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# An Institution-independent Generalization of Tarski's Elementary Chain Theorem

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## Abstract

We prove an institutional version of Tarski's *elementary chain theorem* applicable to a whole plethora of 'first-order-accessible' logics, which are, roughly speaking, logics whose sentences can be constructed from atomic formulae by means of classical first-order connectives and quantifiers. These include the *unconditional equational*, *positive*,  $(\Pi \cup \Sigma)_n^0$  and *full* first-order logics, as well as less conventional logics, used in computer science, such as *hidden* or *rewriting logic*.

*Keywords:* Institution, elementary morphism, elementary chain property.

## 1 Introduction

The notion of elementary embedding is an important one in classical first-order model theory [4]. Elementary chains (i.e. chains of elementary embeddings) are known to be a fundamental proof tool for results regarding preservation of satisfaction, axiomatizability, Robinson consistency, Craig interpolation [4], saturated models, stability, categoricity in power [30] and many others. The extension of elementary embeddings to infinitary logics [18, 21, 17] reveals the need in mathematical logic for accommodating this notion in other logical systems too. And the monograph [18] actually shows that this is a natural and very fruitful thing to do.

The present paper introduces and studies abstract notions of elementary embedding and elementary chain, in the framework of institutions [14], and points out many particular cases. Two aspects motivate and justify our study:

- The mentioned importance of elementary embeddings in model theory; and
- The logic-independent status of our concepts and results.

Besides its intrinsic abstract model-theoretic contribution, our study might be of interest for the theory of formal specifications, where a logic-independent view is desirable for as long as possible in the specification process and where structural properties usually approachable by means of elementary chains, such as Craig interpolation or axiomatizability, are crucial.

Institutions are abstract logical frameworks that provide a category of signatures (languages) and signature morphisms (language translations), and, for each signature, a set of sentences, a category of models and a satisfaction relation. Sentences have *translations*, and models have *reducts*, along signature morphisms; the translations and reducts express the sentence and model modifications under *change of notation* from one language to another. Satisfaction is required to be *invariant* under change of notation. More abstract than the *general logics* of [1], institutions were introduced as frameworks for building model theory for computer science, in a logic-independent way. Thus, general institutional results were applicable to the diversity of logical systems used in computer science. Besides their

great generality, another important feature of institutions, not present, or poorly present, in other abstract frameworks, is the flexible support for language translations. This feature, particularly useful in formal specification and the semantics of programming languages, is also interesting from a logical point of view. As shown in [34, 35, 7, 10, 8, 6] and other places, signature morphisms (language translations) turn out to be a very insightful tool for finding *concrete structure* in the core of abstract logic, for example, any institution hides inside a ‘first-order logic’, which can be uncovered by means of basic sentences, logical connectives, and quantifications over some signature morphisms (see Sections 2 and 3).<sup>1</sup>

Our institutional notion of elementary embedding (that we call ‘elementary morphism’) also uses signature morphisms in an essential way, by defining elementarity as *preservation of satisfaction in expansions along certain signature morphisms*. Recall that, classically, an elementary embedding between two models  $A$  and  $B$  of the same language is a model-embedding  $A \xrightarrow{h} B$  such that for each formula  $e(x_1, \dots, x_n)$  and each sequence  $a_1, \dots, a_n \in A$ ,  $A \models e(a_1, \dots, a_n)$  iff  $B \models e(h(a_1), \dots, h(a_n))$ . Because of the existence of negations, the ‘iff’ in the preceding can be replaced by ‘implies’. In order to abstract away this concept, we follow an idea originating in [34] that treats (non-closed)  $\Sigma$ -formulae as sentences in signature ‘extensions’  $\varphi : \Sigma \rightarrow \Sigma'$ . And we use quasi-representable signature morphisms [6] to capture the requirement that the ‘extension’ only adds ‘constants’. ( $\varphi$  is quasi-representable if, for each  $A' \in |\text{Mod}(\Sigma')|$ , the canonical functor  $A'/\text{Mod}(\varphi) : A'/\text{Mod}(\Sigma') \rightarrow A' \upharpoonright_{\varphi}/\text{Mod}(\Sigma)$  is an isomorphism of categories; hence, giving a  $\Sigma'$ -morphism of source  $A'$  is equivalent to giving a  $\Sigma$ -morphism of source  $A' \upharpoonright_{\varphi}$ . This situation has the following intuitive explanation: all the ‘extra items’ of  $\Sigma'$  with respect to  $\Sigma$  being constant symbols, a  $\Sigma$ -morphism  $A \xrightarrow{h} B$  can have only one  $\Sigma'$ -expansion of given source  $A'$ .) Quasi-representability of signature morphisms is a weakening of the concept of representability introduced in [10]. By fixing a class  $\mathcal{Q}$  of quasi-representable signature morphisms, we call a  $\Sigma$ -morphism  $A \xrightarrow{h} B$   *$\mathcal{Q}$ -elementary* if for each signature morphism  $\varphi : \Sigma \rightarrow \Sigma'$  in  $\mathcal{Q}$ , each  $\varphi$ -expansion  $A' \xrightarrow{h'} B'$  of  $h$  and each  $\Sigma$ -sentence  $e'$ ,  $A' \models e'$  implies  $B' \models e'$ . (See Section 4 for a detailed motivation of this definition.)

One can alternatively define elementary morphisms by elementary diagrams. Classically, the *elementary diagram* [4]  $EDg(A)$  of a model  $A$  is the set of all sentences in  $\Sigma(A)$  (the language  $\Sigma$  of  $A$  extended with all elements of  $A$  as constants) that are true in  $A$ . Then a model inclusion  $A \xrightarrow{h} B$  is elementary iff  $EDg(A) \subseteq EDg(B)$ . Thanks to a recent concept of institutional diagram [8], we can also define institutionally *elementary morphisms by diagrams* (we abbreviate these as *d-elementary morphisms*). The diagrams of [8] provide, for each signature  $\Sigma$  and  $\Sigma$ -model  $A$ , a parameterized signature extension  $\iota_{\Sigma}(A) : \Sigma \rightarrow \Sigma_A$  and a self-parameterized  $\iota_{\Sigma}(A)$ -expansion of  $A$ . In addition, the diagrams are ‘functorial’, i.e. they have corresponding structure for signature and model morphisms; in particular, any  $\Sigma$ -morphism  $A \xrightarrow{h} B$  yields a signature morphism  $\iota_{\Sigma}(h) : \Sigma_A \rightarrow \Sigma_B$  such that  $\iota_{\Sigma}(A); \iota_{\Sigma}(h) = \iota_{\Sigma}(B)$ . We call  $h$  *d-elementary* if, for all  $\Sigma_A$ -sentences  $e'$ ,  $A_A \models e'$  implies  $B_B \models \iota_{\Sigma}(h)(e')$ . *d-elementarity* is expressible more compactly than  $\mathcal{Q}$ -elementarity, but requires an amount of rather evolved extra structure on institutions.

Here is the structure of this article. Section 2 recalls some notions regarding categories, institutions and diagrams. Section 3 discusses and exemplifies the concepts, central in this

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<sup>1</sup> Compare this natural appearance of a first-order sublogic with the need to explicitly postulate the existence of such a sublogic in the context of general logics of [1].

article, of (finitely) representable and (finitely) quasi-representable signature morphism. Section 4 introduces elementary (model) morphisms with respect to a class of quasi-representable signature morphisms. Section 5 proves an institutional version of Tarski's elementary chain theorem, in the following slightly stronger form: elementary morphisms are closed under directed colimits. Section 6 introduces an alternative, diagrammatic version of elementary morphism and shows its equivalence to the previous notion under certain mild conditions (diagrams being in the considered class of quasi-representable morphisms and another condition that we call normality). Some concluding remarks end the article.

## 2 Preliminaries

### 2.1 Categories

We assume that the reader is familiar with basic categorical notions like functor, natural transformation, colimit, comma category, etc. A standard textbook on the topic is [19]. We are going to use the terminology from there, with a few exceptions that we point out in the following text. We use both the terms 'morphism' and 'arrow' to refer morphisms of a category. Composition of morphisms and functors is denoted using the symbol ';' and is considered in diagrammatic order.

Let  $\mathcal{C}$  and  $\mathcal{C}'$  be two categories. Given an object  $A \in |\mathcal{C}|$ , the *comma category of objects in  $\mathcal{C}$  under  $A$*  is denoted  $A/\mathcal{C}$ . Recall that the objects of this category are pairs  $(h, B)$ , where  $B \in |\mathcal{C}|$  and  $A \xrightarrow{h} B$  is a morphism in  $\mathcal{C}$ . Throughout the article we might let either  $(A \xrightarrow{h} B, B)$  or  $(h, B)$  indicate objects in  $A/\mathcal{C}$ . A morphism in  $A/\mathcal{C}$  between two objects  $(h, B)$  and  $(g, D)$  is just a morphism  $B \xrightarrow{f} D$  in  $\mathcal{C}$  such that  $h; f = g$  in  $\mathcal{C}$ . There exists a canonical forgetful functor between  $A/\mathcal{C}$  and  $\mathcal{C}$ , mapping each  $(h, B)$  to  $B$  and each  $f: (h, B) \rightarrow (g, D)$  to  $f: B \rightarrow D$ . Also, if  $F: \mathcal{C}' \rightarrow \mathcal{C}$  is a functor,  $A \in |\mathcal{C}|$ ,  $A' \in |\mathcal{C}'|$ , and  $A \xrightarrow{u} F(A')$  is in  $\mathcal{C}$ , then there exists a canonical functor  $u/F: A'/\mathcal{C}' \rightarrow A/\mathcal{C}$  mapping each  $(A' \xrightarrow{h} B, B)$  to  $(u; F(h), F(B))$  and each  $f: (h, B) \rightarrow (g, D)$  to  $F(f): (u; F(h), F(B)) \rightarrow (u; F(g), F(D))$ . If  $\mathcal{C} = \mathcal{C}'$  and  $F$  is the identity functor  $1_{\mathcal{C}}$ , we write  $u/\mathcal{C}$  instead of  $u/F$ ; and if  $F(A') = A$  and  $u = 1_A$ , we write  $A'/F$  instead of  $u/F$ .

Let  $\mathcal{C}$  and  $\mathcal{S}$  be two categories such that  $\mathcal{S}$  is small. A functor  $D: \mathcal{S} \rightarrow \mathcal{C}$  is also called a *diagram*. We usually identify a diagram  $D: \mathcal{S} \rightarrow \mathcal{C}$  with its image in  $\mathcal{C}$ ,  $D(\mathcal{S})$ . A *cocone* of  $D$  is a natural transformation  $\mu: D \Rightarrow V$  between the functor  $D$  and the constant functor mapping all objects to  $V$  and all morphisms to  $1_V$ ;  $V$  is an object in  $\mathcal{C}$ , the *vertex* of the colimit, and the components of  $\mu$  are the *structural morphisms* of the colimit. Any partially ordered set  $(I, \leq)$  can be regarded as a category in the obvious way, with the arrows being pairs  $i \leq j$ . A non-empty partially ordered set  $(I, \leq)$  is said to be *directed* if for all  $i, j \in I$ , there exists  $k \in I$  such that  $i \leq k$  and  $j \leq k$ , and is called a *chain* if the order  $\leq$  is total. A diagram defined on a directed set (on a chain) shall be called *directed diagram* (*chain diagram*) and a colimit of such a diagram, *directed colimit* (*chain colimit*). A *final subset* of a directed partially ordered set  $(I, \leq)$  is a subset  $K$  of  $I$  such that for all  $i \in I$ , there exists  $k \in K$  such that  $i \leq k$ . For instance, given  $i \in I$ ,  $I_i = \{j \in I \mid i \leq j\}$  is a final subset of  $(I, \leq)$ . A *sub-diagram* of a directed diagram  $D: (I, \leq) \rightarrow \mathcal{C}$  is the restriction of  $D$  to  $(K, \leq)$ , where  $K$  is a subset of  $(I, \leq)$ ; the sub-diagram is said to be *final* if  $K$  is final. An object  $A$  in a category  $\mathcal{C}$  is called *finitely presented* if for each directed diagram  $D: (I, \leq) \rightarrow \mathcal{C}$  with colimit  $\{Di \xrightarrow{\mu_i} B\}_{i \in I}$ , and for each morphism  $A \xrightarrow{h} B$ , there exists  $j \in I$  and  $A \xrightarrow{g} D_j$  such that  $g; \mu_j = h$ .

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Let  $\mathcal{C}'$  be a subcategory of  $\mathcal{C}$ .  $\mathcal{C}'$  is called a *broad subcategory* if it contains all the objects of  $\mathcal{C}$ .  $\mathcal{C}'$  is said to be *closed under directed colimits (chain colimits)* if for any directed diagram (chain diagram)  $D : (I, \leq) \rightarrow \mathcal{C}$  such that  $D(i \leq j)$  is in  $\mathcal{C}'$  for all  $i \leq j$ , any colimit  $\{Di \xrightarrow{\mu} B\}_{i \in I}$  of  $D$  has all the structural morphisms  $\mu_i$  in  $\mathcal{C}'$ .  $\mathcal{C}'$  is said to be *closed under pushouts* if for each pushout  $(A_2 \xleftarrow{h_2} A \xrightarrow{h_1} A_1, A_2 \xrightarrow{h'_1} A' \xleftarrow{h'_2} A_1)$  in  $\mathcal{C}$ ,  $h'_1$  is in  $\mathcal{C}'$  whenever  $h_1$  is in  $\mathcal{C}'$ . Note that the notion of 'closed under' that we adopt is stronger for pushouts than for directed or chain colimits. The following lemma is proved in [19].

LEMMA 1

Let  $(I, \leq)$  be a directed set,  $\mathcal{C}$  a category,  $D : (I, \leq) \rightarrow \mathcal{C}$  a diagram, and  $\{Di \xrightarrow{\mu} A\}_{i \in I}$  its colimit. If  $K$  is a final subset of  $(I, \leq)$ , then  $\{Di \xrightarrow{\mu} A\}_{i \in K}$  is a colimit of the corresponding final sub-diagram of  $D$ .

### 2.2 Institutions

Institutions were introduced in [14] with the original goal of providing an abstract, logic-independent framework for algebraic specifications of computer science systems. By isolating the essence of a logical system in the abstract *satisfaction relation*, these structures achieve an appropriate level of generality for the development of abstract model theory, as shown by a whole series of (old and new) papers: [34–36, 31, 32, 7, 8, 10, 9, 13, 27]. See also [26] for an up-to-date discussion on institutions as abstract logics.

An institution [14] consists of:

- (1) A category *Sign*, whose objects are called *signatures*;
- (2) A functor  $Sen : Sign \rightarrow Set$ , providing for each signature a set whose elements are called  $(\Sigma\text{-})$ *sentences*;
- (3) A functor  $Mod : Sign \rightarrow Cat^{op}$ , providing for each signature  $\Sigma$  a category whose objects are called  $(\Sigma\text{-})$ *models* and whose arrows are called  $(\Sigma\text{-})$ *morphisms*; and
- (4) A relation  $\models_{\Sigma} \subseteq |Mod(\Sigma)| \times Sen(\Sigma)$  for each  $\Sigma \in |Sign|$ , called  $(\Sigma\text{-})$ *satisfaction*, such that for each morphism  $\varphi : \Sigma \rightarrow \Sigma'$  in *Sign*, the *satisfaction condition*

$$M' \models_{\Sigma'} Sen(\varphi)(e) \text{ iff } Mod(\varphi)(M') \models_{\Sigma} e$$

holds for all  $M' \in |Mod(\Sigma')|$  and  $e \in Sen(\Sigma)$ . Following the usual notational conventions, we sometimes let  $\lrcorner_{\varphi}$  denote the reduct functor  $Mod(\varphi)$  and  $\varphi$  denote the sentence translation  $Sen(\varphi)$ . When  $M = M' \lrcorner_{\varphi}$ , we say that  $M'$  is a  $\varphi$ -*expansion* of  $M$  and that  $M$  is the  $\varphi$ -*reduct* of  $M'$ , and similarly for model morphisms.

For all the following concepts related to institutions that we recall in the following text, the reader is referred to [14] unless some other place is explicitly indicated.

Let  $\Sigma$  be a signature. Then,

- For each  $E \subseteq Sen(\Sigma)$ , let  $E^* = \{M \in |Mod(\Sigma)| \mid M \models_{\Sigma} e \text{ for all } e \in E\}$ .
- For each class  $\mathcal{M}$  of  $\Sigma$ -models, let  $\mathcal{M}^* = \{e \in Sen(\Sigma) \mid M \models e \text{ for all } M \in \mathcal{M}\}$ .

With no danger of confusion, we let  $\bullet$  denote any of the two compositions  $**$  of the two operators  $*$ . Each of the two bullets is a closure operator. When  $E$  and  $E'$  are sets of sentences of the same signature  $\Sigma$ , we let  $E \models_{\Sigma} E'$  denote the fact that  $E^* \subseteq E'^*$ . The relation  $\models_{\Sigma}$  between sets of sentences is called the  $(\Sigma\text{-})$ *semantic consequence relation* (notice that it is

written just like the satisfaction relation). If  $E' = \{e'\}$ , we might write  $E \models_{\Sigma} e'$ . In order to simplify notation, we usually write  $\models$  instead of  $\models_{\Sigma}$  for both the satisfaction relation and the semantic consequence relation. Two sentences  $e$  and  $e'$  are called *equivalent*, denoted  $e \equiv e'$ , if  $\{e\}^* = \{e'\}^*$ . Dually, two models  $M$  and  $M'$  are called *elementary equivalent*, denoted  $M \equiv M'$ , if  $\{M\}^* = \{M'\}^*$ .

An institution is called *semi-exact* [22] if the model functor  $Mod : Sign \rightarrow Cat^{op}$  preserves pushouts. Semi-exactness implies the following *amalgamation property* for any pushout of signature morphisms  $(\Sigma_2 \xleftarrow{\varphi_2} \Sigma \xrightarrow{\varphi_1} \Sigma_1, \Sigma_2 \xrightarrow{\varphi'_1} \Sigma' \xleftarrow{\varphi_2} \Sigma_1)$ : for any  $M_1 \in |Mod(\Sigma_1)|$ ,  $M_2 \in |Mod(\Sigma_2)|$  such that  $M_1 \upharpoonright_{\varphi_1} = M_2 \upharpoonright_{\varphi_2}$ , there exists a unique model  $M' \in |Mod(\Sigma')|$  such that  $M' \upharpoonright_{\varphi'_1} = M_1$  and  $M' \upharpoonright_{\varphi_2} = M_2$ . An analogous property is implied about model morphisms. An institution is called *liberal on signature morphisms* if the functor  $Mod(\varphi)$  has a left adjoint for each signature morphism  $\varphi$ .

A *presentation* is a pair  $(\Sigma, E)$ , where  $E \subseteq Sen(\Sigma)$ . A *theory* is a presentation  $(\Sigma, E)$  with  $E$  closed, i.e. with  $E^* = E$ . One usually calls ‘presentation’ or ‘theory’ only the set  $E$ , and not the whole pair  $(\Sigma, E)$ . A *presentation morphism*  $\varphi : (\Sigma, E) \rightarrow (\Sigma', E')$  is a signature morphism  $\varphi : \Sigma \rightarrow \Sigma'$  such that  $\varphi(E) \subseteq E'^*$ . A presentation morphism between theories is called *theory morphism*. For a presentation  $(\Sigma, E)$ , we let  $Mod(\Sigma, E)$  denote the category of all  $\Sigma$ -models  $A$  such that  $A \models E$ .

A sentence  $\rho \in Sen(\Sigma)$  is called *basic* [7] if there exists a  $\Sigma$ -model  $M_{\rho}$  such that, for all  $\Sigma$ -models  $M$ ,  $M \models \rho$  iff there exists a morphism  $M_{\rho} \rightarrow M$ . If, in addition,  $M_{\rho}$  is a finitely presented object in  $Mod(\Sigma)$ ,  $\rho$  is called *finitary basic* [10]. Basic sentences tend to be the starting building blocks for sentences in concrete institutions. For instance, in the institution of first-order predicate logic, FOPL (see the following text the examples of institutions), conjunctions of ground atoms are basic. In this article, we shall be interested in institutions whose sentences are accessible from basic sentences by means of several first-order constructs; hence, it suffices to identify *enough* basic sentences, like the conjunctions of ground atoms above, in order to ensure accessibility. However, the concept of basic sentence turns out to be quite comprehensive in concrete cases; for instance, existentially quantified atoms are also basic in FOPL. The attribute ‘finitary’ is usually equivalent, in concrete cases, to the property that the sentence has only a finite number of symbols. All basic sentences in FOPL are also finitary basic, because FOPL is a ‘finitary’ logic; this is not the case of the institution of infinitary first-order predicate logic, IFOPL, where basic sentences with an infinite number of symbols can be constructed by means of infinite conjunctions of atoms.

The sentences of an institution  $\mathcal{I}$  can be naturally extended with first-order-like constructions [34]: if  $\varphi : \Sigma \rightarrow \Sigma'$ ,  $\rho, \delta \in Sen(\Sigma)$ ,  $\rho' \in Sen(\Sigma')$ , and  $E \subseteq Sen(\Sigma)$ , one can build the sentences  $\neg\rho$ ,  $\delta \wedge \rho$ ,  $\delta \vee \rho$ ,  $\bigwedge E$ ,  $\bigvee E$ ,  $(\forall\varphi)\rho'$ ,  $(\exists\varphi)\rho'$  by means of *negation*, *conjunction*, *disjunction*, *arbitrary conjunction*, *arbitrary disjunction*, *universal and existential quantification (over signature morphisms)*, respectively, with the following semantics for each  $\Sigma$ -model  $M$ :

- $M \models \neg\rho$  iff  $M \not\models \rho$ ;
- $M \models \delta \wedge \rho$  iff  $M \models \delta$  and  $M \models \rho$ ;
- $M \models \delta \vee \rho$  iff  $M \models \delta$  or  $M \models \rho$ ;
- $M \models \bigwedge E$  iff  $M \models e$  for each  $e \in E$ ;
- $M \models \bigvee E$  iff  $M \models e$  for some  $e \in E$ ;
- $M \models (\forall\varphi)\rho'$  iff  $M' \models \rho'$  for each  $\varphi$ -expansion  $M'$  of  $M$ ;
- $M \models (\exists\varphi)\rho'$  iff  $M' \models \rho'$  for some  $\varphi$ -expansion  $M'$  of  $M$ .

It might be the case that the newly constructed sentences are equivalent to some existing sentences in  $\mathcal{I}$ . The notion of a class of sentences *closed under* either one of the aforementioned constructions (e.g. under conjunction, or under universal quantification over a morphism  $\varphi$ ) should be clear. An institution is said to *admit negation* if the class of all its sentences is closed under negation.

### 2.2.1 Examples of institutions

- (1) FOPL — the institution of (many-sorted) first-order predicate logic (with equality). The signatures are triplets  $(S, F, P)$ , where  $S$  is the set of sorts,  $F = \{F_{w,s}\}_{w \in S^*, s \in S}$  is the  $(S^* \times S)$ -indexed set of operation symbols and  $P = \{P_w\}_{w \in S^*}$  is the  $(S^*$ -indexed) set of relation symbols. If  $w = \lambda$ , an element of  $F_{w,s}$  is called a constant symbol, or a *constant*. By a slight notational abuse, we let  $F$  and  $P$  also denote  $\bigcup_{(w,s) \in S^* \times S} F_{w,s}$  and  $\bigcup_{w \in S^*} P_w$ , respectively. A signature morphism between  $(S, F, P)$  and  $(S', F', P')$  is a triplet  $\varphi = (\varphi^{\text{sort}}, \varphi^{\text{op}}, \varphi^{\text{rel}})$ , where  $\varphi^{\text{sort}} : S \rightarrow S'$ ,  $\varphi^{\text{op}} : F \rightarrow F'$ ,  $\varphi^{\text{rel}} : P \rightarrow P'$  such that  $\varphi^{\text{op}}(F_{w,s}) \subseteq F'_{\varphi^{\text{sort}}(w), \varphi^{\text{sort}}(s)}$  and  $\varphi^{\text{rel}}(P_w) \subseteq P'_{\varphi^{\text{sort}}(w)}$  for all  $(w, s) \in S^* \times S$ . When there is no danger of confusion, we may let  $\varphi$  denote each of  $\varphi^{\text{sort}}$ ,  $\varphi^{\text{rel}}$  and  $\varphi^{\text{op}}$ . Given a signature  $\Sigma = (S, F, P)$ , a  $\Sigma$ -model  $A$  is a triplet  $A = (\{A_s\}_{s \in S}, \{A_{w,s}(\sigma)\}_{(w,s) \in S^* \times S, \sigma \in F_{w,s}}, \{A_w(R)\}_{w \in S^*, R \in P_w})$  interpreting each sort  $s$  as a set  $A_s$ , each operation symbol  $\sigma \in F_{w,s}$  as a function  $A_{w,s}(\sigma) : A^w \rightarrow A_s$  (where  $A^w$  stands for  $A_{s_1} \times \dots \times A_{s_n}$  if  $w = s_1 \dots s_n$ ), and each relation symbol  $R \in P_w$  as a relation  $A_w(R) \subseteq A^w$ . When there is no danger of confusion we may let  $A_\sigma$  and  $A_R$  denote  $A_{w,s}(\sigma)$  and  $A_w(R)$ , respectively. Morphisms between models are the usual  $\Sigma$ -homomorphisms, i.e.  $S$ -sorted functions that preserve the structure. The  $\Sigma$ -sentences are obtained from atoms, i.e. equality atoms  $t_1 = t_2$ , where  $t_1, t_2 \in (T_F)_s$ <sup>2</sup> or relational atoms  $R(t_1, \dots, t_n)$ , where  $R \in P_{s_1 \dots s_n}$  and  $t_i \in (T_F)_{s_i}$  for each  $i \in \{1, \dots, n\}$ , by applying for a finite number of times:
  - negation, conjunction, and disjunction;
  - universal or existential quantification over finite sets of constants (variables).

Satisfaction is the usual first-order satisfaction and is defined using the natural interpretations of ground terms  $t$  as elements  $A_t$  in models  $A$ . The definitions of functors  $Sen$  and  $Mod$  on morphisms are the natural ones: for any signature morphism  $\varphi : \Sigma \rightarrow \Sigma'$ ,  $Sen(\varphi) : Sen(\Sigma) \rightarrow Sen(\Sigma')$  translates sentences symbolwise, and  $Mod(\varphi) : Mod(\Sigma') \rightarrow Mod(\Sigma)$  is the forgetful functor.

- (2)  $(\Pi \cup \Sigma)_n^0$  — the fragment of FOPL containing only sentences that are equivalent to sentences in prenex normal form that have at most  $n$  alternated blocks of quantifiers (universal and existential). Within a given signature, the mentioned set of sentences actually puts together two well-known types of first-order sentences:  $\Pi_n^0$  and  $\Sigma_n^0$  [4].
- (3) PFOPL [6] — the institution of partial first-order predicate logic, an extension of FOPL whose signatures  $\Sigma = (S, F, F', P)$  contain, besides relation and (total) operation symbols (in  $F$  and  $P$ ), partial operation symbols too in  $F'$ . Models of course interpret the symbols in  $F'$  as partial operations of appropriate ranks.  $\Sigma$ -model morphisms  $h : A \rightarrow B$  are  $S$ -sorted functions that commute with the total operations and relations in the usual way, and with the partial operations  $\sigma \in F'_{s_1 \dots s_n, s}$  in the following way: for each  $(a_1, \dots, a_n) \in A_{s_1 \dots s_n}$ , if  $A_\sigma(a_1, \dots, a_n)$  is defined, then so is  $B_\sigma(h_{s_1}(a_1), \dots, h_{s_n}(a_n))$ ,

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<sup>2</sup>  $T_F$  is the ground term algebra over  $F$ .

and in this case the latter is equal to  $h_s(A_\sigma(a_1, \dots, a_n))$ . A signature morphism between  $(S, F, F', P)$  and  $(S_0, F_0, F'_0, P_0)$  is a FOPL-signature morphism  $\varphi : (S, F \cup F', P') \rightarrow (S_0, F_0 \cup F'_0, P'_0)$  such that, for each  $\sigma \in F$ , it holds that  $\varphi^{\text{op}}(\sigma) \in F_0$ . Thus, signature morphisms are allowed to map partial operation symbols to total operation symbols, but not vice versa. There exist two kinds of atoms: (*existential*) *equality atoms*  $t = t'$  and relational atoms  $R(t_1, \dots, t_n)$ , having syntax just like at FOPL. An equality atom  $t = t'$  holds in a model  $A$  when both terms are defined and equal ( $A_t = A_{t'}$ ). A relational atom  $R(t_1, \dots, t_n)$  holds when all terms  $t_i$  are defined and their interpretations  $A_{t_i}$  stay in relation  $A_R$ . The sentences are obtained from atoms just like in the case of FOPL (quantification over variables is allowed in the usual sense, which corresponds to considering the quantified variables as new total constants). Note that other kinds of sentences usually considered in partial algebraic frameworks can be expressed here: *definedness*,  $t \downarrow$ , as  $t = t$ ; *strong equality*,  $t =_s t'$  (either both  $t$  and  $t'$  are undefined, or both are defined and equal), as  $(\neg t \downarrow \wedge \neg t' \downarrow) \vee t = t'$ ; *weak equality*,  $t =_w t'$  (if both  $t$  and  $t'$  are defined, then they are equal), as  $(\neg t \downarrow) \vee (\neg t' \downarrow) \vee t = t'$ . The functor *Mod* is defined similarly to the case of FOPL.

- (4) PA — the institution of partial algebra, a fragment of PFOPL having signatures without relation symbols. Partial algebras and their applications were extensively studied in [28] and [3].
- (5) IFOPL — the institution of infinitary first-order logic, an infinitary extension of FOPL, which allows conjunction on arbitrary sets of sentences. This logical system is known under the name  $L_{\infty, \omega}$  [21, 20] and plays an important role in categorical logic.
- (6) IFOPL $_\alpha$  (where  $\alpha$  is an infinite cardinal) — a fragment of IFOPL, admitting only conjunction on sets of sentences with cardinal smaller than  $\alpha$ . This logical system is usually called  $L_{\alpha, \omega}$  [18]. Note that IFOPL $_\omega$  is FOPL.
- (7) PosFOPL — the institution of positive first order predicate logic, a fragment of FOPL, with sentences constructed by means of  $\wedge, \vee, \forall, \exists$ , but not negation  $\neg$ . Here  $\vee$  and  $\exists$  are no longer reducible to  $\wedge$  and  $\forall$  or vice versa. Positive sentences are defined and studied for example in [4, 25].
- (8) EQL — the institution of equational logic [14], a fragment of FOPL, with no relation (only operation) symbols, and with sentences constructed from atoms only by means of universal quantification (no logical connectives).
- (9) EQLN — a minimal extension of EQL with negation, allowing sentences obtained from atoms and negations of atoms through only one round of quantification, either universal or existential, over a set of variables. More precisely, all sentences have the form  $(Q X)t_1 k t_2$ , where  $Q \in \{\forall, \exists\}$  and  $k \in \{=, \neq\}$ . Note that this institution admits negation.
- (10) RWL — the institution of (unconditional) rewriting logic. It has the same signatures as EQL, but models have in addition a preorder relation on each sort carrier, compatible with the operations, and model morphisms have to be increasing with respect to these preorders. The sentences are usual equations as in EQL and transitions  $(\forall X)t \rightarrow t'$ , with  $\rightarrow$  interpreted as the model preorder. This logic cannot be seen as a fragment of FOPL, due to the built-in nature of the preorder on models. Rewriting logic was introduced in [23] with models having a categorical structure where arrows express different transitions between states; a simplified and more amenable version of this logic, which forces this categorical structure to be a preorder, is used in specification languages such as CafeObj [11] or Maude [5]; this simplified version was considered here.

(11) OSL — the institution of order-sorted (equational) logic [29], an extension of EQL where each signature has a partial order on the set of sorts. Thus a signature is a triplet  $(S, \leq, F)$ , where  $(S, \leq)$  is a partially ordered set and  $(S, F)$  is an EQL-signature. A  $(S, \leq, F)$ -model is an  $(S, F)$ -model in EQL subject to two additional requirements:

- For each  $s, s' \in S$  with  $s \leq s'$ , it holds that  $A_s \subseteq A_{s'}$ ;
- For each  $(w, s), (w', s') \in S^* \times S$  such that  $w = s_1 \dots s_n$ ,  $w' = s'_1 \dots s'_n$ ,  $s_i \leq s'_i$ , and  $s \leq s'$ , and each  $\sigma \in F_{w,s} \cap F_{w',s'}$ , it holds that  $A_{w,s}(\sigma) : A^w \rightarrow A_s$  restricts and corestricts  $A_{w',s'}(\sigma) : A^{w'} \rightarrow A_{s'}$ .

A  $(S, \leq, F)$ -morphism between  $A$  and  $B$  is a  $(S, F)$ -morphism in EQL,  $h : A \rightarrow B$ , such that for all  $s, s' \in S$  with  $s \leq s'$ , it holds that  $h_s$  restricts and corestricts  $h_{s'}$ . Given a signature  $\Sigma = (S, \leq, F)$ , one can construct the ground term  $\Sigma$ -algebra  $T_\Sigma$  similarly to the case of EQL, just that one needs to consider the subsort relationship  $\leq$  too. An  $(S, \leq, F)$ -sentence is an equation  $(\forall X)t = t'$ , where  $X$  is an  $S$ -sorted set of variables and  $t, t' \in T_{(S, \leq, F, X)}$ .<sup>3</sup> Satisfaction of a sentence by a model is defined in the obvious way. The functor *Mod* acts just like in the case of EQL.

(12) ML — the institution of (unconditional) membership equational logic, an extension of EQL, which calls the usual sorts ‘kinds’, and allows on each kind a set of sorts that are to be interpreted, on models, as subsets of the kind carrier. Thus a signature is a triplet  $(K, \{S_k\}_{k \in K}, F)$ , where  $(K, F)$  is an EQL-signature and for each kind  $k \in K$ ,  $S_k$  is the set of sorts for this kind. Besides equations, this logic also has membership assertions:  $(\forall X)t : s$ , where  $t \in (T_\Sigma(X))_k$  and  $s \in S_k$ , meaning that ‘ $t$  is of sort  $s$ ’. This logic, introduced in [24], can be seen as a fragment of FOPL, which only uses unary relation symbols and has only universally quantified atoms as sentences. As shown in [24], ML naturally embeds (a variation of) OSL.

(13) EHL — the institution of extended hidden logic. The signatures are triplets  $(H, V, F)$ , where:

- $H$  is the set of *hidden sorts*;
- $V$  is the set of *visible sorts*,  $V \cap H = \emptyset$ ;
- $(H \cup V, F)$  is an EQL-signature (i.e.  $F$  is an  $(H \cup V)^* \times (H \cup V)$ -indexed set of operation symbols).

The  $(H, V, F)$ -models are the usual  $(H \cup V, F)$ -models from EQL. For a model  $A$ , one defines its behavioral equivalence  $\equiv_A$  to be the least congruence on  $A$ , which is an identity on visible sorts. The  $(H, V, F)$ -morphisms are the  $(H \cup V, F)$ -morphisms from EQL that preserve behavioral equivalence. There are two kinds of atoms: (usual) *equality atoms*  $t = t'$  and *behavioral equality atoms*  $t \equiv t'$ . Satisfaction of equality atoms is the usual first-order satisfaction. For a  $(H, V, F)$ -model  $A$ ,  $A \models t \equiv t'$  iff  $A_{c[x_1 \leftarrow a_1, \dots, x_n \leftarrow a_n, z \leftarrow t]} = A_{c[x_1 \leftarrow a_1, \dots, x_n \leftarrow a_n, z \leftarrow t']}$  for each sort  $v \in V$ , each sequence  $x_1, \dots, x_n$  of variables of various sorts, each context  $c$  in  $T_F(\{x_1, \dots, x_n, z\})$  of sort  $v$ , and each sequence  $a_1, \dots, a_n$  of elements in  $A$  of appropriate sorts. (Here a context is a term with only one occurrence of the variable  $z$ ;  $z$  is assumed to have the same sort as  $t$  and  $t'$ . Also, for instance  $c[x_1 \leftarrow a_1, \dots, x_n \leftarrow a_n, z \leftarrow t]$  is a ground  $(H \cup V, F)$ -term parameterized by elements in  $A$ , and  $A_{c[x_1 \leftarrow a_1, \dots, x_n \leftarrow a_n, z \leftarrow t]}$  is its natural interpretation as an element in  $A$ .) All sentences are constructed from atoms by means of first-order

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<sup>3</sup> The variables in  $X$  are interpreted as new constants.



connectives and quantifiers just like in the case of FOPL (quantification is allowed over variables of both hidden and visible sorts). A signature morphism between  $(H, V, F)$  and  $(H', V', F')$  is an EQL-signature morphism  $\varphi : (H \cup V, F) \rightarrow (H' \cup V', F')$  such that:

- $\varphi(V) \subseteq V', \varphi(H) \subseteq H'$ ;
- For each  $(w', s') \in (H' \cup V')^* \times (H' \cup V')$  such that  $w'$  contains a sort in  $\varphi^{\text{sort}}(H)$ , and  $s' \in F'_{w', s'}$ , there exists  $\sigma \in F$  such that  $\varphi^{\text{op}}(\sigma) = s'$ .

On signature morphisms, the functors *Mod* and *Sen* act as in the case of EQL. The preceding description of EHL was adapted from [8]. See [15] for details about hidden logic, and [2] for the description of (a variation of) full first-order hidden logic.

- (14) HL — [16, 12] the institution of hidden logic, a fragment of EHL, with sentences constructed from atoms only by means of universal quantification (no logical connectives).

### 3 Representable and quasi-representable signature morphisms

The institutional notions of *representable* and *quasi-representable* signature morphisms are abstract concepts meant to capture the phenomena of quantification over (sets of) first-order variables. Both notions start from the fact that semantics of quantification in first-order-like logics can be given in terms of signature extensions:  $M \models_{(S, F, P)} (\forall X)e$  ( $M \models_{(S, F, P)} (\exists X)e$ ) iff  $M' \models_{(S, F \cup X, P)} e$  for each (for some)  $(S, F \cup X, P)$ -expansion  $M'$  of  $M$ . Thus, in order to reach first-order quantification institutionally, one needs to define somehow what ‘injective signature morphism that only adds constant symbols’ (such as  $\iota : (S, F, P) \rightarrow (S, F \cup X, P)$ ) means.

DEFINITION 2

A signature morphism  $\varphi : \Sigma \rightarrow \Sigma'$  is called:

- Representable [7], if there exists a  $\Sigma$ -model  $M_\varphi$  (called the representation of  $\varphi$ ) and an isomorphism of categories  $I_\varphi : \text{Mod}(\Sigma') \rightarrow M_\varphi / \text{Mod}(\Sigma)$  such that  $I_\varphi; U = \text{Mod}(\varphi)$ , where  $U : M_\varphi / \text{Mod}(\Sigma) \rightarrow \text{Mod}(\Sigma)$  is the usual forgetful functor;
- Finitely representable [7], if it is representable and  $M_\varphi$  is a finitely presented object in  $\text{Mod}(\Sigma)$ ;
- Quasi-representable [6], if for each  $A' \in |\text{Mod}(\Sigma')|$ , the canonical functor  $A' / \text{Mod}(\varphi) : A' / \text{Mod}(\Sigma') \rightarrow A' \downarrow_\varphi / \text{Mod}(\Sigma)$  is an isomorphism of categories; and
- Finitely quasi-representable [6], if it is quasi-representable and for each colimit  $(A_i \xrightarrow{\mu_i} A)_{i \in I}$  of a directed diagram of  $\Sigma$ -models  $(A_i \xrightarrow{\mu_{ij}} A_j)_{i, j \in I, i \leq j}$  and each  $\varphi$ -expansion  $A'$  of  $A$ , there exists an index  $i \in I$  and a  $\varphi$ -expansion  $\mu'_i$  of  $\mu_i$ .

The notion of representability is built on the intuition that, in FOPL, an expansion of a  $\Sigma = (S, F, P)$ -model  $A$  over a signature inclusion  $\iota : \Sigma \rightarrow \Sigma' = (S, F \cup X, P)$  that only adds constants can be viewed as a pair  $(M, \nu)$ , where  $\nu : X \rightarrow M$  is a function interpreting the new constants in  $X$ , and furthermore as a pair  $(M, \bar{\nu})$ , where  $\bar{\nu} : T_\Sigma(X) \rightarrow M$  is a model-morphism.<sup>4</sup> Hence  $\iota$  is represented by  $T_\Sigma(X)$ . And  $\iota$  is finitely representable, i.e.  $T_\Sigma(X)$  is finitely presented in  $\text{Mod}(\Sigma)$ , if  $X$  is finite.

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<sup>4</sup>  $T_\Sigma(X)$  is the term algebra over variables  $X$  and operations in  $F$ , with all relations in  $P$  empty.

On the other hand, quasi-representability follows the intuition that the aforementioned signature inclusion  $\iota$  does not allow multiple expansions of  $\Sigma$ -morphisms  $A \xrightarrow{h} B$  having a fixed source  $A'$  (where  $A'$  is a  $\iota$ -expansion of  $A$ ). This is because  $A'$  already ‘indicates’, via  $h$ , how the constants in  $X$  should be interpreted in the target model  $B'$  of a presumptive  $\iota$ -expansion  $h'$  of  $h$ ; and of course  $h'$  has to be identical, as a function, to  $h$ . Thus,  $\iota$  is also quasi-representable. And again,  $\iota$  is finitely quasi-represented if  $X$  is finite. Intuitively, if we regard directed colimits of  $\Sigma$ -models as ‘unions’, all the interpretations in the ‘union model’  $A$  of the finite number of constants in  $X$  will eventually be reached by one of the members  $A_i$  of the union; hence the ‘inclusion’ of  $A_i$  into  $A$  has a  $\iota$ -expansion.

For most concrete institutions (at least for those admitting initial objects in the categories of models, like our examples 1–14), the notions of representability and quasi-representability coincide, as shown by the following lemma.

LEMMA 3 [6]

Let  $\varphi: \Sigma \rightarrow \Sigma'$  be a signature morphism. Then  $[\varphi$  is (finitely) quasi-representable and  $Mod(\Sigma)$  has an initial object] iff  $\varphi$  is (finitely) representable.

It is shown in [6] that quasi-representable signature morphisms create directed colimits. Throughout the article, we are going to use intensively a similar property:

LEMMA 4

Let  $\varphi: \Sigma \rightarrow \Sigma'$  be a quasi-representable signature morphism and let  $(A_i \xrightarrow{f_{i,j}} A_j)_{i,j \in I, i \leq j}$  be a directed diagram in  $Mod(\Sigma)$  with colimit  $(A_i \xrightarrow{\mu_i} A)_{i \in I}$ . Also, let  $k_j \in I$ ,  $I_k = \{i \in I \mid k \leq i\}$ , and  $B$  a  $\varphi$ -expansion of  $A_k$ . Then there exists a directed diagram  $(A'_i \xrightarrow{f'_{i,j}} A'_j)_{i,j \in I_k, i \leq j}$  in  $Mod(\Sigma')$ , with colimit  $(A'_i \xrightarrow{\mu'_i} A')_{i \in I_k}$ , such that:

- $A'_k = B$ ;
- For each  $i, j \in I_k$  with  $i \leq j$ ,  $f'_{i,j}$  is a  $\varphi$ -expansion of  $f_{i,j}$ ;
- For each  $i \in I_k$ ,  $\mu'_i$  is a  $\varphi$ -expansion of  $\mu_i$ .

PROOF. For each  $i \in I_k$ , we define  $(f'_{k,i} : B \rightarrow A'_i, A'_i)$  to be  $(A_k/Mod(\varphi))^{-1}(f_{k,i} : A_k \rightarrow A_i, A_i)$ . In particular, for  $i=k$ , we have  $A'_k = B$  and  $f'_{k,k} = 1_B$ . Also, for each  $i, j \in I_k$  with  $k < i \leq j$ , we define  $f'_{i,j}$  to be  $(A_k/Mod(\varphi))^{-1}(f_{i,j} : (f_{k,i}, A_i) \rightarrow (f_{k,j}, A_j))$ . Then  $Dg' = (A'_i \xrightarrow{f'_{i,j}} A'_j)_{i,j \in I_k, i \leq j}$  is a directed diagram in  $Mod(\Sigma')$  and its  $\varphi$ -reduct is  $Dg = (A_i \xrightarrow{f_{i,j}} A_j)_{i,j \in I_k, i \leq j}$ . Now we define  $(\mu'_k : B \rightarrow A', A')$  to be  $(A_k/Mod(\varphi))^{-1}(\mu_k : A_k \rightarrow A, A)$  and for each  $i \in I_k$ ,  $\mu'_i$  to be  $(A_k/Mod(\varphi))^{-1}(\mu_i : (f_{k,i}, A_i) \rightarrow (\mu_k, A))$ . Then  $CC' = (A'_i \xrightarrow{\mu'_i} A')_{i \in I_k}$  is a cocone of  $Dg'$ , and the  $\varphi$ -reduct of  $CC'$  is  $CC = (A_i \xrightarrow{\mu_i} A)_{i \in I_k}$ . The fact that  $CC'$  is an actual colimit for  $Dg'$  follows at once by the quasi-representability of  $\varphi$ : for any cocone  $(A'_i \xrightarrow{v_i} A')_{i \in I_k}$  of  $Dg'$ , we get that its reduct  $(A'_i \downarrow_{\varphi} v_i \downarrow_{\varphi} \rightarrow A'' \downarrow_{\varphi})_{i \in I_k}$  is a cocone of  $Dg$ ; thus if one takes  $u$  to be the universal arrow from  $CC$  to  $(A'_i \downarrow_{\varphi} v_i \downarrow_{\varphi} \rightarrow A'' \downarrow_{\varphi})_{i \in I_k}$  (according to Lemma 2), then  $(A_k/Mod(\varphi))^{-1}(u : (\mu_k, A) \rightarrow (v'_k \downarrow_{\varphi}, A'' \downarrow_{\varphi}))$  is the desired universal arrow from  $CC'$  to  $(A'_i \downarrow_{\varphi} v_i \downarrow_{\varphi} \rightarrow A'')_{i \in I_k}$ . ■

The next lemma shows that (quasi-)representable signature morphisms behave well under composition and pushouts.

LEMMA 5 [6]

- (1) (Finitely) quasi-representable signature morphisms form a subcategory of *Sign*.
- (2) (Finitely) representable signature morphisms form a subcategory of *Sign*.

- (3) If the institution is semi-exact, then the class of (finitely) quasi-representable signature morphisms is closed under pushouts.
- (4) If the institution is semi-exact and liberal on signature morphisms, then the class of (finitely) representable signature morphisms is closed under pushouts.

Of course, representability and quasi-representability are only abstract approximations for ‘injective morphisms that only add constants’. What will be relevant for the results of this paper is that in all our examples 1–14 of institutions, (quasi-)representable signature morphism include the desired types of morphisms. Formally, let  $\varphi: \Sigma \rightarrow \Sigma'$  be a signature morphism in any of the examples 1–14 of institutions. We say that  $\varphi$  is an injective signature morphism that only adds constants if the following conditions hold:  $\varphi^{\text{sort}}$  is bijective,  $\varphi^{\text{rel}}$  is bijective,  $\varphi^{\text{op}}$  is injective and  $F' - \varphi^{\text{op}}(F)$  contains only (total) operation symbols. (Here,  $F$  and  $F'$  stand for the sets of all (partial and total) operation symbols of  $\Sigma$  and  $\Sigma'$ , respectively.) If in addition  $F' - \varphi^{\text{op}}(F)$  is finite, we say that  $\varphi$  is an *injective signature morphism that only adds finitely many constants*.

PROPOSITION 6

In any of the examples 1–14 of institutions, all injective signature morphisms that only add (finitely many) constants are (finitely) representable, hence also (finitely) quasi-representable. Moreover, in each case, the (broad) subcategory of *Sign* of such morphisms is closed under pushouts.

PROOF. The fact that such morphisms are (finitely) representable can be shown using arguments very similar to the ones for FOPL. The only slightly more exotic cases are PFOPL, PA, OSL, RL, HL, and EHL; however, in each case the algebra freely generated by a set of (total) variables exists in any signature. The cases of HL and EHL actually require a small separate discussion. Let  $(H, V, F)$  be a signature in either of the two institutions. Since each free algebra  $T_F(X)$  has its behavioral equivalence equal to the identity, every EQL-morphism with source  $T_F(X)$  is also an EHL- and HL-morphism, thus an EHL- or HL-signature morphism that only adds (finitely many) constants  $X$  is indeed (finitely) represented by  $T_F(X)$ .

As for closure under pushouts, this follows easily from the fact that, in the category of sets and functions, the subcategory of injective functions is closed under pushouts. ■

Although not strictly needed in this article, but helpful for getting an idea on how close the aforementioned approximation is, we recall a concrete characterization of representable (and quasi-representable) signature morphisms in FOPL.

PROPOSITION 7

[33] Let  $\varphi: \Sigma = (S, F, P) \rightarrow \Sigma' = (S', F', P')$  be a signature morphism in FOPL. Then the following are equivalent:

- (1)  $\varphi$  is representable;
- (2)  $\varphi$  is quasi-representable and
- (3)  $\varphi^{\text{sort}}$  and  $\varphi^{\text{rel}}$  are bijective, and for all  $(w', s') \in S'^* \times S'$  with  $w' \neq \lambda$ , for all  $\sigma' \in F'_{w', s'}$ , there exists a unique  $\sigma \in F$  such that  $\varphi^{\text{op}}(\sigma) = \sigma'$  (in other words,  $\varphi$  is bijective with respect to all items except constant symbols).

Moreover, the preceding three conditions stay equivalent if we add the word ‘finitely’ to the first two and add the requirement that  $F' - \varphi^{\text{op}}(F)$  be finite to the third.

#### 4 Elementary morphisms

In classical first-order logic [4], an injective model morphism  $A \xrightarrow{h} B$  is called an *elementary embedding* if one of the following equivalent conditions holds:

- (1) For each formula  $e(x_1, \dots, x_n)$  and each sequence  $a_1, \dots, a_n \in A$ ,  $A \models e(a_1, \dots, a_n)$  iff  $B \models e(h(a_1), \dots, h(a_n))$ ;
- (2) For each formula  $e(x_1, \dots, x_n)$  and each sequence  $a_1, \dots, a_n \in A$ ,  $A \models e(a_1, \dots, a_n)$  implies  $B \models e(h(a_1), \dots, h(a_n))$ .

The notion of elementary embedding has an immediate generalization to the many-sorted case *FOPL*; and it was extended to cope with infinitary first-order logics too [18, 17]. Our next institutional generalization reads the concept of elementary embedding in the following way: the morphism  $h$  preserves sentences in any language extending with constants the original language, regardless of the actual interpretation of these constants. Notice that the two alternative definitions of elementary embeddings listed here are equivalent thanks to the existence of negations in full first-order logic; however, this is not the case in less expressive logics, such as *PosFOPL* or *EQL*. We prefer to consider the second variant and interpret elementarity as a *sentence preservation* property rather than a refinement of elementary equivalence. This subjective choice is motivated by our belief that taking into consideration the direction of the arrow  $h$  in the definition is a more fruitful approach. In an institution, the ‘languages extended with constants’ are captured by quasi-representable signature morphisms  $\varphi$  having as source the given language/signature, and satisfaction inside such an extended language is captured by usual satisfaction by  $\varphi$ -expansions. To keep the discussion general and to avoid certain intricacies in the particular cases resulting from considering all quasi-representable, or representable, signature morphisms, we parameterize our definition by a class  $\mathcal{Q}$  of quasi-representable signature morphisms. Thus let  $A \xrightarrow{h} B$  be a  $\Sigma$ -morphism. Formulae  $e(x_1, \dots, x_n)$  are expressed by usual sentences  $e'$  in signatures  $\Sigma'$ , where  $\varphi: \Sigma \rightarrow \Sigma'$  is a signature morphism in  $\mathcal{Q}$ . Satisfaction of such sentences  $e' \in \text{Sen}(\Sigma')$  makes sense in  $\varphi$ -expansions of  $A$  and  $B$ , expansions which are to be seen as models together with some designated constants.<sup>5</sup> However, asking that ‘ $A' \models e'$  implies  $B' \models e'$ ’ for all  $\varphi$ -expansions  $A'$  and  $B'$  of  $A$  and  $B$  and for all  $e' \in \text{Sen}(\Sigma')$  is not appropriate, since the quoted implication should be required only about constants in  $A$  and  $B$  connected through  $h$ ; the connection is realised by first considering  $\varphi$ -expansions  $h'$  of  $h$ .

The injectivity assumption in the definition of elementary embedding for classical first-order logic is superfluous. We did not consider it in the preceding discussion; this could be seen as yet another subjective choice, meant to emphasize once more the idea of sentence preservation, this time to the prejudice of the algebraic property of model embedding. This choice has an important terminological consequence: we define and study ‘elementary morphisms’, and not ‘elementary embeddings’, although the elementary morphisms yield in particular the FOPL-elementary embeddings.

For the whole section, we fix an institution  $\mathcal{I}$  and a broad subcategory  $\mathcal{Q}$  of *Sign* (i.e. a class of signature morphisms containing all identity morphisms and closed under composition), consisting of quasi-representable signature morphisms. In particular, by taking further

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<sup>5</sup> Recall from Section 3 the connection between quasi-representable signature morphisms and first-order variables/constants.

mild assumptions on  $\mathcal{I}$  such as semi-exactness or liberality on signature morphisms, according to Lemma 5, possible choices for  $\mathcal{Q}$  are given by either of the following four types of signature morphisms: quasi-representable, finitely quasi-representable, representable, finitely representable.

DEFINITION 8

Let  $\Sigma$  be a signature. A  $\Sigma$ -morphism  $A \xrightarrow{h} B$  is called  $\mathcal{Q}$ -elementary if for all signature morphisms  $\varphi : \Sigma \rightarrow \Sigma'$  in  $\mathcal{Q}$ ,  $\varphi$ -expansions  $A' \xrightarrow{h'} B'$  of  $h$ , and sentences  $e' \in \text{Sen}(\Sigma')$ , it holds that  $A' \models e'$  implies  $B' \models e'$ .

REMARK 9

- (1) Because each morphism in  $\mathcal{Q}$  is quasi-representable, in Definition 8 the  $\varphi$ -expansion  $h'$  of  $h$  is uniquely determined by the  $\varphi$ -expansion  $A'$  of  $A$ .
- (2) If the institution admits negations, then the condition in Definition 8 can be equivalently stated by replacing ' $A' \models e'$  implies  $B' \models e'$ ' with ' $A' \models e'$  iff  $B' \models e'$ '.

Let us see what our general concept of  $\mathcal{Q}$ -elementary morphism becomes for our examples 1–14 of institutions. In what follows, for all these institutions, we shall simply call *elementary morphisms* the  $\mathcal{Q}$ -elementary morphisms with  $\mathcal{Q}$  being the category of injective signature morphisms that only add constants. (Note that in logics with finite sentences, such as FOPL and all its fragments, elementarity with respect to arbitrary signature morphisms that only add constants is equivalent to elementarity with respect to morphisms that only add *finitely many* constants; this is because just a finite set of the newly added constants are contained in a given sentence.) Known and relatively well-studied cases are the following:

- For FOPL, the elementary embeddings from (the many-sorted version of) classical model theory [4];
- For PA, the elementary embeddings of partial algebras [3];
- For IFOPL and IFOPL $_{\alpha}$ , the  $L_{\infty, w^-}$  and  $L_{\alpha, w^-}$  elementary embeddings from infinitary model theory [18, 21, 17];
- For  $(\Pi \cup \Sigma)_n^0$ , the existentially closed embeddings [17] and
- For  $(\Pi \cup \Sigma)_n^0$ , the  $\Sigma_n^0$ -extensions [4].

Up to our knowledge, elementary embeddings for the other examples of institutions were not considered so far in the literature. However, such notions are meaningful instances of the logic-independent concept of elementary morphism that we propose here. In each case, an elementary morphism is one that preserves satisfaction of all sentences with elements of the source model as parameters. The next proposition gives some expected properties of  $\mathcal{Q}$ -elementary morphisms.

PROPOSITION 10

Let  $\chi : \Sigma \rightarrow \Sigma'$  be a signature morphism. Then the following hold;

- The  $\mathcal{Q}$ -elementary morphisms in  $\text{Mod}(\Sigma)$  form a subcategory of  $\text{Mod}(\Sigma)$ ;
- Assume that the institution has pushouts of signatures and is semi-exact and that  $\mathcal{Q}$  is closed under pushouts. If  $A' \xrightarrow{h'} B'$  is a  $\mathcal{Q}$ -elementary morphism in  $\text{Mod}(\Sigma')$ , then  $h'|_{\chi}$  is also  $\mathcal{Q}$ -elementary;
- If  $\chi$  is in  $\mathcal{Q}$ ,  $A \xrightarrow{h} B$  is a  $\mathcal{Q}$ -elementary morphism in  $\text{Mod}(\Sigma)$ , and  $A' \xrightarrow{h'} B'$  is a  $\chi$ -expansion of  $h$ , then  $h'$  is also  $\mathcal{Q}$ -elementary.

PROOF.

- (1) If  $A \in |Mod(\Sigma)|$ , then  $1_A$  is  $\mathcal{Q}$ -elementary because any expansion of  $1_A$  along a quasi-representable signature morphism is an identity model-morphism itself. Let now  $A \xrightarrow{h} B$  and  $BC$  be two  $\mathcal{Q}$ -elementary morphisms in  $Mod(\Sigma)$  and let  $f = h;g$ . In order to show that  $f$  is  $\mathcal{Q}$ -elementary, let  $\varphi : \Sigma \rightarrow \Sigma_0$  be a signature morphism in  $\mathcal{Q}$  and  $A_0 f_0 \rightarrow C_0$  a  $\varphi$ -expansion of  $f$ . Let  $h_0 : A_0 \rightarrow B_0$  such that  $(A_0, h_0) = (A_0/Mod(\varphi))^{-1}(A, h)$ . Let  $g_0 = (A_0/Mod(\varphi))^{-1}(g : (A, h) \rightarrow (A, f))$ . Then  $h_0;g_0 = f_0$ . Since  $h$  and  $g$  are  $\mathcal{Q}$ -elementary, it follows that  $\{A_0\}^* \subseteq \{B_0\}^* \subseteq \{C_0\}^*$ .
- (2) Let  $A \xrightarrow{h} B$  denote the  $\chi$ -reduct of  $h'$ . In order to prove  $h$  elementary, let  $\varphi : \Sigma \rightarrow \Sigma_0$  be a signature morphism in  $\mathcal{Q}$  and  $A_0 \xrightarrow{h_0} B_0$  a  $\varphi$ -expansion of  $h$ . Consider the pushout  $\Sigma' \varphi' \rightarrow \Sigma_1 \chi' \Sigma_0$  of the signature morphism span  $(\Sigma' \chi \Sigma \varphi \rightarrow \Sigma_0)$ . Then  $\varphi'$  is also in  $\mathcal{Q}$ . By semi-exactness, since  $h'$  and  $h_0$  have a common reduct (that is,  $h$ ), they also have a common expansion  $A_1 \xrightarrow{h_1} B_1$  in  $Mod(\Sigma_1)$ . Because  $h'$  is elementary,  $\{A_1\}^* \subseteq \{B_1\}^*$ . Finally, using the satisfaction condition, we get  $\{A_0\}^* \subseteq \{B_0\}^*$ .
- (3) Immediate from the definition of  $\mathcal{Q}$ -elementary morphisms and the fact that  $\mathcal{Q}$  is closed under composition. ■

## 5 Elementary chain property

Throughout this section, we again fix an institution  $\mathcal{I}$  and a broad subcategory  $\mathcal{Q}$  of  $Sign$  consisting of quasi-representable signature morphisms.

A  *$\mathcal{Q}$ -elementary chain* is a chain diagram in  $Mod(\Sigma)$  for some signature  $\Sigma$ , such that all its morphisms are  $\mathcal{Q}$ -elementary. The *elementary chain property* (parameterized by  $\mathcal{Q}$  and abbreviated  $\mathcal{Q}$ -ECP) asks that, for each colimit of each  $\mathcal{Q}$ -elementary chain, all the structural morphisms be  $\mathcal{Q}$ -elementary. In other words, it asks that for each signature  $\Sigma$ , the subcategory of  $Mod(\Sigma)$  of  $\mathcal{Q}$ -elementary morphisms be closed under chain colimits. We are going to prove that, under appropriate accessibility assumptions on sentences,  $\mathcal{Q}$ -ECP holds in an arbitrary institution. But first we need to consider some technical concepts and results.

We say that a sentence  $e \in Sen(\Sigma)$  for some signature  $\Sigma$  is *preserved (reflected)* by directed colimits of  *$\mathcal{Q}$ -elementary morphisms, abbreviated  $\mathcal{Q}$ -preserved (reflected)*, if for each directed diagram of  $\mathcal{Q}$ -elementary  $\Sigma$ -morphisms  $(A_i \xrightarrow{f_{ij}} A_j)_{i,j \in I, i \leq j}$  with colimit  $(A_i \xrightarrow{\mu_i} A)_{i \in I}$  and each  $k \in I$ ,  $A_k \models e$  implies  $A \models e$  ( $A \models e$  implies  $A_k \models e$  respectively).

PROPOSITION 11

The class of sentences preserved by directed colimits of  $\mathcal{Q}$ -elementary morphisms

- (1) Contains all basic sentences,
- (2) Is closed under arbitrary conjunction and disjunction,
- (3) Is closed under existential quantification over morphisms in  $\mathcal{Q}$  and
- (4) Is closed under universal quantification over finitely quasi-representable morphisms in  $\mathcal{Q}$ .

PROOF. Let  $e \in Sen(\Sigma)$ ,  $(A_i \xrightarrow{f_{ij}} A_j)_{i,j \in I, i \leq j}$  a directed diagram of  $\mathcal{Q}$ -elementary  $\Sigma$ -morphisms, with colimit  $(A_i \xrightarrow{\mu_i} A)_{i \in I}$ , and let  $k \in I$ . Assume that  $A_k \models e$ . We need to prove that  $A \models e$ .

- (1) Assume  $e$  is a basic sentence. Since  $A_k \models e$ , there exists a  $\Sigma$ -morphism  $M_e \rightarrow A_k$ , hence, by composition with  $\mu_k$ , we find a morphism  $M_e \rightarrow A$ , implying  $A \models e$ .
- (2) Assume  $e$  is equivalent to  $\bigwedge E$ , where  $E \subseteq Sen(\Sigma)$  such that for all  $e' \in E$ ,  $e'$  is  $\mathcal{Q}$ -preserved. Since  $A_k \models e$ , it holds that  $A_k \models e'$  for all  $e' \in E$ , hence  $A \models e'$  for all  $e' \in E$ , hence  $A \models e$ . The proof for disjunction goes similarly.

- (3) Assume  $e$  is equivalent to  $(\exists\varphi)e'$ , where  $\varphi: \Sigma \rightarrow \Sigma'$  is in  $\mathcal{Q}$  and  $e'$  is  $\mathcal{Q}$ -preserved. Then there exists a  $\varphi$ -expansion  $A'_k$  of  $A_k$  such that  $A'_k \models e'$ . By Lemma 4, there exists a directed diagram  $(A'_i \xrightarrow{f_{i,j}} A'_j)_{i,j \in I_k, i \leq j}$  in  $\text{Mod}(\Sigma')$ , with colimit  $(A'_i \xrightarrow{\mu'_i} A')_{i \in I_k}$  such that, for each  $i, j \in I_k$  with  $i \leq j$ ,  $f_{i,j}$  is a  $\varphi$ -expansion of  $f_{i,j}$  and  $\mu'_i$  is a  $\varphi$ -expansion of  $\mu_i$ . In particular,  $A' \upharpoonright_\varphi = A$ . According to Proposition 10.(3), each  $f_{i,j}$  is  $\mathcal{Q}$ -elementary. Applying the fact that  $e'$  is  $\mathcal{Q}$ -preserved, we obtain that  $A' \models e'$ , hence  $A \models (\exists\varphi)e'$ , i.e.  $A \models e$ .
- (4) Assume  $e$  is equivalent to  $(\forall\varphi)e'$ , where  $\varphi: \Sigma \rightarrow \Sigma'$  is a finitely quasi-representable signature morphism in  $\mathcal{Q}$  and  $e'$  is  $\mathcal{Q}$ -preserved. Let  $A'$  be a  $\varphi$ -expansion of  $A$ . We need to show that  $A' \models e'$ . Because  $\varphi$  is finitely quasi-representable, there exists  $q \in I$  and a  $\varphi$ -expansion  $\xi'_q: A'_q \rightarrow A'$  of  $\mu_q$ . Since  $(I, \leq)$  is directed, there exists  $p \in I$  such that  $q \leq p$  and  $k \leq p$ . Thus, because  $f_{k,p}$  is  $\mathcal{Q}$ -elementary, we get  $A_p \models e$ . Define  $(A'_q \rightarrow A'_p, A'_p)$  to be  $(A_q/\text{Mod}(\varphi))^{-1}(A_q \xrightarrow{f_{q,p}} A_p, A_p)$  and  $v'$  to be  $(A_q/\text{Mod}(\varphi))^{-1}(\mu_p: (A_q, f_{q,p}) \rightarrow (A_p, \mu_q))$ . Note that  $v': A'_q \rightarrow A'_p$ . By Lemma 4 applied to the index  $p$ , there exists a directed diagram  $(A'_i \xrightarrow{f_{i,j}} A'_j)_{i,j \in I_p, i \leq j}$  in  $\text{Mod}(\Sigma')$ , with colimit  $(A'_i \xrightarrow{\mu'_i} B')_{i \in I_p}$ , such that, for each  $i, j \in I_p$  with  $i \leq j$ ,  $f_{i,j}$  is a  $\varphi$ -expansion of  $f_{i,j}$  and  $\mu'_i$  is a  $\varphi$ -expansion of  $\mu_i$ . Again, Proposition 10.(3) assures us that each  $f_{i,j}$  is  $\mathcal{Q}$ -elementary. Since both  $(v', A')$  and  $(\mu'_p, B')$  are equal to  $(A_p/\text{Mod}(\varphi))^{-1}(\mu_p, A)$ , it follows that  $A' = B'$  and  $v' = \mu'_p$ . Finally, since  $e'$  is  $\mathcal{Q}$ -preserved and  $A'_p \models e'$  ( $A'_p$  being a  $\varphi$ -expansion of  $A_p$ ), we obtain that  $A' \models e'$ .

PROPOSITION 12

Assume that the institution admits negation. Then the class of sentences preserved and reflected by directed colimits of  $\mathcal{Q}$ -elementary morphisms

- (1) Contains all finitary basic sentences,
- (2) Is closed under arbitrary conjunction and disjunction, and under negation and
- (3) Is closed under universal and existential quantification over finitely quasi-representable morphisms in  $\mathcal{Q}$ .

PROOF. Let  $e \in \text{Sen}(\Sigma)$ ,  $(A_i \xrightarrow{f_{i,j}} A_j)_{i,j \in I, i \leq j}$  a directed diagram of  $\mathcal{Q}$ -elementary  $\Sigma$ -morphisms, with colimit  $(A_i \xrightarrow{\mu_i} A)_{i \in I}$ , and let  $k \in I$ . We need to prove  $[A_k \models e \text{ iff } A \models e]$ .

- Assume  $e$  is a finitary basic sentence. That  $A_k \models e$  implies  $A \models e$  follows from Proposition 11(1). Therefore let us suppose  $A \models e$ , i.e. there exists a  $\Sigma$ -morphism  $g: M_e \rightarrow A$ , and let us show that  $A_k \models e$ . Because  $M_e$  is finitely presentable, there exists  $j \in I$  and  $\Sigma$ -morphism  $h: M_e \rightarrow A_j$  such that  $h; \mu_j = g$ . Because  $(I, \leq)$  is directed, there exists  $i \in I$  such that  $k \leq i$  and  $j \leq i$ . Then, by the existence of the morphism  $M_e \rightarrow A_i$ , it follows that  $A_i \models e$ . Moreover, since  $f_{k,i}: A_k \rightarrow A_i$  is  $\mathcal{Q}$ -elementary and the institution admits negation, we obtain  $A_k \models e$ .
- Similar to the proof of Proposition 11(2) for conjunction and disjunction. For negation, the property is obvious thanks to its symmetry.
- Because the institution admits negation, universal and existential quantifications are mutually definable. Therefore, let us focus on existential quantification. Assume  $e$  is equivalent to  $(\exists\varphi)e'$ , where  $\varphi: \Sigma \rightarrow \Sigma'$  is a finitely quasi-representable morphism in  $\mathcal{Q}$  and  $e'$  is a  $\Sigma'$ -sentence  $\mathcal{Q}$ -[preserved and reflected]. That  $A_k \models e$  implies  $A \models e$  follows from Proposition 11(3). Let us now suppose  $A \models (\exists\varphi)e'$  and let us show that  $A_k \models (\exists\varphi)e'$ . We have that  $A \models \neg(\forall\varphi)\neg e'$ , which means  $A \not\models (\forall\varphi)\neg e'$ . By point (2),  $\neg e'$  is  $\mathcal{Q}$ -[preserved and reflected]. Thus, by Proposition 11.(4),  $(\forall\varphi)\neg e'$  is  $\mathcal{Q}$ -preserved, hence  $A_k \models (\forall\varphi)\neg e'$  would imply  $A \models (\forall\varphi)\neg e'$ , which is a contradiction. Thus  $A_k \models \neg(\forall\varphi)\neg e'$ , i.e.  $A_k \models (\exists\varphi)e'$ . ■

## PROPOSITION 13

Assume that all sentences of the institution are preserved by directed colimits of  $\mathcal{Q}$ -elementary morphisms. Then for each signature  $\Sigma$ , the subcategory of  $Mod(\Sigma)$  of  $\mathcal{Q}$ -elementary morphisms is closed under directed colimits.

PROOF. Let  $(A_i \xrightarrow{f_{i,j}} A_j)_{i,j \in I, i \leq j}$  be a directed diagram in  $Mod(\Sigma)$  such that each  $f_{i,j}$  is  $\mathcal{Q}$ -elementary, and let  $(A_i \rightarrow A)_{i \in I}$  be its colimit. Let  $k \in I$ . In order to prove that  $\mu_k$  is  $\mathcal{Q}$ -elementary, let  $v' : A'_k \rightarrow A'$  be a  $\varphi$ -expansion of  $\mu_k$  and let  $e' \in Sen(\Sigma')$  such that  $A'_k \models e'$ . By Lemma 4, there exists a directed diagram  $(A'_i \xrightarrow{f'_{i,j}} A'_j)_{i,j \in I_k, i \leq j}$  in  $Mod(\Sigma')$ , with colimit  $(A'_i \rightarrow B')_{i \in I_k}$ , such that, for each  $i, j \in I_k$  with  $i \leq j$ ,  $f'_{i,j}$  is a  $\varphi$ -expansion of  $f_{i,j}$  and  $\mu'_i$  is a  $\varphi$ -expansion of  $\mu_i$ . Just like in the proof of Proposition 11(4), one obtains that each  $f'_{i,j}$  is  $\mathcal{Q}$ -elementary and that  $A' = B'$  and  $v' = \mu'_k$ . Thus, according to our hypothesis,  $A' \models e'$ .

## DEFINITION 14

An institution  $\mathcal{I}$  is called  $\mathcal{Q}$ -first-order-accessible if one of the two following properties holds:

- (1) All sentences of  $\mathcal{I}$  are (equivalent to ones) obtained from basic sentences by applying a finite number of times the following rules:
  - Arbitrary conjunction;
  - Arbitrary disjunction;
  - Existential quantification over morphisms in  $\mathcal{Q}$  and
  - Universal quantification over finitely quasi-representable morphisms in  $\mathcal{Q}$ .
- (2)  $\mathcal{I}$  admits negation and all sentences of  $\mathcal{I}$  are (equivalent to ones) obtained from finitary basic sentences by applying a finite number of times the following rules:
  - Arbitrary conjunction;
  - Arbitrary disjunction;
  - Negation;
  - Existential quantification over finitely quasi-representable morphisms in  $\mathcal{Q}$  and
  - Universal quantification over finitely quasi-representable morphisms in  $\mathcal{Q}$ .

## PROPOSITION 15

All the examples 1–14 of institutions are  $\mathcal{Q}$ -first-order-accessible, where  $\mathcal{Q}$  is each time the category of injective signature morphisms that only add constants.

PROOF. Let us first see that IFOPL and all its fragments are  $\mathcal{Q}$ -first-order accessible. Indeed, for each signature  $\Sigma = (S, F, P)$ , an equality atom  $t = t'$  is finitary basic thanks to the model  $T_\Sigma/t = t'$ , that is, the  $(S, F)$ -algebra  $T_F/t = t'$  (a quotient of the ground term algebra over  $F$ ) with all the relations in  $P$  empty, while a relational atom  $R(t_1, \dots, t_n)$  is basic, thanks to the model consisting of  $T_F$  together with all relations in  $P$  empty, except  $R$ , which is the singleton  $\{(t_1, \dots, t_n)\}$ . Moreover, quantification over finite or infinite sets of variables are particular cases of quantification over signature morphisms in  $\mathcal{Q}$ . Thus, PosFOPL, EQL and ML fall into case 1 of Definition 14, and FOPL,  $(\Pi \cup \Sigma)_n^0$ , IFOPL, IFOPL $\alpha$  and EQLN into case 2.

A similar argument as the preceding holds for PFOPL and PA too, since, for instance given a PA-signature and a set of equality atoms, there exists the initial algebra in the category of algebras satisfying these atoms [3]. And similarly for RWL and OSL.

As for EHL and HL, one has to notice mainly two things. First, all usual equality atoms are basic; indeed, the algebra  $T_F/t = t'$  has the property that, for each  $(H, V, F)$ -model  $A$  satisfying the usual equality atom  $t = t'$ , the unique EQL-morphism between  $T_F/t = t'$  and  $A$  preserves behavioral equivalence; hence, it is also an EHL- and HL-morphism.



Second, the behavioural equality atoms are equivalent to (infinite) conjunctions of universally quantified usual equality atoms; indeed, it holds that  $A \models t \equiv t'$  iff  $A \models \bigwedge \{(\forall \{x_1, \dots, x_n\}) c[z \leftarrow t] = c[z \leftarrow t'] \mid v \in V, x_1, \dots, x_n \text{ variables, } c \in T_F(\{x_1, \dots, x_n, z\})_v, \text{context}\}$ . Thus HL falls into case 1 of Definition 14. Moreover, EHL falls into case 2, since although behavioural equality atoms are not finitary, they are nevertheless obtainable from finitary basic sentences by means of the rules of universal quantification over finitely quasi-representable morphisms in  $\mathcal{Q}$  (i.e. over finite sets of variables) and arbitrary conjunction.

**THEOREM 16**

(Elementary Chain Theorem) Assume that the institution is  $\mathcal{Q}$ -first-order-accessible. Then for each signature  $\Sigma$ , the subcategory of  $Mod(\Sigma)$  of  $\mathcal{Q}$ -elementary morphisms is closed under directed colimits. In particular, the institution enjoys the  $\mathcal{Q}$ -ECP.

**PROOF.** Follows immediately: for case 1 of Definition 14 from Propositions 11 and 13, and for case 2 from Propositions 12 and 13.

The separation on two cases in Definition 14 covers mainly the following situations: the institution  $\mathcal{I}$  either admits negation, or has no negation—intermediate cases are not covered. Some important examples of institutions to which our Theorem 16 does not apply are all variations of Horn logic—in fact, for those institutions, we conjecture that the elementary chain property does not hold.

**COROLLARY 17**

All the examples 1–14 of institutions enjoy the  $\mathcal{Q}$ -ECP.

Note that Theorem 16 is applicable to a whole variety of other logics resulted from other different combinations of connectives and quantifiers. An interesting example which takes full advantage of Proposition 11 is a version of positive infinitary first-order logic admitting arbitrary conjunction and disjunction, existential quantification over arbitrary sets of variables, and universal quantification over finite sets of variables. Moreover, the case of *fragments* of languages (over transitive sets) in infinitary first-order logic [18] also seems to fall into our framework, provided that one takes the trouble of formalizing this as an institution.

## 6 Elementary morphisms by diagrams

An alternative definition of elementary embeddings in classical model theory is given in terms of elementary diagrams [4, 25]. There, the *elementary diagram*  $EDg(A)$  of a model  $A$  is the set of all sentences in  $\Sigma(A)$  (the language  $\Sigma$  of  $A$  extended with all elements of  $A$  as constants) that are true in  $A$ . Then, an embedding  $A \xrightarrow{h} B$  is elementary iff  $h(EDg(A)) \subseteq EDg(B)$ , where  $h(EDg(A))$  is the obvious translation through  $h$  of the sentences in  $EDg(A)$ . The main difference to the original definition (discussed at the beginning of Section 4) is that a language which includes parameter symbols for the source model,  $\Sigma(A)$ , is *a priori* given and the desired property is stated locally, in that fixed language. By adapting an existing institutional concept of diagram, we can discuss this alternative definition in a logic-independent framework.

### 6.1 Institutional diagrams

Diagrams are a basic concept in classical model theory [4]. They were first generalized to the institutional framework in [35, 36]; there, it is defined the concept of *abstract*

*algebraic institution*, which is an institution subject to some additional natural requirements (like finite-exactness, existence of direct products of models, etc.) and enriched with a system of diagrams. The reason for introducing diagrams there was *making all algebras accessible*, for specification purposes. In this article, we need a more elaborated notion of institutional diagram, defined in [8], which takes into consideration not only models, but also model morphisms. The concept was introduced under the name ‘elementary diagram’. For reasons that will be pointed out soon, here we prefer to use, like in [35], the name ‘positive diagram’ instead.

An institution  $\mathcal{I} = (\text{Sign}, \text{Sen}, \text{Mod}, \models)$  is said to have positive diagrams [8] if

- (1) For each signature  $\Sigma$  and  $\Sigma$ -model  $A$  there exists a signature morphism  $\iota_\Sigma(A) : \Sigma \rightarrow \Sigma_A$  and a set  $E_A$  of  $\Sigma_A$ -sentences (called the *positive diagram* of  $A$ ) such that  $\text{Mod}(\Sigma_A, E_A)$  and  $A/\text{Mod}(\Sigma)$  are isomorphic by an isomorphism  $i_{\Sigma, A}$  making the following diagram commutative:

$$\begin{array}{ccc} \text{Mod}(\Sigma_A, E_A) & \xrightarrow{i_{\Sigma, A}} & A/\text{Mod}(\Sigma) \\ & \searrow^{-1_{\iota_\Sigma(A)}} & \swarrow^{\text{forgetful}} \\ & & \text{Mod}(\Sigma) \end{array}$$

- (2)  $\iota$  is ‘functorial’, i.e. for each signature morphism  $\varphi : \Sigma \rightarrow \Sigma'$ , each  $A \in |\text{Mod}(\Sigma)|$ ,  $A' \in |\text{Mod}(\Sigma')|$  and  $h : A \rightarrow A' \upharpoonright_\varphi$  in  $\text{Mod}(\Sigma)$ , there exists a presentation morphism  $\iota_\varphi(h) : (\Sigma_A, E_A) \rightarrow (\Sigma'_{A'}, E_{A'})$  making the following diagram commutative:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\iota_\Sigma(A)} & \Sigma_A \\ \varphi \downarrow & & \downarrow \iota_\varphi(h) \\ \Sigma' & \xrightarrow{\iota_{\Sigma'}(A')} & \Sigma'_{A'} \end{array}$$

- (3)  $i$  is natural, i.e. for each signature morphism  $\varphi : \Sigma \rightarrow \Sigma'$ , each  $A \in |\text{Mod}(\Sigma)|$ ,  $A' \in |\text{Mod}(\Sigma')|$  and  $h : A \rightarrow A' \upharpoonright_\varphi$  in  $\text{Mod}(\Sigma)$ , the following diagram is commutative:

$$\begin{array}{ccc} \text{Mod}(\Sigma_A, E_A) & \xrightarrow{i_{\Sigma, A}} & A/\text{Mod}(\Sigma) \\ -1_{\iota_\varphi(h)} \uparrow & & \uparrow h/\text{Mod}(\varphi) \\ \text{Mod}(\Sigma'_{A'}, E_{A'}) & \xrightarrow{i_{\Sigma', A'}} & A'/\text{Mod}(\Sigma') \end{array}$$

Here are some notational conventions that we hope will make the reader’s life easier. Let  $\varphi : \Sigma \rightarrow \Sigma'$  be a signature morphism,  $A' \in |\text{Mod}(\Sigma')|$  and  $h : A \rightarrow B$  in  $\text{Mod}(\Sigma)$ . We write  $\iota_\Sigma(h)$  instead of  $\iota_{1_\Sigma}(h)$  and  $\iota_\varphi(A' \upharpoonright_\varphi)$  instead of  $\iota_\varphi(1_{(A' \upharpoonright_\varphi)})$ . Let  $A$  be a fixed object in  $\text{Mod}(\Sigma)$  and let  $B, C \in |\text{Mod}(\Sigma)|$  and  $f : A \rightarrow B$ ,  $g : A \rightarrow C$ ,  $u : B \rightarrow C$  morphisms in  $\text{Mod}(\Sigma)$  such that  $f; u = g$ . Then  $(f, B)$  and  $(g, C)$  are objects in  $A/\text{Mod}(\Sigma)$  and  $u$  is also a morphism in  $A/\text{Mod}(\Sigma)$  between  $(f, B)$  and  $(g, C)$ . We establish the following notations:  $B_f = i_{\Sigma, A}^{-1}(f, B)$  (and, similarly,  $C_g = i_{\Sigma, A}^{-1}(g, C)$ ),  $u_{f, g} = i_{\Sigma, A}^{-1}((f, B) \xrightarrow{u} (g, C))$ . Thus, for instance, let  $f : A \rightarrow B$  be a  $\Sigma$ -model morphism. Then  $f_{1_A, f}$  is the image through  $i_{\Sigma, A}^{-1}$  of the morphism  $f : (1_A, A) \rightarrow (f, B)$  in  $A/\text{Mod}(\Sigma)$ , and has source  $A_{(1_A)}$  and target  $B_f$ . We shall write  $A_A$  instead of  $A_{(1_A)}$  and  $f_{A, f}$  instead of  $f_{1_A, f}$ .

In classical model theory,  $\Sigma_A$  is the signature  $\Sigma$  enriched with all the elements of  $A$  as constants,  $\iota_\Sigma(A) : \Sigma \rightarrow \Sigma_A$  is the inclusion of signatures, and  $E_A$  is a set of parameterized sentences which hold in  $A$ , depending on the considered type of morphism between models. If arbitrary model homomorphisms are allowed as morphisms, like in FOPL, one gets the ‘positive diagram’; if just model embeddings are considered, one gets the ‘diagram’; and only if just elementary embeddings are allowed, one gets what is classically called ‘elementary diagram’ (see [4] for the pointed standard terminology in classical model theory). Thus, the preceding institutional definition of diagrams particularises to elementary diagrams for classical first-order logic only if a notion of elementary morphism is assumed as *previously* defined. However, it is precisely the latter notion that we want to capture using diagrams. Therefore, we prefer to use the term ‘positive diagram’, in accordance to the particularization of the concept to the concrete institution FOPL, widely accepted as the institution of first-order logic. Thus we view the set of sentences  $E_A$  as the positive, rather than elementary, diagram of  $A$ , but of course keeping for it the same understanding as in [8]: that  $E_A$  axiomatizes the class of  $\Sigma$ -morphisms with source  $A$ . And we use the term elementary diagram of  $A$  for the set  $\{A_A\}^*$ , of all sentences satisfied by the self-parameterized extension  $A_A$  of  $A$ .

In [8], there are presented positive diagrams for FOPL, RWL, PA, and HL. Most institutions that were built starting from ‘working’ logical systems tend to have elementary diagrams. We next recall the system of positive diagrams for FOPL. Let  $\Sigma = (S, F, P)$  be a FOPL-signature and  $A \in |\text{Mod}(\Sigma)|$ . Define  $\Sigma_A = (S, F_A, P)$ , where  $F_A$  extends  $F$  by adding, for each  $s \in S$ , all elements of  $A_s$  as constants of sort  $s$ . Further, we define:

- (1)  $A_A \in |\text{Mod}(\Sigma_A)|$ , as the  $\Sigma_A$ -expansion of  $A$  which interprets each constant  $a \in A$  by  $a$ ;
- (2)  $E_A$ , as the set of all atoms in  $\text{Sen}(\Sigma_A)$  satisfied by  $A_A$ ;
- (3)  $\iota_\Sigma(A)$ , as the signature inclusion of  $\Sigma$  into  $\Sigma_A$ ;
- (4) The functor  $i_{\Sigma, A} : \text{Mod}(\Sigma_A, E_A) \rightarrow A/\text{Mod}(\Sigma)$ , as:

- $i_{\Sigma, A}(B') = (A \xrightarrow{h} B, B)$ , where  $B = B' \upharpoonright_{\iota_\Sigma(A)}$  and, for each  $s \in S$  and  $a \in A_s$ ,  $h_s(a) = B'_a$ .
- $i_{\Sigma, A}(f) = f$ .

Let  $\varphi : \Sigma = (S, F, P) \rightarrow \Sigma' = (S', F', P')$  be a signature morphism,  $A \in |\text{Mod}(\Sigma)|$ ,  $B \in |\text{Mod}(\Sigma')|$  and  $h : A \rightarrow B \upharpoonright_\varphi$  in  $\text{Mod}(\Sigma)$ . Then the natural presentation morphism  $\iota_\varphi(h) : (\Sigma_A, E_A) \rightarrow (\Sigma'_B, E_B)$  from the definition of positive diagrams is the following: if  $e \in \text{Sen}(\Sigma_A)$ , then  $\iota_\varphi(h)(e)$  is obtained from  $e$  by symbolwise translation, mapping:

- Each  $\sigma \in F$  into  $\varphi^{\text{op}}(\sigma)$ ,
- Each  $R \in P$  into  $\varphi^{\text{rel}}(R)$ ,
- For all  $s \in S$ , each  $a \in A_s$  into  $h_s(a)$ ,
- For all  $s \in S$ , each variable  $x:s$  of sort  $s$  into a variable  $x:\varphi^{\text{sort}}(s)$  and
- Each other symbol  $u$  that appears in  $e$  (e.g. logical connectives and quantifiers) into  $u$ .

As a general rule easily seen to hold about positive diagrams, one has that:

- If an institution  $\mathcal{I}'$  extends an institution  $\mathcal{I}$  and has the same category  $\text{Sign}$  and functor  $\text{Mod}$  (thus only adds new sentences), then positive diagrams are inherited by  $\mathcal{I}'$  from  $\mathcal{I}$ ;
- If an institution  $\mathcal{I}'$  restricts an institution  $\mathcal{I}$ , has the same category  $\text{Sign}$  and functor  $\text{Mod}$  (thus only restricts the sets of sentences), but  $\mathcal{I}'$  still has all the sentences in the positive diagrams  $E_A$  of  $\mathcal{I}$ , then positive diagrams are inherited by  $\mathcal{I}'$  from  $\mathcal{I}$ .

Thus, the earlier described positive diagrams of FOPL are inherited by IFOPL, IFOLPL $\alpha$ , PosFOPL, EQL, EQLN, ML. The positive diagrams for the other mentioned institutions can be constructed with a similar pattern as those of FOPL; as remarked in [8], the sentences  $E_A$  are always the basic sentences satisfied by the model  $A$  expanded to  $\Sigma_A$  with constants in  $A$  pointing to themselves.

For what follows, we fix an institution with positive diagrams,  $\mathcal{I}$ .

DEFINITION 18

Let  $\Sigma$  be a signature.

- Given a  $\Sigma$ -model  $A$ , the elementary diagram of  $A$  is the set  $\{A_A\}^*$  (of  $\Sigma_A$ -sentences satisfied by  $A_A$ ).<sup>6</sup>
- A  $\Sigma$ -morphism  $A \xrightarrow{h} B$  is called elementary by diagrams (d-elementary) if one of the following two equivalent conditions holds:
  - $A_A^* \subseteq B_h^*$ ;
  - $\iota_\Sigma(h)(A_A^*) \subseteq B_B^*$ .

That the two conditions in point (2) of the preceding definition are equivalent follows from the satisfaction condition together with the fact that, by the naturality of  $i$ ,  $B_B \upharpoonright_{\iota_\Sigma(h)} = B_h$ .

Thus, we defined elementary morphisms by means of elementary diagrams. We can spell out this definition as follows:  $A \xrightarrow{h} B$  is d-elementary if the elementary diagram of  $A$  is embedded, via  $h$ , into the elementary diagram of  $B$ .

## 6.2 The relationship between $\mathcal{Q}$ -elementary and d-elementary

The notion of d-elementary morphism is more compact than that of  $\mathcal{Q}$ -elementary morphism, but the former needs a lot of further structure on top of the plain institutional structure. We next provide conditions under which the two concepts are equivalent. For all this section, we fix a broad subcategory  $\mathcal{Q}$  of *Sign* consisting of representable signature morphisms.<sup>7</sup>

PROPOSITION 19

The positive diagrams of  $\mathcal{I}$  are said to be  $\mathcal{Q}$ -normal if for each representable signature morphism  $\varphi : \Sigma \rightarrow \Sigma'$  (represented by  $M_\varphi$ ) there exists a signature morphism  $\chi : \Sigma' \rightarrow \Sigma_{M_\varphi}$  such that  $\varphi; \chi = \iota_\Sigma(M_\varphi)$  and  $Mod(\chi); I_\varphi = i_{\Sigma, M_\varphi}$ .<sup>8</sup>

In examples 1–14 of institutions, for the usual choice of  $\mathcal{Q}$ , i.e. to consist of all injective signature morphisms that only add constants, the signature morphisms  $\iota_\Sigma(A)$  of the positive diagrams are all in  $\mathcal{Q}$ . Moreover, in each case, the positive diagrams are also  $\mathcal{Q}$ -normal. Indeed, for example, in FOPL, given an injective signature morphism  $\varphi$  that only adds constants, which we can assume without loss of generality to be an inclusion  $\Sigma = (S, F, P) \rightarrow \Sigma' = (S, F \cup X, P)$ , represented by the  $\Sigma$ -model  $T_\Sigma(X)$ , the desired morphism  $\chi$  such that  $\varphi; \chi = \iota_\Sigma(T_\Sigma(X))$  is the signature inclusion  $(S, F \cup X, P) \rightarrow (S, F \cup T_\Sigma(X), P)$  given by the set inclusion  $X \rightarrow T_\Sigma(X)$ . In order to see that the corresponding condition on model

<sup>6</sup>Since  $A_A \models E_A$ , the positive diagram of  $A$  is included in the elementary diagram of  $A$ .

<sup>7</sup>Note that we require more than usual for the subcategory  $\mathcal{Q}$ , namely representability instead of quasi-representability.

<sup>8</sup>Here, we made the slight notational abuse of letting  $Mod(\chi)$  denote the restriction of  $Mod(\chi) : Mod(\Sigma_{M_\varphi}) \rightarrow Mod(\Sigma')$  to  $Mod(\Sigma_{M_\varphi}, E_{M_\varphi})$ .

categories holds, let  $N$  be a  $\Sigma_{T_\Sigma(X)}$ -model that satisfies  $E_{T_\Sigma(X)}$ . Then  $i_{\Sigma, T_\Sigma(X)}(N) = (h : T_\Sigma(X) \rightarrow N|_{\iota_\Sigma(T_\Sigma(X))}, N|_{\iota_\Sigma(T_\Sigma(X))})$ , where  $h(t) = N_t$  for all  $t \in T_\Sigma(X)$ . On the other hand,  $I_\varphi(N|_\chi) = (g : T_\Sigma(X) \rightarrow N|_{\varphi; \chi}, N|_{\varphi; \chi}) = (g : T_\Sigma(X) \rightarrow N|_{\iota_\Sigma(T_\Sigma(X))}, N|_{\iota_\Sigma(T_\Sigma(X))})$ , where  $g$  is the unique  $\Sigma$ -morphism extending the mapping  $v : X \rightarrow N$  defined by  $v(x) = N_x$  for all  $x \in X$ . Thus, by the freeness of  $T_\Sigma(X)$ ,  $g = h$ . Hence, the functors  $Mod(\chi); I_\varphi$  and  $i_{\Sigma, M_\varphi}$  coincide on objects. That they coincide on morphisms too follows at once from  $\varphi; \chi = \iota_\Sigma(T_\Sigma(X))$ . Normality of the positive diagrams for the other examples of institutions can be shown similarly to the case of FOPL.

PROPOSITION 20

If the positive diagrams are normal and have each signature morphism  $\iota_\Sigma(A)$  in  $\mathcal{Q}$ , then any model morphism is  $\mathcal{Q}$ -elementary iff it is d-elementary.

PROOF. Let  $\Sigma$  be a signature and  $A \xrightarrow{h} B$  a  $\Sigma$ -morphism.

Assume first that  $h$  is  $\mathcal{Q}$ -elementary. Then, since  $\iota_\Sigma(A)$  is in  $\mathcal{Q}$  and  $A_A \xrightarrow{h_A} B_h$  is a  $\iota_\Sigma(A)$ -expansion of  $h$ , we get  $\{A_A\}^* \subseteq \{B_h\}^*$ . Thus,  $h$  is d-elementary.

Conversely, assume that  $h$  is d-elementary. Let  $\varphi : \Sigma \rightarrow \Sigma'$  be a signature morphism in  $\mathcal{Q}$  and  $A' \xrightarrow{h'} B'$  a  $\varphi$ -expansion of  $h$ . Let  $(M_{\varphi A} \rightarrow A, A) = I_\varphi(A')$  and  $(M_{\varphi B} \rightarrow B, B) = I_\varphi(B')$ . By the naturality of  $i$  we have  $h_{A, h'}|_{\iota_\Sigma(A)} = h_{a, b}$ .<sup>9</sup> Since  $\{A_A\}^* \subseteq \{B_h\}^*$ , by the satisfaction condition, it follows that  $\{A_a\}^* \subseteq \{B_b\}^*$ . Now, by the normality of diagrams, there exists  $\chi : \Sigma' \rightarrow \Sigma_{M_\varphi}$  such that  $\varphi; \chi = \iota_\Sigma(M_\varphi)$  and  $Mod(\chi); I_\varphi = i_{\Sigma, M_\varphi}$ . Then  $Mod(\chi) = i_{\Sigma, M_\varphi}; I_\varphi^{-1}$ , thus  $h_{a, b}|_\chi = I_\varphi^{-1}(i_{\Sigma, M_\varphi}(h_{a, b})) = I_\varphi^{-1}(h) = h'$ . Hence,  $A_a|_\chi = A'$  and  $B_b|_\chi = B'$ . Finally, by the satisfaction condition, we get  $\{A'\}^* \subseteq \{B'\}^*$ .

COROLLARY 21

In all the examples 1–14 of institutions (with their mentioned diagrams), a model morphism is elementary iff it is d-elementary.

## 7 Concluding remarks

We outline the contributions of the present article:

- Introduced an abstract notion of elementary morphism, parameterized by a class of signature morphisms;
- Studied the connection between elementary morphisms and positive diagrams in an arbitrary institution, by giving an alternative diagrammatic definition of elementarity and
- Showed how the general results particularize to many concrete cases of logical systems, yielding different known results in a unitary fashion but also some new results; in particular, the less conventional cases of partial algebra, hidden logic and rewriting logic fall into our framework.

An open problem that we consider worthwhile is the institutional relationship between elementary morphisms and model embeddings. Given the fact that classically elementary morphisms are also embeddings, a result stating that, under certain assumptions on the expressive power of sentences, all elementary morphisms are embeddings (where ‘embeddings’

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<sup>9</sup>  $h_{A, h'} : A_A \rightarrow B_h$  is a morphism in  $\Sigma_A$  and  $h_{a, b} : A_a \leftarrow B_b$  is a morphism in  $\Sigma_{M_\varphi}$ —recall the notational conventions about elementary diagrams.

can be defined either strictly categorically, as subobjects, or by means of inclusion or factorization systems), would be very desirable.

A more in depth study of elementary morphisms in some particular cases might also prove to be interesting. Take for instance the equational framework. In EQL and EQLN, the elementary morphisms do not seem very amenable. It is not clear to us how they look like. Note that a surjective morphism is always elementary in EQL, and an elementary morphism has to be injective in EQLN. The case of HL is even more intricate, and the notion of ‘elementary behavioral morphism’, complementing that of bisimulation, is potentially fruitful in the algebraic study of systems and behaviour.

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