1)(n-2)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} \leq \frac{3!a(a+1)b(b|+1)}{c(c+1)(1)_3}. \tag{25}$$

Noting that $(a)_n = a(a+1)_{n-1}$, and then applying (4), we may express left-hand side of (25)

$$\sum_{n=4}^{\infty} n(n-1)(n-2)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} = \frac{3!a(a+1)b(b+1)}{c(c+1)(1)_3}\sum_{n=1}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n}.$$

$$\sum_{n=4}^{\infty} n(n-1)(n-2)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n}$$

$$= \frac{3!a(a+1)b(b+1)}{c(c+1)(1)_3}\left[\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)} - 1\right]. \tag{26}$$

But the expression (26) is bounded above by $\frac{3!a(a+1)b(b+1)}{c(c+1)(1)_3}$, if (24) holds, which completes the proof of Theorem 4.3. □

Fixing $b = a$, we can state the assertion of Theorem 4.3 as follows.

COROLLARY 4.4. *If $a > 0$ and $c > 2a + 2$, then a sufficient condition for $G_1(a, a; c; z)$ to be in $\mathcal{T}_2$ is that the inequality*

$$\frac{\Gamma(c+2)\Gamma(c-2a-2)}{[\Gamma(c-a)]^2} \leq 2 \tag{27}$$

holds true.

5. CONCLUDING REMARKS

We have obtained the sufficient conditions for the functions $F(a, b; c; z)$, $H_{a,b,c}(f)$ to be in the classes $\mathcal{S}_1$, $\mathcal{T}_1, \mathcal{T}_m$. More importantly, we obtained the mapping properties of Integral operator $G(a, b; c; z)$ and $G_1(a, b; c; z)$ and a result for the particular integral operator acting on $F(a, b; c; z)$. The results derived in this paper are general in nature and expected to find certain applications in the theory of special functions. Moreover, one can determine new results for the subclasses presented in this article allied with certain probability distribution series [12] and Mittag-Leffler functions [13].

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