SCOALA NORMALA SUPERIOARA BUCURESTI

MASTER'S THESIS

Dispersion property for discrete Schrödinger equations on networks

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Rezumat

In aceasta lucrare este tratata analiza ecuatiei Schrödinger pe retele constituind arbori pentru care muchiile din ultima generatie sunt infinite:

$$
(0.1) \quad
\begin{cases}
i\mathbf{u}_t(t,v) + \Delta\mathbf{u}(t,v) = 0, & t \neq 0, v \in V, \deg(v) = 2, \\[2mm]
\Delta\mathbf{u}(t,v) = 0, & t \neq 0, v \in V, \deg(v) \geq 3, \\[2mm]
\mathbf{u}(0,v) = \mathbf{u}_0(v), & v \in V, \deg(v) = 2.
\end{cases}
$$

Acest model corespunde unui sistem de ecuatii Schrödinger liniare discrete pe fiecare muchie a arborelui, care sunt cuplate in varfurile retelei printr-o cuplare de tip *Kirchhoff*.

Aceasta problema a fost ridicata de profesorul si coordonatorul meu, Liviu Ignat. Noi am abordat acest proiect pe baza rezultatelor obtinute anterior de Liviu Ignat, Valeria Banica si Diana Stan.

Principalul rezultat obtinut (si pe care ne asteptam sa il publicam in curand) este urmatorul, din care rezulta estimari Strichartz:

Theorem 0.1. *Pentru orice $\mathbf{u}_0 \in l^2(\Gamma)$, sistemul are o unica solutie $\mathbf{u} \in C(\mathbb{R}, l^2(\Gamma))$. In plus, exista $C > 0$ astfel incat*

$$
(0.2) \quad \|\mathbf{u}(t)\|_{l^\infty(\Gamma)} \leq \frac{C}{(1+|t|)^{1/3}} \|\mathbf{u}_0\|_{l^1(\Gamma)} \qquad \forall\, \mathbf{u}_0 \in l^1(\Gamma).
$$

Lucrarea incepe prin analiza proprietatilor spectrului laplacianul pe graf, in particular spectrul este $\sigma(\Delta) = [-4, 0]$ si rezolvanta $R_\lambda = (\Delta - \lambda)^{-1}$ este

$$
\begin{cases}
u_n^\alpha = u_0^\alpha \cdot r_1^n + S_n^\alpha, & \text{pentru } \alpha \text{ muchie infinita} \\[2mm]
u_n^\beta = a^\beta \cdot r_1^n + b^\beta \cdot r_2^n + S_n^\beta, & \text{pentru } \beta \text{ muchie finita,}
\end{cases}
$$

pentru $R_\lambda f = u$ si

$$
S_j^\alpha = \frac{1}{r_1 - \frac{1}{r_1}} \sum_{k \in I_\alpha} \left[r_1^{|j-k|} - r_1^{j+k} \right] f_k^\alpha.
$$

Apoi, putem reprezenta solutia prin calcul functional, folosind teorema lui Stone si Teorema lui Cauchy - pentru a determina masurile spectrale, sub forma:

$$
u(t)_n = \left(e^{it\Delta}\varphi \right)_n = \frac{1}{2\pi i} \int_I e^{it\lambda} \left[R_{\lambda-} - R_{\lambda+} \right] \varphi_n \, d\lambda.
$$

unde $R_{\lambda-} - R_{\lambda+}$ are o forma explicita.

Pentru a demonstra inegalitatile dispersive trebuie sa demonstram inegalitati de forma:

$$\left| \int_I e^{it\lambda} a_-^\alpha r^n \, \mathrm{d}\lambda \right| \le \frac{C}{\sqrt[3]{1+|t|}} \|\varphi\|_{l^1}$$

Coeficientii $a_-^\alpha = x_i = \frac{D_n^i}{D_n}$ sunt solutiile unui sistem liniar obtinut din conditiile de cuplare si putem determina prin recurenta expresiile

$$D_n = (r_1 - 1)^n \cdot (r_1 + 1)^{n-1} \cdot h(r_1)$$

unde h este o functie rarionala care nu se anuleaza pe $\mathbb{T}$.

$$D_n^i = (r_1 - 1)^{n-1} \cdot (r_1 + 1)^{n-2} \cdot \left[\sum_{\alpha \in E} S_1^\alpha \cdot q_1^{\alpha,i}\left(r_1, \frac{1}{r_1}\right) + \sum_{\alpha - finite} S_{N_\alpha}^\alpha \cdot q_{N_\alpha}^{\alpha,i}\left(r_1, \frac{1}{r_1}\right) \right]$$

pentru polinoame $q_1^{\alpha,i}$ si $q_{N_\alpha}^{\alpha,i}$. De aici se obtine

$$x_i = \sum_{\alpha \in E} \frac{S_1^\alpha \cdot q_1^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1 - 1)(r_1 + 1)h(r_1)} + \sum_{\alpha - finite} \frac{S_{N_\alpha}^\alpha \cdot q_{N_\alpha}^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1 - 1)(r_1 + 1)h(r_1)}.$$

In final, folosind aceaste expresii, se demonstreaza inegalitatile cu lema lui Van der Corput, folosind o teorema a lui Wiener.

As vrea sa ii multumesc calduros coordonatorului meu, L. Ignat de la IMAR, pentru tot sprijinul din acesti ani. De asemenea, le multumesc tuturor profesorilor care au tinut cursuri la SNSB.

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Summary

This thesis treats the analysis of the Schrödinger equation on networks consisting of a tree with infinite edges in the last generation:

$$(0.3) \qquad \begin{cases} i\mathbf{u}_t(t,v) + \Delta\mathbf{u}(t,v) = 0, & t \neq 0, v \in V, \deg(v) = 2, \\ \Delta\mathbf{u}(t,v) = 0, & t \neq 0, v \in V, \deg(v) \geq 3, \\ \mathbf{u}(0,v) = \mathbf{u}_0(v), & v \in V, \deg(v) = 2. \end{cases}$$

This model corresponds to a system of discrete linear Schrödinger equations on each edge of the tree coupled at the vertives of the network by a *discrete Kirchhoff's-type coupling*. This problem was posed by my supervisor, Prof. Liviu Ignat, and we have been working on this project motivated by earlier results of Liviu Ignat, Valeria Banica and Diana Stan.

The main result of our work (which we expect to publish soon) is one, from which Strichartz estimates follow:

Theorem 0.2. *For every $\mathbf{u}_0 \in l^2(\Gamma)$, this system has a unique solution $\mathbf{u} \in C(\mathbb{R}, l^2(\Gamma))$. Additionally, there exists $C > 0$ for which*

$$(0.4) \qquad \|\mathbf{u}(t)\|_{l^\infty(\Gamma)} \leq \frac{C}{(1+|t|)^{1/3}} \|\mathbf{u}_0\|_{l^1(\Gamma)} \qquad \forall\, \mathbf{u}_0 \in l^1(\Gamma).$$

DISPERSION PROPERTY FOR DISCRETE SCHRÖDINGER EQUATIONS ON NETWORKS

1. INTRODUCTION

Let us consider the linear Schrödinger equation (LSE):

$$(1.1) \qquad \begin{cases} iu_t + u_{xx} = 0, \ x \in \mathbb{R}, \ t \neq 0, \\ u(0, x) = \varphi(x), \ x \in \mathbb{R}. \end{cases}$$

The linear equation (1.1) is solved by $u(t, x) = S(t)\varphi$, where $S(t) = e^{it\Delta}$ is the free Schrödinger operator. The linear semigroup has two important properties. First, the conservation of the L^2-norm:

$$(1.2) \qquad \|S(t)\varphi\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}$$

and a dispersive estimate of the form:

$$(1.3) \qquad |(S(t)\varphi)(x)| \leq \frac{1}{(4\pi|t|)^{1/2}} \|\varphi\|_{L^1(\mathbb{R})}, \ x \in \mathbb{R}, \ t \neq 0.$$

More general estimates known as Strichartz's estimates has been obtained. These estimates have been successfully applied to obtain well-posedness results for the nonlinear Schrödinger equation (see [3], [12] and the reference therein).

In the case of the Schrödinger equation on trees similar properties have been obtained in [5] and [2]. In [5] the case of regular tree has been analyzed. The main idea has been the connexion between the Schrödinger equations on regular trees and the 1-D laminar Schrödinger equation considered in [1]. The extension of these result to the case of a general tree has been considered in [2].

The main goal of this article to consider problems similar to those above but in the discrete case. Let us recall some previous results in the discrete framework.

Let us now consider the following system of difference equations

$$(1.4) \qquad \begin{cases} iu_t + \Delta_d u = 0, \ \ j \in \mathbb{Z}, t \neq 0, \\ u(0) = \varphi, \end{cases}$$

where Δ_d is the discrete laplacian defined by

$$(1.5) \qquad (\Delta_d u)(j) = u_{j+1} - 2u_j + u_{j-1}, \quad j \in \mathbb{Z}.$$

Concerning the long time behavior of the solutions of system (1.4) in [11] the authors have proved that a decay property similar to the one obtained for the continuous Schrödinger equation holds:

$$(1.6) \qquad \|u(t)\|_{l^\infty(\mathbb{Z})} \leq C(|t| + 1)^{-1/3} \|\varphi\|_{l^1(\mathbb{Z})}, \quad \forall \ t \neq 0.$$

The proof of (1.6) consists in writing the solution u of (1.4) as the convolution between a kernel K_t and the initial data φ and then estimate K_t by using Van der Corput's lemma. In [11] the authors apply these estimates on the linear semigroup to prove that in some cases the semilinear discrete Schrödinger equation decay like the free solution in the corresponding $l^p(\mathbb{Z})$-norms.

In a recent paper [9] the authors use some modifications of the stationary phase method to obtain improved $l^1 - l^p$ decay estimates for the linear discrete Schrödinger equation and better than in [11] results for the nonlinear case.

More general models have been considered in [6] where the authors couple two discrete Schrödinger equations posed on two infinite strips:

$$(1.7) \quad \begin{cases} iu_t(t,j) + b_1^{-2}(\Delta_d u)(t,j) = 0 & j \leq -1, \ t \neq 0, \\ iv_t(t,j) + b_2^{-2}(\Delta_d v)(t,j) = 0 & j \geq 1, \ t \neq 0, \\ u(t,0) = v(t,0), & t \neq 0, \\ b_1^{-2}(u(t,-1) - u(t,0)) = b_2^{-2}(v(t,0) - v(t,1)), & t \neq 0, \\ u(0,j) = \varphi(j), & j \leq -1, \\ v(0,j) = \varphi(j), & j \geq 1. \end{cases}$$

In the above system $u(t,0)$ and $v(t,0)$ have been artificially introduced to couple the two equations on positive and negative integers. The third condition in the above system requires continuity along the interface $j = 0$ and the fourth one can be interpreted as the continuity of the flux along the interface. We will make evidence in the paper the connection between this result and the models considered here.

In this paper we prove dispersion inequality similar to (1.6) for the linear discrete Schrödinger equation defined on a network formed by a tree $\Gamma = (V, E)$ (connected graph without closed paths), V being the set of vertices and E of edges, with all the edges having length an integer number greater than two and with the external edges infinite. Once we fixed the tree Γ as in Fig. 1 we discretize all its edges as in Fig. 2 obtaining a new graph $\Gamma^d = (V^d, E^d)$. Now we consider a discrete Laplace operator on each of the edges. The presentation of the discrete Laplace operator Δ_d will be given in the next section. Let us just say here that Δ_d acts as the usual discrete Laplacian (1.5) at each internal node of each edge, and at the vertices of the tree we consider a coupling condition.

We now discretize all the edges of the tree choosing the mesh size to be one. Since the lengths of the edges has been chosen to be integer numbers greater than two this is always possible and moreover there is at least one internal vertex on each edge of the considered tree. In this way we have vertices with degree at least three that come form the way that we have considered the tree and vertices with their degree of order two that come from the discretization. In Fig. 1 we have a such structure. The discrete Laplacian of a function $\mathbf{u}$ on such structure Γ^d is given as follows:
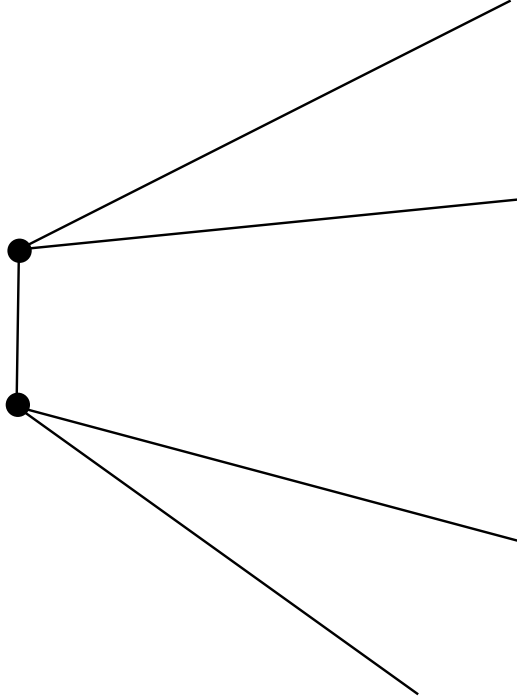
FIGURE 1. A tree consisting in two vertices, one finite edge and four infinite edges.

$$(1.8) \qquad (\Delta_d \mathbf{u})(v) = \sum_{w \in E_v} (\mathbf{u}(w) - \mathbf{u}(v)),$$

where the sum is taken over all the neighbors $w \in \Gamma^d$ of the vertex v.

We now can state the main results of this paper. Let us consider the equation

$$(1.9) \qquad \begin{cases} i\mathbf{u}_t(t, v) + \Delta \mathbf{u}(t, v) = 0, & t \neq 0, v \in V, \deg(v) = 2, \\ \Delta \mathbf{u}(t, v) = 0, & t \neq 0, v \in V, \deg(v) \geq 3, \\ \mathbf{u}(0, v) = \mathbf{u}_0(v), & v \in V, \deg(v) = 2. \end{cases}$$

This model corresponds to a system of discrete linear Schrödinger equations on each edge of the tree coupled at the vertices of the network by a *discrete Kirchhoff's-type coupling*. We will later explain this type of coupling. We point out that the function $\mathbf{u}$ should be considered only at the internal nodes $V^d \setminus V$.

Another model of interest is the case of the "combinatorial Laplacian". Here we have no coupling condition and the Schrödinger equation should be satisfies even at the vertices with degree greater than three.
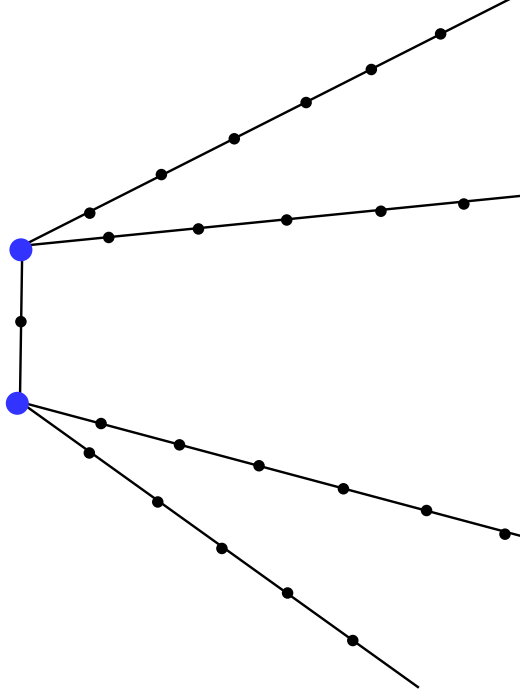
FIGURE 2. The discrete graph obtained from the previous one in Fig.1. At the blue points we have coupling conditions, at the rest classical discrete Laplacian

$$(1.10) \qquad \begin{cases} i\mathbf{u}_t(t,v) + \Delta\mathbf{u}(t,v) = 0, & t \neq 0, v \in V, \\ \mathbf{u}(0,v) = \mathbf{u}_0(v), & v \in V. \end{cases}$$

In some sense this model consider a dynamic coupling at the original vertices of the tree Γ.

This two type of models has been already observed in the simpler case of a system formed by two discrete Schrödinger equations on two infinite half-lines coupled at the origin [6].

The main result of this paper is the following one:

Theorem 1.1. *For every* $\mathbf{u}_0 \in l^2(\Gamma)$, *system* (1.9) *has a unique solution* $\mathbf{u} \in C(\mathbb{R}, l^2(\Gamma))$. *Additionally, there exists* $C > 0$ *for which*

$$(1.11) \qquad \|\mathbf{u}(t)\|_{l^\infty(\Gamma)} \leq \frac{C}{(1+|t|)^{1/3}}\|\mathbf{u}_0\|_{l^1(\Gamma)} \qquad \forall\, \mathbf{u}_0 \in l^1(\Gamma).$$

An immediate consequence (see [7] for an abstract result) are the following space-time estimates for the solutions of the linear problem.

Theorem 1.2. *For every* $\mathbf{u}_0 \in l^2(\Gamma)$, *the solution* $\mathbf{u}$ *of system* (1.9) *satisfies*

$$(1.12) \qquad \|u\|_{L^q(\mathbb{R}, l^r(\Gamma))} \leq C_{q,r}\|\mathbf{u}_0\|_{l^2(\Gamma)}$$

for any pair (q, r) satisfying

$$\frac{1}{q} \leq \frac{1}{3}(\frac{1}{2} - r).$$

2. Notations and Preliminaries

In this section we present some generalities about graphs and introduce the discrete Laplace operator on such structure. Let $\Gamma = (V, E)$ be a graph where V is a set of vertices and E the set of edges. For each $v \in V$ we denote $E_v = \{e \in E : v \in e\}$. We assume that Γ is a connected locally finite graph, i.e. the degree of each vertex v of Γ is finite: $d(v) = |E_v| < \infty$. The edges could be of finite length and then their ends are vertices of V or they have infinite length and then we assume that each infinite edge is a ray with a single vertex belonging to V (see [8] for more details on graphs with infinite edges).

We fix an orientation of Γ and for each oriented edge e, we denote by $I(e)$ the initial vertex and by $T(e)$ the terminal one. Of course in the case of infinite edges we have only initial vertices. We assume that the considered tree has all the vertices, except the origin, with degree at least three.

We identify every edge e of Γ with an interval I_e, where $I_e = [0, N_e + 1]$, N_e an integer, if the edge is finite and $I_e = [0, \infty)$ if the edge is infinite.

Let v be a vertex of V and e be an edge in E_v. We set for finite edges e

$$j(v, e) = \begin{cases} 0 & \text{if} \quad v = I(e), \\ l_e & \text{if} \quad v = T(e) \end{cases}$$

and

$$j(v, e) = 0, \text{ if } v = I(e)$$

for infinite edges.

We now discretize all the edges of the tree choosing the mesh size to be one. Since the lengths of the edges has been chosen to be integer numbers greater than two this is always possible and moreover there is at least one internal vertex on each edge of the considered tree. In this way we have vertices with degree at least three that come form the way that we have considered the tree and vertices with their degree of order two that come from the discretization.

We identify any function $\mathbf{u}$ on Γ with a collection $\{u^e\}_{e \in E}$ of functions u^e defined on the vertices of the edges e of Γ. Each u^e can be considered as a function on the set $I_e \cap \mathbb{Z}$. In fact, we use the same notation u^e for both the function on the edge e and the function on the interval I_e identified with e. For a function $\mathbf{u} : \Gamma \to \mathbb{C}$, $\mathbf{u} = \{u^e\}_{e \in E}$, we denote by $f(\mathbf{u}) : \Gamma \to \mathbb{C}$ the family $\{f(u^e)\}_{e \in E}$, where $f(u^e) : e \to \mathbb{C}$.

Each edge $\alpha \in E$ is parametrized by I_α, which is $\{1, 2, \ldots N_\alpha\}$ or $\{1, 2, \ldots\}$.

Each vertex $v \in V$ is the initial point for m_v edges: α_i^v ($1 \leq i \leq m_v$) and (except for the origin) the final point for a single edge, α_0^v.

We indentify the funtions defined on the graph by the families $(u^\alpha)_{\alpha \in E}$ such that $u^\alpha : I_\alpha \to \mathbb{C}$ with the convention that

$$u^e(j(v, e)) = u^{e'}(j(v, e')), \quad \forall e, e' \in E_v$$

If the v vertex is: the final point of the α edge and the origin of the $\alpha_1, \ldots, \alpha_m$ vertices, then we extend these functions by setting

$$(2.1) \qquad u^\alpha \left(N_\alpha + 1 \right) = u^{\alpha_i}(0) = \frac{u^\alpha(N_\alpha) + u^{\alpha_1}(1) + \ldots u^{\alpha_m}(1)}{m+1}, \qquad \forall\, 1 \le i \le m.$$

In the origin of the graph we extend the functions by the same formula, but without the α term, and divide by m instead.

We denote l^2 the space of functions $(u^\alpha)_{\alpha \in E}$ for which

$$\|u\|_2^2 = \sum_{\alpha \in E} \sum_{j \in I_\alpha} |u^\alpha(j)|^2 < \infty.$$

Then l^2 is a Hilbert space with the inner product

$$\langle u, v \rangle = \sum_{\alpha \in E} \sum_{j \in I_\alpha} u^\alpha(j) \overline{v^\alpha(j)}.$$

In the usual way, and by analogy with l^2, we also consider the l^p spaces for $1 \le p \le \infty$.

We will use both of the following notations: $u^\alpha(j) = u_j^\alpha$.

Of course, the functions u^α obey the continuity and coupling conditions defined in (2.1).

3. The spectrum and the resolvent of the discrete laplacian

In this section we determine the spectrum and the resolvent of the considered discrete laplacian on the graph Γ.

Theorem 3.1. *The spectrum of the discrete laplacian on Γ is $\sigma(\Delta) = [-4, 0]$.*

Proof. By the triangle inequality, we immediately get $\|\Delta u\|_2 \le 4\|u\|_2 \,\forall u \in l^2$, so $\|\Delta\|_{B(l^2)} \le 4$. We see that Δ is self-adjoint and $-\Delta \ge 0$ because $-\langle \Delta u, u \rangle$ is a sum of squares and these show that $\sigma(\Delta) \subset [-4, 0]$.

It is known that the discrete laplacian on $l^2(\mathbb{N}^*)$ has the spectrum $[-4, 0]$. This means that for every $\lambda \in [-4, 0]$ there is a sequence $(u_m)_{m \ge 0}$ such that for every m we have $u_m(\cdot) \in l^2(\mathbb{N}^*)$ and $\|u_m\|_2 = 1$, $\|(\Delta - \lambda)u_m\|_2 \to 0$.

We now use it to construct a sequence $v_m^\alpha(\cdot) \in l^2(\Gamma)$, for $\lambda \in [-4, 0]$, in the following way: we chose an infinite edge α_0 and define $v_m^{\alpha_0}(j) = u_m(j)$ for $j \ge 1$, and $v_m^\alpha(k) = 0$ otherwise, except for $v_m^{\alpha_0}(0) = \frac{v_m^{\alpha_0}(1)}{t+1}$ so that (1.1) is verified (t is the number of edges that start from the same vertex as α_0).

It is clear then that $\|v_m^\alpha\|_2 = 1$ and $\|(\Delta - \lambda)v_m^\alpha\|_2 \to 0$, which proves that $\sigma(\Delta) = [-4, 0]$. $\qquad\square$

We now consider the resolvent $R_\lambda = (\Delta - \lambda)^{-1} \in B(l^2)$ for $\lambda \in \mathbb{C} \setminus [-4, 0]$.

Theorem 3.2. *For any $\lambda \in \mathbb{C} \setminus [-4, 0]$ and $f \in l^2(\Gamma^d)$, the resolvent $R_\lambda f$ is given by*

$$(3.1) \qquad\qquad (R_\lambda f)_n^\alpha = a^\alpha \cdot r_1^n + b^\alpha \cdot r_2^n + S_n^\alpha, \qquad \alpha \in E, n \in I_\alpha$$

where $|r_1| < 1 < |r_2|$ are the roots of the second order equation

$$r^2 - (2 + \lambda)r + 1 = 0$$

and

$$(3.2) \qquad S_j^\alpha = \frac{1}{r_1 - r_1^{-1}} \sum_{k \in I_\alpha} \left[r_1^{|j-k|} - r_1^{j+k} \right] f_k^\alpha.$$

Remark 1. *We emphasize that on infinite edges $a^\alpha = u_0^\alpha$ and $b^\alpha = 0$.*

Proof. Denoting $u = R_\lambda f$ we have

$$u_{j+1}^\alpha - (2 + \lambda) u_j^\alpha + u_{j-1}^\alpha = f_j^\alpha, \quad j \in I_\alpha.$$

This is a second order recurrence relation for which S_j^α is a particular solution. The general solution is $u_j^\alpha = T_j^\alpha + S_j^\alpha$, where T^α is the general solution of the homogenous equation

$$(3.3) \qquad v_{j+1}^\alpha - (2 + \lambda) v_j^\alpha + v_{j-1}^\alpha = 0.$$

The set of solutions of (3.3) forms a two-dimensional vector space, so for any α,

$$T_j^\alpha = a^\alpha r_1^j + b^\alpha r_2^j$$

where r_1, r_2 are the solutions of $r^2 - (2 + \lambda)r + 1 = 0$ with $|r_1| < 1$.
    On the infinite edges we must have $T^\alpha \in l^2(I_\alpha)$, so in that case $b^\alpha = 0$. $\qquad\square$

In the following proposition we prove a limiting absorption principle. In particular, it shows that $R_{x-} - R_{x+}$ is well-defined as an operator in $B(l^1, l^\infty)$.

Proposition 3.1. *Let $x \in [-4, 0]$ and $\varepsilon > 0$. For any $f \in l^2$ and any $\alpha \in E$ we have*
(3.4)
$$[R_{x-i\varepsilon} - R_{x+i\varepsilon}] f_n^\alpha = \frac{1}{2i} \operatorname{Im}[a^\alpha r_1^n + b^\alpha r_2^n] + \frac{1}{2i} \sum_{k \in I_\alpha} f_k^\alpha \operatorname{Im} \left[\frac{1}{r_1 - r_1^{-1}} \left(r_1^{|n-k|} - r_1^{n+k} \right) \right], \quad n \in I_\alpha$$

where $|r_1| < 1 < |r_2|$ are the roots of the equation $r^2 - (2 + \lambda)r + 1 = 0$ for $\lambda = x - i\varepsilon$.
    For any $\varphi \in l^1$ and $a, b \in \mathbb{R}$ we have

$$(3.5) \qquad \lim_{\varepsilon \to 0+} \int_a^b [R_{x-i\varepsilon} - R_{x+i\varepsilon}] \varphi_n^\alpha \, dx = \int_a^b [R_{x-} - R_{x+}] \varphi_n^\alpha \, dx, \quad n \in I_\alpha$$

where the last term in the RHS is given by

$$(3.6) \qquad 2i [R_{x-} - R_{x+}] \varphi_n^\alpha = \operatorname{Im}[a_-^\alpha r^n + b_-^\alpha \bar{r}^n] + \sum_{k \in I_\alpha} \varphi_k^\alpha \operatorname{Im} \left[\frac{1}{r - r^{-1}} \left(r^{|n-k|} - r^{n+k} \right) \right]$$

such that r is the solution with $\operatorname{Im} r \geq 0$ of the equation $r^2 - (2 + x)r + 1 = 0$
    (Remark: a_-^α, b_-^α means $\lim_{(x-i\varepsilon \to x-0)}$ of these functions.)

Proof. The formulas (3.4) and (3.5) are only computation and then we use the dominated convergence theorem. Checking the necessary conditions to apply it is done identically as Step 1 from the Lemma 4.4 of [6]. $\qquad\square$

4. The main result

We return to the equation

$$
(4.1) \qquad
\begin{cases}
i\partial_t u^\alpha(t,j) + \Delta u^\alpha(t,j) = 0, & t \neq 0 \\[2mm]
u^\alpha(0,j) = \varphi(j), & \alpha \in E, \quad j \in I_\alpha.
\end{cases}
$$

Theorem 4.1. *For any $\varphi \in l^2$, there exists a unique solution $u = (u^\alpha)_{\alpha \in E} \in C(\mathbb{R}, l^2)$ of the system (4.1) which satisfies $\|u(t)\|_{l^2} = \|\varphi\|_{l^2}$. Moreover, if $\varphi \in l^1$ then u can be represented as*

$$
(4.2) \qquad u(t)_n = \left(e^{it\Delta}\varphi\right)_n = \frac{1}{2\pi i} \int_I e^{it\lambda} \left[R_{\lambda-} - R_{\lambda+}\right] \varphi_n \, d\lambda.
$$

Proof. We now use the Hille-Yosida theorem, or it's particular case, Stone's theorem (theorems 13.37, 13.36 of [10]) to show that there exists a unique solution $u \in C(\mathbb{R}, l^2)$ of equation (6.1), given by

$$
(4.3) \qquad u(t) = e^{it\Delta}\varphi,
$$

where the operators $e^{it\Delta}$ are defined by power series or by functional calculus. They are unitary operators, so

$$
\|u(t)\|_{l^2} = \|\varphi\|_{l^2}.
$$

Before proving (4.2) let us recall that as a consequence of Cauchy's formula (see [4] Th. X.6.1), for any bounded and self-adjoint operator T, the resolution of the identity E is given by

$$
E(a,b) = \lim_{\delta \to 0+} \lim_{\varepsilon \to 0+} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} R_{(x-i\varepsilon)}T - R_{(x+i\varepsilon)}T \, dx
$$

where the limit is in the strong-operator topology sense. Let us define

$$
(4.4) \qquad \mu_{\varphi,n}(a,b) = E(a,b)\varphi_n = \lim_{\delta \to 0+} \lim_{\varepsilon \to 0+} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} \left[R_{x-i\varepsilon} - R_{x+i\varepsilon}\right] \varphi_n \, dx, .
$$

Using functional calculus (see [10], Th. 12.23 and Th. 12.24) the solution of equation (4.1) satisfies

$$
(4.5) \qquad u(t)_n = \left(e^{it\Delta}\varphi\right)_n = \int_{\sigma(\Delta)} e^{it\lambda} \, d\mu_{\varphi,n}, .
$$

From now on, the integral on $I = [-4, 0]$ means $\int_I = \lim_{\delta \to 0+} \int_{-4+\delta}^{-\delta}$. Using Proposition 3.1 we get

$$
\lim_{\varepsilon \to 0+} \int_a^b \left[R_{x-i\varepsilon} - R_{x+i\varepsilon}\right] \varphi_n \, dx = \int_a^b \left[R_{x-} - R_{x+}\right] \varphi_n \, dx.
$$

Thus, from (4.5) and (4.4) we get

$$
u(t)_n = \left(e^{it\Delta}\varphi\right)_n = \frac{1}{2\pi i} \int_I e^{it\lambda} \left[R_{\lambda-} - R_{\lambda+}\right] \varphi_n \, d\lambda,
$$

which finishes the proof. □

Before we prove the main result, we state the key lemma which will be proved in the following sections.

Lemma 4.1. *For any $\varphi \in l^1$ and any $t \in \mathbb{R}$, $n \in \mathbb{Z}$ the following inequalities are true:*

$$(4.6) \qquad \left| \int_I e^{it\lambda} a_-^\alpha r^n \, d\lambda \right| \leq \frac{C}{\sqrt[3]{1+|t|}} \|\varphi\|_{l^1}, \qquad \left| \int_I e^{it\lambda} b_-^\alpha \bar{r}^n \, d\lambda \right| \leq \frac{C}{\sqrt[3]{1+|t|}} \|\varphi\|_{l^1}$$

Proof of the main result:

We use (3.6) and (4.2) and observe that, in order to get the inequality (1.11), it suffices to check that for all n and k

$$(4.7) \qquad \left| \int_I e^{it\lambda} \operatorname{Im} a_-^\alpha r^n \, d\lambda \right| \leq \frac{C}{\sqrt[3]{1+|t|}} \|\varphi\|_{l^1}, \qquad \left| \int_I e^{it\lambda} \operatorname{Im} b_-^\alpha \bar{r}^n \, d\lambda \right| \leq \frac{C}{\sqrt[3]{1+|t|}} \|\varphi\|_{l^1}$$

$$(4.8) \qquad \left| \int_I e^{it\lambda} \operatorname{Im} \left[\frac{1}{r - \frac{1}{r}} \left(r^{|n-k|} - r^{n+k} \right) \right] d\lambda \right| \leq \frac{C}{\sqrt[3]{1+|t|}}.$$

We recall that r is defined in Proposition 3.1, and $u_{0,-}^\alpha$, b_-^α means $\lim_{(x-i\varepsilon \to x-0)}$ of u_0^α si b^α as functions of λ.

Inequality (4.8) was proved in [6], (4.13) , in a more general case by using Van der Corput's lemma.

It is sufficient to prove (4.7) without the imaginary part Im because, by chosing $-t$ instead of t and conjugating both, we get the inequalities with additional Im. So, by using the lemma stated before, the result follows. $\square$

Let's note that what we have obtained thus far did not use the structure of the graph. To prove (4.6) we must use the structure of the graph to estimate u_0^α, a^α and b^α using the continuity and coupling conditions (2.1).

5. A PARTICULAR CASE

In this subsection, in order to better understand the difficulties of determining the coefficients in Lemma 4.1, we first consider the particular case of a tree as in Fig. 3.

In the considered case the set or vertices is given by $V = \{v_0, \ v_1\}$. From v_0 starts a finite edge $\alpha = 1$ on which we have the function $u^\alpha = u^1 : I_1 = \{1, 2, \ldots N\} \to \mathbb{C}$ and two infinite edges u^2 and u^3. From v_1, which is the endpoint of $\alpha = 1$, start two infinite edges: $\alpha \in \{11, 12\}$.

We are in the case where $\lambda \in I$, but the sum (3.2) is also defined there for $r_1 = r$.

Looking at (3.1), we see that we have to determine: $u_0^{11}, u_0^{12}, u_0^2, u_0^3, a = a^1$ and $b = b^1$. Writing the coupling conditions at vertices v_0 and v_1 it follows that

$$(5.1) \qquad \begin{cases} u_0^1 = a + b = u_0^2 = u_0^3 = \frac{u_1^1 + u_1^2 + u_1^3}{3}, \\[2mm] u_{N+1}^1 = u_0^{11} = u_0^{12} = \frac{u_N^1 + u_1^{11} + u_1^{12}}{3}. \end{cases}$$
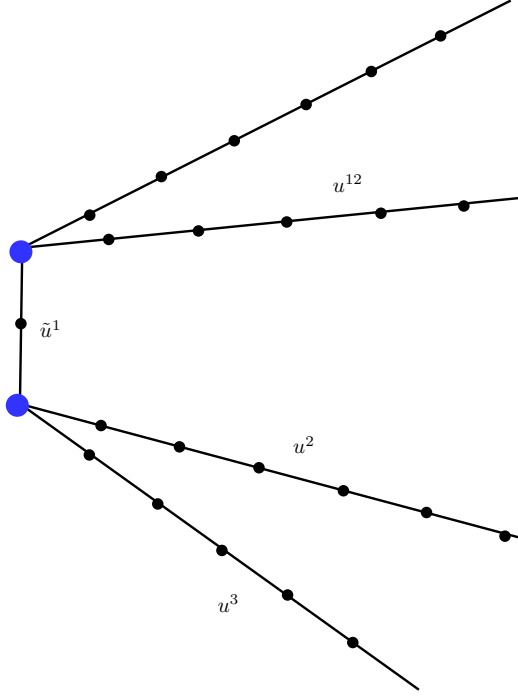
FIGURE 3. A particular case. At the blue points we have coupling conditions, at the rest classical discrete Laplacian

Using the representation formula given by Theorem 3.2 we obtain that

$$
(5.2) \quad
\begin{cases}
u_0^2 - u_0^3 = 0, \\[4pt]
u_0^2 - a - b = 0, \\[4pt]
(r-1)u_0^2 + (r-1)u_0^3 + (r-1)a + (r_2-1)b = -\left(S_1^1 + S_1^2 + S_1^3\right) = K_3, \\[4pt]
r^{N+1}a + r_2^{N+1}b - u_0^{11} = 0, \\[4pt]
u_0^{11} - u_0^{12} = 0, \\[4pt]
r^N a + r_2^N b + \left(r - \tfrac{3}{2}\right)u_0^{11} + \left(r - \tfrac{3}{2}\right)u_0^{12} = -\left(S_N^1 + S_1^{11} + S_1^{12}\right) = K_6.
\end{cases}
$$

In matrix formulation we have $Mx = K$ where $x^T = [x_i] = [u_0^2, u_0^3, a, b, u_0^{11}, u_0^{12}]$, $K^T = [0, 0, K_3, 0, 0, K_6]$ and

$$M = \begin{bmatrix} 1 & -1 & 0 & | & 0 & 0 & 0 \\ 1 & 0 & -1 & | & -1 & 0 & 0 \\ r_1 - 1 & r_1 - 1 & r_1 - 1 & | & r_2 - 1 & 0 & 0 \\ --- & --- & --- & | & & & \\ 0 & 0 & r_1^{N+1} & r_2^{N+1} & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & r_1^N & r_2^N & r_1 - \frac{3}{2} & r_1 - \frac{3}{2} \end{bmatrix}$$

The determinant marked in the corner comes from the case where there are only three infinite edges and it equals $3(r_1 - 1)$. Each x_i is given by $x_i = D_i / \det M$ where D_i is the determinant of the matrix obtained from M by inserting column K in the place of column i of M.

Lemma 5.1. *For any $i = 1, \ldots, 6$ and every $\alpha \in E$ there is a polynomial $q_1^{\alpha,i}(x, y)$ and another polynomial $q_{N_\alpha}^{\alpha,i}(x, y)$ when α is finite such that*

$$(5.3) \qquad x_i = \sum_{\alpha \in E} \frac{S_1^\alpha \cdot q_1^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1 - 1)(r_1 + 1)h(r_1)} + \sum_{\substack{\alpha - finite}} \frac{S_{N_\alpha}^\alpha \cdot q_{N_\alpha}^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1 - 1)(r_1 + 1)h(r_1)},$$

where h is a rational function that does not vanishes on the unit circle.

Proof. Using that $x_i = D_i / \det M$ we have to give some representation for D_i and to compute $\det M$. Observe that

$$\det M = (r_1 - 1)^2 \cdot r_2^{N+1} \left[9 - r_1^{2N}(2r_1 - 1)^2 \right].$$

Denoting $\varphi(x) = 9 - x^{2N}(2x - 1)^2$, it satisfies $\varphi(-1) = 0$ and $\varphi'(-1) \neq 0$. Thus, there is a polynomial function $g(x)$ which does not vanish on $\mathbb{T}$ such that $\varphi(x) = (x + 1)g(x)$. Denoting now $h(r_1) = r_2^{N+1}g(r_1)$, we obtain that

$$(5.4) \qquad \det M = (r_1 - 1)^2 \cdot (r_1 + 1)h(r_1).$$

Let us now analyze the form of the determinants D_i. We want to prove that for every $1 \leq i \leq 6$ and every $\alpha \in E$ there is a polynomial $q_1^{\alpha,i}(x, y)$ and another polynomial $q_{N_\alpha}^{\alpha,i}(x, y)$ when α is finite such that

$$(5.5) \qquad D_i(r_1) = (r_1 - 1) \left[\sum_{\alpha \in E} S_1^\alpha \cdot q_1^{\alpha,i}\left(r_1, \frac{1}{r_1}\right) + \sum_{\substack{\alpha - finite}} S_{N_\alpha}^\alpha \cdot q_{N_\alpha}^{\alpha,i}\left(r_1, \frac{1}{r_1}\right) \right].$$

Recall that $\lambda \in I$, $r_2 = 1/r_1$ since they are the solutions of the equation

$$x^2 - (2 + \lambda)x + 1 = 0.$$

By inserting the column K in the position of column i in M and expanding the determinant we obtain that D_i is a polynomial in the variables (r_1, r_2). Setting $r_2 = 1/r_1$, there exists

$p \in \mathbb{N}$ such that

$$D_i(r_1) = \frac{P(r_1)}{r_1^p}.$$

We prove that, when $r_1 = 1$, $D_i(r_1) = 0$. We distinguish two cases. If $i \notin \{3, 4\}$ then the third and fourth columns are equal. When $i \in \{3, 4\}$ we easily make a linear combination of columns that vanishes. For example, for $i = 3$ we have: $C_1 + C_2 + C_4 + C_5 + C_6 = 0$. This proves the existence of a polynomial $Q(x)$ such that

$$D_i(r_1) = (r_1 - 1)\frac{Q(r_1)}{r_1^p}.$$

By observing that $K_3 = -(S_1^1 + S_1^2 + S_1^3)$ and $K_6 = -(S_N^1 + S_1^{11} + S_1^{12})$ are the elements in K, and considering the expansion by that column, we get the desired formula for D_i.

Using now (5.4) and 5.5 we get the desired expression for x_i. $\qquad\square$

6. THE GENERAL CASE

Like in the particular case, we now have to determine the coefficients in Lemma 4.1.

Starting from a graph with n vertices Γ_n, we consider it as a graph with $n - 1$ vertices Γ_{n-1} to which we truncate an infinte edge U at the $N + 1$ point - where we add a new vertex v, from which m new edges start: $u^1, \ldots, u^m$. The edge U starts from vertex w, from which t edges start in total.

In the case of Γ_2 we have a truncated edge, $m = 2$ new edges, and $t = 3$ because 3 edges start from the origin.

For Γ_{n-1}, we denote by V_0 the finite edges, by W_0 the infinite edges except for U, and consider U separately.

The expression of the resolvent in (5.3) is:

$$(6.1) \qquad \begin{cases} u_n^\alpha = u_0^\alpha \cdot r_1^n + S_n^\alpha, & \alpha \in W_0 \\[2mm] u_n^\beta = a^\beta \cdot r_1^n + b^\beta \cdot r_2^n + S_n^\beta, & \beta \in V_0 \\[2mm] U_n = U_0 \cdot r_1^n + S_n^U. \end{cases}$$

For Γ_n, the former edge U, now restricted at $\{1, \ldots, N\}$, is denoted by V. The resolvent is:

$$(6.2) \qquad \begin{cases} u_n^\alpha = u_0^\alpha \cdot r_1^n + S_n^\alpha, & \alpha \in W_0 \\[2mm] u_n^\beta = a^\beta \cdot r_1^n + b^\beta \cdot r_2^n + S_n^\beta, & \beta \in V_0 \\[2mm] V_n = a \cdot r_1^n + b \cdot r_2^n + S_n^V \\[2mm] u_n^i = u_0^i \cdot r_1^n + S_n^i, & 1 \le i \le m \end{cases}$$

The continuity and coupling conditions (1.1) form a system, like in the previous case. For the moment, we are only interested in the matrix of the system's coefficients, which we denote by M_n for Γ_n.

The matrix M_{n-1} characterises the system corresponding to Γ_{n-1}, but written in such a way (by permuting the lines and the order of the variables) that U_0 is the last variable and the last two equations correspond to the vertex w in (1.1) (see the matrix below or the particular case).

To get the relation between M_{n-1} and M_n we see that, except for a - which formerly coresponeded to U_0, we have $m+1$ new variables: $[b, u_0^1, \ldots u_0^m]$.

By using the system obtained for Γ_{n-1} and the new coupling conditions, we get the system for Γ_n given by the matrix:

(6.3)

$$
M_n =
\begin{bmatrix}
\cdots & \cdots & a & & b & u_0^1 & u_0^2 & \cdots & u_0^{m-1} & u_0^m \\
\hline
 & & \cdots & & 0 & & & & & \\
 & M_{n-1} & \cdots & & 0 & & & & & \\
 & & \cdots & -1 & & -1 & 0 & 0 & & 0 \\
 & & \cdots & r_1 - \frac{t+1}{t} & & r_2 - \frac{t+1}{t} & 0 & 0 & & 0 \\
\hline
 & & r_1^{N+1} & & r_2^{N+1} & -1 & & & & \\
 & & & & & 1 & -1 & & & \\
 & & & & & & 1 & -1 & & \\
 & 0 & \cdots & & \cdots & \cdots & & & & \\
 & & \cdots & & \cdots & \cdots & & & & \\
 & & & & & & & & 1 & -1 \\
 & & r_1^N & & r_2^N & r_1 - \frac{m+1}{m} & \cdots & & \cdots & r_1 - \frac{m+1}{m}
\end{bmatrix}
$$

The matrix marked in the up-left corner is M_{n-1} (the first line just lists the variables). (An exception occurs when the vertex w is the origin and the term $\frac{t+1}{t}$ is replaced by 1.)

We denote by $\tilde{M}_n$ the matrix M_n in which we have inserted in the last position $r_2 - \frac{m+1}{m}$ instead of $r_1 - \frac{m+1}{m}$.

We denote these determinants by D_n and $\tilde{D}_n$.

Lemma 6.1. *We have*

(6.4)
$$
\frac{\tilde{D}_n}{D_n} = \frac{\frac{m-r_2}{m+1} - \frac{m-1-2r_2}{m+1} \cdot \left(\frac{r_1}{r_2}\right)^{N+1} \cdot \frac{\tilde{D}_{n-1}}{D_{n-1}}}{1 - \left(\frac{r_1}{r_2}\right)^{N+1} \cdot \frac{m-r_2}{m+1} \cdot \frac{\tilde{D}_{n-1}}{D_{n-1}}}.
$$

For $-1 \neq r_1 \in \mathbb{T}$ we have $\left|\frac{\tilde{D}_n}{D_n}\right| < 1$, and in $r_1 = -1$ we have $\frac{\tilde{D}_n}{D_n} = 1$.

Proof. By expanding the determinants on the first lines until the marked line, we get the following recurrence relation:

$$
D_n = D_{n-1}\left[r_2^{N+1}(m \cdot r_1 - m - 1) + r_2^N\right] - \tilde{D}_{n-1}\left[r_1^{N+1}(m \cdot r_1 - m - 1) + r_1^N\right].
$$

$$(6.5) \qquad D_n = D_{n-1} \cdot r_2^{N+1} \cdot (m+1) \cdot (r_1 - 1) \cdot \left[1 - \left(\frac{r_1}{r_2} \right)^{N+1} \frac{m - r_2}{m+1} \frac{\tilde{D}_{n-1}}{D_{n-1}} \right].$$

Analogously

$$\tilde{D}_n = D_{n-1} \cdot \left[r_2^{N+1} \left((m-1)(r_1 - \frac{m+1}{m}) + r_2 - \frac{m+1}{m} \right) + r_2^{N} \right] -$$

$$- \tilde{D}_{n-1} \cdot \left[r_1^{N+1} \left((m-1)(r_1 - \frac{m+1}{m}) + r_2 - \frac{m+1}{m} \right) + r_1^{N} \right].$$

$$(6.6) \;\; \tilde{D}_n = D_{n-1} \cdot r_2^{N+1} \cdot (m+1) \cdot (r_1 - 1) \cdot \left[\frac{m - r_2}{m+1} - \frac{m - 1 - 2r_2}{m+1} \cdot \left(\frac{r_1}{r_2} \right)^{N+1} \cdot \frac{\tilde{D}_{n-1}}{D_{n-1}} \right].$$

Observe that D_n and $\tilde{D}_n$ have the same factor before the large bracket, and this factor cancels when we divide them, obtaining (6.4).

We prove the other statement by induction on n. For $n = 1$ we have a single vertex and m infinite edges and

$$\frac{\tilde{D}_1}{D_1} = \frac{(m-1)(r_1 - 1) + r_2 - 1}{m(r_1 - 1)} = \frac{m - 1 - r_2}{m}$$

so the statement is true in this case.

Suppose that $-1 \neq r_1, r_2 \in \mathbb{T}$ and denote $z = \left(\frac{r_1}{r_2} \right)^{N+1} \frac{\tilde{D}_{n-1}}{D_{n-1}}$ so that we can assume that $|z| < 1$ for proving that

$$|m - r_2 - (m - 1 - 2r_2)z| < |(m+1) - (m - r_2)z|.$$

By squaring this inequality, simplifing and collecting the terms, it is equivalent to

$$(r_2 + \bar{r}_2 + 2) \left[(m-1)|z-1|^2 + 1 - |z|^2 \right] > 0$$

which is true and completes the induction step for this case.

Suppose now that $r_1 = r_2 = -1$. The ratios $\frac{\tilde{D}_{n-1}}{D_{n-1}}$ are rational functions and we write

$$\frac{\tilde{D}_n}{D_n} = \frac{f(x)}{g(x)} = \frac{f_1(x)}{g_1(x)}, \quad f(x) = (x+1)f_1(x) = mx - 1 - x^{2N+2}[(m-1)x - 2]\frac{\tilde{D}_{n-1}}{D_{n-1}}(x)$$

$$g(x) = (x+1)g_1(x) = (m+1)x - x^{2N+2}(mx - 1)\frac{\tilde{D}_{n-1}}{D_{n-1}}(x).$$

Then, for $x \in \mathbb{T}$,

$$f(x) - g(x) = (x+1)\left(x^{2N+2}\frac{\tilde{D}_{n-1}}{D_{n-1}}(x) - 1 \right) \;\Rightarrow\; f_1(x) - g_1(x) = x^{2N+2}\frac{\tilde{D}_{n-1}}{D_{n-1}}(x) - 1.$$

$$\frac{\tilde{D}_n}{D_n}(-1) = 1 \iff f_1(-1) - g_1(-1) = 0 \iff \frac{\tilde{D}_{n-1}}{D_{n-1}}(-1) = 1.$$

$\square$

Lemma 6.2. *We have*

(6.7) $$D_n = (r_1 - 1)^n \cdot (r_1 + 1)^{n-1} \cdot h(r_1)$$

for h a rational function which does not vanish on $\mathbb{T}$.

Proof. The $(r_1 - 1)^n$ factor is clear from (6.5) and, denoting the bracket from (6.5) by

$$\varphi(x) = 1 - x^{2N+1} \frac{mx - 1}{m + 1} \frac{\tilde{D}_{n-1}}{D_{n-1}}(x)$$

we see from Lemma 6.1 that $\varphi(-1) = 0$, so that $D_n = (r_1 - 1)^n \cdot (r_1 + 1)^{n-1} \cdot h(r_1)$, for a rational function h.

To prove that h does not vanish on $\mathbb{T}$, by Lemma 6.1, we only need to prove that $\varphi'(-1) \neq 0$.

$$\varphi'(-1) = \frac{m(2N + 2) + 2N + 1}{m + 1} - \frac{\tilde{D}_{n-1}}{D_{n-1}}'(-1).$$

We prove by induction that $\forall n \geq 1 : \frac{\tilde{D}_n}{D_n}'(-1) \in (0, 1)$, which clearly implies $\varphi'(-1) \neq 0$.

For $n = 1$, we have $\frac{\tilde{D}_1}{D_1}(x) = \frac{m-1}{m} - \frac{1}{mx}$ for which $\frac{\tilde{D}_1}{D_1}'(-1) \in (0, 1)$.

For $n > 1$ we use the functions f, g, f_1, g_1 from Lemma 6.1 for which

$$\frac{\tilde{D}_n}{D_n} = \frac{f_1(x)}{g_1(x)} \quad and \quad f_1(x) - g_1(x) = x^{2N+2} \frac{\tilde{D}_{n-1}}{D_{n-1}}(x) - 1.$$

$$\frac{\tilde{D}_n}{D_n}'(-1) = \frac{f_1' g_1 - f_1 g_1'}{g_1^2}(-1) = \frac{f_1' - g_1'}{g_1}(-1) = \frac{2N + 2 - \frac{\tilde{D}_{n-1}}{D_{n-1}}'(-1)}{(2N + 2)(m + 1) + m - (m + 1)\frac{\tilde{D}_{n-1}}{D_{n-1}}'(-1)}.$$

The last ratio is always in the interval $(0, 1)$ using the induction hypothesis. For the denominator we have used that $g_1(-1) = g'(-1)$. $\square$

We write the coefficients from Lemma 4.1 in the vector $x = [\ldots, a, b, u_0^1, u_0^2, \ldots, u_0^{m-1}, u_0^{m-1}] = [x_i]$. Replacing the column i from M_n by column K, like in the particular case, we get a determinant which we denote by D_n^i. Then, the elements in the vector x are written $x_i = \frac{D_n^i}{D_n}$.

Lemma 6.3. *We have*

(6.8) $$D_n^i = (r_1 - 1)^{n-1} \cdot (r_1 + 1)^{n-2} \cdot \left[\sum_{\alpha \in E} S_1^\alpha \cdot q_1^{\alpha,i}\left(r_1, \frac{1}{r_1}\right) + \sum_{\alpha - finite} S_{N_\alpha}^\alpha \cdot q_{N_\alpha}^{\alpha,i}\left(r_1, \frac{1}{r_1}\right) \right]$$

where $q_1^{\alpha,i}$ si $q_{N_\alpha}^{\alpha,i}$ are polynomials. And

$$(6.9) \qquad x_i = \sum_{\alpha \in E} \frac{S_1^\alpha \cdot q_1^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1 - 1)(r_1 + 1)h(r_1)} + \sum_{\alpha-finite} \frac{S_{N_\alpha}^\alpha \cdot q_{N_\alpha}^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1 - 1)(r_1 + 1)h(r_1)},$$

Proof. The column K contains sums of elements S_1^α and $S_{N_\alpha}^\alpha$, like in the particular case in Lemma 5.1 and the same reasoning applies here. So the formula (6.8) is true, provided that we prove that the following derivatives are zero:

$$D_n^i(1) = 0, \ D_n^i(1)' = 0, \ \ldots \ , D_n^i(1)^{(n-2)} = 0$$

$$D_n^i(-1) = 0, \ D_n^i(-1)' = 0, \ \ldots, D_n^i(-1)^{(n-3)} = 0$$

(We write the number of derivatives in brackets placed up).

We use the following known rule for differentiating a determinant with columns $c_1, c_2, \ldots, c_r$: if $D(x) = |c_1(x), \ldots, c_r(x)|$ then:

$$D'(x) = |c_1'(x), c_2(x), \ldots, c_r(x)| + \cdots + |c_1(x), c_2(x), \ldots, c_r'(x)|.$$

So, by iterating this rule and denoting $c_1, c_2, \ldots, c_r$ the columns of D_n^i, it is sufficient to prove:

$$\left| c_1^{(i_1)}, c_2^{(i_2)}, \ldots, c_r^{(i_r)} \right| (1) = 0 \qquad \forall \ i_1 + i_2 + \cdots + i_r \leq n - 2$$

$$\left| c_1^{(j_1)}, c_2^{(j_2)}, \ldots, c_r^{(j_r)} \right| (-1) = 0 \qquad \forall \ j_1 + j_2 + \cdots + j_r \leq n - 3.$$

Note that, in the same way as in the particular case Γ_2 we had (in M) the columns 3 and 4 equal in $r_1 = 1$ and $r_1 = -1$, now, for M_n , we have $n - 1$ pairs of columns and in each pair, the two columns are equal for $r_1 = r_2 = \pm 1$. This is true because, if M_{n-1} has $n - 2$ such pairs, M_n will have the additional pair (a, b).

Now, for $r_1 = 1$, in D_n^i we also insert the column K:

- If K does not enter in one of these pairs of columns, as we only differentiate at most $n - 2$ columns, for at least one of these pairs the two columns will be identical in $r_1 = 1$.
- If K enters in one of the columns from the $n - 1$ pairs, as we differentiate at most $n - 2$ of the others, considering the same reasoning as above, the only chance for the determinant to be non-zero is if we differentiate one column from each of the remaining pairs.

  In this instance: we make a linear combination of columns (different from K) to be zero by induction: assuming we have such a combination from $M_{n-1} \setminus K_{n-1}$, we use it such that (in addition to it) we only have to make a linear combination of the columns $a, b, u_0^1, u_0^2, \ldots, u_0^{m-1}, u_0^{m-1}$ below the marked line, which is trivial.

For $r_1 = r_2 = -1$ it is simpler because we have at most $n - 3$ derivatives, so we can always find a pair of columns which are equal.

Equality (6.9) follows from (6.8) and (6.7). □

Proof of Lemma 4.1. We recall that we have to prove the following estimate

$$\left| \int_I e^{it\lambda} x_i r_1^n \, d\lambda \right| \le \frac{C}{\sqrt[3]{1+|t|}} \|\varphi\|_{l^1} \qquad (n \in \mathbb{Z})$$

where x_i are now written as in Lemma 6.3:

$$x_i = \sum_{\alpha \in E} \frac{S_1^\alpha \cdot q_1^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1-1)(r_1+1)h(r_1)} + \sum_{\alpha-finite} \frac{S_{N_\alpha}^\alpha \cdot q_{N_\alpha}^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1-1)(r_1+1)h(r_1)}.$$

Using (3.2) we have that S_1^α and $S_{N_\alpha}^\alpha$ have the form

$$S_1^\alpha = -\sum_{k \in I_\alpha} r_1^k \varphi_k^\alpha, \quad S_{N_\alpha}^\alpha = -r_1^{N_\alpha} \sum_{k \in I_\alpha} \frac{r_1^k - r_1^{-k}}{r_1 - r_1^{-1}} \varphi_k^\alpha.$$

Hence

(6.10)
$$\left| \int_I e^{it\lambda} x_i r_1^n \, d\lambda \right| \le \sum_{\alpha \in E} \sum_{k \in I_\alpha} |\varphi_k^\alpha| \left| \int_I e^{it\lambda} \frac{r_1^k \cdot q_1^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1-1)(r_1+1)h(r_1)} r_1^n \, d\lambda \right|$$
$$+ \sum_{\alpha-finite} \sum_{k \in I_\alpha} |\varphi_k^\alpha| \left| \int_I e^{it\lambda} \frac{r_1^{N_\alpha} \frac{r_1^k - r_1^{-k}}{r_1 - r_1^{-1}} \cdot q_{N_\alpha}^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1-1)(r_1+1)h(r_1)} r_1^n \, d\lambda \right|.$$

It is then sufficient to prove the following estimates

(6.11)
$$\left| \int_I e^{it\lambda} \frac{q_1^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1-1)(r_1+1)h(r_1)} r_1^m \, d\lambda \right| \le \frac{C}{\sqrt[3]{1+|t|}}$$

and for finite edges α

(6.12)
$$\left| \int_I e^{it\lambda} \frac{\frac{r_1^k - r_1^{-k}}{r_1 - r_1^{-1}} \cdot q_{N_\alpha}^{\alpha,i}(r_1, \frac{1}{r_1})}{(r_1-1)(r_1+1)h(r_1)} r_1^m \, d\lambda \right| \le \frac{C}{\sqrt[3]{1+|t|}}.$$

We claim that it is sufficient to prove the following one:

(6.13)
$$\left| \int_I e^{it\lambda} \frac{r_1^m}{(r_1-1)(r_1+1)h(r_1)} \, d\lambda \right| \le \frac{C}{\sqrt[3]{1+|t|}}.$$

Indeed, $q_1^{\alpha,i}(x,y)$ are a finite number of polynomials, since $\alpha \in E$- a finite set and we get the same constant for all of them. Hence inequality (6.11) clearly follows from (6.13). In the case of (6.12), since $k \in I_\alpha$, α is a finite edge and the graph has a finite number of edges, it follows that $|I_\alpha| \le K$, where the constant K depends only on the graph, and so $k \le K$. By expanding

$$\frac{r_1^k - r_1^{-k}}{r_1 - r_1^{-1}} = r_1^{k-1} + r_1^{k-3} + \cdots + r_1^{-(k-1)}$$

and using the fact that $q_{N_\alpha}^{\alpha,i}$ are polynomials, inequality (6.12) also follows from (6.13).

Putting now the estimates (6.10), (6.11) and (6.12) together we obtain that

$$\left| \int_I e^{it\lambda} x_i r_1^n \, \mathrm{d}\lambda \right| \leq \frac{C}{\sqrt[3]{1+|t|}} \left[\sum_{\alpha \in E} \sum_{k \in I_\alpha} |\varphi_k^\alpha| + \sum_{\alpha - finite} \sum_{k \in I_\alpha} |\varphi_k^\alpha| \right] \leq \frac{C}{\sqrt[3]{1+|t|}} \|\varphi\|_{l^1}.,$$

which proves the desired inequality.

Let us now prove (6.13).

Since h is a rational function, it has an absolutely and uniformly convergent Laurent series on an annulus and this series becomes a fourier series if restricted to $\mathbb{T}$. The function h does not vanish on the unit circle so, using Wiener's theorem (see [10]), $1/h$ has an absolutely and uniformly convergent Fourier series

$$\frac{1}{h(r_1)} = \sum_{p \in \mathbb{Z}} a_p r_1^p, \qquad \sum_{p \in \mathbb{Z}} |a_p| < A.$$

This reduces the proof of (6.13) to the following estimate

$$\left| \int_I e^{it\lambda} \frac{r_1^m}{(r_1 - 1)(r_1 + 1)} \, \mathrm{d}\lambda \right| \leq \frac{C}{\sqrt[3]{1+|t|}}. \qquad (m \in \mathbb{Z}).$$

We now write $r_1 = e^{i\theta}$ and hence $\lambda = 2(\cos\theta + 1)$ and $(r_1 - 1)(r_1 + 1) = i \sin\theta \exp(i\theta)$. By making a change of variable such that we integrate over $\theta \in [0, \pi]$, the last integral becomes

$$\left| \int_0^\pi e^{2it(\cos\theta - 1)} e^{im\theta} \, \mathrm{d}\theta \right| \leq \frac{C}{\sqrt[3]{1+|t|}} \qquad (m \in \mathbb{Z}).$$

This last estimate has been proved in [6] by using Van der Corput's Lemma.

The proof if now complete. $\qquad\qquad\square$

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