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Departamentul de Matematică

LUCRARE DE DIZERTAŢIE

Puncte de neramificare pentru rezolvante de nuclee

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Bucureşti, septembrie 2011

CONTENTS

Introduction

The excessive functions with respect to a right continuous Markov process are related to the generator of the process (or to its inverse, the potential kernel) in the same way as the classical superharmonic functions on an Euclidean open set are related to the Laplace operator on that set (or to the Newtonian kernel).

The aim of this work is to present systematically several properties of the excessive functions with respect to a transition function (a semigroup of sub-Markovian kernels), possible associated to a Markov process. We follow essentially the first Chapter from the monograph [BeBo 04], the monograph [Sh 88], and the article [St 89] of J. Steffens; cf. also [Be 11b] and [BeBoRö 06]. The main property we characterize is the stability to the pointwise infimum of the convex cone of all excessive functions. From the probabilistic point of view, this property is precisely the fact that all the points of the state space of the process are *nonbranch point*.

The final aim is to present an application to the construction of Right Markov processes in infinite dimensional situations.

The plan of the work is the following.

In the first section (Preliminaries) we present some useful basic results on the Gaussian semigroup in $\mathbb{R}^d$, transition functions, the associated resolvent of kernels, the infinitesimal generator, the definition of the Markov processes, and the Brownian motion as an example.

In Ssection 2 we give results on the excessive functions with respect to a sub-Markovian resolvent of kernels: Hunt's Approximation Theorem (Theorem 2.1), the C_0-resolvent of contractions on L^p-spaces induced by such a resolvent (Theorem 2.2). The main result is Theorem 2.5, stating the above mentioned characterization of the min-stability of the excessive functions. It turns out that this characterization of the property that all the points are nonbranch points is an esential step in the construction of the measure-valued branching Markov processes; cf. [Be 11a] for the case of continuous branching, using several potential theoretical tools. We expect that a similar procedure will be efficient in the case of the discrete branching processes, as treated in the Addendum of [BeOp11]; see also [INW 68], [Si68] and [Na 76].

The announced application is developed in Section 3. More precisely, we prove in Theorem 3.2 that the min-stability property of the excessive functions is preserved by subordination with a convolution semigroup, the so called Bochner subordination; we follow the notations and the approach from [BlHa 86]. We complete in this way results from the recent article [BeRö 11], where this property was supposed to be satisfied by the subordinate resolvent, in order to associate to it a cadlag Markov process; see the Example following Corollary 5.4 from [BeRö 11].

1 Preliminaries

The n-dimensional Brownian motion:
the Gaussian semigroup and the Newtonian potential kernels

For each $f \in p\mathcal{B}(\mathbb{R}^n)$ and $t > 0$, the Gaussian kernel P_t on $\mathbb{R}^n$ is defined by:

$$P_t f(x) = \frac{1}{(2\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{2t}} f(y) dy,$$

where $p\mathcal{B} = p\mathcal{B}(\mathbb{R}^n)$ denotes the set of all positive, real-valued Borelian functions on $\mathbb{R}^n$.

The family $(P_t)_{t\geq 0}$, $P_0 = Id$ is called the *Gaussian semigroup* on $\mathbb{R}^n$.

Let g_t, $g_t : \mathbb{R}^n \longrightarrow \mathbb{R}$, be the density of the Gaussian kernel P_t,

$$g_t(x) = \frac{1}{(2\pi t)^{n/2}} e^{\frac{-|x|^2}{2t}}.$$

Clearly, the kernel P_t is defined by a convolution,

$$P_t f(x) = g_t * f(x) = \int_{\mathbb{R}^n} g_t(x-y) f(y) dy$$

for all $f \in p\mathcal{B}$.

Proposition 1.1. (i) *For each $t > 0$ the kernel P_t is Markovian, i.e., $P_t 1 = 1$. In particular, for every $x \in \mathbb{R}^n$ the measure $f \longmapsto P_t f(x)$ is a probability on $\mathbb{R}^n$.*

(ii) *P_t is a linear operator on the space $b\mathcal{B}$ of all bounded Borel measurable functions on $\mathbb{R}^n$ and if $f \geq 0$ then $P_t f \geq 0$.*

(iii) *The family $(P_t)_{t\geq 0}$ is a semigroup of kernel on $\mathbb{R}^n$, i.e., $P_s \circ P_t = P_{t+s}$ for all $s, t \geq 0$.*

Proof. (i) If $n = 1$, then we have:

$$P_t 1(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{2t}} \, dy = \frac{\sqrt{2}t}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-z^2} \, dz = 1.$$

The case $n > 1$ follows by Fubini's Theorem:

$$P_t 1(x) = \frac{1}{(2\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{2t}} \, dy = \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x_i-y_i)^2}{2t}} \, dy_i \right] = 1.$$

(*ii*) Let $f \in b\mathcal{B}$, $|f| \leq M$. Then $|P_t f| \leq P_t(|f|) \leq P_t M = M \cdot P_t 1 = M$. Consequently $P_t f \in b\mathcal{B}$.

(*iii*) Since $P_t f = g_t * f$ and $P_t \circ P_s(f) = g_t * g_s * f$ for all $f \in b\mathcal{B}$, it follows that in order to prove the semigroup property of $(P_t)_{t\geq 0}$ we have to show that $g_t * g_s = g_{t+s}$. We check the above equality in the case $n = 1$:

$$g_s * g_t(x) = \frac{1}{\sqrt{2\pi t}} \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2s}} e^{-\frac{y^2}{2t}} \, dy$$

$$= \frac{1}{2\pi\sqrt{ts}} e^{-\frac{x^2}{2s}} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{s+t}{2st} y^2 + \frac{x}{s} y} \, dy$$

$$= \int_{-\infty}^{\infty} e^{-\frac{s+t}{2st}\left(y - \frac{t}{s+t}x\right)^2} \, dy = \frac{1}{\sqrt{2\pi(s+t)}} e^{-x^2/2(s+t)} = g_{s+t}(x).$$

$\square$

Let E be a metrizable Lusin topological space and $\mathcal{B}$ the Borel σ-algebra on E

Transition function. A family of kernels $(P_t)_{t\geq 0}$ on $(E, \mathcal{B})$ which are sub-Markovian (i.e., $P_t 1 \leq 1$ for all $t \geq 0$), such that $P_0 f = f$ and $P_s(P_t f) = P_{s+t} f$ for all $s, t \geq 0$ and $f \in p\mathcal{B}$ is called *transition function on E*.

We assume further that for all $f \in bp\mathcal{B}$ the real-valued function $(t, x) \longmapsto P_t f(x)$ is $\mathcal{B}\big([0, \infty)\big) \otimes \mathcal{B}$-measurable.

Resolvent of kernels. The *resolvent of kernels associated with the transition function* $(P_t)_{t\geq 0}$ is the family $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ defined by

$$U_\alpha f := \int_0^\infty e^{-\alpha t} P_t f \, \mathrm{d}t \, .$$

The following two properties hold for the resolvent of kernels associated with a transition function.

• The family $\mathcal{U} = (U_\alpha)_{\alpha>0}$ satisfies the *resolvent equation*, i.e., for all $f \in bp\mathcal{B}$ we have

$$U_\alpha = U_\beta + (\beta - \alpha) U_\alpha U_\beta \quad \text{for all } \alpha, \beta > 0.$$

Note that in particular we have: $U_\alpha U_\beta = U_\beta U_\alpha$ for all $\alpha, \beta > 0$.

• The resolvent family $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is *sub-Markovian*, i.e., $\alpha U_\alpha 1 \leq 1$ for all $\alpha > 0$. Indeed, we have: $U_\alpha 1 = \int_0^\infty e^{-\alpha t} P_t 1 dt \leq \int_0^\infty e^{-\alpha t} dt = \frac{1}{\alpha}$.

Right continuous Markov process. A system $X = (\Omega, \mathcal{G}, \mathcal{G}_t, X_t, \theta_t, P^x)$ is called *right continuous Markov process with state space E, with transition function* $(P_t)_{t\geq 0}$ provided that the following conditions are satisfied:

a) $(\Omega, \mathcal{G})$ is a measurable space, $(\mathcal{G}_t)_{t\geq 0}$ is a family of sub σ-algebras of $\mathcal{G}$ such that $\mathcal{G}_s \subseteq \mathcal{G}_t$ if $s < t$; for all $t \geq 0$

b) $X_t : \Omega \longrightarrow E_\Delta$ is a $\mathcal{G}_t/\mathcal{B}_\Delta$-measurable map such that $X_t(\omega) = \Delta$ for all $t > t_0$ if $X_{t_0}(\omega) = \Delta$, where Δ is a *cemetery state* adjoined to E as an isolated point of $E_\Delta := E \cup \{\Delta\}$ and $\mathcal{B}_\Delta$ is the Borel σ-algebra on E_Δ.

c) $\zeta(\omega) := \inf\{t \mid X_t(\omega) = \Delta\}$ (the *lifetime* of X)

d) For each $t \geq 0$, the map $\theta_t : \Omega \longrightarrow \Omega$ is such that $X_s \circ \theta_t = X_{s+t}$ for all $s > 0$

(*ii*) (Markov property). For all $x \in E_\Delta$, P^x is a probability measure on $(\Omega, \mathcal{G})$ such that $x \longmapsto P^x(F)$ is universally $\mathcal{B}$-measurable for all $F \in \mathcal{G}$, $E^x(f \circ X_0) = f(x)$, and

$$E^x(f \circ X_{s+t} \cdot G) = E^x(P_t^\Delta f \circ X_s \cdot G)$$

for all $x \in E_\Delta$, $s, t \geq 0$, $f \in p\mathcal{B}_\Delta$ and $G \in p\mathcal{G}_s$, where P_t^Δ is the Markovian kernel on $(E_\Delta, \mathcal{B}_\Delta)$ such that $P_t^\Delta 1 = 1$ and $P_t^\Delta|_E = P_t$.

Brownian motion. A (right) continuous Markov process $(B_t)_{t\geq 0}$ with state space $\mathbb{R}^n$ is called n-dimensional *Brownian motion* provided that its transition function is the Gaussian semigroup:

$$P^x(B_t \in A) = \frac{1}{(2\pi t)^{n/2}} \int_A e^{-\frac{|x-y|^2}{2t}} dy, \quad \text{for all } \; A \in \mathcal{B}(\mathbb{R}^n).$$

The generator. Let F be a Banach space and $(P_t)_{t\geq 0}$ be a semigroup of contractions on F. We define

$$D(L) := \left\{ u \in F : \text{ there exists } \lim_{t\searrow 0} \frac{P_t u - u}{t} \in F \right\}.$$

For $u \in D(L)$ we define:

$$Lu := \lim_{t\searrow 0} \frac{P_t u - u}{t},$$

The linear operator $(L, D(L))$ is called the *infinitesimal operator (or generator)* of the semigroup $(P_t)_{t\geq 0}$.

Example. The infinitesimal operator of the Gaussian semigroup $(P_t)_{t\geq 0}$ (regarded, e.g., as a C_0-semigroup of contractions on $F = L^2(\mathbb{R}^n, \lambda)$) is the Laplace operator, we write $P_t = e^{t\Delta}$. More precisely, if $u \in C_0^2(\mathbb{R})$, then we have in $L^2(\mathbb{R}^n, \lambda)$: $\lim_{t\searrow 0} \frac{P_t u - u}{t} = \Delta u$.

2 Sub-Markovian resolvent of kernels

Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a sub-Markovian resolvent of kernels on the Lusin measurable space $(E, \mathcal{B})$. We shall denote by U the initial kernel of $\mathcal{U} : U = \sup_{\alpha>0} U_\alpha$.

If $q > 0$, then the family $\mathcal{U}_q = (U_{q+\alpha})_{\alpha>0}$ is also a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$, having U_q as (bounded) initial kernel.

Excessive function. A function $v \in p\mathcal{B}$ is called *$\mathcal{U}$-supermedian* if $\alpha U_\alpha v \le v$ for all $\alpha > 0$.

A $\mathcal{U}$-supermedian function v is named *$\mathcal{U}$-excessive* if in addition $\sup_{\alpha>0} \alpha U_\alpha v = v$. We denote by $\mathcal{E}(\mathcal{U})$ (resp. $\mathcal{S}(\mathcal{U})$) the set of all $\mathcal{B}$-measurable $\mathcal{U}$-supermedian functions. It is easy to check that $\mathcal{S}(\mathcal{U})$ and $\mathcal{E}(\mathcal{U})$ are convex cones.

If $v \in \mathcal{S}(\mathcal{U})$ then the function $\widehat{v} := \sup_{\alpha>0} \alpha U_\alpha v$ is $\mathcal{U}$-excessive and the set $M = [v \ne \widehat{v}]$ is $\mathcal{U}$-neglijable, i.e., $U_\beta(1_M) = 0$ for some (and hence all) $\beta > 0$. In addition the following assertions hold:

(2.1) If $u \in \mathcal{S}(\mathcal{U})$ then: $u \in \mathcal{E}(\mathcal{U})$ if and only if $u = \widehat{u}$.

(2.2) If $(u_n)_n$ is a sequence of $\mathcal{U}$-supermedian functions which is pointwise increasing to u, then the function u is also $\mathcal{U}$-supermedian and the sequence $(\widehat{u_n})_n$ increases to $\widehat{u}$.

In particular,

(2.3) if $(u_n)_n$ is a sequence of $\mathcal{U}$-excessive functions which is pointwise increasing to u, then the function u is also a $\mathcal{U}$-excessive.

A first main results on the $\mathcal{U}$-excessive function is the following approximation result of G.A. Hunt.

Theorem 2.1. Hunt's Approximation Theorem. *Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a sub-Markovian resolvent of kernels on the measurable space $(E, \mathcal{B})$ and let us fix $q > 0$. Then for each $v \in \mathcal{E}(\mathcal{U}_q)$ there exists a sequence $(f_n)_n$ in $bp\mathcal{B}$ such that $U_q f_n$ is bounded for all n and the sequence $(U_q f_n)_n$ is pointwise increasing to v.*

Proof. Let $v_n := \widehat{\inf(v, nU_q 1)}$. Note first that if $x \in E$ is such that

$U_q 1(x) = 0$ and $v \in \mathcal{E}(\mathcal{U}_q)$, then $v(x) = 0$. Indeed, we have $v = \sup_n \inf(v, n)$ and so $U_{q+\alpha}(\inf(v, n))(x) \le U_q(n)(x) = 0$, $U_{q+\alpha}v(x) = \sup_n U_{q+\alpha}(\inf(v, n))(x) = 0$. Therefore $v(x) = \sup_\alpha \alpha U_{q+\alpha} v(x) = 0$.

By (2.3) we deduce that the sequence $(v_n)_n$ is increasing and $\sup_n v_n = \sup_n \widehat{\inf(v, nW_q 1)} = \widehat{v} = v$. Since $v \in \mathcal{E}(\mathcal{U}_q)$ it follows that $\sup_n nU_{q+n} v_n = v$.
If we set

$$f_n := n(v_n - nU_{q+n} v_n),$$

then $U_q f_n = nU_{q+n} v_n$. We conclude that the sequence $(U_q f_n)_n$ is increasing and $\sup_n U_q f_n = \sup_n nU_{q+n} v_n = v$. $\square$

Excessive measure. Recall that a σ-finite measure ξ on $(E, (B))$ is called $\mathcal{U}$-excessive provided that $\xi \circ \alpha U_\alpha \le \xi$ for all $\alpha > 0$. We denote by $\mathrm{Exc}(\mathcal{U})$ the set of all $\mathcal{U}$-excessive measures.

Let further m be a fixed $\mathcal{U}$-excessive measure.

Notation: We denote by $p\mathcal{B} \cap L^p(E, m)$ the set of all $\mathcal{B}$-measurable functions which belong to $L^p(E, m)$.

If $f \in L^p(E, m)$ and $f' \in p\mathcal{B} \cap L^p(E, m)$ is a m-version of f then clearly $U_\alpha f$ is the element of $L^p(E, m)$ having the function $U_\alpha f'$ as m-version. Usually we shall identify a function from $p\mathcal{B} \cap L^p(E, m)$ with its class in $L^p(\mu)$. For instance if $f \in p\mathcal{B} \cap L^p(E, m)$ then $U_\alpha f$ denotes in the same time a function from $p\mathcal{B} \cap L^p(E, m)$ and the element from $L^p(E, m)$ having $U_\alpha f$ as m-version.

Theorem 2.2. *Assume that the set $\mathcal{E}(\mathcal{U}_q)$ of all $\mathcal{B}$-measurable $\mathcal{U}_q$-excessive functions generates the σ-algebra $\mathcal{B}$. If $1 < p < \infty$ is fixed then the following assertions hold.*

i) If $f \in p\mathcal{B}$ and $f = 0$ m-a.e. then $U_\alpha f = 0$ m-a.e. for all $\alpha > 0$.

ii) If $\alpha > 0$, $1 < p < \infty$ and $f \in L^p(E, m)$ then $U_\alpha f \in L^p(E, m)$ and $||\alpha U_\alpha f||_p \le ||f||_p$.

iii) For every $f \in L^p(E, m)$ we have $\lim_{\alpha \to \infty} ||\alpha U_\alpha f - f||_p = 0$ Consequently, the family $(U_\alpha)_{\alpha > 0}$ becomes a C_0-resolvent of sub-Markovian contractions on $L^p(E, m)$.

Proof. By hypothesis we get $\int \alpha U_\alpha f dm \leq \int f dm$ for all $f \in p\mathcal{B}$ and consequently if $f = 0$ m-a.e. then $U_\alpha f = 0$ m-a.e. for all $\alpha > 0$. Also if $0 \leq f \leq 1$ m-a.e. then $\alpha U_\alpha f \leq \alpha U_\alpha 1 \leq 1$ m-a.e. and thus U_α becomes a continuous linear operator on $L^\infty(E,m)$ and $L^1(E,m)$ respectively, such that $||\alpha U_\alpha||_{L^\infty(E,m)} \leq 1$ and $||\alpha U_\alpha||_{L^1(E,m)} \leq 1$.

ii) If $\alpha > 0$ and $x \in E$ then $\alpha U_\alpha f(x) \leq (\alpha U_\alpha(f^p)(x))^{\frac{1}{p}} (\alpha U_\alpha 1(x))^{\frac{1}{p'}}$ $\leq (\alpha U_\alpha (f^p)(x))^{\frac{1}{p}}$, where p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$. So, if $f \in L^p(E,m)$ then $\int |\alpha U_\alpha f|^p dm \leq \int \alpha U_\alpha(|f|^p) dm \leq \int |f|^p dm$. We conclude that $U_\alpha f \in L^p(E,m)$ and $||\alpha U_\alpha||_p \leq 1$.

iii) Because $||\alpha U_\alpha f||_p \leq 1$ for all $\alpha > 0$, it follows that the set $\mathcal{A} := \{f \in L^p(E,m) / \lim_{\alpha\to\infty} ||\alpha U_\alpha f - f||_p = 0\}$ is a closed subspace of $L^p(E,m)$. If $q > 0$ and $v \in b\mathcal{E}(\mathcal{U}_q) \cap L^p(E,m)$ then $v \in \mathcal{A}$. Indeed, from $\alpha U_{q+\alpha} v \nearrow v$ we get $\lim_{\alpha\to\infty} ||v - \alpha U_\alpha v||_p = \lim_{\alpha\to\infty} ||v - \alpha U_{q+\alpha} v||_p = 0$. Let now $g \in L^{p'}(E,m)$ be such that $\langle g, f\rangle = 0$ for all $f \in \mathcal{A}$. Particularly, we have $\int g^- U_q f dm = \int g^+ U_q f dm$ for all $f \in L^p_+(E,m)$. Thus the last equality holds for all $f \in p\mathcal{B}$. By the mass uniqueness we get $g^+ \cdot dm = g^- \cdot dm$, i.e., $g = 0$, hence $\mathcal{A} = L^p(E,m)$ (Hahn-Banach Theorem). □

Nonbranch point. A point $x \in E$ is called *nonbranch point with respect to* $\mathcal{U}$ provided that

$(N1)$ $\quad \inf(u,v)(x) = \widehat{\inf(u,v)}(x)$ for all $u, v \in \mathcal{E}(\mathcal{U})$

and

$(N2)$ $\quad \widehat{1}(x) = 1$

We denote by $\mathcal{D}_\mathcal{U}$ the set of all nonbranch points with respect to $\mathcal{U}$.

(2.4) By Hunt's approximation theorem and using (2.2) and (2.3), one can easyly see that: a point $x \in E$ is a nonbranch point with respect to $\mathcal{U}$ if and only if $(N2)$ holds and $(N1)$ is verified for all bounded functions $u, v \in \mathcal{E}(\mathcal{U})$ of the form $u = Uf$ and $v = Ug$ with $f, g \in b\mathcal{B}$.

A $\mathcal{U}$-excessive measure of the form $\mu \circ U$ (where μ is a σ-finite measure) is called *potential*. We denote by $\mathrm{Pot}(\mathcal{U})$ the convex cone af all potential $\mathcal{U}$-excessive measuares.

Further let $L: \operatorname{Exc}(\mathcal{U}) \times \mathcal{E}(\mathcal{U}) \longrightarrow \bar{R}_+$ be the *energy functional* (associated with $\mathcal{U}$) defined by

$$L(\xi, v) := \sup\{\mu(v), \operatorname{Pot}(\mathcal{U}) \ni \mu \circ U \leq \xi\}$$

for all $\xi \in \operatorname{Exc}(\mathcal{U})$ and $v \in \mathcal{E}(\mathcal{U})$. The energy functional associated with $\mathcal{U}_q$ will be denoted by L_q.

For the rest of the section we assume that $q = 0$.

Let $\mathcal{P}$ denotes the transition function $(P_t)_{t \geq 0}$ and define

$$\mathcal{E}(\mathcal{P}) := \{v : E \longrightarrow \overline{\mathbb{R}}_+,\ P_t v \leq v \text{ for all } t > 0 \text{ and } \lim_{t \to 0} P_t v = v\}$$

Proposition 2.3. *We have $\mathcal{E}(\mathcal{U}) = \mathcal{E}(\mathcal{P})$.*

Proof. Let $u \in \mathcal{E}(\mathcal{P})$. From $P_t u \leq u$ for all $t > 0$ we obtain

$$U_\alpha u = \int_0^\infty e^{-\alpha t} P_t u\, dt \leq \int_0^\infty e^{-\alpha t} u\, dt = u \int_0^\infty e^{-\alpha t}\, dt = \frac{u}{\alpha}$$

and so, $u \in \mathcal{S}(\mathcal{U})$.

Because $u \in \mathcal{E}(\mathcal{P})$, the map $t \longmapsto P_t u$ is descreasing and there exists the pointwise limit

$$\lim_{t \searrow 0} P_t u = \sup_{t > 0} P_t u = \lim_{n \to \infty} P_{t_n} u = u,$$

where $(t_n)_n$ is a sequence of positive numbers decreasing to zero. We have

$$\alpha U_\alpha u = \alpha \int_0^\infty e^{-\alpha t} P_t u\, dt = \int_0^\infty e^{-s} P_{s/\alpha} u\, ds.$$

For each fixed $s > 0$ we have $P_{s/\alpha} u \longrightarrow u$ as $\alpha \to \infty$, therefore by dominated convergence we get

$$\lim_{\alpha \to \infty} \int_0^\infty e^{-s} P_{s/\alpha} u\, ds = \int_0^\infty e^{-s} u\, ds = u \int_0^\infty e^{-s} ds = u.$$

It follows that

$$\widehat{u} = \lim_{\alpha \to \infty} \alpha U_\alpha u = \lim_{\alpha \to \infty} \int_0^\infty e^{-s} P_{s/\alpha} u\, ds = u.$$

Hence $u \in \mathcal{E}(\mathcal{U})$ and we conclude that $\mathcal{E}(\mathcal{P}) \subset \mathcal{E}(\mathcal{U})$.

Let now $u \in \mathcal{E}(\mathcal{U})$. By Hunt's approximation Theorem there exists a sequence $(f_n)_n \in bp\mathcal{B}$ such that $Uf_n \nearrow u$. Consequently, to prove that the function v belongs to $\mathcal{E}(\mathcal{P})$, it is enough to show that $P_tUf \leq Uf$ for all $t > 0$ and that $\lim_{t \to 0} P_tUf = Uf$. We have

$$P_tUf = \int_0^\infty P_{t+s}f\,ds = \int_t^\infty P_sf\,ds \leq \int_0^\infty P_sf\,ds = Uf$$

and

$$\lim_{t \to 0} P_tUf = \lim_{t \to 0} \int_t^\infty P_sf\,ds = \int_0^\infty P_sf\,ds = Uf.$$

□

Proposition 2.4. *The following assertions hold for a resolvent of kernels* $\mathcal{U} = (U_\alpha)_{\alpha > 0}$ *on* $(E, \mathcal{B})$.

(i) *The following two conditions are equivalent.*

($i.a$) *All the points of* E *are nonbranch points with respect to* $\mathcal{U}$.

($i.b$) *The convex cone* $\mathcal{E}(\mathcal{U})$ *is min-stable and contains the positive constant functions, i.e., for all* $u, v \in \mathcal{E}(\mathcal{U})$ *we have* $\inf(u, v) \in \mathcal{E}(\mathcal{U})$ *and* $1 \in \mathcal{E}(\mathcal{U})$.

(ii) *If* $\mathcal{U}$ *is the resolvent of right Markov process with state space* E*, then all the points of* E *are nonbranch points with respect to* $\mathcal{U}$.

Proof. Since $\mathcal{U}$ is sub-Markovian, we clearly have that $1 \in \mathcal{S}(\mathcal{U})$. Let $u, v \in b\mathcal{E}(\mathcal{U})$ and set $w := \inf(u, v)$. Then clearly $w \in \mathcal{S}(\mathcal{U})$

(i) The equivalence between ($i.a$) and ($i.b$) is a direct consequence of (2.1).

(ii) Let X be the right Markov process having $\mathcal{U}$ as associated resolvent. Then every $\mathcal{U}$-excessive function is a.s. right continuous along the paths of X, i.e.,

(2.5) If $u \in \mathcal{E}(\mathcal{U})$ then the function $t \longmapsto u \circ X_t$ is a.s. right continuous on $[0, \infty)$.

We already noted that the constant function 1 is $\mathcal{U}$-supermedian. If $x \in E$ then $P_t1(x) = E^x([t < \zeta])$ and since $E^x(X_0 = x) = 1$ we deduce that a.s. $\zeta > 0$ and therefore $\lim_{t \searrow 0} P_t1(x) = E^x([0 < \zeta]) = 1$, hence $1 \in \mathcal{E}(\mathcal{U})$ since by Proposition 2.2 we have $\mathcal{E}(\mathcal{U}) = \mathcal{P}(\mathcal{U})$.

We also noted that if $u, v \in b\mathcal{E}(\mathcal{U})$ then the function $w = \inf(u, v)$ is $\mathcal{U}$-supermedian and by (2.4) w is also right continous along the paths of X. By dominated convergence and again since $E^x(X_0 = x) = 1$ we get

$$\lim_{t \searrow 0} P_t w(x) = \lim_{t \searrow 0} P^x(w \circ X_t) = P^x(w \circ X_0) = w(x).$$

It follows that $w \in \mathcal{P}(\mathcal{U})$, so, again by Proposition 2.2, it is $\mathcal{U}$-excessive, hence $(i.b)$ holds and therefore also $(i.a)$ is verified. We conclude that all the points of E are nonbranch points with respect to $\mathcal{U}$ and the proof is complete. □

We can state now the central result of this section.

Theorem 2.5. *Let $(U_\alpha)_{\alpha>0}$ be a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$, $q > 0$ be fixed, and assume that the σ-algebra generated generated by $\mathcal{E}(\mathcal{U}_q)$ is precisely $\mathcal{B}$. Then the following three assertions are equivalent.*

(i) All the points in E are nonbranch points with respect to $\mathcal{U}$.

(ii) The following two properties hold.
(UC) **Uniqueness of charges***: If μ, ν are two finite measures such that $\mu \circ U_q = \nu \circ U_q$ then $\mu = \nu$.*
(SSP) **Specific solidity of potentials***: If $\xi, eta \in Exc(\mathcal{U}_q)$ such that $\xi + \eta = \mu \circ U_q$, then there exists a measure ν on E such that $\xi = \nu \circ U_q$.*

(iii) The linear space $[b\mathcal{E}(\mathcal{U}_q)]$ spanned by $b\mathcal{E}(\mathcal{U}_q)$ is an unitary algebra.

Proof. We show first that

(2.6) If $v : E \longrightarrow \mathbb{R}_+$ and $\varphi : I \longrightarrow \mathbb{R}_+$is an increasing concave function, where I is an interval, $0 \in I$, such that $Im(v) \subset I$, and if $v \in \mathcal{S}(\mathcal{U}_q)$, then $\varphi \circ v \in \mathcal{S}(\mathcal{U}_q)$. Particularly, the vector space $[b\mathcal{S}(\mathcal{U}_q)]$ spanned by $\mathcal{S}(\mathcal{U}_q)$ is an algebra.

The first assertion follows by Jensen inequality, applied to the sub-probability $\mu_x := \alpha U_{q+\alpha}(x, dy)$ for all $x \in E$. Indeed, for all $x \in E$ we have

$$\alpha U_{q+\alpha}(\varphi \circ v)(x) = \int \varphi \circ v \, d\mu_x \leq \varphi(\mu_x(v)) = \varphi(\alpha U_{q+\alpha} v(x)) \leq \varphi(v(x)),$$

where the last inequality holds because φ is incresing and note that $\mu_x(v) \in I$ because $0 \leq \mu_x(v) \leq v(x)$.

To prove that $[b\mathcal{S}(\mathcal{U}_q)]$ is an algebra, it is sufficient to show that $v^2 \in [b\mathcal{S}(\mathcal{U}_q)]$, for every $v \in b\mathcal{S}(\mathcal{U}_q)$. We may assume that $v \leq 1$ and let $\varphi : [0, 1] \longrightarrow \mathbb{R}_+$ defined by $\varphi(x) = 2x - x^2$. Then φ is concave and increasing, hence $\varphi \circ v \in b\mathcal{S}(\mathcal{U}_a)$ and therefore $v^2 \in [b\mathcal{S}(\mathcal{U}_q)]$.

$(i) \Longrightarrow (iii)$. As before, to prove that $[b\mathcal{E}(\mathcal{U}_q)]$ is an algebra, it is sufficient to show that $v^2 \in [b\mathcal{E}(\mathcal{U}_q)]$ for every $v \in b\mathcal{E}(\mathcal{U}_q)$. We may assume that $v \leq 1$. By (2.6) it follows that v^2 belongs to $[b\mathcal{S}(\mathcal{U}_q)]$, $v^2 = 2v - w$ with $w := 2v - v^2 \in b\mathcal{S}(\mathcal{U}_q)$. It remains to show that $w \in \mathcal{E}(\mathcal{U}_q)$. But w is a finely continuous $\mathcal{U}_q$-supermedian function, hence it is $\mathcal{U}_q$-excessive.

$(iii) \Longrightarrow (i)$. Let $\mathcal{A}$ be the closure of $[b\mathcal{S}(\mathcal{U}_q)]$ in the supremum norm, it is a Banach algebra and therefore a lattice with respect to the pointwise order relation. Since $\lim_{\alpha\to\infty} \alpha U_{q+\alpha} v = v$, pointwise for all $v \in \mathcal{E}(\mathcal{U}_q)$ it follows that the same property holds for all $v \in \mathcal{A}$. Consequently, since $1 \in \mathcal{A}$, we have $\widehat{1} = 1$ and if $u_1, u_2 \in \mathcal{E}(\mathcal{U}_q)$ then the $\mathcal{U}_q$-supermedian function $v = \inf(u_1, u_2)$ belongs to $\mathcal{A}$ and therefore $\widehat{v} = v$, $D_{\mathcal{U}_q} = E$.

$(i) \Longrightarrow (ii)$. We show that if μ, ν are two measures on $(E, \mathcal{B})$ such that their potentials $\mu \circ U_q$ and $\nu \circ U_q$ are σ-finite and

$$\mu \circ U_q = \nu \circ U_q,$$

then $\mu = \nu$.

Indeed, the resolvent equation implies that if $\beta > 0$ then the measures $\mu \circ U_{q+\beta}$ and $\mu \circ U_q U_{q+\beta}$ are σ-finite, hence

$$\mu \circ U_{q+\beta} = \nu \circ U_{q+\beta} \text{ for all } \beta > 0. \tag{2.7}$$

Let further $g \in bp\mathcal{B}$, $g > 0$, be such that $\mu \circ U_q(g) = \nu \circ U_q(g) < \infty$ and set $h := U_q g$, so $0 < h \in L^1(E, \mu) \cap L^1(E, \nu)$. If $f \in [b\mathcal{E}(\mathcal{U}_q)]$, $0 \leq f \leq 1$, then $fh \in [b\mathcal{E}(\mathcal{U}_q)]$ (because by the already proved implication $(i) \Longrightarrow (iii)$ it is an algebra) and therefore $lim_n n U_{q+n}(fh) = fh$.

Since $nU_{q+n}(fh) \leq nU_{q+n}h \leq h \in L^1(E, \mu) \cap L^1(E, \nu)$, by (2.7) and the dominated convergence, we obtain that $\mu(fh) = \nu(fh)$ for all $f \in [b\mathcal{E}(\mathcal{U}_q)]$ (which is an algebra of bounded functions generating the σ-algebra $\mathcal{B}$). By the monotone class theorem we conclude that $\mu = \nu$. Hence the unqueness of charges property (UC) holds.

We prove now that the specific solidity of potentials property (SSP) also holds. Let $\xi, \eta mu \circ U_q \in Exc(\mathcal{U}_q)$such $\xi + \eta = \mu \circ U_q$. We may assume that the measure μ is finite. Consider the functional $\varphi_\xi : b\mathcal{E}(\mathcal{U}_q) \longrightarrow \mathbb{R}_+$ defined as

$$\varphi_\xi(v) := L_q(\xi, v) \text{ for all } v \in b\mathcal{E}(\mathcal{U}_q).$$

Note that $L_q(\xi, v) \leq L_q(\mu \circ U_q, v) = \mu(v) < \infty$. We may extand φ_ξ to a real valued linear functional on $[b\mathcal{E}(\mathcal{U}_q)]$ and we get

$$\varphi_\xi(f) + \varphi_\eta(f) = \mu(f) \text{ for all } f \in [b\mathcal{E}(\mathcal{U}_q)]. \tag{2.8}$$

Note that φ_ξ is positive, i.e.

$$\varphi_\xi(f) \geq 0 \text{ provided that } f \in [b\mathcal{E}(\mathcal{U}_q)] \text{ is positive.} \tag{2.9}$$

This follows because (by the properties of the energy functional L_q) φ_ξ is increasing as a functional on $\mathcal{E}(\mathcal{U}_q)$: if $u, v \in \mathcal{E}(\mathcal{U}_q)$ and $u \leq v$, then $\varphi_\xi(u) \leq \varphi_\eta(v)$. We claim taht if $(f_n)_n \subset [b\mathcal{E}(\mathcal{U}_q)]$ is decreasing poinwise to zero then the sequence $(\varphi_\xi(f_n))_n$ also decreases to zero.Note first that by monotene convergence we have $\lim_n \mu(f_n) = 0$. From (2.8) and (2.9) it follows that $0 \leq \varphi_\xi(f_n) \leq \mu(f_n)$ for all n and thus

$$0 \leq \lim_n \varphi_\xi(f_n) \leq \lim_n \mu(f_n) = 0.$$

We can apply now Daniell's theorem on the vector lattice $[b\mathcal{E}(\mathcal{U}_q)]$, for the functional φ_ξ. Hence there exists a positive measure ν on $\mathcal{B}$ such that

$$\varphi_\xi(f) = \nu(f) \text{ for all } f \in bp\mathcal{B}.$$

(Recall again that the σ-algebra generated by $[b\mathcal{E}(\mathcal{U}_q)]$ is precisely $\mathcal{B}$.) Taking $f = U_q g$ with $g \in bp\mathcal{B}$, we get $\xi(g) = L_q(\xi, U_q g) = \varphi_\xi(f) = \nu(U_q g)$ for all $g \in bp\mathcal{B}$, so $\xi = \nu \circ U_q$.

For the proof of $(ii) \Longrightarrow (i)$ see [St 89]. $\square$

3 Subordination by convolution semigroups

A family $(\mu_t)_{t>0}$ of measures on $\mathbb{R}_+^*$ is called a (vaguely continuous) *convolution semigroup* on $\mathbb{R}_+^*$ if the following conditions are satisfied:

(i) $\mu_t(\mathbb{R}_+^*) \leq 1$ for all $t > 0$,

(ii) $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t > 0$,

(iii) $\lim\limits_{t\to 0} \mu_t = \epsilon_0$ (vaguely).

Note that (i) and (iii) imply that $\lim_{t\to 0} \mu_t(f) = f(0)$ for every $f \in C_b(\mathbb{R}_+)$.

In the sequel we fix a transition function $\mathbb{P} = (P_t)_{t>0}$ on $(E, \mathcal{B})$ and a convolution semigroup $(\mu_t)_{t>0}$ on $\mathbb{R}_+^*$.

For each $t > 0$ we define the kernel kernel P_t^μ on $(E, \mathcal{B})$ by

$$P_t^\mu f := \int_0^\infty P_s f \mu_t(ds) \quad \text{for all } f \in bp\mathcal{B}.$$

Proposition 3.1. *The family $\mathbb{P}^\mu = (P_t^\mu)_{t\geq 0}$ is a sub-Markovian semigroup of kernels on $(E, \mathcal{B})$ and $\mathcal{E}(\mathbb{P}) \subset \mathcal{E}(\mathbb{P}^\mu)$. The semigroup $\mathbb{P}^\mu = (P_t^\mu)_{t>0}$ is called the sub-Markovian* semigroup subordinated to $\mathbb{P}$ by means of $(\mu_t)_{t>0}$.

Proof. Since for all $t_1, t_2 > 0$, and $f \in bp\mathcal{B}$ we have

$$P_{t_1}^\mu P_{t_1}^\mu f = \int_0^\infty P_{s_1}(P_{t_2}^\mu f)\mu_{t_1}(ds_1) = \int_0^\infty \left(\int_0^\infty P_{s_1} P_{s_2} f \mu_{t_2}(ds_2) \right) \mu_{t_1}(ds_1)$$

$$= \int_0^\infty \int_0^\infty P_{s_1+s_2} f \mu_{t_2}(ds_2)\mu_{t_1}(ds_1) = \int_0^\infty P_s f(\mu_{t_1} * \mu_{t_2}(ds)$$

$$\int_0^\infty P_s f(\mu_{t_1+t_2}(ds) = P_{t_1+t_2}^\mu,$$

it follows that the family of kernels $\mathbb{P}^\mu$ is indeed a semigroup which is certainly sub-Markovian.

We prove now that

$$\mathcal{E}(\mathbb{P}) \subset \mathcal{E}(\mathbb{P}^\mu) :$$

Fix $u \in \mathcal{E}(\mathbb{P})$. Then obviously $P_t^\mu u \le u$ for every $t > 0$. Let $x \in X$, $a < u(x)$ and $0 < b < 1$. Then there exist $s_0 > 0$ and $t_0 > 0$ such that $P_s u(x) > a$ for every $0 < s < s_0$ and $\mu_{t_0}(]0, s_0[) > b$, hence

$$P_{t_0}^\mu u(x) \ge \int_0^{s_0} P_s u(x) \mu_{t_0}(ds) \ge ab.$$

This implies that $u \in \mathcal{E}(\mathbb{P}^\mu)$. □

Theorem 3.2. *Assume that the resolvent $\mathcal{U}$ is proper and that all the points of E are nonbranch points with respect to $\mathcal{U}$. Then the same property holds for the resolvent $\mathcal{U}^\mu$ associated with $\mathbb{P}^\mu$.*

Proof. By Theorem 2.5 we have to show that conditions (UC) and (SSP) are verified by the resolvent $\mathcal{U}^\mu$.

Let ν_1 and ν_2 be two positive finite measures on E such that $\nu_1 \circ U_q^\mu = \nu_2 \circ U_q^\mu$. Using Theorem 2.1 (Hunt's Approximation Theorem) it follows that $\nu_1(v) = \nu_2(v)$ for all $v \in \mathcal{E}(\mathcal{U}_q^\mu)$. Because $\mathcal{E}(\mathbb{P}) \subset \mathcal{E}(\mathbb{P}^\mu) = \mathcal{E}(\mathcal{U}^\mu) \subset \mathcal{E}(\mathcal{U}_q^\mu)$, we get that $\nu_1(v) = \nu_2(v)$ for all $v \in \mathcal{E}(\mathcal{U})$. In particular, we have $\nu_1(Uf) = \nu_2(Uf)$ for all $f \in bp\mathcal{B}$, hence $\nu_1 \circ U = \nu_2 \circ U$. Because by hypothesis all the points of E are nonbranch points with respect to $\mathcal{U}$, by Theorem 2.5 we deduce that the uniqueness of charges property holds for $\mathcal{U}$. It follows that $\nu_1 = \nu_2$ and we conclude that (UC) also holds for $\mathcal{U}^\mu$.

We check now that the specific solidity of potentials property (SSP) holds with respect to $\mathcal{U}^\mu$. Let ξ, η, and $\nu \circ U_q^\mu$ be $\mathcal{U}_q^\mu$-excessive measures such that

$$(3.1) \quad \xi + \eta = \nu \circ U_q^\mu.$$

We may assume that ν is a finite measure, consequently the measures ξ and η are also finite. We define the positive measures ξ' and η' on E by

$$\xi'(f) := L_q^\mu(\xi, Uf), \quad \eta'(f) := L_q^\mu(\eta, Uf) \quad \text{for all } f \in bp\mathcal{B}.$$

We claim that ξ' and η' are $\mathcal{U}$-excessive measures. Indeed, if $\alpha > 0$ then

$$\xi' \circ \alpha U_\alpha(f) = L_q^\mu(\xi, \alpha U_\alpha Uf) \le L_q^\mu(\xi, Uf) = \xi'(f).$$

We show now that ξ' is a σ-finite measure. If $f_o \in bp\mathcal{B}$, $f_o > 0$, is such that $Uf_o \le 1$, then

$$\xi'(f_o) = L_q^\mu(\xi, Uf_o) \le L_q^\mu(\nu \circ U_q^\mu, Uf_o) = \nu(Uf_o) \le \nu(1) < \infty.$$

Hence the measure ξ' is σ-finite. We conclude that ξ' is $\mathcal{U}$-excessive and analogously one gets that η' is also a $\mathcal{U}$-excessive measure.

Using (3.1), we have for every $f \in bp\mathcal{B}$

$$\xi'(f) + \eta'(f) = L_q^\mu(\xi, Uf) + L_q^\mu(\eta, Uf) =$$

$$L_q^\mu(\xi + \eta, Uf) = L_q^\mu(\nu \circ U_q^\mu, Uf) = \nu(U_q f).$$

We obtained that the following equality of $\mathcal{U}$-excessive measures holds:

$$\xi' + \eta' = \nu \circ U.$$

Since by hypothesis the property (SSP) holds for $\mathcal{U}$, we deduce from the last equality that there exists a measure λ on E such that $\xi' = \lambda \circ U$. It follows that

$$L_q^\mu(\xi, Uf) = \xi'(f) = \lambda(Uf) = L_q^\mu(\lambda \circ U_q^\mu, Uf),$$

hence

$$L_q^\mu(\xi, Uf) = L_q^\mu(\lambda \circ U_q^\mu, Uf) \quad \text{for all } f \in bp\mathcal{B}.$$

In particular, taking $f = U_q^\mu g$, with $g \in bp\mathcal{B}$, and since $UU_q^\mu = U_q^\mu U$, it follows that for all $g \in bp\mathcal{B}$ we have

$$\xi(Ug) = L_q^\mu(\xi, UU_q^\mu g) = L_q^\mu(\lambda \circ U_q^\mu, U_q^\mu Ug) = \lambda \circ U_q^\mu(Ug)$$

Note that in addition we have

$$\xi(Ug) \le \nu(U_q^\mu Ug) \le \frac{1}{q}\nu(Ug).$$

In particular, the measures $\xi \circ U$ and $(\lambda \circ U_q^\mu) \circ U$ are σ-finite and equal. Because by the resolvent equation we have $U_q g = U(g - qU_q g)$ for all $g \in bp\mathcal{B}$, $g \le f_o$, it follows that $\xi \circ U_q g = (\lambda \circ U_q^\mu) \circ U_q g$, and therefore $\xi \circ U_q = (\lambda \circ U_q^\mu) \circ U_q$. By the uniqueness of charges propert (UC) for the resolvent $\mathcal{U}$ (see the proof of the implication $(i) \Longrightarrow (ii)$ in the proof of Theorem 2.5) we conclude that $\xi = \lambda \circ U_q^\mu$, so, the property (SSP) holds with respect to $\mathcal{U}^\mu$. $\square$

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