A SPECIAL SEQUENCE AND PRIMORIAL NUMBERS

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In this paper, we study a class of functions defined recursively on the set of natural numbers in terms of the greatest common divisor algorithm of two numbers and requiring a minimality condition. These functions are permutations, products of infinitely many cycles that depend on certain breaks in the natural numbers involving the primes, and some special products of primes with a density of approximately 29.4%. We show that these functions split into only two equivalence classes (modulo the natural equivalence relation of eventually identical maps): one is the class of the identity map and the other is generated by a map whose discrete derivative is almost periodic with "periods" of the primorial numbers.

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1. INTRODUCTION

The following problem was proposed by the first author in Crux Mathematicorum [2]:

Let $f : \mathbb{N} \to \mathbb{N}$ *with* $f(1) = 1$, $f(2) = a$ *for some* $a \in \mathbb{N}$ *and, for each positive integer* $n \geq 3$, $f(n)$ *is the smallest value not assumed at lower integers that is coprime with* $f(n-1)$. *Prove that* f *is onto.*

In what follows, we use the conventional notation for the greatest common divisor of two natural numbers: for $m, n \in \mathbb{N}$ this is denoted by $\gcd(m, n)$.

If a is irrelevant, we refer to the sequence by f. When a plays an important role, we refer to the sequence by f_a. For example, f_1 is not much different from f_2 which is the identity map on $\mathbb{N}$, so we assume that $a \geq 2$ from now on.

A special case of the sequence was introduced for $a = 3$, in OEIS (The On-Line Encyclopedia of Integer Sequences) by Reinhard Zumkeller in 2003 [14] but with an intrinsic definition: "Smallest natural number x_n which is coprime to n and to x_{n-1}, and is not yet in the sequence ($x_1 = 1$)." It is not obvious that this definition is equivalent to our definition above and we show this as a

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corollary of Theorem 2.3. A slightly different definition is given for [9] which coincides with our sequence f_3, for all indices $n \geq 4$.

It turns out that the sequence can be easily computed (a short code in Python is provided on OEIS) and the first 10,000 terms of f_3 are also available. We prove that f_a is not only a surjection but also an injection. So, we are dealing with permutations of $\mathbb{N}$. In particular, [14] is f_3^{-1}.

There are a few results that are mentioned in OEIS by Michael De Vlieger (April 13th, 2022) concerning properties of f_3 (see [12]).

PROPOSITION 1.1. *$f_3(2k+1) = 2k$ for all $k > 0$.*

We show that this follows from Theorem 2.3.

PROPOSITION 1.2. *$f_3(3k+1) = 3k$ for all $k > 1$.*

Let us observe that for $k = 2m$ this follows from Proposition 1.1. Also, from Proposition 1.1 we see that the terms of f_3 are following the pattern:

$$1, 3, 2, 5, 4, 7, 6, \boxed{?}, 8, \boxed{?}, 10, \boxed{?} 12, \boxed{?}, 14, \boxed{?} 16, \boxed{?}, \ldots$$

Let us assume for the moment that f_3 is a surjection. Then, if we look at 9, it cannot fit in the first box since $\gcd(6, 9) = 3$, and so it should go into the second by its minimality. That means $f(10) = 9$. Then 15 cannot go into the first, the third, or the fourth box since $\gcd(12, 15) = 3$ and then by minimality it has to go into the fifth which means $f(16) = 15$. This argument can be finished by induction showing that $f(6m+4) = 6m+3$ for $m \geq 1$ proving the claim for $k = 2m+1$.

The list of the first 24 terms in f_3 is included next:

n	1	2	3	$\boxed{4}$	5	$\boxed{6}$	7	$\boxed{8}$	9	10	11	$\boxed{12}$
$f(n)$	1	3	2	(5)	4	(7)	6	(11)	8	9	10	(13)

n	13	$\boxed{14}$	15	$\boxed{16}$	17	$\boxed{18}$	19	$\boxed{20}$	21	22	23	$\boxed{24}$
$f(n)$	12	(17)	14	15	16	(19)	18	(23)	20	21	22	(25)

In [12] there is a mention of a concept named *record*. If we look at the above table, we observe some bigger jumps when the sequence goes up more than 1 from the previous value in the sequence. In the next section, we introduce a slightly different term, that of a *turning point*. The numbers in the boxes are turning points and their values (circled) are records. The smallest composite value for a turning point is $f_3(24) = 25$ and the smallest record that has at least two prime factors is $f_3(54) = 55$. From Proposition 1.1 and Proposition 1.2, we see that every record is an odd number and 3 cannot divide a record. Hence, all records must be of the form $6k \pm 1$. We show that every prime $p \geq 5$ is a record. So, this function contains good information about primes having the advantage that the terms can be recursively calculated only using the gcd function. Also, one can compute a section of the sequence without knowing all of the terms up to that particular starting input.

Here is a list of all the non-prime records less than 100 and their jumps, i.e., $j(r) = r - f_3^{-1}(r)$.

$$\big\{[25,1],\ [49,1],\ [55,1],\ [77,3],\ [85,1],\ [91,1]\big\}.$$

There are a few important questions here related to the records (especially the ones which are composite numbers), say $\{\overline{R}_j\}$, $\overline{R}_1 = 25$, $\overline{R}_2 = 49$, $\overline{R}_3 = 55$, etc. What is their distribution? What is their distribution within the records, or equivalently, what is the distribution of the primes within the set of records?

In general, a permutation of a finite set is a product of cycles. In our case, f_a is a permutation of the infinite set $\mathbb{N}$. However, we show that f_a is still a product of finite cycles. We use the usual convention of denoting a cycle by $(c_1, c_2, \ldots, c_n)$ meaning the permutation which maps c_1 into c_2, c_2 into $c_3, \ldots$, and c_n into c_1. Cycles of length one are usually left out. This way we can write

$$f_3 = (3,2)(5,4)(7,6)(11,10,9,8)(13,12)$$
$$(17,16,15,14)(19,18)(23,22,21,20)(25,24)\ldots$$

So, essentially f_3 is defined by the sequence of records.

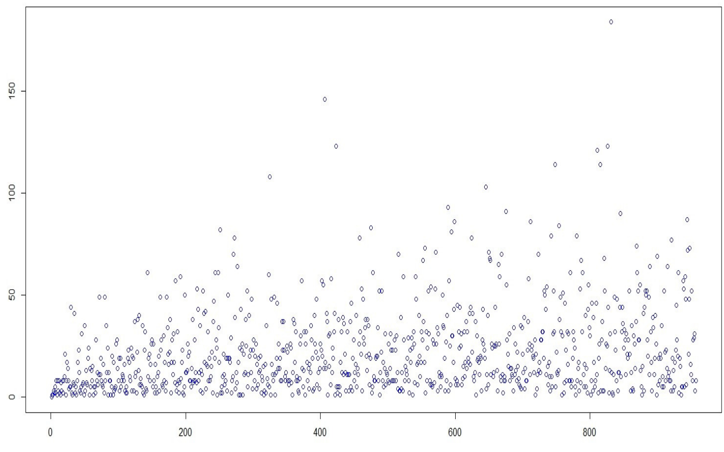

Figure 1 – Twin primes distribution into cycles.

We can permute these cycles in whatever order we want. But we assume the set of these cycles to be ordered in terms of the numbers in it (non-decreasing). For each record t, denote by $C(t)$ the cycle number. For instance, $C(23) = 8$ and $C(25) = 9$.

In Figure 1, we included the values of $C(m_{j+1}) - C(M_j)$ where (m_j, M_j) is a twin pair of primes. We formulate a few conjectures about this data in Section 3. We notice that $j(r) = 1$ is an indication that r is the biggest of a twin pair. This happens for a lot of records which are not primes. However, let us call these records *twin records*.

Let us point out at least one connection with primorial numbers. If p_n is the n-th prime, the n-th primorial number (see [13]) is defined by

$$p_n\# = \prod_{k=1}^{n} p_k.$$

The values of $p_n\#$ for $n = 1, 2, \ldots$, are 2, 6, 30, 210, 2310, 30030, 510510, ... ([11]).

In Figure 2, we included the values of

$$g(t) := f(t) - f(t-1),\ t = 1, 2, \ldots, 12000 \ \text{ for } f_3.$$

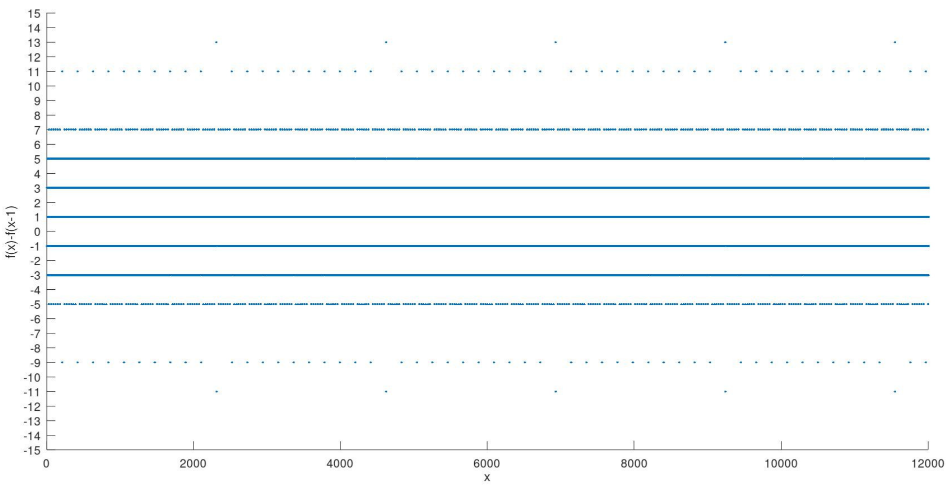

Figure 2 – Discrete derivative of f.

We show that $|g(t)|$ is unbounded by proving that for infinitely many $k \in \mathbb{N}$, we have

$$g(kp_n\# + 1) \geq 2n + 1.$$

Similar sequences have been studied in [1, 8, 5, 6, p. 94]. The sequence in [1] is [15] in the OEIS. Very similar results are shown, including the proof of the one-to-one correspondence with the natural numbers.

2. RESULTS AND PROOFS

Definition 2.1. A *turning point* is a natural number having either of the following two properties:

i) $t > 3$ and $f(t) - f(t-1) > 1$.

ii) $t = 3$ and $f(3) \neq \min \mathbb{N} \setminus \{1, a\}$.

The value $f(t)$ for t a turning point is called a *record*.

For example, if $a = 4$, the few terms of the sequence are:

$$1, 4, 3, 2, 5, 6, 7, 8, \dots$$

and so 3 is a turning point and $t = 5$ is also a turning point since

$$f_4(5) - f_4(4) = 5 - 2 = 3 > 1.$$

Since $f_4(5) = 5$, this is called a fixed point and we see that lots of fixed points follow, making the sequence less interesting. In fact, for $a = 2$, we have no turning point since $f_2(n) = n$ for all n. We are not going to consider these functions and assume further that $a \geq 3$.

Definition 2.2. An *essential turning point* (or ETP) is a turning point t having in addition the following three properties:

i) $t > a$ and $f(t) \neq t$.

ii) $f(t-1) = t - 2$.

iii) $\{1, 2, 3, \dots, t-1\} = \{f(1), f(2), \dots, f(t-1)\}$.

For example, if $a = 7$, the first 12 terms are given in the table below:

n	1	2	3	4	5	6	7	$\boxed{8}$	9	10	11	$\boxed{12}$
$f(n)$	1	7	2	3	4	5	6	(11)	8	9	10	(13)

We observe that $t_1 = 8$ is an essential turning point. Also, $t_2 = 12$ is an ETP and the list continues. The two corresponding records are the primes 11 and 13. We prove the following theorem about ETPs.

THEOREM 2.3. *If t is an ETP then $T := f(t) + 1$ is the next ETP and there are no turning points in the interval (t, T).*

Proof. Since $t > a$, we may assume that $t > 3$, and define $p := f(t) - f(t-1)$. Hence, we have $p > 1$ since t is a turning point. Hence,

$$T = f(t) + 1 = p + f(t-1) + 1 = p + t - 2 + 1 = p - 1 + t > t.$$

By the definition of f, we have

$$1 = \big(f(t), f(t-1)\big) = \big(f(t-1)+p, f(t-1)\big) = \big(p, f(t-1)\big).$$

So p is the smallest natural number $p > 1$, with the property $(p, f(t-1)) = 1$, such that $p + f(t-1) = p + t - 2 \geq t$ is not one of the values $f(m)$ with $m \leq t-2$. The last condition is obviously satisfied since

$$\big\{f(1), f(2), \ldots, f(t-2)\big\} \subset \{1, 2, \ldots, t-1\}$$

by the definition of ETP.

We claim that $(f(t), f(t-1)+2) = 1$. By way of contradiction, suppose that

$$\big(f(t), f(t-1)+2\big) = d > 1.$$

Then d divides $f(t) - (f(t-1)+2) = p-2 < p$. But $(f(t), f(t-1)) = 1$ and since d divides $f(t)$ we must have $1 = (d, f(t-1)) = (d + f(t-1), 1)$. The minimality of p that was pointed out above shows that $d = p$, a contradiction. It remains that $d = 1$ and then by the definition of $f(t+1)$, $f(t+1) = f(t-1)+2 = t-2+2 = t$.

As result of this, $f(t+2) = t+1, \ldots, f(t+j) = t+j-1$, as long as $t+j-1 < f(t) = T-1$. None of the values $t+j$ are turning points. For j such that $t+j-1 = f(t) = T-1$ or $j = j_0 = T-t$, we have

$$\begin{aligned}&\{f(1), f(2), \ldots, f(t-1), f(t), \ldots, f(T-1)\}\\ &\quad = \{1, 2, \ldots, t-1, T-1, t, t+1, \ldots, T-3, f(T-1)\}\\ &\quad = \{1, 2, \ldots, t-1, t, t+1, \ldots, T-2, T-1\}\end{aligned}$$

which shows that T is the next ETP provided that $f(T) - f(T-1) = f(T) - T + 2 > 1$. Because $f(T)$ is forced to be more than or equal to T the last constraint is satisfied. Therefore, the next ETP is T. □

Remark 2.4. If we have at least one ETP, say t_0 (may as well assume it is the smallest one), then we can generate them all by using the recursion

$$t_n = f(t_{n-1}) + 1 \ for\ n \geq 1.$$

The sequence t_n is strictly increasing, making it unbounded. The property (iii) of an ETP shows that f_a is then onto. So, the problem we started with in the Introduction is proven if we show the existence of at least one ETP. In general, for some values of a, f_a does not have any ETP. However, in that case, it is easy to show that f_a is a bijection.

Remark 2.5. Let us observe that f is actually one-to-one. Indeed, let us assume that $1 \leq m < n$. If $n = 2$ then $m = 1$ and so $f(1) = 1 \neq f(2) = a$ by the assumption in the Introduction. If $n \geq 3$, by definition $f(n)$ is not in the set

$$\{f(1), f(2), \ldots, f(m), \ldots, f(n-1)\}$$

and so $f(n) \neq f(m)$. We have seen that

$$f_3 = (3,2)(5,4)(7,6)(11,10,9,8)(13,12)(17,16,15,14)(19,18)(23,22,21,20)(25,24)\ldots$$

with the beginning of each cycle an ETP (except for 2). Similarly, we have the writing

$$f_7 = (7,6,5,4,3,2)(11,8,9,10)(13,12)\ldots$$

and clearly, we observe that f_3 and f_7 have the same cycles eventually. Also, another interesting situation appears if a is a multiple of 6:

$$\begin{aligned} f_6 &= (6,5,2,3,4) \\ f_{12} &= (12,5,2,3,4,7,6,11,8,9,10) \\ f_{18} &= (18,5,2,3,4,7,6,11,8,9,10,13,12,17,14,15,16) \\ &\vdots \end{aligned}$$

in which case $f(n) = n$ eventually (for big enough n), and f consists of only one nontrivial cycle.

Definition 2.6. Let us call two permutations f_a and f_b *EI-permutations* (eventually identical) if there exists m which depends on a and b such that $f_a(n) = f_b(n)$ for all $n > m$. This (equivalence) relation partitions the set of these bijections into equivalence classes, $\mathcal{C}$.

Observation suggests we have only two classes so let $\mathcal{C} = \mathcal{C}_3 \cup \mathcal{ID}$ where $\mathcal{C}_3$ is the class of f_3 and $\mathcal{ID}$ is the class of f_2 or eventually the identity maps. For a such that $f_a \in \mathcal{ID}$ let us denote by M_a the smallest natural number with the property $f_a(n) = n$ for all $n \geq M_a$. Also, we use the notation

$$\mathcal{A} := \{a \in \mathbb{N} | f_a \quad \text{is in} \quad \mathcal{ID}\}.$$

The set $\mathcal{A}$ appears to be nontrivial but we see that most of the numbers which are multiples of 6 are in $\mathcal{A}$. There are some exceptions such as $a = 216 = 210 + 6$. We observe that $210 = 2 \cdot 3 \cdot 5 \cdot 7$ and $6 = 2 \cdot 3$ which are primorial numbers. We prove next Proposition 1.1 from the Introduction and that every prime is a record for f_3.

Corollary 2.7. *$f_3(2k+1) = 2k$ for all $k \geq 1$ and if p is a prime greater than or equal to 5, p is a record of f_3.*

Proof. It is clear that the first ETP for f_3 is $t_1 = 4$, the second is $t_2 = f(t_1) + 1 = 6$, and so on. We see that between two consecutive turning points, as in the above proof, the sequence continues as:

$$\frac{k \quad t_n - 1 \quad t_n \quad t_n + 1 \;\; t_n + 2, \ldots \;\; T_n - 1 \quad T_n \quad T_n + 1}{f_3(k) \;\; t_n - 2 \;\; T_n - 1 \quad t_n \quad t_n + 1, \ldots \;\; T_n - 2 \;\; f(T_n) \;\; T_n, \ldots}, \;\; T_n = f(t_n) + 1.$$

Using induction on n we see that every t_n must be even and every record $R_n = f(t_n)$ must be odd. So the sequence of values that are even goes in increasing order and $f_3(2k + 1) = 2k$ for each $k \geq 1$.

The second claim in the corollary is obviously true for $p = 5$. Let us assume by way of contradiction that $p > 5$ is not a record. So, it appears in the sequence (f_3 is a bijection) between two EPTs as above. Then $p < T_n - 1 = f(t_n)$ and p is relatively prime with $t_n - 2 < p$, contradicting the choice of $f(t_n)$. Then every prime (except 2 and 3) is a record. □

Remark 2.8. We see that the proof above works if we assume that for the prime p there exists an ETP t such that $p > t$. In particular, it is true for every a such that f_a has at least one ETP and p a prime big enough.

Next, let us show that $|g(t)|$ is unbounded, where $g(t) = f(t + 1) - f(t)$ for $t \in \mathbb{N}$.

PROPOSITION 2.9. *$g(kp_n\# + 1) \geq 2n + 1$ for infinitely many $k \in \mathbb{N}$.*

Proof. By Dirichlet's theorem on arithmetic progressions, $q = kp_n\# + 1$ is a prime for infinitely many $k \in \mathbb{N}$. Choose k so that $q > 5$. This means that q is an ETP for f_3 by Corollary 2.7. Then we have

$$g(kp_n\#+1) = f_3(kp_n\#+2) - f_3(kp_n\#+1) = f_3(kp_n\#+2) - kp_n\# := m+1-q$$

where $m := f_3(kp_n\# + 2)$ is relatively prime with $q - 1 = kp_n\#$ and is bigger than q. Then m must not be divisible by any of the prime factors of $q - 1$. So $m \neq q + 1$ as $q + 1$ is even. So we are done if $n = 1$.

Suppose $n \geq 2$. Observe that $\gcd(q - 1, q + 2r)$ is divisible by at least one of the p_i, $i = 1, 2, \ldots, n$ due to the obvious inequality $p_n \geq 2n - 1$. Also, as $q - 1$ is even, m must be odd and hence $m - q$ is even. So $m - q \geq 2n$ which completes the proof. □

THEOREM 2.10. *If a is odd and t is an ETP then t is an even number and $f_a \in \mathcal{C}_3$*

Proof. If a is an odd number then $f(3) = 2$, $f(4) = 3$, and so on, until $f(a) = a - 1$, and then $f(a + 1)$ is $a + 2$ or bigger, turning $a + 1$ into an EPT. This is the first EPT. Clearly, $t_1 = a + 1$ is even and so $f(t_1)$ must be odd, otherwise $(f(a), f(a+1)) = (a-1, f(t_1)) \geq 2$, a contradiction. This shows that $t_2 = f(t_1) + 1$ (by Theorem 2.3) is even. Inductively, we see that all of ETP's must be even. Supposed that we take a prime $p > t_2 = f(t_1)+1$ and also $p \geq 5$. As in the proof of Corollary 2.7, p must be a record or $f(t_k) = p$ for some k. Then $t_{k+1} = f(t_k) + 1 = p + 1$ is a EPT for f_a but also for f_3. Therefore, from this point on, $f_3(n) = f_a(n)$ for all $n \geq p + 1$ since the definitions of the two functions are recursively in terms of the same data. $\square$

Remark 2.11. The result obtained in Theorem 2.10 can be clearly improved by only assuming that f_a is a function which does have an ETP.

THEOREM 2.12. *Assume a is even and a multiple of* 6. *Then f_a is either in $\mathcal{ID}$ or in $\mathcal{C}_3$ in which case every ETP is even.*

Proof. Let us assume that $a > 4$ and define $\kappa := f(3) \neq 3$ which must be an odd number and in addition $(\kappa, a) = 1$. Since $a-1$ is odd and $(a-1, a) = 1$, by the minimality of $f(3)$, we see that $\kappa \leq a - 1$.

Clearly, κ is an odd number greater than or equal to 5 and then $f(4) = 2$, $f(5) = 3, \ldots, f(\kappa+1) = \kappa - 1$ an even number. Then, $f(\kappa + 2)$ should skip κ since it is already in the list. Then the next candidate is $\kappa + 1$ but this is also even so we need to move up to $\kappa + 2$, i.e., $f(\kappa + 2) = \kappa + 2$. This is possible for lots of values of a, ($a \in \{6, 12, 18, 24, 36, \ldots\}$). If we have

$$\{1, 2, \ldots, \kappa + 2\} = \big\{f(1), f(2), \ldots, f(\kappa + 2)\big\}$$

then $f(n) = n$ for all $n \geq \kappa + 2$. In this case, we have no turning point and no ETP. This is exactly what happens if $a = 6$ and only if $a = 6$ (but it is not necessary to prove this at this point). Hence, we assume that $a \geq 12$ from here on. So, if $f(\kappa + 2) = \beta$ for some odd number $\beta \geq \kappa + 2$ with $(\beta, \kappa - 1) = 1$, $\kappa + 2$ becomes a turning point. We look at f_{36} as a generic example:

n	1	2	$\boxed{3}$	4	5	6	$\boxed{7}$	8	$\boxed{9}$	10	11	12	$\boxed{13}$	14	$\boxed{15}$	16	17	18	$\boxed{19}$	20
$f(n)$	1	36	5	2	3	4	7	6	11	8	9	10	13	12	17	14	15	16	19	18

n	$\boxed{21}$	22	23	24	$\boxed{25}$	26	$\boxed{27}$	28	29	30	$\boxed{31}$	32	$\boxed{33}$	34	35	36	37	38
$f(n)$	23	20	21	22	25	24	29	26	27	28	31	30	37	32	33	34	35	38

This shows that $M_{36} = 38$. We notice that in this example $f(2k) = 2k-2$ for all $k \in \{2, 3, \ldots, 18\}$. This rule breaks at $k = 19$ and also 38 is a turning point and the last one.

Let us denote by s, the largest turning point with the property that $s < a$ and we let $q = f(s)$.

We claim that if $q+1=a$ then $q+2=a+1$ is a turning point which is equal to M_a. If $q+1>a$, then $q+1$ is a turning point which is equal to M_a or it is the first ETP and f_a is in $\mathcal{C}_3$.

In general, for $a \in \{6, 12, \ldots\}$, we already know that $f(4)=2$, $f(5)=3$, $f(6)=4$, and so on. First, let us prove that

$$(1) \qquad f(2k) = 2k-2 \quad \text{for all} \quad k \text{ such that } 4 \le 2k < q+1.$$

We have already shown that (a is at least 12) $f(\kappa+1)=\kappa-1$ and $f(\kappa+2)=\beta \ge \kappa+2$. We claim that $f(\kappa+3)=\kappa+1$ which means the property (1) holds for all k such that $2k \le \kappa+3$. The list

$$\big[f(1), f(2), \ldots, f(\kappa+2)\big] = [1, a, \kappa, 2, 3, 4, \ldots, \kappa-1, \beta]$$

shows that if

$$(\kappa+1, \beta) = 1$$

then $f(\kappa+3)=\kappa+1$. The proof of this is done by way of contradiction as in the proof of Theorem 2.3. Hence, the property (1) holds for $2k=\kappa+3$. This allows us to continue the above list

$$\big[f(1), f(2), \ldots, f(\kappa{+}2), \ldots, f(\beta{+}1)\big] = \big[1, a, \kappa, 2, 3, 4, \ldots, \kappa{-}1, \beta, \kappa{+}1, \ldots, \beta{-}1\big]$$

making $\beta+2$ the next turning point. This list continues until we get to the last turning point less than a, which is s. Note that all the turning points are odd numbers. The list above becomes

$$\begin{aligned}\big[f(1), f(2), \ldots, f(\kappa+2), \ldots, f(\beta+1), f(\beta+2), \ldots, f(s), \ldots, f(q+1)\big]\\ = \big[1, a, \kappa, 2, 3, 4, \ldots, \kappa-1, \beta, \kappa+1, \ldots, \beta-1, \beta', \ldots, q, \ldots, q-1\big].\end{aligned}$$

We observe that $q+2<a$ is not possible since s is the greatest turning point less than a. So, $q+2 \ge a$, but since $q+2$ is odd we must have $q+2 \ge a+1$. If we have equality, $q=a-1$, then from the above equality of lists we conclude that $f(q+2)=a+1=q+2$ ($\gcd(q+2, q-1)=\gcd(3, q-1)=\gcd(3, a-2)=1$) and for $n \ge q+2$, we have $f(n)=n$. That makes $a+1=M_a$ and there are no EPT.

In the second situation, $q+1>a$ and the above equality of lists need to be corrected. So

$$\begin{aligned}\big[f(1), f(2), \ldots, f(\kappa+2), \ldots, f(\beta+1), f(\beta+2), \ldots, f(s), \ldots, f(q), f(q+1)\big]\\ = \big[1, a, \kappa, 2, 3, 4, \ldots, \kappa-1, \beta, \kappa+1, \ldots, \beta-1, \beta', \ldots, q, \ldots, q-2, q+1\big]\end{aligned}$$

because in the process of writing the even numbers we had to skip over a if $\gcd(q-2, q+1)=1$. This places f_a in $\mathcal{ID}$. If $\gcd(q-2, q+1)>1$, we get an ETP and we have $f_a \in \mathcal{C}_3$. $\square$

Let us denote by R the set of records for f_3.

THEOREM 2.13. *The set $\mathcal{A}$ has the following description*

(2)
$$\mathcal{A} = \{2,4\} \cup \{a | a \equiv 0 \pmod 6 \text{ s. t. there exists a record } r \in R, |r-a| \le 1\}$$

Remark 2.14. In the last proof, if $a = 216$ then in the last step $q = 221$ and $\gcd(q-2, q+1) = 3$. That places $f_{216} \in \mathcal{C}_3$.

Let us denote by R the set of records for f_3.

THEOREM 2.15. *The set $\mathcal{A}$ has the following description*

(3)
$$\mathcal{A} = \{2,4\} \cup \{a | a \equiv 0 \pmod 6 \text{ s. t. there exists a record } r \in R, \ |r-a| \le 1\}.$$

Proof. We are denoting the right side of (3) by B. First, we show that $B \subset \mathcal{A}$. We have already observed that 2, 4 and 6 are in $\mathcal{A}$. Consider p the smallest prime less than a such that p doesn't divide a ($\gcd(p, a) = 1$). This prime exists since $a \ge 12$. Then $f_a(3) \le p$ and 3 is a turning point for f_a. Using the same arguments as before, we see that eventually, the records of f_a are going to overlap with the records of f_3 (p in particular has to be a record of f_a also). Because p is less than a, we may assume that this happens before a, i.e., the records of f_a which are also records of f_3 start at a value less than a (this does not happen if $a = 6$). Then it makes sense to define L be the largest record (in R) less than a and S be the smallest one (in R) bigger than a. We observe that L and R are records for f_a too (by earlier observations). Let us analyze the two cases.

We assume first that $L = a-1$. The sequence f_a takes the values

$$1, a, \ldots, L, k, k+1, k+2, \ldots, L-1 = a-2 \quad \text{where } f(k-1) = k$$

and by the maximality of L, we must have used all of the values in the set $\{1, 2, 3, \ldots, a\}$. Therefore, we must have $f(a) = a-2$ and then $f(a+1) = a+1$ ($\gcd(a+1, a-2)$ can be at most 3 and since 6 divides a, 3 does not divide $a-2$). Now it is easy to see that $f_a \in \mathcal{ID}$.

We assume next that $L < a-1$, and $S = a+1$ ($a = 36$ is the first with such a property). Then the sequence f_a takes the values

$$1, a, \ldots, L, k, k+1, k+2, \ldots, L-1 < a-2, S, m, m+1, \ldots, S-2 = a-1$$

where $f(k-1) = k$ and $f(m-1) = S$, which cover all the values in the set $\{1, 2, \ldots, a, S\}$. Therefore, we must have $f(a+1) = a-1$ and so $f(a+2) = a+2$ ($\gcd(a+2, a-1)$ can be at most 3 and since 6 divides a, 3 does not divide $a-1$). Now it is easy to see that $f_a \in \mathcal{ID}$. This shows that $B \subset \mathcal{A}$.

Suppose that $a \notin B$. First, we assume that a is such that $L < a-1$ and $S > a+1$. Because a is even, $\gcd(a-1, a+1) = 1$ and so the sequence f_a

takes the values

$$1, a, \ldots, L, k, k+1, k+2, \ldots, L-1 < a-2, S, m, m+1, \ldots, a-1, a+1, \ldots, S-1$$

which cover all the values in the set $\{1, 2, \ldots, a, \ldots, S-1, S\}$. This makes $f(S) = S - 1$ and so $S + 1$ is an ETP placing $f_a \in \mathcal{C}_3$ and so $a \notin \mathcal{A}$.

Next, we assume that a is not a multiple of 6. If a is odd we have already seen that $f_a \in \mathcal{C}_3$ and so $a \notin \mathcal{A}$. If a is even then $a = 6\ell \pm 2$ for some ℓ. So, let $a = 6\ell + 2$ and $L = a - 1$. The previous argument still works the same way because $\gcd(a-2, a+1) = 3$ and so we avoid the situation $f_a \in \mathcal{ID}$. Finally, if $a = 6\ell + 4$ and $S = a + 1$, then $\gcd(a+2, a-1) = 3$ and again, we avoid $f_a \in \mathcal{ID}$ as in the case above. In each case $a \notin \mathcal{A}$, showing the other inclusion, i.e., $\mathcal{A} \subset B$. □

For the rest of the paper, we are simply using f for f_3 and the results and conjecture are going to be concerned with this case.

THEOREM 2.16. *Let $p_n\#$ denote the n^{th} primorial number. Then for $n \geq 2$, $p_n\# \pm 1$ and $2p_n\# \pm 1$ are records.*

Proof. If $q = p_n\# - 1$ is a prime then we are done because every prime is a record. So, we may assume that q is composite. Let r_l denote the largest record less than q. Then the next record is the smallest number larger than r_l but coprime to $r_l - 1$. If q is not a record then $r_l - 1$ must have a common factor d with q. Now q does not have $p_1, p_2, \ldots, p_n$ as prime factors so $d \geq p_{n+1}$. Hence $q - (r_l - 1) \geq p_{n+1}$ or $r_i - 1 + p_{n+} \leq q$. Now $r_l - 1$ cannot have each of $p_1, p_2, \ldots, p_n$ as prime factors as it is smaller than q. As a result, it must not be divisible by some p_k for $k \leq n$. Then $r_l - 1 + p_k$ is a record as it is coprime with $r_l - 1$. But $r_l - 1 + p_k$ lies between r_l and $r_l + p_n < r_l - 1 + p_{p+1} \leq q$ which contradicts the maximality of r_l. Hence q must be a record.

Now, since $f(p_n\# - 1) = p_n\# - 2$ and $3 \mid p_n\#$, $p_n\# - 2$ and $p_n\# + 1$ must be coprime. This implies that $p_n\# + 1$ is also a record.

The other part of the theorem's statement follows because there is always a prime between $p_n\# + 1$ and $2p_n\#$ which is always a record. So the largest record less than $2p_n\# - 1$, say $r_{l'}$, is larger than $p_n\# + 1$. Hence $r_{l'} - 1$ is larger than $p_n\#$ and less than $2p_n\#$. This means that $r_{l'} - 1$ cannot be divisible by each of the first n primes and a similar argument as above applies. □

We notice that this shows that $\{p_n\#-1, p_n\#+1\}$ and $\{2p_n\#-1, 2p_n\#+1\}$ are twin records. This fact is exploited in the next theorem.

THEOREM 2.17.

$$f(p_n\# + k) = f(k) + p_n\# \tag{4}$$

for $k \in [p_{n+1}, 2p_n\#]$.

Proof. We first prove (4) for $k \in [p_n\#, 2p_n\#]$. It suffices to establish a one-to-one correspondence between the records in $I_1 := [p_n\#, 2p_n\#]$ and the ones in $I_2 := [2p_n\#, 3p_n\#]$, i.e., show that r is a record in I_1 if and only if $r + p_n\#$ is a record in I_2. This is because between records the values taken by f are determined by the records and their corresponding turning points. The values in between are constructed in a pattern that is compatible with the translation by $p_n\#$. So,

$$p_n\# + 1 = r_1 < r_2 < r_3 < \cdots < r_s$$

is all of the records in I_1, by Theorem 2.16.

We proceed inductively on r_j $(j = 1, 2, \ldots, s)$. The first record in I_1 is $f(p_n\#) = p_n\# + 1$ and the corresponding record in I_2 is $f(2p_n\#) = 2p_n\# + 1$, by Theorem 2.16. Now, the next record in I_1 is the smallest number larger than $p_n\# + 1$ but coprime to $p_n\#$ which is $r_2 = p_n\# + p_{n+1}$ and the next one in I_2 is $2p_n\# + p_{n+1}$. This is true because there is always a prime p' between $p_n\#$ and $2p_n\#$ which is a record hence r_2 has to lie in the interval I_1. (This shows that $p_{n+1} < p_n\#$ which we use later). We proceed in this manner to the point where $r_m + p_n\#$ is a record in I_2 and r_m is a record in I_1 $(m < s)$. We let $r_m = p_n\# + r$, with $r < p_n\#$. Then r_{m+1} is the smallest number coprime to $p_n\# + r - 1$ and bigger than $p_n\# + r$. Now $r - 1$ cannot be divisible by all the primes $p_1, p_2, \ldots, p_n$. Let p_i $(i \le n)$ be the smallest prime not dividing $r-1$. As $p_i \mid p_n\#$ the smallest number coprime to $p_n\# + r - 1$ and bigger than $p_n\# + r$ is $p_n\# + r - 1 + p_i = r_m - 1 + p_i$. A similar argument shows that $2p_n\# + r - 1 + p_i = r_m - 1 + p_n\# + p_i$ is the next record in I_2. Not only that, but every record in I_2 appears as a translation of the corresponding record in I_1.

Let us define next $I = [p_{n+1}, p_n\#]$. We have just established that the record of f just after $p_n\# + 1$ is $p_n\# + p_{n+1}$. We also know that p_{n+1} is a record. Now a proof similar to the one given in the previous paragraph holds, by replacing the I_1 with I and I_2 with I_1. $\square$

Remark 2.18. As $p_{n+1} < p_n\#$, we have $2p_n\# + p_{n+1} \in [2p_n\#, 3p_n\#]$ the largest record less than $3p_n\# - 1$ is at least $2p_n\# + p_{n+1}$. Let the largest record be r_l. We have $r_l - 1 > 2p_n\#$ and thus this cannot be divisible by all of $p_1, p_2, \ldots, p_n$. An argument similar to that of Theorem 2.7 gives $3p_n\# \pm 1$ are records. Now, once we have this we can prove that for k in $[2p_n\#, 3p_n\#]$, $f(k + p_n\#) = f(k) + p_n\#$ (proof similar to Theorem 2.8). So, proceeding similarly we get the following theorem whose proof is clear from this remark.

Theorem 2.19. *The numbers $rp_n\# \pm 1$ are records for*

$$r \in \{1, 2, 3, \ldots, p_{n+1} - 1\}.$$

Furthermore

$$f(k + p_n\#) = f(k) + p_n\#$$

for $k \in [p_{n+1}, (p_{n+1} - 1)p_n\#]$.

For example, one can check that

$$f(30 + k) = f(k) + 30 \text{ for all } k \in [7, 181],$$
$$f(210 + k) = f(k) + 210 \text{ for all } k \in [9, 2101].$$

f is not periodic or additive, but the property above suggests an almost periodic nature of the sequence.

Remark 2.20. The multiples of the prime p_n appearing as records tend to show a particular pattern because records are translated over large regions by $p_n\#$ giving a pattern to the multiples of that prime appearing as record. This also suggests that getting a good idea of these multiples of primes can give a sieve-like primality test of a number based on the records of this function.

In Theorem 2.15, we gave a characterization of the set $\mathcal{A}$. Now, from the material developed afterwards, we can give a better characterization of the set $\mathcal{A}$ which is included in the following theorem.

Theorem 2.21. *The set $\mathcal{A}$ introduced in* 2.6 *has the precise description in terms of primes*

$$\mathcal{A} = \{2, 4\} \cup \Big\{ a | a = 6k, k \in \mathbb{N}, a \neq m \cdot p_n\# + 6t, \quad \text{where } n > 3,\ m, t \in \mathbb{N}, 1 \leq t \leq \left\lfloor \frac{p_{n+1} - 2}{6} \right\rfloor \Big\}. \tag{5}$$

Proof. We say that a natural number a is "nice" if $a = 6k, k \in \mathbb{N},\ a \neq p_n\#m + 6t,$ where $n > 3, m, t \in \mathbb{N}, 1 \leq t \leq \lfloor \frac{p_{n+1}-2}{6} \rfloor$. We show that a natural number a is "nice" if and only if there is a record r such that $|r - a| \leq 1$ and then we are done by Theorem 2.15.

Let us assume that $a \in \mathbb{N}$ is "nice" and consider n such that $p_n\# \leq a < p_{n+1}\#$. If $a = mp_n\#$ for $m = 1, 2, \ldots, p_{n+1} - 1$ then we are done because $mp_n\# + 1$ is a record. Also, if $p_{n+1} \equiv 1 \pmod 6$ and $a = mp_n\# + p_{n+1} - 1$, then we are also done because $mp_n\# + p_{n+1}$ is a record. Thus, we may assume $a \in [mp_n\# + p_{n+1}, (m + 1)p_n\# - 6]$ where $m \in \{1, 2, \ldots, p_{n+1} - 1\}$. Hence, there is a record r such that $|r - a| \leq 1$ if and only if there is a record r' with $|r' - a - mp_n\#| \leq 1$. Now, $p_k\# \leq a - mp_n\# < p_{k+1}\#$ where $k < n$. We can repeat this process until we get x such that $a - x < p_4\#$ and after that one can easily check that for any multiple of 6 less than $p_4\#$ there is a record r'', with $r'' = x \pm 1$, which is either one larger or less than x.

Now let us assume that a is "not nice". Then $a = mp_n\# + 6t$ where $1 \le t \le \lfloor \frac{p_{n+1}-2}{6} \rfloor$. Now, if $p_{n+1} \equiv -1 \pmod 6$ then $a \in [mp_n\#+6, mp_n\#+p_{n+1}-5]$ and as there is no record between $mp_n\# + 1$ and $mp_n\# + p_{n+1}$ we must have $a \notin \mathcal{A}$. Similarly, if $p_{n+1} \equiv 1 \pmod 6$ then $a \in [mp_n\# + 6, mp_n\# + p_{n+1} - 7]$ and for the same reason as before $a \notin \mathcal{A}$. □

Corollary 2.22. *The density of the set of "not nice" numbers, say $\mathcal{A}'$, is given by the expression*

$$\sum_{k\ge 4}\left(\left\lfloor\frac{p_{k+1}-2}{6}\right\rfloor - \left\lfloor\frac{p_k-2}{6}\right\rfloor\right)\frac{1}{p_k\#}. \tag{6}$$

Proof. We observe that the set

$$\mathcal{A}' := \left\{a | a = p_n\#m + 6t \text{ where } n > 3,\ m,t \in \mathbb{N}, 1 \le t \le \left\lfloor\frac{p_{n+1}\# - 2}{6}\right\rfloor\right\}$$

is just the union of the sets

$$\mathcal{A}'_n := \left\{a | a = mp_n\# + 6t \text{ where } m,t \in \mathbb{N}, 1 \le t \le \left\lfloor\frac{p_{n+1}\# - 2}{6}\right\rfloor\right\}$$

for $n = 4, 5, \ldots$ These sets are disjoint but because $p_{n+1}\# = p_{n+1} \cdot p_n\#$ and $p_{n+1} > p_n$ each element of the set $\mathcal{A}'_n$ with $m \ge p_{n+1}$ and $t \le \lfloor\frac{p_{n+1}\#-2}{6}\rfloor$ appears in $\mathcal{A}'_{n+1}$ and in all of the subsequent sets $\mathcal{A}'_k$ with $k \ge n+1$. Hence, we can write $\mathcal{A}'$ us union of disjoint sets

$$\mathcal{B}_n := \left\{a | a = mp_n\# + 6t \text{ where } m,t \in \mathbb{N}, \left\lfloor\frac{p_n\# - 2}{6}\right\rfloor < t \le \left\lfloor\frac{p_{n+1}\# - 2}{6}\right\rfloor\right\}.$$

As a result, the density of $\mathcal{A}'$ is the sum of the densities of each $\mathcal{B}_n$. The density of $\mathcal{B}_n$ is

$$\left(\left\lfloor\frac{p_{n+1}-2}{6}\right\rfloor - \left\lfloor\frac{p_n-2}{6}\right\rfloor\right)\frac{1}{p_n\#}$$

because of the periodicity of the elements in B_n modulo $p_n\#$. This gives the formula (6). □

Let us denote the set of all records by $\mathcal{R}$. Written in non-decreasing order gives essentially the sequence from [10]. The definition is the following.

Definition 2.23. $a_{n+1} = a_n + p - 1$, $a_1 = 1$ and p is the smallest prime number that is not a factor of $a_n - 1$. The equivalence between two concepts is contained in Theorem 2.3.

We are interested in the following limit

$$\kappa := \lim_{n\to\infty} \frac{\#\{r|r \in \mathcal{R}, r \le n\}}{n}. \tag{7}$$

One can see that we can use the property in (4) to say more about this limit. We are going to denote by s_n the number of records between p_n and p_{n+1}. More precisely, we let

$$s_n = \#\{r|r \in \mathcal{R}, p_n \le r < p_{n+1}\}.$$

We notice that $s_1 = 0$, $s_2 = 1$, $s_3 = 1, \ldots, s_9 = 2$, $s_{10} = 1, \ldots, s_{16} = 2$, ...So, this sequence is mostly equal to 1, and when it is greater than 1, the difference is the number of composite records between the respective consecutive primes.

THEOREM 2.24. *The limit* (7) *exists and we have*

$$\frac{3}{10} - \sum_{k=4}^{\infty} \frac{p_{k+1} - p_k}{2p_k\#} \le \kappa \le \frac{3}{10} - \sum_{k=4}^{\infty} \frac{1}{p_k\#}.$$

Proof. Let us first show that (7) exists for a subsequence, namely, the limit

$$\lim_{n\to\infty} \frac{\#\{r|r \in \mathcal{R}, r \le p_n\# + 1\}}{p_n\# + 1} \tag{8}$$

exists. To simplify notation, we let $q_n := p_n\# + 1$ and

$$w_n = \#\{r|r \in \mathcal{R}, p_{n+1} \le r \le q_n\}.$$

We observe that

$$w_1 = \#\{r|r \in \mathcal{R}, 3 \le r \le 3\} = 1, \quad w_2 = \#\{r|r \in \mathcal{R}, 5 \le r \le 7\} = \#\{5, 7\} = 2,$$

$$w_3 = \#\{r|r \in \mathcal{R}, 7 \le r \le 31\} = \#\{7, 11, 13, 17, 19, 23, 25, 29, 31\} = 9,$$

and so on. We can use Theorem 2.17 and the proof of Theorem 2.16, to see, for instance, that the records in $(32, 61]$ are the records in

$$R_3 := \{7, 11, 13, 17, 19, 23, 25, 29, 31\}$$

translated with 30. Hence their number is the same as w_3. Similarly, we can use the same theorems to conclude that the number of records in $(62, 91]$ is also w_3. This extends all the way to the interval $(182, 211]$ and so we can say that

$$w_4 = 7 \cdot 9 - 1 = 62,$$

as we can see from Table 1 of the records in the interval $[7, 211]$.

We can make this argument in general and derive the formula

$$w_{n+1} = w_n p_{n+1} - s_{n+1}. \tag{9}$$

Table 1

7	11	13	17	19	23	25	29	31
37	41	43	47	49	53	55	59	61
67	71	73	77	79	83	85	89	91
97	101	103	107	109	113	115	119	121
127	131	133	137	139	143	145	149	151
157	161	163	167	169	173	175	179	181
187	191	193	197	199	203	205	209	211

Hence, after dividing by $p_{n+1}\#$ we get

$$\frac{w_{n+1}}{p_{n+1}\#} - \frac{w_n}{p_n\#} = -\frac{s_{n+1}}{p_{n+1}\#}.$$

Summing up these identities, we get

$$0 < \frac{w_{n+1}}{p_{n+1}\#} = \frac{w_3}{p_3\#} - \sum_{k=3}^{n} \frac{s_{k+1}}{p_{k+1}\#} \le \frac{w_3}{p_3\#} - \sum_{k=3}^{n} \frac{1}{p_{k+1}\#}.$$

This shows that $\frac{w_n}{p_n\#}$ is decreasing (hence, convergent) and we have an estimate from above for the limit. For the other estimate, we can use the obvious inequality $s_n \le \frac{1}{2}(p_{n+1} - p_n)$ giving the maximum number of odd integers in the interval $[p_n, p_{n+1})$. The limit in (7) exists (in general) and it is the same as the one for $\frac{w_n}{p_n\#}$. $\square$

We observe that in Table 1, all the records in red and blue generate composite records in the subsequent intervals, i.e., $[p_n\#, p_{n+1}\#]$. These records are multiples of 7, one on each column and multiples of 5 all on column 7, a total of 15. There are four records that may or may not turn into primes later on, but it is not clear what happens for 121, 169, 187 and 209. This leaves us with $63 - 9 - 7 + 1 - 4 = 44$ primes.

Remark 2.25. We can use Theorem 2.24 to get numeric bounds on κ. The series of reciprocal of primorial numbers is a convergent series whose value is known to be an irrational number (see [4] and [7]). The first few terms of the decimal expansion are given by 0.7052301717918009... Hence, we can write $0.704 \le \prod_{n=1}^{\infty} \frac{1}{p_n\#} \le 0.706$. This gives us

$$\kappa \le 0.3 - 0.704 + \frac{1}{2} + \frac{1}{6} + \frac{1}{30} = 0.296 = \frac{296}{1000}.$$

Also as $p_{k+1} \leq 2p_k$, we have

$$\kappa \geq 0.3 - \prod_{n=3}^{\infty} \frac{1}{p_n\#} \geq 0.3 - 0.706 + \frac{1}{2} + \frac{1}{6} = \frac{782}{3000} \approx 0.26067.$$

Using the Prime Number Theorem, we can tell what is the asymptotic density of primes within the number of records.

Corollary 2.26. *We have the following limit*

$$\lim_{n\to\infty} \frac{\#\{r \in \mathcal{R} | r \quad \textit{is prime and } r < n\}}{\#\{r \in \mathcal{R} | r < n\}} \ln n = \frac{1}{\kappa}.$$

This ratio is included in the next figure for the first 1000 records.

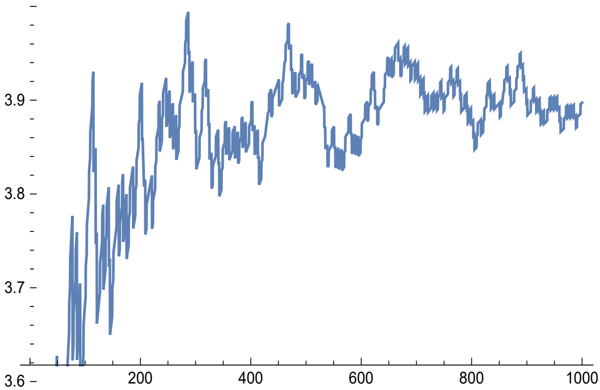

Figure 3 – $\frac{\#\{r|r \quad prime r<n\}}{\#\{r|r<n\}} \ln n$

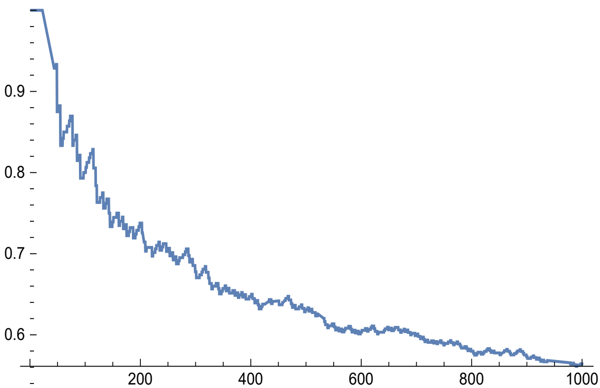

Figure 4 – Number of primes within the records

3. CONJECTURES, QUESTIONS, PROBLEMS AND CONECTIONS

(i) We defined in the Introduction the function $h(j) = C(M_{j+1}) - C(m_j)$, $j \in \mathbb{N}$, where (m_j, M_j) is a twin prime pair (or we can simply refer to only records that form a pair). Numerical evidence suggests that the range of h is $\{k \in \mathbb{Z} | k \geq 0\}$. So, we are making this conjecture which relates to the twin prime conjecture.

(ii) Is there a good description for the set $\overline{\mathcal{R}}$, the subset of records which are composite?

We know that all the prime numbers appear as records in f_3. So it is important to characterize the composite numbers that appear as a record in f_3, to differentiate the primes. For that, let us look at the first few multiples of 5 that appear as a record:

$$25, 55, 85, 115, 145, 175, 205, 235, 265, 295, 325, 355, 385, 415, 445, 475, \ldots$$

Note that consecutive pairs of these differ by 30.

Next, let us look at the multiples of 7 appearing as a record. They seem to be more interesting:

$$49, 77, 91, 119, 133, 161, 175, 203, 259, 287, 301, 329, 343, 371, 385, 413, 469, \ldots$$

Here the multiples occur on an interval of 28 and 14 alternately until 203 where it takes a jump of 56. Then it continues this pattern taking alternate jumps of 28 and 14 until in $413 = 203 + 210$ where it again takes a jump of 56. We guess that this pattern continues until $623 = 413 + 210$ where it again takes a jump of 56. So these multiples of 7 appearing as records seem to be very predictable. Similar observations can be made regarding other primes. The first few multiples of 11 appearing as records are given below:

$$55, 77, 121, 143, 187, 209, 253, 319, 341, 385, 407, 451, 473, 517, 539, 583, \ldots$$

Here the jumps are alternately 22 and 44.

4. AN AD HOC PROOF OF THE SURJECTIVITY OF f_a

Let us show that every prime p is in the range of f. If $a = p$ we are done. Otherwise, let i be the greatest index such that $f(i) < p$. This index exists because f is one-to-one (so for $M > 0$ there exits n large enough that $f(m) \geq M$ for all $m > n$), and $f(1) = 1 < p$. If $i = 1$ then it must be the case that $1 < p < a = f(2)$ and none of the numbers $2, 3, \ldots, p-1$ appear in the sequence $\{f(n)\}_n$ after a by the definition of i. Hence, if $\gcd(a,p)=1$ we have $f(3)=p$ and we are done. If not, $a=pa'$ and then $f(3)$ must not contain p in its prime factorization. But $\gcd(f(3),p)=1$ which forces $f(4)=p$ because of the assumption on i. Therefore, we may assume $i\geq 2$ and we can apply the definition on $f(i+1)$. Namely, if p is in the set $\{f(1),f(2),\dots,f(i)\}$ we are done. If not, then $f(i+1)=s$ implies $s\geq p$ (by the definition of i) and since $\gcd(p,f(i))=1$ we must have $s=p$.

By way of contradiction, we assume that f is not onto. This means there are values in $\mathbb{N}$ which are not in the range of f. Let k be the smallest such number which is not in the range of f (this exists because of the Well-Ordering Principle for $\mathbb{N}$).

Then all $m\in\{1,2,\dots,k-1\}:=A$ must be in the range of f. Then the set $B:=f^{-1}(A)=\{f^{-1}(m)|m\in A\}$ has at least $k-1$ elements, but because f is one-to-one B must have exactly $k-1$ elements. So, we let in order $x_1=1,x_2=a,\dots,x_{k-1}=j$ $(x_1<x_2<\cdots<j)$. We let $\ell=f(j)$. If $\ell=k-1$ then $f(j+1)=k$ and we have a contradiction. A similar argument goes for the situation in which ℓ is relatively prime with k.

Suppose that the primes in the decomposition of k are $q_1,q_2,\dots,q_s$ (all distinct primes).

Then let us look at $\ell_1:=f(j+1)$ which is relatively prime with ℓ and since it is not in A, we must have $\ell_1>k$. If $\gcd(\ell_1,k)=1$ then by definition we must have $f(j+2)=k$ which is not possible. It remains that $\gcd(\ell_1,k)>1$ so ℓ_1 and k must have some of the previous primes in common, or in other words, at least one of the primes q_i must divide ℓ_1.

By induction, we can show that $\ell_n:=f(j+n)$ is a number that must have some prime factor q_i for every $n\geq 1$. This is in contradiction with the fact that all primes must be in the range of f.

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