Qualitative properties for the solutions of Scrödinger equations

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Introduction

The Schrödinger equation is the fundamental equation of physics for describing quantum mechanical behavior. It is also often called the Schrödinger wave equation, and is a partial differential equation that describes how the wave function of a physical system evolves over time.

The purpose of this work is to give a presentation of some basic results concerning the continuous Schödinger equation and some new results for the discrete Schödinger equation.

In the first chapter we recall some basic properties of the Fourier Transform and some classical and new results on oscillatory integrals.

In Chapter 2 we establish some important properties of the linear equation

$$\begin{cases} iu_t + u_{xx} = 0,\ x \in \mathbb{R}^d,\ t \neq 0, \\ u(0,x) = \varphi(x),\ x \in \mathbb{R}^d. \end{cases}$$

We recall the dispersive properties of this model and present some of its applications to nonlinear models.

The core of this thesis is Chapter 3. In this chapter we study some models involving discrete Schrodinger equations focusing on the long time behavior of the solutions and Strichartz-like properties. These results are contained in the paper *Dispersive properties for discrete Schrödinger Equation* [5]. We now resume the result of Chapter 3. For completeness we first present the long time behavior of the solutions for the equation

$$\begin{cases} iu_t + \Delta_d u = 0, \quad j \in \mathbb{Z}, t \neq 0, \\ \\ u(0) = \varphi, \end{cases}$$

where Δ_d is the discrete laplacian defined by

$$(\Delta_d u)(j) = u_{j+1} - 2u_j + u_{j-1}, \quad j \in \mathbb{Z}.$$

The main result consist of proving dispersive estimates for the system formed by two coupled Schrodinger equations:

$$\begin{cases} iu_t(t,j) + b_1^{-2}(\Delta_d u)(t,j) = 0, & j \leq -1,\ t \neq 0, \\ iv_t(t,j) + b_2^{-2}(\Delta_d v)(t,j) = 0, & j \geq 1,\ t \neq 0, \\ \\ u(t,0) = v(t,0), & t \neq 0, \\ b_1^{-2}(u(t,-1) - u(t,0)) = b_2^{-2}(v(t,0) - v(t,1)), & t \neq 0, \\ \\ u(0,j) = \varphi(j), & j \leq -1, \\ v(0,j) = \varphi(j), & j \geq 1. \end{cases}$$

We obtain estimates for the resolvent of the discrete operator and prove that it satisfies the limiting absorption principle. The decay of the solutions is proved by using classical and some new results on oscillatory integrals.

Finally, I would like to thank my advisor, professor Liviu Ignat from IMAR, for his support.

Chapter 1

The Fourier Transform

1.1 Main properties

In this section we will present some basic properties of the Fourier Transform.

Definition 1.1.1. The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$, denoted by $\hat{f}$, is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i (x\cdot\xi)} dx, \quad \text{for } \xi \in \mathbb{R}^n,$$

where $x \cdot \xi = x_1\xi_1 + ... + x_n\xi_n$.

We list some basic properties of the Fourier transform in $L^1(\mathbb{R}^d)$.

Theorem 1.1.1. *Let $f \in L^1(\mathbb{R}^d)$. Then:*

1. *$f \mapsto \hat{f}$ is a linear transformation from $L^1(\mathbb{R}^d)$ into $L^\infty(\mathbb{R}^d)$ with*

$$\|\hat{f}\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)}.$$

2. *$\hat{f}$ is continuous.*

3. *$\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$ (Riemann Lebesgue).*

4. *If $\tau_h f(x) = f(x - h)$ denotes the translation by $h \in \mathbb{R}^d$, then*

$$\widehat{(\tau_h f)}(\xi) = e^{-2\pi i (h\cdot\xi)} \hat{f}(\xi),$$

and

$$\widehat{e^{-2\pi i(x\cdot h)}} = (\tau_{-h}\hat{f})(\xi).$$

5. *If $\delta_a f(x) = f(ax)$ denotes the dilatation by $a > 0$, then*

$$\widehat{(\delta_a f)}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

6. *Let $g \in L^1(\mathbb{R}^d)$ and $f * g$ be the convolution of f and g. Then*

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

7. *Let $g \in L^1(\mathbb{R}^d)$. Then*

$$\int_{\mathbb{R}^d} \hat{f}(y)g(y)dy = \int_{\mathbb{R}^d} f(y)\hat{g}(y)dy.$$

8. *Suppose $x_k f \in L^1(\mathbb{R}^d)$, where x_k denotes the kth coordinate of x. Then $\hat{f}$ is differentiable with respect to ξ_k and*

$$\frac{\partial \hat{f}}{\partial \xi_k}(\xi) = (-\widehat{2\pi x_k f}(x))(\xi).$$

In other words, the Fourier transform of the product $x_k f(x)$ is equal to a multiple of the partial derivative of $\hat{f}(\xi)$ with respect to the k- variable.

9. *Let $f \in L^1(\mathbb{R}^d)$. Then*

$$\widehat{\left(\frac{\partial f}{\partial x_k}\right)}(\xi) = 2\pi i \xi_k \hat{f}(\xi).$$

10. *Let $f, \hat{f} \in L^1(\mathbb{R}^d)$. Then*

$$f(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} \hat{f}(\xi) d\xi, \quad a.e.\ x \in \mathbb{R}^d.$$

Using the fact that $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is a dense subset of $L^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ we can define the Fourier Transform for $L^2(\mathbb{R}^d)$-functions.

Theorem 1.1.2. ***(Plancherel)*** *For any $f \in L^2(\mathbb{R}^d)$, $\hat{f} \in L^2(\mathbb{R}^d)$ and*

$$\|\hat{f}\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}.$$

The following examples will be needed in the next sections.

Example 1.1.1. 1. $\widehat{(e^{-\pi|x|^2})}(\xi) = e^{-\pi|\xi|^2}$

2. $\widehat{(e^{-4\pi^2 it|x|^2})}(\xi) = \frac{e^{i|\xi|^2/4t}}{(4\pi it)^{d/2}}$.

1.2 The discrete Fourier Transform

The discrete Fourier Transform is defined for functions in $l^1(h\mathbb{Z})$ and it has similar properties to the continuous one.

For a fixed number $h > 0$ we define the discrete Fourier transform of a function $u \in l^1(h\mathbb{Z})$ by

$$\hat{u} : \mathbb{R} \to \mathbb{C}, \quad \hat{u}(\xi) = h \sum_{j \in \mathbb{Z}} e^{-ijh\xi} u(jh), \xi \in \mathbb{R}.$$

These are some basic properties of the discrete Fourier transform:

Theorem 1.2.1. *Let $u \in l^1(h\mathbb{Z})$. Then*

1. *The discrete Fourier transform is periodic of period $\frac{2\pi}{h}$. It is sufficient to define it on an interval of length $\frac{2\pi}{h}$: $[-\frac{\pi}{h}, \frac{\pi}{h}]$.*

2. *We can recuperate the function u from its discrete Fourier transform by:*

$$u(jh) = \frac{1}{2\pi h} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \hat{u}(\xi) e^{ijh\xi} d\xi.$$

3. *The discrete Fourier transform of the discrete convolution is the product of the discrete Fourier transforms: $\widehat{u \star v} = \hat{u} \cdot \hat{v}$;*

4. ***(Plancherel)** For $u \in l^2(h\mathbb{Z})$ we define its $l^2(h\mathbb{Z})$-norm as follows:*

$$\|u\|^2_{l^2(h\mathbb{Z})} = h \sum_{j \in \mathbb{Z}} |u(jh)|^2.$$

Then

$$\|u\|_{l^2(h\mathbb{Z})} = \|\hat{u}\|_{L^2(-\frac{\pi}{h}, \frac{\pi}{h})}.$$

1.3 Oscillatory Integrals

In many problems and applications the following question arises: what is the asymptotic behavior of $I(\lambda)$ when $\lambda \to \infty$ where

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)} f(x) dx,$$

ϕ is a smooth real valued function, called the *phase function*, and f is a smooth complex-valued function? We shall see that this asymptotic behavior is determined by the critical points of the phase function, i.e. the points $\overline{x}$ where the derivative of ϕ vanishes, $\phi'(\overline{x}) = 0$.

Proposition 1.3.1. *Let $f \in C_0^\infty([a,b])$ and ϕ a smooth real valued function such that $\phi'(x) \neq 0$, for any $x \in [a,b]$. Then, for any $k \in \mathbb{Z}^+$*

$$|I(\lambda)| \leq C(k,\phi,f)\lambda^{-k}, \text{ for } \lambda \text{ big enough.}$$

Proof. We consider the differential operator $\mathcal{L} = \frac{1}{i\lambda\phi'}\frac{df}{dx}$. His adjoint $\mathcal{L}^*$ is the operator that satisfies

$$\int_a^b \mathcal{L}(f)g = \int_a^b f\mathcal{L}^*(g)$$

We will prove that

$$\mathcal{L}^*(g) = -\frac{d}{dx}\left(\frac{g}{i\lambda\phi'}\right).$$

Indeed, using integration by parts, we have that

$$\int_a^b \frac{1}{i\lambda\phi'}\frac{df}{dx}g = f\frac{g}{i\lambda\phi'}\Big|_a^b - \int_a^b f\frac{d}{dx}\left(\frac{g}{i\lambda\phi'}\right) = \int_a^b f\frac{-d}{dx}\left(\frac{g}{i\lambda\phi'}\right).$$

It is easy to see that $\mathcal{L}(e^{i\lambda\phi}) = e^{i\lambda\phi}$ and, moreover, $\mathcal{L}^k(e^{i\lambda\phi}) = e^{i\lambda\phi}$.

Using integration by parts it follows that

$$\int_a^b e^{i\lambda\phi}f dx = \int_a^b \mathcal{L}^k(e^{i\lambda\phi})f dx = \int_a^b e^{i\lambda\phi}\left(\mathcal{L}^*\right)^k(f)dx.$$

This implies that

$$\begin{aligned}\left|\int_a^b e^{i\lambda\phi}f dx\right| &\leq \int_a^b \left|e^{i\lambda\phi}\left(\mathcal{L}^*\right)^k(f)\right|dx = \int_a^b \left|\left(\mathcal{L}^*\right)^k(f)\right|dx \\ &\leq (b-a)\left\|(\mathcal{L}^*)^k(f)\right\|_{L^\infty(a,b)} \\ &\leq (b-a)\lambda^{-k}C(f,f',...,f^{(k)},\phi',...,\phi^{(k)}) \\ &\leq C(a,b,k,f,\phi)\lambda^{-k}.\end{aligned}$$

The proof is now finished. □

Proposition 1.3.2. *Let $k \in \mathbb{Z}^+$ and assume that function ϕ satisfies $\left|\phi^{(k)}(x)\right| \geq 1$ for any $x \in [a,b]$ with $\phi'(x)$ monotonic in the case $k=1$. Then*

$$\left|\int_a^b e^{i\lambda\phi(x)}dx\right| \leq C_k\lambda^{-1/k}, \tag{1.1}$$

where the constant C_k is independent of a and b.

Proof. For $k=1$ we have that

$$\int_a^b e^{i\lambda\phi(x)}dx = \int_a^b \mathcal{L}(e^{i\lambda\phi(x)})dx = \frac{1}{i\lambda\phi'}e^{i\lambda\phi(x)}\Big|_a^b - \int_a^b e^{i\lambda\phi(x)}\frac{1}{i\lambda}\frac{d}{dx}\left(\frac{1}{\phi'}\right)dx.$$

Clearly, the first term of the right hand side is bounded by $2\lambda^{-1}$. On the other hand, the hypothesis of monotonicity on ϕ' guarantees that

$$\left|\int_a^b e^{i\lambda\phi(x)}\frac{1}{i\lambda}\frac{d}{dx}\left(\frac{1}{\phi'}\right)dx\right| \leq \frac{1}{\lambda}\int_a^b\left|\frac{d}{dx}\left(\frac{1}{\phi'}\right)\right|dx = \frac{1}{\lambda}\left|\frac{1}{\phi'(b)}-\frac{1}{\phi'(a)}\right| \leq \frac{2}{\lambda}.$$

This yields the proof in the case $k=1$.

For the proof in the case $k\geq 2$ we will use induction in k. Assuming the result for k, we shall prove it for $k+1$. By hypothesis $\left|\phi^{(k+1)}(x)\right| \geq 1$. Let $x_0\in[a,b]$ be such that

$$\left|\phi^k(x_0)\right| = \min_{a\leq x_0\leq b}|\phi^{(k)}(x)|.$$

If $\phi^{(k)}(x_0)=0$, outside the interval $(x_0-\delta, x_0+\delta)$ one has $|\phi^{(k)}|\geq\delta$ with ϕ' monotonic if $k=1$. Splitting the domain of integration and using the hypothesis we obtain that

$$\left|\int_a^{x_0-\delta} e^{i\lambda\phi(x)}dx\right| + \left|\int_{x_0+\delta}^b e^{i\lambda\phi(x)}dx\right| \leq C_k(\lambda\delta)^{-1/k}.$$

A simple computation shows that

$$\left|\int_{x_0-\delta}^{x_0+\delta} e^{i\lambda\phi(x)}dx\right| \leq 2\delta.$$

Thus

$$\left|\int_a^b e^{i\lambda\phi(x)}dx\right| \leq c_k(\lambda\delta)^{-1/k}+2\delta.$$

Is $\phi^{(k)}(x_0)\neq 0$, then $x_0=a$ or b and a similar argument provides the same bound. Finally, taking $\delta=\lambda^{-1/(k+1)}$ we complete the proof. □

The result can be also stated for a phase function ϕ such that is minimum is positive.

Corrolary 1.3.1. *Let $k\in\mathbb{Z}^+$ and $\min_{x\in[a,b]}\left|\phi^{(k)}(x)\right|>0$, with $\phi'(x)$ monotonic in the case $k=1$. Then*

$$\left|\int_a^b e^{i\lambda\phi(x)}dx\right| \leq C_k\left(\lambda\cdot\min_{x\in[a,b]}\left|\phi^{(k)}(x)\right|\right)^{-1/k}, \tag{1.2}$$

where the constant C_k is independent of a and b.

Proof. Let us set $m=\min_{x\in[a,b]}\left|\phi^{(k)}(x)\right|$. The proof follows immediately by considering the function ϕ/m instead of ϕ and the number λm instead of λ. □

Corrolary 1.3.2. *(**Van der Corput**) Under the hypothesis of Proposition 1.3.2,*

$$\left|\int_a^b e^{i\lambda\phi(x)}f(x)dx\right| \leq c_k\lambda^{-1/k}(\|f\|_{L^\infty(a,b)}+\|f'\|_{L^1(a,b)}), \tag{1.3}$$

with c_k independent of a and b.

Proof. Define

$$G(x) = \int_a^x e^{i\lambda\phi(y)} dy.$$

By (1.1) one has that

$$|G(x)| \leq c_k \lambda^{-1/k}.$$

Now, using integration by parts, we obtain

$$\begin{aligned}\left|\int_a^b e^{i\lambda\phi(x)} f(x)dx\right| = \left|\int_a^b G'(x)f(x)dx\right| \leq |G(x)f(x)|\Big|_a^b + \left|\int_a^b G(x)f'(x)dx\right| \\ \leq c_k\lambda^{-1/k}(\|f\|_{L^\infty(a,b)} + \|f'\|_{L^1(a,b)}).\end{aligned}$$

□

We shall present now some applications of this result.

Example 1.3.1. *For any $a, b \in \mathbb{R}$*

$$\left|\int_a^b e^{i\lambda x^2} dx\right| \leq C\lambda^{-1/2}.$$

Proof. The non-identically vanishing derivatives of the function $\phi(x) = x^2$ are $\phi'(x) = 2x$ and $\phi''(x) = 2$. If the interval $[a, b]$ contains the origin then the first derivative ϕ' vanish at $x = 0$. Applying Corollary 1.3.1 with $k = 2$

$$|I(\lambda)| = \left|\int_a^b e^{i\lambda x^2} dx\right| \leq C\left(\lambda \inf_{x\in[a,b]} \phi''(x)\right)^{-1/2} = C\lambda^{-1/2}.$$

If the point 0 belongs to the interval $[a, b]$ then this estimate cannot be improved. If $0 \notin [a, b]$ then the last nonzero derivative of ϕ is ϕ' and $\inf_{x\in[a,b]} |\phi'(x)| = 2\min\{|a|, |b|\}$. Applying Corollary 1.3.1 with $k = 2$ we obtain a better estimate for large λ

$$I(\lambda) \leq C(2\lambda \min\{|a|, |b|\})^{-1},$$

since $(2\lambda \min\{|a|, |b|\})^{-1} \leq C\lambda^{-1/2}$, which finishes the proof. □

A first improvement of Van der Corput's Lemma has been obtained in [9] where the authors analyze the smoothing effect of some dispersive equations. We will present here a particular case of the results in [9], that will be sufficient for our purposes. In the sequel Ω will be a bounded interval. We consider class $\mathcal{A}_2$ of real functions $\phi \in C^3(\overline{\Omega})$ satisfying the following conditions:

1) Set $S_\phi = \{\xi \in \Omega : \phi'' = 0\}$ is finite,

2) If $\xi_0 \in S_\phi$ then there exist constants ϵ, c_1, c_2 and $\alpha \geq 2$ such that for all $|\xi - \xi_0| < \epsilon$,

$$c_1|\xi - \xi_0|^{\alpha-2} \leq |\phi''(\xi)| \leq c_2|\xi - \xi_0|^{\alpha-2},$$

3) ϕ'' has a finite number of changes of monotonicity.

Lemma 1.3.1. *Let Ω be a bounded interval, $\phi \in \mathcal{A}_2$ and*

$$I(x,t) = \int_\Omega e^{i(t\phi(\xi)-x\xi)}|\phi''(\xi)|^{1/2}d\xi.$$

Then for any $x, t \in \mathbb{R}$

$$|I(x,t)| \leq c_\phi |t|^{-1/2}, \tag{1.4}$$

where c_ϕ depends only on the constants involved in the definition of class $\mathcal{A}_2$.

Remark 1.3.1. *The results of [9] are more general that the one we presented here allowing functions with vertical asymptotics, finite union of intervals or infinite domains.*

As a corollary we also have [9]:

Corrolary 1.3.3. *If $\phi \in \mathcal{A}_2$ then*

$$\Big|\int_\Omega e^{i(t\phi(\xi)-x\xi)}|\phi''(\xi)|^{1/2}\psi(\xi)d\xi\Big| \leq C_\phi|t|^{-1/2}\Big(\|\psi\|_{L^\infty(\Omega)} + \int_\Omega |\phi'(\xi)|d\xi\Big),$$

holds for all $x, t \in \mathbb{R}$.

In the proof of our main result we will need a result similar to Lemma 1.3.1 but with $|p'''|^{1/3}$ instead of $|p''|^{1/2}$ in the definition of $I(x,t)$. We define class $\mathcal{A}_3$ of real functions $\phi \in C^4(\overline{\Omega})$ satisfying the following conditions:
1) Set $S_\phi = \{\xi \in \Omega : \phi''' = 0\}$ is finite,
2) If $\xi_0 \in S_\phi$ then there exist constants ϵ, c_1, c_2 and $\alpha \geq 3$ such that for all $|\xi - \xi_0| < \epsilon$,

$$c_1|\xi-\xi_0|^{\alpha-3} \leq |\phi'''(\xi)| \leq c_2|\xi-\xi_0|^{\alpha-3}, \tag{1.5}$$

3) ϕ''' has a finite number of changes of monotonicity.

Lemma 1.3.2. *Let Ω be a bounded interval, $\phi \in \mathcal{A}_3$ and*

$$I(x,t) = \int_\Omega e^{i(t\phi(\xi)-x\xi)}|\phi'''(\xi)|^{1/3}d\xi.$$

Then for any $x, t \in \mathbb{R}$

$$|I(x,t)| \leq c_\phi |t|^{-1/3}, \tag{1.6}$$

where c_ϕ depends only on the constants involved in the definition of class $\mathcal{A}_3$.

In the following we will write $a \lesssim b$ if there exists a positive constant C such that $a \leq Cb$. Similar for $a \gtrsim b$. Also we will write $a \sim b$ if $C_1 b \leq a \leq C_2 b$ for some positive constants C_1 and C_2.

Proof. We observe that since Ω is bounded we only need to consider the case when t is large.

Case 1: $0 < m \leq |\phi'''(\xi)| \leq M$.

We apply Van der Corput's Lemma with $k = 3$ to the phase function $\phi(\xi) - x\xi/t$ and to $\psi = |\phi'''|^{1/3}$. Then

$$|I(x,t)| \leq C(tm)^{-\frac{1}{3}}(\|\psi\|_{L^\infty(\Omega)} + \|\psi'\|_{L^1(\Omega)}).$$

Since ϕ''' has a finite number of changes of monotonicity we deduce that $\phi^{(4)}$ changes the sign finitely many times and then

$$\|\psi'\|_{L^1(\Omega)} = \frac{1}{3}\int_\Omega \left|(\phi'''(\xi))^{-\frac{2}{3}}\phi^{(4)}(\xi)\right| d\xi \leq \frac{1}{3}m^{-\frac{2}{3}}\int_\Omega |\phi^{(4)}(\xi)|d\xi \leq C(m,M).$$

Hence

$$|I(x,t)| \leq C(M,m)t^{-\frac{1}{3}}.$$

Case 2: $0 \leq |\phi'''(\xi)| < M$.

Using the assumptions on ϕ we can assume that there exists only one point $\xi_0 \in \overline{\Omega}$ such that $\phi'''(\xi_0) = 0$. Notice that if $\phi \in \mathcal{A}_3$, then any translation and any linear perturbation of ϕ (i.e. $\phi(\xi - \xi_0) + a\xi + b$) is still in $\mathcal{A}_3$ and the conditions in the definition of set $\mathcal{A}_3$ are verified with the same constants as ϕ. Therefore we can assume that $\xi_0 = 0$ and $\phi'(\xi_0) = 0$. Moreover let us assume that as $\xi \sim 0$, $|\phi'(\xi)| \sim |\xi|^\alpha$ and $|\phi'''(\xi)| \sim |\xi|^\beta$ for some numbers $\alpha \geq 2$ and $\beta > 0$.

We distinguish now two cases depending on the behavior of ϕ' near $\xi = 0$. If $\alpha \geq 4$ then $|\phi^{(k)}(\xi)| \sim |\xi|^{\alpha-k}$ as $\xi \sim 0$ for $k = 2, 3$ and, in particular $\beta = \alpha - 3$. The case $\alpha = 3$ cannot appear since then $\beta = \alpha - 3$ and ϕ''' does not vanish at $\xi = 0$. For $\alpha = 2$, $|\phi'(\xi)| \sim |\xi|$, $|\phi''(\xi)| \sim 1$ as $\xi \sim 0$ and the third derivative satisfies $|\phi'''(\xi)| \sim |\xi|^\beta$ as $\xi \sim 0$ for some positive integer β. This last case occurs for example when $\phi'(\xi) = \xi + \xi^3$. In all cases $\beta \geq \alpha - 3$.

We split Ω as follows

$$I(x,t) = \int_{|\xi|\leq\epsilon} e^{i(t\phi(\xi)-x\xi)}|\phi'''(\xi)|^{\frac{1}{3}}d\xi + \int_{|\xi|\geq\epsilon} e^{i(t\phi(\xi)-x\xi)}|\phi'''(\xi)|^{\frac{1}{3}}d\xi = I_1 + I_2.$$

Since $\xi = 0$ is the only point where the third derivative vanishes we have that outside an interval that contains the origin ϕ''' does not vanish. Thus I_2 can be treated as in the first case.

Let us now estimate the first term I_1. We define $\Omega_j, 1 \leq j \leq 3$, as follows

$$\Omega_1 = \{\xi \in \Omega \,|\, |\xi| \leq \min(\epsilon, |t|^{-1/\alpha})\},$$
$$\Omega_2 = \left\{\xi \in \Omega - \Omega_1 \,|\, |\xi| \leq \epsilon, \text{ and } \left|\phi'(\xi) - \frac{x}{t}\right| \leq \frac{1}{2}\left|\frac{x}{t}\right|\right\},$$
$$\Omega_3 = \{\xi \in \Omega - (\Omega_1 \cup \Omega_2) \,|\, |\xi| \leq \epsilon\}.$$

In the case of Ω_1 we use that for some $\beta \geq 1$, the third derivative of ϕ satisfies $c_1|\xi|^\beta \leq |\phi'''(\xi)| \leq c_2|\xi|^\beta$ for $|\xi| < \epsilon$. We get

$$\int_{\Omega_1} |\phi'''(\xi)|^{\frac{1}{3}} d\xi \leq c_2^{\frac{1}{3}} \int_{\Omega_1} |\xi|^{\frac{\beta}{3}} d\xi \leq C|\Omega_1|t^{-\frac{\beta}{3\alpha}} \leq C|t|^{-\frac{1}{\alpha}-\frac{\beta}{3\alpha}} \leq C|t|^{-1/3},$$

where the last inequality holds since $\alpha \leq \beta + 3$ and $|t| \geq 1$.

In the case of the integral on Ω_2 we assume that $x \neq 0$ since otherwise Ω_2 has measure zero. Observe that for $\xi \in \Omega_2$ we have

$$\pm|\phi'(\xi)| \mp \left|\frac{x}{t}\right| \leq \left|\phi'(\xi) - \frac{x}{t}\right| \leq \frac{1}{2}\left|\frac{x}{t}\right|,$$

which implies that

$$\frac{1}{2}\left|\frac{x}{t}\right| \leq |\phi'(\xi)| \leq \frac{3}{2}\left|\frac{x}{t}\right|.$$

Since $|\phi'(\xi)| \sim |\xi|^{\alpha-1}$ we have that $|\xi| \sim |x/t|^{\frac{1}{\alpha-1}}$. Then $|\phi'''(\xi)| \sim |\xi|^\beta \sim |x/t|^{\frac{\beta}{\alpha-1}}$ and

$$\min_{\xi\in\Omega_2} |\phi'''(\xi)| > 0.$$

Applying Van der Corput's Lemma with $k = 3$ and using that $\phi^{(4)}$ changes the sign finitely many times we obtain that

$$\begin{aligned}\left|\int_{\Omega_2} e^{i(t\phi(\xi)-x\xi)}|\phi'''(\xi)|^{\frac{1}{3}} d\xi\right| &\leq C(\min_{\xi\in\Omega_2} |\phi'''(\xi)||t|)^{-\frac{1}{3}}\left(\||\phi'''(\xi)|^{\frac{1}{3}}\|_{L^\infty(\Omega_2)} + \|(|\phi'''(\xi)|^{\frac{1}{3}})'\|_{L^1(\Omega_2)}\right)\\ &= C(\min_{\xi\in\Omega_2} |\phi'''(\xi)|)^{-\frac{1}{3}}|t|^{-\frac{1}{3}}\left(\max_{\xi\in\Omega_2} |\phi'''(\xi)|^{\frac{1}{3}} + \frac{1}{3}\int_{\Omega_2} |\phi'''(\xi)|^{-\frac{2}{3}}|\phi^{(4)}(\xi)|d\xi\right)\\ &\leq C(\min_{\xi\in\Omega_2} |\phi'''(\xi)|)^{-\frac{1}{3}} \max_{\xi\in\Omega_2} |\phi'''(\xi)|^{\frac{1}{3}}|t|^{-\frac{1}{3}}.\end{aligned}$$

Since on Ω_2, $|\phi'''(\xi)| \sim |x/t|^{\frac{\beta}{\alpha-1}}$, there exists a positive constant C such that

$$\max_{\xi\in\Omega_2} |\phi'''(\xi)|^{\frac{1}{3}} \leq C(\min_{\xi\in\Omega_2} |\phi'''(\xi)|)^{\frac{1}{3}},$$

which gives us the desired estimates on the integral on Ω_2.

Now, we estimate the integral on Ω_3. Observe that we have to consider the case $|t|^{-1/\alpha} < \epsilon$, otherwise $\Omega_2 = \Omega_3 = \emptyset$. In particular, for $\xi \in \Omega_3$, we have $|t|^{-1/\alpha} < \xi < \epsilon$. Integrating by parts the integral on Ω_3 satisfies

$$\Big|\int_{\Omega_3} e^{i(t\phi(\xi)-x\xi)}|\phi'''(\xi)|^{\frac{1}{3}}d\xi\Big| = \frac{1}{|t|}\Big|\int_{\Omega_3}(e^{i(t\phi(\xi)-x\xi)})'\frac{|\phi'''(\xi)|^{\frac{1}{3}}}{\phi'(\xi)-\frac{x}{t}}d\xi\Big| \tag{1.7}$$
$$\le \frac{1}{|t|}\Big|\pm e^{i(t\phi(\xi)-x\xi)}\frac{|\phi'''(\xi)|^{\frac{1}{3}}}{\phi'(\xi)-\frac{x}{t}}\big|_{\partial\Omega_3}\Big|$$
$$+\frac{1}{|t|}\Big|\int_{\Omega_3} e^{i(t\phi(\xi)-x\xi)}\frac{\frac{1}{3}|\phi'''(\xi)|^{-\frac{2}{3}}\phi^{(4)}(\xi)(\phi'(\xi)-\frac{x}{t})-|\phi'''(\xi)|^{\frac{1}{3}}\phi''(\xi)}{\big(\phi'(\xi)-\frac{x}{t}\big)^2}d\xi\Big|$$
$$\le \frac{2}{|t|}\max_{\xi\in\Omega_3}\frac{|\phi'''(\xi)|^{\frac{1}{3}}}{\big|\phi'(\xi)-\frac{x}{t}\big|}+\frac{1}{3|t|}\int_{\Omega_3}\frac{|\phi'''(\xi)|^{-\frac{2}{3}}|\phi(4)(\xi)|}{\big|\phi'(\xi)-\frac{x}{t}\big|}+\frac{1}{|t|}\int_{\Omega_3}\frac{|\phi'''(\xi)|^{\frac{1}{3}}|\phi''(\xi)|}{\big(\phi'(\xi)-\frac{x}{t}\big)^2}d\xi.$$

In the following we obtain upper bounds for all terms in the right hand side of (1.7). Since on Ω_3, $|\phi'(\xi)-x/t|\ge|x/2t|$, there exists a positive constant c such that

$$\Big|\phi'(\xi)-\frac{x}{t}\Big| > c|\phi'(\xi)|\ge c|\xi|^{\alpha-1},\ \forall\xi\in\Omega_3.$$

In the case of the first term

$$\frac{1}{|t|}\sup_{\xi\in\Omega_3}\frac{|\phi'''(\xi)|^{\frac{1}{3}}}{\big|\phi'(\xi)-\frac{x}{t}\big|}\le\frac{C}{|t|}\sup_{\xi\in\Omega_3}\frac{|\xi|^{\frac{\beta}{3}}}{|\xi|^{\alpha-1}}=\frac{C}{|t|}\sup_{\xi\in\Omega_3}|\xi|^{\frac{\beta}{3}-\alpha+1}\le|t|^{-1/3}, \tag{1.8}$$

since $|\xi|\le\epsilon\le1$ and $|\xi|^{\beta/3-\alpha+1}\le|\xi|^{(\alpha-3)/3-\alpha+1}=|\xi|^{-2\alpha/3}\le|t|^{2/3}$.

The second term satisfies

$$\frac{1}{|t|}\int_{\Omega_3}\frac{\frac{1}{3}|\phi'''(\xi)|^{-\frac{2}{3}}|\phi^{(4)}(\xi)|}{\big|\phi'(\xi)-\frac{x}{t}\big|}d\xi\le\frac{C}{|t|}\int_{\Omega_3}\frac{|\xi|^{-2\beta/3}}{|\xi|^{\alpha-1}}|\phi^{(4)}(\xi)|d\xi\le\frac{C}{|t|}\int_{\Omega_3}|\xi|^{\frac{-2\beta}{3}-\alpha+1}|\phi^{(4)}(\xi)|d\xi.$$

Integrating by parts, applying the triangle inequality and using the definition of Ω_3 we get

$$\int_{\Omega_3}|\xi|^{\frac{-2\beta}{3}-\alpha+1}|\phi^{(4)}(\xi)|d\xi\lesssim\sup_{\Omega_3}|\xi|^{\frac{-2\beta}{3}-\alpha+1}|\phi'''(\xi)|+\int_{\Omega_3}|\xi|^{\frac{-2\beta}{3}-\alpha}|\phi'''(\xi)|d\xi$$
$$\lesssim\sup_{\Omega_3}|\xi|^{\frac{\beta}{3}-\alpha+1}+\int_{\Omega_3}|\xi|^{\frac{\beta}{3}-\alpha}d\xi$$
$$\lesssim\sup_{\Omega_3}|\xi|^{\frac{\beta}{3}-\alpha+1}\le|t|^{2/3},$$

where the last inequality follows as in (1.8).

The last term in (1.7) can be estimated as follows

$$\int_{\Omega_3}\frac{|\phi'''(\xi)|^{\frac{1}{3}}|\phi''(\xi)|}{\big(\phi'(\xi)-\frac{x}{t}\big)^2}d\xi\lesssim\int_{\Omega_3}\frac{|\xi|^{\beta/3+\alpha-2}}{|\xi|^{2(\alpha-2)}}=\int_{\Omega_3}|\xi|^{\beta/3-\alpha}\lesssim\sup_{\Omega_3}|\xi|^{\frac{\beta}{3}-\alpha+1}\le|t|^{2/3}.$$

Putting together the estimates for the terms in the right hand side of (1.7) we obtain that the integral on Ω_3 also decays as $|t|^{-1/3}$.

The proof is now finished. $\square$

Chapter 2

The Schrodinger Equation

In this chapter we first present some classical facts about the linear Schrodinger equation. We analyze the long time behavior of solutions and state some space-time estimates known as Strichartz estimates. Using these estimates we obtain estimates for the solutions of non-homogeneous Schrödinger equations and apply them to the well-posedness of solutions of some nonlinear Schrödinger equations.

2.1 The linear Scrödinger Equation

In this section we will study the asymptotic behavior of the solution of the initial value problem

$$\begin{cases} u_t(t,x) = i\Delta u(t,x),\ x \in \mathbb{R}^d,\ t \neq 0, \\ u(0,x) = \varphi(x),\ x \in \mathbb{R}^d. \end{cases} \tag{2.1}$$

Proposition 2.1.1. *The solution of the linear equation* (2.1) *with initial data φ is denoted by $u(t,x) = e^{it\Delta}\varphi$ and it has the following important properties.*

1. *For all $t \in \mathbb{R}$, $e^{it\Delta} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is an isometry; which implies*

$$\|e^{it\Delta}\varphi\|_{L^2(\mathbb{R}^d)} = \|\varphi\|_{L^2(\mathbb{R}^d)}.$$

2. $e^{it\Delta}e^{it'\Delta} = e^{i(t+t')\Delta}$ *with* $\left(e^{it\Delta}\right)^{-1} = e^{-it\Delta} = \left(e^{it\Delta}\right)^*$.

3. $e^{i0\Delta} = 1$.

4. *Fixing $\varphi \in L^2(\mathbb{R}^d)$, the function $\phi_\varphi : \mathbb{R} \to L^2(\mathbb{R})$ defined by $\phi_\varphi(t) = e^{it\Delta}\varphi$ is a continuous function; i.e. describes a curve in $L^2(\mathbb{R}^d)$.*

Proof. Applying the Fourier transform (with respect to the spatial variable) to equation (2.1) we get that

$$\begin{cases} \widehat{u}_t(t,\xi) = i\widehat{\Delta u}(t,\xi),\ \xi \in \mathbb{R}^d,\ t \neq 0, \\ \widehat{u}(0,\xi) = \widehat{\varphi}(\xi),\ \xi \in \mathbb{R}^d. \end{cases}$$

Using the properties of the Fourier Transform we have that $\widehat{\Delta u}(t,\xi) = -4\pi^2|\xi|^2\widehat{u}(\xi,t), \xi \in \mathbb{R}^d, t \neq 0$. Thus function $\widehat{u}$ verifies the system:

$$\begin{cases} \widehat{u}_t(t,\xi) = -4\pi^2 i|\xi|^2\widehat{u}(t,\xi),\ \xi \in \mathbb{R}^d,\ t \neq 0, \\ \widehat{u}(0,\xi) = \widehat{\varphi}(\xi),\ \xi \in \mathbb{R}^d. \end{cases}$$

For ξ fixed, this is an ordinary differential equation and has the solution

$$\widehat{u}(t,\xi) = e^{-4\pi^2 it|\xi|^2}\widehat{\varphi}(\xi). \tag{2.2}$$

Now, we consider the function $K_t(\xi)$ defined by means of its Fourier transform

$$\widehat{K_t}(\xi) = e^{-4\pi^2 it|\xi|^2}, \quad \xi \in \mathbb{R}^d, t \neq 0.$$

Using Example 1.1.1 from Section 1.1 we deduce that

$$K_t(x) = \frac{e^{\frac{i|x|^2}{4t}}}{(4\pi it)^{d/2}}, \quad x \in \mathbb{R}^d, t \neq 0.$$

It implies that solution u of equation (2.8) is given by

$$u(t,x) = K_t(x) * \varphi(x) = (4\pi it)^{-d/2}\int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}}\varphi(y)dy.$$

Using (2.2) the $L^2(\mathbb{R}^d)$ norm of u satisfies

$$\|u(t)\|_{L^2(\mathbb{R}^d)} = \|\widehat{u(t)}\|_{L^2(\mathbb{R}^d)} = \|e^{-4\pi^2 it|\xi|^2}\widehat{\varphi}\|_{L^2(\mathbb{R}^d)} = \|\widehat{\varphi}\|_{L^2(\mathbb{R}^d)} = \|\varphi\|_{L^2(\mathbb{R}^d)}.$$

Properties $2-4$ follow by using property (2.2). ☐

We recall a known result about interpolation of operators.

Theorem 2.1.1. *(Riesz-Thorin) Let $p_0 \neq p_1, q_0 \neq q_1$. Let T be a bounded linear operator from $L^{p_0}(X,\mathcal{A},\mu)$ to $L^{q_0}(Y,\mathcal{B},\nu)$ with norm M_0 and from $L^{p_1}(X,\mathcal{A},\mu)$ to $L^{q_1}(Y,\mathcal{B},\nu)$ with norm M_1. Then T is bounded from $L^{p_\theta}(X,\mathcal{A},\mu)$ to $L^{q_\theta}(Y,\mathcal{B},\nu)$ with norm M_θ such that*

$$M_\theta \leq M_0^{1-\theta}M_1^{\theta},$$

with

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},\ \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},\ \theta \in (0,1).$$

As a corollary we have the following well-known inequality:

Corrolary 2.1.1. *(Young's Inequality) Let $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, and $g \in L^1(\mathbb{R}^d)$. Then $f * g \in L^p(\mathbb{R}^d)$ and*

$$\|f * g\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}. \tag{2.3}$$

Now, we establish the properties how the group $\left\{e^{it\Delta}\right\}_{t=-\infty}^{\infty}$ acts on the $L^p(\mathbb{R}^d)$-spaces. For any $p \in [1, \infty)$ we set p' by the rule $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

Proposition 2.1.2. *For any $t \neq 0$ and $p \geq 2$, $e^{it\Delta}$ maps continuously $L^{p'}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ and*

$$\|e^{it\Delta}\varphi(x)\|_{L^p(\mathbb{R}^d)} \leq C|t|^{-d/2(1/p'-1/p)} \|\varphi\|_{L^{p'}(\mathbb{R}^d)}, \quad \textit{where } \frac{1}{p} + \frac{1}{p'} = 1. \tag{2.4}$$

Proof. From Proposition 2.1.1 we have that

$$e^{it\Delta} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

is an isometry; that is,

$$\|e^{it\Delta}\varphi\|_{L^2(\mathbb{R}^d)} = \|\varphi\|_{L^2(\mathbb{R}^d)}.$$

Using Young's Inequality (2.3), we have

$$\begin{aligned}
\|e^{it\Delta}\varphi\|_{L^\infty(\mathbb{R}^d)} &= \frac{C_n}{\sqrt{|4\pi i t|^d}} \|e^{i|\cdot|^2/4t} * \varphi\|_{L^\infty(\mathbb{R}^d)} \\
&\leq \frac{C_n}{\sqrt{|4\pi i t|^d}} \|e^{i|\cdot|^2/4t}\|_{L^\infty(\mathbb{R}^d)} \|\varphi\|_{L^1(\mathbb{R}^d)} \\
&\leq c|t|^{-d/2} \|\varphi\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

Combining these inequalities with the Riesz-Thorin interpolation theorem 2.1.1, we obtain that, for any $p \geq 2$, the operator $e^{it\Delta}$ maps $L^{p'}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ and

$$\left\|e^{it\Delta}\varphi\right\|_{L^p(\mathbb{R}^d)} \leq (c|t|^{-d/2})^{1-\theta} \|\varphi\|_{L^{p'}(\mathbb{R}^d)},$$

where

$$\frac{1}{p} = \frac{\theta}{2} \text{ and } 1 - \theta = 1 - \frac{2}{p} = \frac{1}{p'} - \frac{1}{p}.$$

The proof is now finished. □

Proposition 2.1.3. *The following hold:*

1. *Given $t_0 \neq 0$ and $p > 2$, there exists $f \in L^2(\mathbb{R}^d)$ such that $e^{it_0\Delta} f \notin L^p(\mathbb{R}^d)$.*

2. *Let $s' > s > 0$ and $f \in H^s(\mathbb{R}^d)$ such that $f \notin H^{s'}(\mathbb{R}^d)$. Then, for all $t \in \mathbb{R}$, $e^{it\Delta} f \in H^s(\mathbb{R}^d)$ and $e^{it\Delta} f \notin H^{s'}(\mathbb{R}^d)$.*

Proof. To show (1) it is enough to choose $g \in L^2(\mathbb{R}^d)$ such that $g \notin L^p(\mathbb{R}^d)$ and take $f = e^{-it_0\Delta} g$.

The statement (2) follows from the fact $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$ is a unitary group in $H^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$ since

$$\begin{aligned}
\|e^{it\Delta} f\|_{s,2} &= \|(I-\Delta)^{s/2}(e^{it\Delta} f)\|_{L^2(\mathbb{R}^d)} = \|e^{it\Delta}((I-\Delta)^{s/2} f)\|_{L^2(\mathbb{R}^d)} \\
&= \|(I-\Delta)^{s/2} f\|_{L^2(\mathbb{R}^d)} = \|f\|_{H^s(\mathbb{R}^d)}.
\end{aligned}$$

Therefore, for any $s_0 > 0$, if $e^{it\Delta} f \in H^{s_0}(\mathbb{R}^d)$ then $f = e^{-it\Delta}(e^{it\Delta}) f \in H^{s_0}(\mathbb{R}^d)$. We then conclude as in the case of L^p-spaces. $\square$

In order to emphasize the optimality of the $L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ estimate in Proposition 2.1.2 we can choose the particular case when the initial data is a gaussian profile $\varphi(x) = e^{-\pi|x|^2}$.

Using formula (2.2) the solution of equation (2.1) is given by

$$\begin{aligned}
u(x,t) &= \left(e^{-4\pi^2 it|\xi|^2}\widehat{\varphi}(\xi)\right)^\vee = \left(e^{-(1+4\pi it)\pi|\xi|^2}\right)^\vee \\
&= \frac{1}{(1+4\pi it)^{d/2}} \exp\left(\frac{-\pi|x|^2}{1+4\pi it}\right) \\
&= (1+4\pi it)^{-d/2} \exp\left(\frac{-\pi|x|^2}{1+16\pi^2 t^2}\right) \exp\left(\frac{4\pi^2 it|x|^2}{1+16\pi^2 t^2}\right).
\end{aligned}$$

Notice that when $t >> 1$ and $|x| < t$ the solution is bounded below by $ct^{-d/2}$ and oscillates for $|x| > t^{1/2}$, but if $|x| > t$ the solution decays exponentially. Moreover,

$$Ct^{-d/2}\chi_{\{|x|<t\}}(x) \leq |u(x,t)| \leq ct^{-d/2}.$$

2.2 Strichartz estimates

In this chapter we present some estimates for the inhomogeneous Scrödinger equation and show how we can apply them to obtain the well-posedness of nonlinear problems.

First, we introduce some notation that will be used in what follows. The mixed Lebesgue spaces $L_t^q L_x^r, 1 \leq q, r \leq \infty$, are defined as the completion of the set of all Schwartz functions $f : \mathbb{R}_+^1 \times \mathbb{R}^d \to \mathbb{C}$ in the norm

$$\|f\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^d))} := \left(\int_0^\infty \left(\int_{\mathbb{R}^d} |f(t,x)|^r dx \right)^{q/r} dt \right)^{1/q}.$$

The next result describes the *global smoothing* property of the group $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$ and these estimates are known as Strichartz estimates(see for examples Linares [11], page 64).

Theorem 2.2.1. *The group* $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$ *satisfies:*

1. $\left(\int_{-\infty}^{\infty} \|e^{it\Delta} f\|_{L^r(\mathbb{R}^d)}^q dt \right)^{1/q} \leq c\|f\|_{L^2(\mathbb{R}^d)},$

2. $\left(\int_{-\infty}^{\infty} \left\| \int_{-\infty}^{\infty} e^{i(t-s)\Delta} g(\cdot, s) ds \right\|_{L^r(\mathbb{R}^d)}^q dt \right)^{1/q} \leq c \left(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{L^{r'}(\mathbb{R}^d)}^{q'} dt \right)^{1/q'},$

3. $\left\| \int_{-\infty}^{\infty} g(\cdot, t) dt \right\|_{L^2(\mathbb{R}^d)} \leq c \left(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{L^{r'}(\mathbb{R}^d)}^{q'} dt \right)^{1/q'},$

with

$$\begin{cases} 2 \leq r \leq \frac{2d}{d-2} \text{ if } d \geq 3 \\ 2 \leq r < \infty \text{ if } d = 2, \\ 2 \leq r \leq \infty \text{ if } d = 1, \end{cases} \tag{2.5}$$

and $\frac{2}{q} = \frac{d}{2} - \frac{d}{r}$, *where* $c = c(r,d)$ *is a constant that depends only on* r *and* d.

We recall here a stronger result due to Keel and Tao [8].

Definition 2.2.1. We say that a pair of exponents (q,r) is σ-*admissible*, if $q, r \geq 2, (q,r,\sigma) \neq (2, \infty, 1)$ and

$$\frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2}. \tag{2.6}$$

If equality holds in (2.6) we say that (q,r) is *sharp* σ-*admissible*, otherwise we say that (q,r) is nonsharp σ-admissible. Note in particular that when $\sigma > 1$ the endpoint

$$P = \left(2, \frac{2\sigma}{\sigma - 1}\right)$$

is sharp σ-admissible.

Theorem 2.2.2. *Let H be a Hilbert space, (X, dx) be a measure space and $U(t) : H \to L^2(X)$ be a one parameter family of mappings, which obey the energy estimate*

$$\|U(t)f\|_{L^2(X)} \leq C\|f\|_H$$

and for some $\sigma > 0$ one of the following decay estimates:

- *for all $t \neq s$ and all $g \in L^1(X)$*

$$||U(t)(U(s))^* g\|_{L^\infty(X)} \leq C|t-s|^{-\sigma}\|g\|_{L^1(X)} \text{(untruncated decay)},$$

- *for all $t, s \in \mathbb{R}$ and $g \in L^1(X)$*

$$||U(t)(U(s))^* g\|_{L^\infty(X)} \leq C(1+|t-s|)^{-\sigma}\|g\|_{L^1(X)} \text{(truncated decay)}. \tag{2.7}$$

Then the following estimates hold for all sharp-σ-admissible pairs $(q, r), (\tilde{q}, \tilde{r})$. Furthermore, if the decay hypothesis is strengthened to (2.7), then these estimates hold for all σ- admissible (q, r) and $(\tilde{q}, \tilde{r})$:

$$\|U(t)f\|_{L^q(\mathbb{R})L^r(X)} \leq C\|f\|_H,$$
$$\left\|\int (U(t))^* F(t, \cdot)dt\right\|_H \leq C\|F\|_{L^{q'}(\mathbb{R}, L^{r'}(X))},$$
$$\left\|\int_0^t U(t)(U(s))^* F(s, \cdot)ds\right\|_{L^q(\mathbb{R}, L^r(X))} \leq C\|f\|_{L^{\tilde{q}'}(\mathbb{R}, L^{\tilde{r}'}(X))}.$$

As a consequence, choosing $H = L^2(\mathbb{R}^d)$ and $X = \mathbb{R}^d$, we obtain that the solution of the inhomogeneous problem

$$\begin{cases} iu_t + \Delta u + F = 0 \\ u(0, x) = \varphi. \end{cases} \tag{2.8}$$

satisfies the following theorem:

Theorem 2.2.3. *Let (q, r) and $(\tilde{q}, \tilde{r})$ both be sharp σ-admissible pairs with $\sigma = 1/2$. Then,*

$$\|e^{it\Delta}f\|_{L^q(\mathbb{R}, L^r(X))} \leq C\|f\|_{L^2(\mathbb{R}^d)},$$

$$\|\textstyle\int e^{it\Delta}F(t, \cdot)dt\|_{L^2(\mathbb{R}^d)} \leq C\|F\|_{L^{q'}(\mathbb{R}, L^{r'}(\mathbb{R}^d))},$$

$$\|\textstyle\int_0^t e^{i(t-s)\Delta}F(s, \cdot)ds\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^d))} \leq C\|f\|_{L^{\tilde{q}'}(\mathbb{R}, L^{\tilde{r}'}(\mathbb{R}^d))}.$$

The solution of equation (2.8) satisfies

$$\|u\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^d))} \leq C\left(\|\varphi\|_{L^2(\mathbb{R}^d)} + \|F\|_{L^{\tilde{q}'}(\mathbb{R}, L^{\tilde{r}'}(\mathbb{R}^d))}\right).$$

2.3 Nonlinear problems

In this section we give some well-posedness results for the semi-linear Schödinger equation:

$$\begin{cases} iu_t + \Delta u + \lambda |u|^p u = 0, t > 0, x \in \mathbb{R} \\ u(0,x) = \varphi, \end{cases} \tag{2.9}$$

where $p > 0$ and $\lambda \in \mathbb{R}$.

Theorem 2.3.1. *Any regular solution of problem (2.9) has the following properties:*

1. *the conservation of the $L^2(\mathbb{R})$-norm : $\|u(t,x)\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}$.*
2. *the conservation of energy:*

$$\frac{1}{2}\int_{\mathbb{R}} |\nabla u(t)|^2 - \frac{\lambda}{p+2}\int_{\mathbb{R}} |u(t)|^{p+2}dx = \frac{1}{2}\int_{\mathbb{R}} |\nabla \varphi|^2 dx - \frac{\lambda}{p+2}\int_{\mathbb{R}} |\varphi|^{p+2}dx, \forall t \in \mathbb{R}.$$

Proof. First, we will show the conservation of the $L^2(\mathbb{R})$-norm.

$$\frac{d}{dt}\int_{\mathbb{R}} |u(t,x)|^2 dx = \int_{\mathbb{R}} \frac{d}{dt}|u(t,x)|^2 dx = 2\int_{\mathbb{R}} \operatorname{Re}(\overline{u}u_t)dx.$$

Since $|u|^2 = u \cdot \overline{u}$ we have

$$\partial_t |u|^2 = u_t\overline{u} + u\overline{u}_t = 2\operatorname{Re}(u_t\overline{u}).$$

We multiplicate the equation by $\overline{u}$ and integrate over $\mathbb{R}$:

$$i\int_{\mathbb{R}} u_t \cdot \overline{u}dx + \int_{\mathbb{R}} \Delta u \cdot \overline{u}dx + \lambda \int_{\mathbb{R}} |u|^p u \cdot \overline{u} = 0$$

thus

$$\int_{\mathbb{R}} u_t \cdot \overline{u}dx = i\int_{\mathbb{R}} \Delta u \cdot \overline{u}dx + i\lambda \int_{\mathbb{R}} |u|^{p+2}dx$$

and

$$\operatorname{Re}\left(\int_{\mathbb{R}} u_t \cdot \overline{u}dx\right) = \operatorname{Re}\left(i\int_{\mathbb{R}} \Delta u \cdot \overline{u}dx\right) + \operatorname{Re}\left(i\lambda \int_{\mathbb{R}} |u|^{p+2}dx\right).$$

But

$$\int_{\mathbb{R}} \Delta u \cdot \overline{u}dx = -\int_{\mathbb{R}} \nabla u \nabla \overline{u}dx = -\int_{\mathbb{R}} |\nabla u|^2 dx \in \mathbb{R}$$

and also $\int_{\mathbb{R}} |u|^{p+2}dx \in \mathbb{R}$. We deduce that $\operatorname{Re}\left(\int_{\mathbb{R}} u_t \cdot \overline{u}dx\right) = 0$, thus $\frac{d}{dt}\int_{\mathbb{R}} |u(t,x)|^2 dx = 0$. From this we obtain that $\int_{\mathbb{R}} |u(t,x)|^2 dx$ is constant with respect to variable t, i.e.:

$$\|u(t,x)\|_{L^2(\mathbb{R})} = \|u(0,x)\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}.$$

Now we will prove conservation of the energy. Let us set

$$\mathcal{E}(t) = \frac{1}{2}\int_{\mathbb{R}} |\nabla u(x,t)|^2 - \frac{\lambda}{p+2}\int_{\mathbb{R}} |u(x,t)|^{p+2}, \text{ for } t \in \mathbb{R}.$$

We have

$$\frac{d}{dt}\mathcal{E}(t) = \frac{1}{2}\int_{\mathbb{R}} |\nabla u|^2 dx - \frac{\lambda}{p+2}\frac{d}{dt}\int_{\mathbb{R}} |u|^{p+2}$$

$$= \text{ Re } \left(\int_{\mathbb{R}} \nabla u \overline{\nabla u_t} dx\right) - \frac{\lambda}{p+2}\int_{\mathbb{R}} (p+2) \text{ Re } \left(|u|^p u \overline{u}_t dx\right).$$

Thus

$$\frac{d}{dt}\mathcal{E}(t) = \text{ Re } \left(\int_{\mathbb{R}} \nabla u \overline{\nabla u_t} dx - \lambda \int_{\mathbb{R}} |u|^p u \overline{u}_t dx\right).$$

We multiplicate equation (2.9) with $\overline{u}_t$, integrate over $\mathbb{R}$ and take the real part of the resulting terms:

$$\text{Re } \left(i\int_{\mathbb{R}} u_t \cdot \overline{u}_t dx + \int_{\mathbb{R}} \Delta u \cdot \overline{u}_t dx + \lambda \int_{\mathbb{R}} |u|^p u \overline{u}_t dx\right) = 0.$$

We have

$$\int_{\mathbb{R}} u_t \cdot \overline{u}_t dx = \int_{\mathbb{R}} |u_t|^2 dx \in \mathbb{R}$$

and thus

$$\text{Re } \left(i\int_{\mathbb{R}} u_t \overline{u}_t dx\right) = 0, \ \int_{\mathbb{R}} \Delta u \cdot \overline{u}_t dx = -\int_{\mathbb{R}} \nabla u \cdot \nabla \overline{u}_t dx.$$

We obtain that

$$\text{Re } \left(-\int_{\mathbb{R}} \nabla u \cdot \nabla \overline{u}_t dx + \lambda \int_{\mathbb{R}} |u|^p \cdot u \cdot \overline{u}_t\right) = 0,$$

which is exactly the derivative of $\mathcal{E}(t)$. It follows that $\mathcal{E}(t)$ is constant:

$$\mathcal{E}(t) = \mathcal{E}(0) = \int_{\mathbb{R}} |\nabla \varphi|^2 dx - \frac{2\lambda}{p+2}\int_{\mathbb{R}} |\varphi|^{p+2} dx.$$

□

We now prove a well-posedness result for $H^1(\mathbb{R})$ solutions of the semilinear problem (2.9). Finding a solution for the equation (2.8) is equivalent to finding a solution for the integral equation

$$u(t) = S(t)\varphi + \int_0^t S(t-s)F(u(s))ds, \tag{2.10}$$

where $F(u(t)) = \lambda |u(t)|^p u(t)$ and $S(t)\varphi = e^{it\Delta}\varphi$ is the semigroup generated by the initial value problem

$$\begin{cases} u_t(t,x) = \Delta u(t,x), & x \in \mathbb{R}, t > 0, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}. \end{cases}$$

Theorem 2.3.2. *For any $\varphi \in H^1(\mathbb{R})$ and $p > 0$ there exists a unique local solution $u \in C([0,T], H^1(\mathbb{R}^d))$ of equation (2.9). Moreover, if $\lambda > 0$ the solution is global.*

Proof. We will prove local existence of the solution using Banach's Fixed Point Theorem. Define the following map

$$\Phi : X \to X, \phi(u) = S(t)\varphi + \int_0^t S(t-s)|u(s)|^p u(s)ds,$$

where

$$X = \left\{ u \in C([0,T], H^1(\mathbb{R})); \max_{t\in[0,T]} \|u(t)\|_{H^1(\mathbb{R})} \leq M \right\},$$

endowed with the norm $\|u\|_X = \max_{t\in[0,T]} \|u(t)\|_{H^1(\mathbb{R})}$. The positive numbers M and T will be chosen later such that Φ to be well defined and contraction.

In order to do this, we first need to show that that function F is locally Lipschitz on $H^1(\mathbb{R})$. We recall the following inequality: there exists a constant $C(p) > 0$ such that

$$\left||a|^{p-1}a - |b|^{p-1}b\right| \leq C\left(|a|^{p-1} + |b|^{p-1}\right)|a-b|, \quad \forall a, b > 0.$$

As a consequence, we obtain that, for any $u, v \in H^1(\mathbb{R})$,

$$\|F(u) - F(v)\|_{H^1(\mathbb{R})} \leq C(p)\left(\|u(t)\|^p_{L^\infty(\mathbb{R})} + \|v(t)\|^p_{L^\infty(\mathbb{R})}\right)\|u - v\|_{H^1(\mathbb{R})}.$$

The embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ gives us that

$$\|F(u) - F(v)\|_{H^1(\mathbb{R})} \leq C(p)\left(\|u(t)\|^p_{H^1(\mathbb{R})} + \|v(t)\|^p_{H^1(\mathbb{R})}\right)\|u - v\|_{H^1(\mathbb{R})}.$$

Indeed,

$$\begin{aligned}
\|F(u(t)) - F(v(t))\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \left(|u(t)|^{p-1}u(t) - |v(t)|^{p-1}v(t)\right)^2 dx \\
&\leq C\int_{\mathbb{R}} \left(|u(t)|^{p-1} + |v(t)|^{p-1}\right)^2 (u(t) - v(t))^2 dx \\
&\leq C\left(\|u(t)\|^{2(p-1)}_{L^\infty(\mathbb{R})} + \|v(t)\|^{2(p-1)}_{L^\infty(\mathbb{R})}\right)\|u(t) - v(t)\|^2_{L^2(\mathbb{R})} \\
&\leq C\left(\|u(t)\|^{2(p-1)}_{H^1(\mathbb{R})} + \|v(t)\|^{2(p-1)}_{H^1(\mathbb{R})}\right)\|u(t) - v(t)\|^2_{H^1(\mathbb{R})},
\end{aligned}$$

where for the last inequality we used the continuous embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Also

$$\|(F(u) - F(v))_x\|_{L^2(\mathbb{R})} = \|F'(u)u_x - F'(v)v_x\|_{L^2(\mathbb{R})} \leq C(p)\|u - v\|_{H^1(\mathbb{R})}.$$

For $u, v \in X$ we have $\|u(t)\|_{H^1(\mathbb{R})} \leq M$, $\|v(t)\|_{H^1(\mathbb{R})} \leq M$ and this implies that

$$\|F(u(t)) - F(v(t))\|_{L^2(\mathbb{R})} \leq C(M)\|u(t) - v(t)\|_{H^1(\mathbb{R})} \leq C(M)\|u(t) - v(t)\|_X.$$

Step 1. Φ is well defined. Let $u \in X$. We have

$$\begin{aligned}\|\Phi(u)\|_X &= \left\| S(t)\varphi + \int_0^t S(t-s)|u(s)|^p u(s)ds \right\|_X \\ &\leq \|S(t)\varphi\|_X + \left\| \int_0^t S(t-s)|u(s)|^p u(s)ds \right\|_X .\end{aligned}$$

Using Proposition 2.1.1, the first term satisfies

$$\|S(t)\varphi\|_X = \max_{t\in[0,T]} \|S(t)\varphi\|_{H^1(\mathbb{R})} = \|\varphi\|_{H^1(\mathbb{R})}.$$

For the second term:

$$\begin{aligned}\left\| \int_0^t S(t-s)|u(s)|^p u(s)ds \right\|_X &= \max_{t\in[0,T]} \left\| \int_0^t S(t-s)|u(s)|^p u(s)ds \right\|_{H^1(\mathbb{R})} \\ &\leq \max_{t\in[0,T]} \int_0^t \||u(s)|^p u(s)\|_{H^1(\mathbb{R})} ds \\ &\leq T \max_{t\in[0,T]} \||u(t)|^p u(t)\|_{H^1(\mathbb{R})}.\end{aligned}$$

Using the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ we have

$$\begin{aligned}\||u(t)|^p u(t)\|_{H^1(\mathbb{R})} &\leq \||u(t)|^p u(t)\|_{L^2(\mathbb{R})} + C\, \||u(t)|^p u_x(t)\|_{L^2(\mathbb{R})} \\ &\leq \|u(t)\|^p_{L^\infty(\mathbb{R})} \|u(t)\|_{L^2(\mathbb{R})} + C\|u(t)\|^p_{L^\infty(\mathbb{R})} \|u_x(t)\|_{L^2(\mathbb{R})} \\ &\leq C_1 \|u(t)\|^{p+1}_{H^1(\mathbb{R})}.\end{aligned}$$

Thus, for any $u \in X$

$$\left\| \int_0^t S(t-s)|u(s)|^p u(s)ds \right\|_X \leq C_1 T \max_{t\in[0,T]} \|u(t)\|^{p+1}_{H^1(\mathbb{R})} \leq C_1 T M^{p+1}.$$

We obtain

$$\|\Phi u\|_X \leq \|\varphi\|_{H^1(\mathbb{R})} + C_1 T M^{p+1}$$

and, if we choose

$$M = 2\|\varphi\|_{H^1(\mathbb{R})} \text{ and } T = \frac{1}{C_1} 2^{-p-1} \|\varphi\|^{-p}_{H^1(\mathbb{R})} \tag{2.11}$$

it follows that

$$\|\Phi u\|_X \leq M.$$

Step 2. Φ is a contraction. For any $u, v \in X$ we have

$$\begin{aligned}
\|\Phi(u)-\Phi(v)\|_X &= \max_{t\in[0,T]} \left\| \int_0^t S(t-s)(|u|^p u - |v|^p v)(s)ds \right\|_{H^1(\mathbb{R})} \\
&\leq \max_{t\in[0,T]} \int_0^t \|(|u|^p u - |v|^p v)(s)\|_{H^1(\mathbb{R})}\, ds \\
&\leq T \max_{t\in[0,T]} \|(|u|^p u - |v|^p v)(t)\|_{H^1(\mathbb{R})} \\
&\leq T \max_{t\in[0,T]} \|u(t)-v(t)\|_{H^1(\mathbb{R})} \left(\|u(t)\|^p_{H^1(\mathbb{R})} + \|v(t)\|^p_{H^1(\mathbb{R})} \right) \\
&\leq T\|u-v\|_X \cdot 2M^p = \frac{1}{2}\|u-v\|_X,
\end{aligned}$$

where M and T are given by formula (2.11).

Thus, $\Phi : X \to X$ being a contraction, it follows that ϕ has a unique fixed point in X, that means there exists solution of equation (2.9). Using similar arguments we can show that the $H^1(\mathbb{R})$-solution of equation (2.9) is unique.

Repeating the above arguments we can extend the solution of problem 2.9 up to a time T_0 where we have the blow-up alternative: if $T_0 < \infty$ then $\lim_{t\uparrow T_0} \|u(t)\|_{H^1(\mathbb{R})} = \infty$. Since the $L^2(\mathbb{R})$-norm is conserved it means that when $T_0 < \infty$ we must have that $\lim_{t\uparrow T_0} \|u_x(t)\|_{L^2(\mathbb{R})} = \infty$. In the case $\lambda > 0$ using the conservation of the energy we have that

$$\|u_x(t)\|_{L^2(\mathbb{R})} \leq \mathcal{E}(t) = \mathcal{E}(0).$$

Thus, the solutions of (2.9) are global in this case, i.e. $u \in C([0,\infty), H^1(\mathbb{R}))$. □

Theorem 2.3.3. *L^2-solutions If $0 < p < 4/d$, then for all $\varphi \in L^2(\mathbb{R}^d)$ there exists unique global solution u of equation* (2.9) *that satisfies*

$$u \in C(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^r_{loc}(\mathbb{R}, L^{p+2}(\mathbb{R}^d)),$$

where $r = 4(p+2)/pd$.

Proof. We first prove the existence of a local solution. We will obtain the existence of a time $T = T(\|\varphi\|_{L^2(\mathbb{R}^d)}, d, \lambda, p)$ such that the integral equation (2.10) has a unique solution

$$u \in C([0,T], L^2(\mathbb{R}^d)) \cap L^r([0,T], L^{p+2}(\mathbb{R}^d)).$$

This argument is standard and we use the Banach Fix Point Theorem in a suitable space. We consider the space

$$\begin{aligned}
E(T,a) = \Big\{ & u \in C([0,T], L^2(\mathbb{R}^d)) \cap L^r([0,T], L^{p+2}(\mathbb{R}^d)) : \\
& |||u|||_T := \sup_{[0,T]} \|u(t)\|_{L^2(\mathbb{R}^d)} + \left(\int_0^T \|u(t)\|^r_{L^{p+2}(\mathbb{R}^d)} dt \right)^{1/r} \leq a \Big\},
\end{aligned}$$

with $0 < p < 4/d$ and $r = 4(p+2)/pd$. $E(T,a)$ is a complete metric space.

For appropriate values of a and $T > 0$ we shall show that

$$\Phi(u)(t) = e^{it\Delta}\varphi + i\lambda \int_0^t e^{i\Delta(t-s)}(|u|^p u)(s)ds \tag{2.12}$$

defines a contraction map on $E(T,a)$.

Without loss of generality we consider only the case $T > 0$.

$$\begin{aligned} \left(\int_0^T \|\Phi(u)(t)\|^r_{L^{p+2}(\mathbb{R}^d)}dt\right)^{1/r} &\leq c\|\varphi\|_{L^2(\mathbb{R}^d)} + c|\lambda| \left(\int_0^T \||u(t)|^{p+1}\|^{r'}_{L^{(p+2)/(p+1)}(\mathbb{R}^d)}dt\right)^{1/r'} \\ &\leq c\|\varphi\|_{L^2(\mathbb{R}^d)} + c|\lambda| \left(\int_0^T \|u(t)\|^{(p+1)r'}_{L^{p+2}(\mathbb{R}^d)}dt\right)^{1/r'}. \end{aligned} \tag{2.13}$$

By hypothesis $0 < p < 4/d$ so $(p+1)r' < r$, since

$$p + 1 < r - 1 = \frac{4(p+2)}{pd} - 1 \text{ which means } (p+1)r' = (p+1)\frac{r}{r-1} < r.$$

Therefore, from (2.13) we deduce that

$$\left(\int_0^T \|\Phi(u)(t)\|^r_{L^{p+2}(\mathbb{R}^d)}dt\right)^{1/r} \leq c\|\varphi\|_{L^2(\mathbb{R}^d)} + c|\lambda|T^\theta \left(\int_0^T \|u(t)\|^r_{L^{p+2}(\mathbb{R}^d)}dt\right)^{(p+1)/r},$$

with $\theta = 1 - pd/4 > 0$. Then, if $u \in E(T,a)$ we have

$$\left(\int_0^T \|\Phi(u)(t)\|^r_{L^{p+2}(\mathbb{R}^d)}dt\right)^{1/r} \leq c\|\varphi\|_{L^2(\mathbb{R}^d)} + c|\lambda|T^\theta a^{p+1}.$$

Using Theorem 2.2.1 and unitary group properties in expression (2.10), we obtain that if $u \in E(T,a)$ then

$$\begin{aligned} \sup_{t\in[0,T]} \|\Phi(u)(t)\|_{L^2(\mathbb{R}^d)} &\leq c\|\varphi\|_{L^2(\mathbb{R}^d)} + c|\lambda| \left(\int_0^T \||u(t)|^{p+1}\|^{r'}_{L^{(p+2)/(p+1)}(\mathbb{R}^d)}dt\right)^{1/r'} \\ &\leq c\|\varphi\|_{L^2(\mathbb{R}^d)} + c|\lambda|T^\theta a^{p+1}, \end{aligned}$$

where constant c depends only on p and dimension d. Hence,

$$|||\Phi|||_T \leq c\|\varphi\|_{L^2(\mathbb{R}^d)} + c|\lambda|T^\theta a^{p+1}.$$

If we fix $a = 2c\|\varphi\|_{L^2(\mathbb{R}^d)}$ and take $T > 0$ such that

$$2^{p+1}c^{p+1}|\lambda|T^\theta\|\varphi\|^p_{L^2(\mathbb{R}^d)} < 1/2 \tag{2.14}$$

it follows that the application Φ is well defined on $E(T,a)$. Now, if $u, v \in E(T,a)$,

$$(\Phi(v) - \Phi(u))(t) = i\lambda \int_0^t e^{i(t-s)\Delta}(|v|^p v - |u|^p u)(s)ds.$$

Thus

$$
\begin{aligned}
&\left(\int_0^T \|(\phi(v)-\phi(u))(t)\|^r_{L^{p+2}(\mathbb{R}^d)}dt\right)^{1/r}\\
&\leq c|\lambda|\left(\int_0^T \||v|^p v-|u|^p u)\|^{r'}_{L^{(p+2)(p+1)}(\mathbb{R}^d)}dt\right)^{1/r'}\\
&\leq c_p|\lambda|\left(\int_0^T (\|v\|^p_{L^{p+2}(\mathbb{R}^d)}+\|u\|^p_{L^{p+2}(\mathbb{R}^d)})^{r'}\|v-u\|^{r'}_{L^{p+2}(\mathbb{R}^d)}(t)dt\right)^{1/r'}\\
&\leq c_p|\lambda|T^\theta\left(\left(\int_0^T \|v\|^r_{L^{p+2}(\mathbb{R}^d)}dt\right)^{p/r}+\left(\int_0^T \|u\|^r_{L^{p+2}(\mathbb{R}^d)}dt\right)^{p/r}\right)\cdot\left(\int_0^T \|v(t)-u(t)\|^r_{L^{p+2}(\mathbb{R}^d)}dt\right)^{p/r}\\
&\leq 2c_p|\lambda|T^\theta a^p\left(\int_0^T \|v(t)-u(t)\|^r_{L^{p+2}(\mathbb{R}^d)}dt\right)^{1/r}.
\end{aligned}
$$

Using the estimates from Theorem 2.2.1 we get

$$\sup_{[0,T]}\|(\Phi(v)-\Phi(u))(t)\|_{L^2(\mathbb{R}^d)}\leq 2c_p|\lambda|T^\theta a^p\left(\int_0^T \|v(t)-u(t)\|^r_{L^{p+2}(\mathbb{R}^d)}\right)^{1/r}.$$

Finally, it follows from the choice of a, that is $a\leq 2c\|\varphi\|_{L^2(\mathbb{R}^d)}$ and inequality (2.14), that

$$2c|\lambda|T^\theta a^p\leq 2^{p+1}c^{p+1}|\lambda|T^\theta\|\varphi\|^p_{L^2(\mathbb{R}^d)}<1.$$

Hence,

$$T\sim\|\varphi\|^\beta_{L^2(\mathbb{R}^d)},\ \text{with}\ \beta=\frac{-4p}{4-pd}. \tag{2.15}$$

Thus, we have proved the existence and uniqueness in an appropriate class of solution of equation (2.10).

This proves the existence of a local solution

$$u\in C([0,T],L^2(\mathbb{R}^d))\cap L^r([0,T],L^{p+2}(\mathbb{R}^d)),$$

where $T=T(\|\varphi\|_{L^2(\mathbb{R}^d)},d,\lambda,p)$. Using the conservation of the $L^2(\mathbb{R}^d)$-norm, we obtain that the solution exists globally. □

Chapter 3

The discrete Schrodinger Equation

In this chapter we will present some results obtained in [5], where we proved dispersive estimates for the system formed by two coupled discrete Schrödinger equations. We obtained estimates for the resolvent of the discrete operator and prove that it satisfies the limiting absorption principle. The decay of the solutions was proved by using classical and some new results on oscillatory integrals.

Let us consider the linear Schrödinger equation (LSE) in dimension 1:

$$\begin{cases} iu_t + u_{xx} = 0, \ x \in \mathbb{R}, \ t \neq 0, \\ u(0,x) = \varphi(x), \ x \in \mathbb{R}. \end{cases} \tag{3.1}$$

Linear equation (3.1) is solved by $u(t,x) = S(t)\varphi$, where $S(t) = e^{it\Delta}$ is the free Schrödinger operator. The linear semigroup has two important properties. First, the conservation of the L^2-norm:

$$\|S(t)\varphi\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})} \tag{3.2}$$

and a dispersive estimate of the form:

$$|(S(t)\varphi)(x)| \leq \frac{1}{(4\pi|t|)^{1/2}} \|\varphi\|_{L^1(\mathbb{R})}, \ \ x \in \mathbb{R}, \ \ t \neq 0. \tag{3.3}$$

The space-time estimate

$$\|S(\cdot)\varphi\|_{L^6(\mathbb{R},\, L^6(\mathbb{R}))} \leq C\|\varphi\|_{L^2(\mathbb{R})}, \tag{3.4}$$

due to Strichartz [14], is deeper. It guarantees that the solutions of system (3.1) decay as t becomes large and that they gain some spatial integrability. Inequality (3.4) was generalized by Ginibre and Velo [3]. They proved the mixed space-time estimates, well known as Strichartz estimates:

$$\|S(\cdot)\varphi\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}))} \leq C(q,r)\|\varphi\|_{L^2(\mathbb{R})} \tag{3.5}$$

for the sharp 1/2-admissible pairs (q, r):

$$\frac{1}{q} = \frac{1}{2}\Big(\frac{1}{2} - \frac{1}{r}\Big), \quad 2 \le q, r \le \infty. \tag{3.6}$$

Similar results can be stated in any space dimension but it is beyond the scope of this article. These estimates have been successfully applied to obtain well-posedness results for the nonlinear Schrödinger equation (see [2], [15] and the reference therein).

Let us now consider the following system of difference equations

$$\begin{cases} iu_t + \Delta_d u = 0, & j \in \mathbb{Z}, t \neq 0, \\ u(0) = \varphi, \end{cases} \tag{3.7}$$

where Δ_d is the discrete laplacian defined by

$$(\Delta_d u)(j) = u_{j+1} - 2u_j + u_{j-1}, \quad j \in \mathbb{Z}.$$

Concerning the long time behavior of the solutions of system (3.7) in [13] the authors have proved that a similar to the continuous Scrödinger equation decay property holds:

$$\|u(t)\|_{l^\infty(\mathbb{Z})} \le C(|t| + 1)^{-1/3} \|\varphi\|_{l^1(\mathbb{Z})}, \quad \forall\, t \neq 0. \tag{3.8}$$

The proof of (3.8) consists in writing solution u of (3.7) as the convolution between a kernel K_t and the initial data φ and then estimate K_t by using Van der Corput's lemma. For the linear semigroup $\exp(it\Delta_d)$, Strichartz like estimates similar to those in (3.5) have been obtained in [13] for a larger class of pairs (q, r), namely 1/3-admissible pairs,

$$\frac{1}{q} \le \frac{1}{3}\Big(\frac{1}{2} - \frac{1}{r}\Big), \quad 2 \le q, r \le \infty. \tag{3.9}$$

We give here the proof of the decay property (3.8) for the solutions of equation (3.7).

Theorem 3.0.4. *For any $\varphi \in l^2(\mathbb{Z})$ there exists a unique solution $u \in C(\mathbb{R}, l^2(\mathbb{Z}))$ of system (3.7). Moreover the solution u satisfies*

1. *the energy identity: $\|u(t)\|_{l^2(\mathbb{Z})} = \|\varphi\|_{l^2(\mathbb{Z})}$,*
2. *the decay estimate: $\|u(t)\|_{l^\infty(\mathbb{Z})} \le C(1 + |t|)^{-1/3} \|\varphi\|_{l^1(\mathbb{Z})}$.*

Proof. The well-posedness is a consequence of the fact that the operator Δ_d is bounded on $l^2(\mathbb{Z})$. Function u being defined on $\mathbb{Z}$ we apply the discrete Fourier transform with $h = 1$:

$$\begin{cases} i\widehat{u}_t(t, \xi) + \widehat{\Delta_d u}(t, \xi) = 0, & j \in \mathbb{Z}, t \neq 0, \xi \in [-\pi, \pi], \\ \hat{u}(0, \xi) = \hat{\varphi}(\xi), & \xi \in [-\pi, \pi], \end{cases} \tag{3.10}$$

We now compute $\widehat{\Delta_d u}$:

$$\widehat{\Delta_d u}(\xi) = \sum_{j\in\mathbb{Z}} e^{-ij\xi}(\Delta u)(j) = \sum_{j\in\mathbb{Z}} e^{-ij\xi}(u(j+1) - 2u(j) + u(j-1))$$

$$= \sum_{j\in\mathbb{Z}} e^{-i(j-1)\xi}u(j) - 2\sum_{j\in\mathbb{Z}} e^{-ij\xi}u(j) + \sum_{j\in\mathbb{Z}} e^{-i(j+1)\xi}u(j)$$

$$= \left(e^{i\xi} + e^{-i\xi} - 2\right)\hat{u}(\xi) = 2(\cos\xi - 1)\hat{u}(\xi) = -4\sin^2\frac{\xi}{2}\hat{u}(\xi).$$

System (3.10) becomes

$$\begin{cases} i\hat{u}_t(\xi) - 4\sin^2\frac{\xi}{2}\hat{u}(\xi) = 0, & \xi\in[-\pi,\pi], t\neq 0, \\ \hat{u}(0,\xi) = \hat{\varphi}(\xi), & \xi\in[-\pi,\pi]. \end{cases}$$

This is an ordinary differential equation with initial data $\hat{\varphi}$ and its solution is given by:

$$\hat{u}(t,\xi) = e^{-4it\sin^2\frac{\xi}{2}}\hat{\varphi}(\xi), \quad \xi\in[-\pi,\pi].$$

Let K_t be such that $\widehat{K_t}(\xi) = e^{-4it\sin^2\frac{\xi}{2}}$. Then $\hat{u}(t,\xi) = \widehat{K_t}(\xi)\hat{\varphi}(\xi)$ and from the properties of the discrete Fourier Transform it follows that $u(t) = K_t * \varphi$, where here $*$ is the discrete convolution:

$$u(t,j) = \sum_{k\in\mathbb{Z}} K_t(j-k)\varphi(k), \quad \forall j\in\mathbb{Z}.$$

The kernel K_t satisfies $\|K_t\|_{l^\infty(\mathbb{Z})} \le 1$, since

$$K_t(j) = \frac{1}{2\pi}\int_{\pi}^{\pi} e^{-4it\sin^2\frac{\xi}{2}}e^{ij\xi}d\xi.$$

Using the properties of the kernel K_t we obtain the following estimates for the solution u of system (3.10):

1.$\|u(t)\|_{l^2(\mathbb{Z})} = \|\hat{u}(t)\|_{L^2(\pi,\pi)} = \left(\int_{-\pi}^{\pi}|e^{-4it\sin^2\frac{\xi}{2}}\hat{\varphi}(\xi)|^2d\xi\right)^{1/2} = \|\hat{\varphi}\|_{L^2(-\pi,\pi)} = \|\varphi\|_{L^2(-\pi,\pi)}.$

2.$\|u\|_{l^\infty(\mathbb{Z})} = \|K_t * \varphi\|_{l^\infty(\mathbb{Z})} \le \|K_t\|_{l^\infty(\mathbb{Z})}\cdot\|\varphi\|_{l^1(\mathbb{Z})} \le \|\varphi\|_{l^1(\mathbb{Z})}.$

We can obtain stronger estimates for $\|u\|_{l^\infty(\mathbb{Z})}$ by using Van der Corput's Lemma. We write the kernel K_t as follows:

$$K_t(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-4it\sin^2\frac{\xi}{2}}e^{ij\xi}d\xi = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{it(-4\sin^2\frac{\xi}{2}+\frac{j\xi}{t})}d\xi.$$

Set

$$p(\xi) = -4\sin^2\frac{\xi}{2} + \frac{j\xi}{t}.$$

The derivatives of p are

$$\begin{cases} p^{'}(\xi) = \frac{j}{t} - 4\sin\frac{\xi}{2}\cos\frac{\xi}{2} = \frac{j}{t} - 2\sin\xi, \\ p^{''}(\xi) = -2\cos\xi, \\ p^{'''}(\xi) = 2\sin\xi. \end{cases}$$

Thus, on the interval $[-\pi,\pi]$ the second and the third derivatives can not vanish at the same point since $p^{''}(\xi) = 0$ only for $\xi \in \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$ and $p^{'''}(\xi) = 0$ only for $\xi \in \{-\pi,\pi\}$. We split the previous integral as follows:

$$\int_{-\pi}^{\pi} e^{itp(\xi)}d\xi = \int_{-\pi}^{-\frac{3\pi}{4}} e^{itp(\xi)}d\xi + \int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} e^{itp(\xi)}d\xi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{itp(\xi)}d\xi + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{itp(\xi)}d\xi \int_{\frac{3\pi}{4}}^{\pi} e^{itp(\xi)}d\xi.$$

Using Van der Corput's Lemma we obtain that

$$\left|\int_{-\pi}^{-\frac{3\pi}{4}} e^{itp(\xi)}d\xi\right| \le |t|^{-\frac{1}{2}}, \quad \left|\int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} e^{itp(\xi)}d\xi\right| \le |t|^{-\frac{1}{3}}, \quad \left|\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{itp(\xi)}d\xi\right| \le |t|^{-\frac{1}{2}},$$

$$\left|\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{itp(\xi)}d\xi\right| \le |t|^{-\frac{1}{3}}, \quad \left|\int_{\frac{3\pi}{4}}^{\pi} e^{itp(\xi)}d\xi\right| \le |t|^{-\frac{1}{2}}.$$

Thus, for $t > 1$ we obtain $|K_t(j)| \le C|t|^{-\frac{1}{3}}, \forall j \in \mathbb{Z}$. Using that $\|K_t\|_{l^\infty(\mathbb{Z})} \le 1$ we get

$$\|K_t\|_{l^\infty(\mathbb{Z})} \le C(|t|+1)^{-1/3}$$

and

$$\|u(t)\|_{l^\infty(\mathbb{Z})} \le C(|t|+1)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z})}, \quad \forall t \ne 0.$$

□

Theorem 3.0.5. *For any* $1/3$*-admissible pair* (q,r) *the solution of the inhomogeneous equation*

$$\begin{cases} iu_t(t,j) + \Delta_d u(t,j) + f(t,j) = 0, & j \in \mathbb{Z}, t \in \mathbb{R} \\ u(0,j) = \varphi(j), & j \in \mathbb{Z} \end{cases} \tag{3.11}$$

satisfies the following estimates:

$$\|u(t)\|_{L^q(\mathbb{R},l^r(\mathbb{Z}))} \le C\left(\|\varphi\|_{l^2(\mathbb{Z})} + \|f(t)\|_{L^{\tilde{q}'}(\mathbb{R},l^{\tilde{r}'}(\mathbb{Z}))}.\right)$$

Proof. We apply Theorem 2.2.2 with $\sigma = 1/3$ since $e^{it\Delta_d}\varphi$ satisfies:

$$\begin{aligned} &\|e^{it\Delta_d}\varphi\|_{l^2(\mathbb{Z})} = C\|\varphi\|_{l^2(\mathbb{Z})} \\ &\|e^{it\Delta_d}(e^{is\Delta_d})^*\varphi\|_{l^\infty(\mathbb{Z})} = \|e^{i(t-s)\Delta_d}\varphi\|_{l^\infty(\mathbb{Z})} \le C(1+|t-s|)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z})}. \end{aligned} \tag{3.12}$$

□

We also mention [6] and [7] where the authors consider a similar equation on $h\mathbb{Z}$ by replacing Δ_d by Δ_d/h^2 and analyze the same properties in the context of numerical approximations of the linear and nonlinear Schrödinger equation.

A more thorough analysis has been done in [10] and [12] where the authors analyze the decay properties of the solutions of equation $iu_t + Au = 0$ where $A = \Delta_d - V$, with V a real-valued potential. In these papers $l^1(\mathbb{Z}) - l^\infty(\mathbb{Z})$ and $l^2_{-\sigma}(\mathbb{Z}) - l^2_\sigma(\mathbb{Z})$ estimates for $\exp(itA)P_{a,c}(A)$ have been obtained where $P_{a,c}(A)$ is the spectral projection to the absolutely continuous spectrum of A and $l^2_{\pm\sigma}(\mathbb{Z})$ are weighted $l^2(\mathbb{Z})$-spaces.

In what concerns the Schödinger equation with variable coefficients we mention the results of Banica [1]. Consider a partition of the real axis as follows: $-\infty = x_0 < x_1 < \cdots < x_{n+1} = \infty$ and a step function $\sigma(x) = b_i^{-2}$ for $x \in (x_i, x_{i+1})$, where b_i are positive numbers. The solution u of the Schrödinger equation

$$\begin{cases} iu_t(t,x) + (\sigma(x)u_x)_x(t,x) = 0, & \text{for } x \in \mathbb{R}, t \neq 0, \\ \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

satisfies the dispersion inequality

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq C|t|^{-1/2}\|u_0\|_{L^1(\mathbb{R})}, \qquad t \neq 0,$$

where constant C depends on n and on sequence $\{b_i\}_{i=0}^n$. We recall that in [4] the above result was used in the analysis of the long time behavior of the solutions of the linear Schödinger equation on regular trees. In the case of discrete equations the corresponding model is given by

$$\begin{cases} iU_t + AU = 0, \ t \neq 0, \\ U(0) = \varphi, \end{cases} \tag{3.13}$$

where the infinite matrix A is symmetric with a finite number of diagonals nonidentically vanishing. Once a result similar to [1] will be obtained for discrete Schrödinger equations with non-constant coefficients we can apply it to obtain dispersive estimates for discrete Schrödinger equations on trees. But as far as we know the study of the decay properties of solutions of system (3.13) in terms of the properties of A is a difficult task and we try to give here a partial answer to this problem. In the case when A is a diagonal matrix these properties are easily obtained by using the Fourier transform and classical estimates for oscillatory integrals.

The main goal of this article is to analyze a simplified model which consists in coupling two DSE by Kirchhoff's type condition:

$$\begin{cases} iu_t(t,j) + b_1^{-2}(\Delta_d u)(t,j) = 0 & j \leq -1,\ t \neq 0, \\ iv_t(t,j) + b_2^{-2}(\Delta_d v)(t,j) = 0 & j \geq 1,\ t \neq 0, \\ u(t,0) = v(t,0), & t \neq 0, \\ b_1^{-2}(u(t,-1) - u(t,0)) = b_2^{-2}(v(t,0) - v(t,1)), & t \neq 0, \\ u(0,j) = \varphi(j), & j \leq -1, \\ v(0,j) = \varphi(j), & j \geq 1. \end{cases} \tag{3.14}$$

In the above system $u(t,0)$ and $v(t,0)$ have been artificially introduced to couple the two equations on positive and negative integers. The third condition in the above system requires continuity along the interface $j = 0$ and the fourth one can be interpreted as the continuity of the flux along the interface.

The main result of this paper is given in the following theorem.

Theorem 3.0.6. *For any $\varphi \in l^2(\mathbb{Z} \setminus \{0\})$ there exists a unique solution $(u,v) \in C(\mathbb{R}, l^2(\mathbb{Z} \setminus \{0\}))$ of system (3.14). Moreover, there exists a positive constant $C(b_1, b_2)$ such that*

$$\|(u,v)(t)\|_{l^\infty(\mathbb{Z}\setminus\{0\})} \leq C(b_1,b_2)(|t|+1)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z}\setminus\{0\})}, \quad \forall t \in \mathbb{R}, \tag{3.15}$$

holds for all $\varphi \in l^1(\mathbb{Z} \setminus \{0\})$.

Using the well-known results of Keel and Tao [8] we obtain the following Strichartz-like estimates for the solutions of system (3.14).

Theorem 3.0.7. *For any $\varphi \in l^2(\mathbb{Z} \setminus \{0\})$ the solution (u,v) of system (3.14) satisfies*

$$\|(u,v)\|_{L^q(\mathbb{R}, l^r(\mathbb{Z}\setminus\{0\}))} \leq C(q,r)\|\varphi\|_{l^2(\mathbb{Z}\setminus\{0\})}$$

for all pairs (q,r) satisfying (3.9).

The paper is organized as follows: In section 3.1 we present some discrete models, in particular system (3.14) in the case $b_1 = b_2$ and show how it is related with problem (3.7). In addition, a system with a dynamic coupling along the interface is presented. In section 3.2 we obtain an explicit formula for the resolvent associated with system (3.14). We prove a limiting absorption principle and we give the proof of the main result of this paper. Finally we present some open problems.

3.1 Some discrete models

In this section in order to emphasize the main differences and difficulties with respect to the continuous case when we deal with discrete systems we will consider two models. In the first case we consider system (3.14) with the two coefficients in the front of the discrete laplacian equal. In the following we denote $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$.

Theorem 3.1.1. *Let us assume that* $b_1 = b_2$. *For any* $\varphi \in l^2(\mathbb{Z}^*)$ *there exists a unique solution* $u \in C(\mathbb{R}, l^2(\mathbb{Z}^*))$ *of system* (3.14). *Moreover there exists a positive constant* $C(b_1)$ *such that*

$$\|u(t)\|_{l^\infty(\mathbb{Z}^*)} \leq C(b_1)(|t|+1)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z}^*)}, \quad \forall\ t \in \mathbb{R}, \tag{3.1}$$

holds for all $\varphi \in l^1(\mathbb{Z}^*)$.

In the particular case considered here we can reduce the proof of the dispersive estimate (3.1) to the analysis of two problems: one with Dirichlet's boundary condition and another one with a discrete Neumann's boundary condition.

Before starting the proof of Theorem 3.1.1 let us recall that in the case of system (3.7) its solution is given by $u(t) = K_t * \varphi$ where $*$ is the standard convolution on $\mathbb{Z}$ and

$$K_t(j) = \int_{-\pi}^{\pi} e^{-4it\sin^2(\frac{\xi}{2})} e^{ij\xi} d\xi, \quad t \in \mathbb{R},\ j \in \mathbb{Z}.$$

In [13] a simple argument based on Van der Corput's lemma has been used to show that for any real number t the following holds:

$$|K_t(j)| \leq C(|t|+1)^{-1/3}, \quad \forall j \in \mathbb{Z}. \tag{3.2}$$

Proof of Theorem 3.1.1. The existence of the solutions is immediate since operator A defined in (3.7) is bounded in $l^2(\mathbb{Z}^*)$. We prove now the decay property (3.1.1). Let us restrict for simplicity to the case $b_1 = b_2 = 1$.

For (u, v) solution of system (3.14) let us set

$$S(j) = \frac{v(j) + u(-j)}{2}, \quad D(j) = \frac{v(j) - u(-j)}{2}, j \geq 0.$$

Observe that u and v can be recovered from S and D as follows

$$(u, v) = ((S - D)(-\cdot), S + D).$$

Writing the equations satisfied by u and v we obtain that D and S solve two discrete Schrödinger equations on $Z^+ = \{j \in \mathbb{Z}, j \geq 1\}$ with Dirichlet, respectively Neumann boundary conditions:

$$\begin{cases} iD_t(t,j) + (\Delta_d D)(t,j) = 0 & j \geq 1,\ t \neq 0, \\ D(t,0) = 0, & t \neq 0, \\ D(0,j) = \frac{\varphi(j) - \varphi(-j)}{2}, & j \geq 1, \end{cases} \tag{3.3}$$

and

$$\begin{cases} iS_t(t,j)+(\Delta_d S)(t,j)=0 & j\geq 1,\ t\neq 0,\\ S(t,0)=S(t,1), & t\neq 0,\\ S(0,j)=\frac{\varphi(j)+\varphi(-j)}{2}, & j\geq 1. \end{cases} \tag{3.4}$$

Making an odd extension of the function D and using the representation formula for the solutions of (3.7) we obtain that the solution of the Dirichlet problem (3.3) satisfies

$$D(t,j)=\sum_{k\geq 1}(K_t(j-k)-K_t(j+k))D(0,k), \quad t\neq 0,\ j\geq 1. \tag{3.5}$$

A similar even extension of function S permits us to obtain the explicit formula for the solution of the Neumann problem (3.4)

$$S(t,j)=\sum_{k\geq 1}(K_t(k-j)+K_t(k+j-1))S(0,k), \quad t\neq 0,\ j\geq 1. \tag{3.6}$$

Using the decay of the kernel K_t given by (3.2) we obtain that $S(t)$ and $D(t)$ decay as $(|t|+1)^{-1/3}$ and then the same property holds for u and v. This finishes the proof of this particular case. □

Observe that our proof has taken into account the particular structure of the equations. When the coefficients b_1 and b_2 are not equal we cannot write an equation verified by functions D or S.

We now write system (3.14) in matrix formulation. Using the coupling conditions at $j=0$ system (3.14) can be written in the following equivalent form

$$\begin{cases} iU_t+AU=0,\\ U(0)=\varphi, \end{cases}$$

where $U=(u,v)^T$, $u=(u(j))_{j\leq -1}$, $v=(v_j)_{j\geq 1}$ and

$$A=\begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0\\ 0 & b_1^{-2} & -2b_1^{-2} & b_1^{-2} & 0 & 0 & 0 & 0\\ 0 & 0 & b_1^{-2} & -b_1^{-2}-\frac{1}{b_1^2+b_2^2} & \frac{1}{b_1^2+b_2^2} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{b_1^2+b_2^2} & -\frac{1}{b_1^2+b_2^2}-b_2^{-2} & b_2^{-2} & 0 & 0\\ 0 & 0 & 0 & 0 & b_2^{-2} & -2b_2^{-2} & b_2^{-2} & 0\\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}. \tag{3.7}$$

In the particular case $b_1 = b_2 = 1$ the operator A can be decomposed as follows

$$A = \Delta_d + B = \begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix} + \begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}.$$

However, we do not know how to use the dispersive properties of $\exp(it\Delta_d)$ and the particular structure of B in order to obtain the decay of the new semigroup $\exp(it(\Delta_d + B))$.

Another model of interest is the following one inspired in the numerical approximations of LSE. Set

$$a(x) = \begin{cases} b_1^{-2}, & x < 0, \\ b_2^{-2}, & x > 0. \end{cases}$$

Using the following discrete derivative operator

$$(\partial u)(x) = u(x + \frac{1}{2}) - u(x - \frac{1}{2})$$

we can introduce the second order discrete operator

$$\partial(a\partial u)(j) = a(j + \frac{1}{2})u(j+1) - \big(a(j + \frac{1}{2}) + a(j - \frac{1}{2})\big)u(j) + a(j - \frac{1}{2})u(j-1), j \in \mathbb{Z}.$$

In this case we have to analyze the following system

$$\begin{cases} iu_t(t,j) + b_1^{-2}(\Delta_d u)(t,j) = 0, & j \leq -1, \ t \neq 0, \\ iu_t(t,j) + b_2^{-2}(\Delta_d u)(t,j) = 0, & j \geq 1, \ t \neq 0, \\ iu_t(t,0) + b_1^{-2}u(t,-1) - (b_1^{-2} + b_2^{-2})u(t,0) + b_2^{-1}u(t,1) = 0, & t \neq 0, \\ u(0,j) = \varphi(j), & j \in \mathbb{Z}. \end{cases} \tag{3.8}$$

In matrix formulation it reads $iU_t + AU = 0$ where $U = (u(j))_{j\in\mathbb{Z}}$, and the operator A is given by the following one

$$A = \begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 \\ 0 & b_1^{-2} & -2b_1^{-2} & b_1^{-2} & 0 & 0 & 0 \\ 0 & 0 & b_1^{-2} & -(b_1^{-2} + b_2^{-2}) & b_2^{-2} & 0 & 0 \\ 0 & 0 & 0 & b_2^{-2} & -2b_2^{-2} & b_2^{-2} & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}. \tag{3.9}$$

Observe that in the case $b_1 = b_2$ the results of [13] give us the decay of the solutions.

Regarding the long time behavior of the solutions of system (3.8) we have the following result.

Theorem 3.1.2. *For any $\varphi \in l^2(\mathbb{Z})$ there exists a unique solution $u \in C(\mathbb{R}, l^2(\mathbb{Z}))$ of system* (3.8)*. Moreover, there exists a positive constant $C(b_1, b_2)$ such that*

$$\|u(t)\|_{l^\infty(\mathbb{Z})} \leq C(b_1, b_2)(|t| + 1)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z})}, \quad \forall t \in \mathbb{R},$$

holds for all $\varphi \in l^1(\mathbb{Z})$.

The proof of this result is similar to the one of Theorem 3.0.6 and we will only sketch it at the end of Section 3.2.

3.2 Proof of the main result

In this section we prove the main result of this paper. In order to do this, we will follow the ideas of [1] in the case of a discrete operator. Let us consider the system

$$\begin{cases} iU_t + AU = 0, \\ U(0) = \varphi, \end{cases} \tag{3.1}$$

where $U(t) = (u(t, j))_{j \neq 0}$ and operator A is given by (3.7). We compute explicitly the resolvent $(A - \lambda I)^{-1}$, we obtain a limiting absorption principle and finally we prove the main result of this paper Theorem 3.0.6.

3.2.1 The resolvent.

We start by localizing the spectrum of operator A and computing the resolvent $R(\lambda) = (A - \lambda I)^{-1}$. We use some classical results on difference equations.

Theorem 3.2.1. *For any b_1 and b_2 positive the spectrum of operator A satisfies*

$$\sigma(A) = [-4\max\{b_1^{-2}, b_2^{-2}\}, 0]. \tag{3.2}$$

Proof. Since A is self-adjoint we have that

$$\sigma(A) \subset [\inf_{\|u\|_{l^2(\mathbb{Z}^*)} \leq 1} (Au, u), \sup_{\|u\|_{l^2(\mathbb{Z}^*)} \leq 1} (Au, u)].$$

Explicit computations show that

$$(Au, u) = -b_1^{-2} \sum_{j \leq -1} (u_j - u_{j-1})^2 - \frac{1}{b_1^2 + b_2^2}(u_{-1} - u_1)^2 - b_2^{-2} \sum_{j \geq 1} (u_{j+1} - u_j)^2.$$

It is easy to see that $(Au, u) \leq 0$ and

$$(Au, u) \geq -2\max\{b_1^{-2}, b_2^{-2}\} \sum_{j \in \mathbb{Z}^*} (u_j^2 + u_{j+1}^2) = -4\max\{b_1^{-2}, b_2^{-2}\} \sum_{j \in \mathbb{Z}^*} u_j^2.$$

In order to prove that the spectrum is continuous we need to prove that for any $\lambda \in [-4\max\{b_1^{-2}, b_2^{-2}\}, 0]$ we can find $u_n \in l^2(\mathbb{Z}^*)$ with $\|u_n\|_{l^2(\mathbb{Z}^*)} \leq 1$ such that $\|(A-\lambda I)u_n\|_{l^2(\mathbb{Z}^*)}$ tends to zero. To fix the ideas let us assume that $b_2 \leq b_1$ and $\lambda \in [-4b_2^{-2}, 0]$. We construct u_n such that all its components $u_{n,j}$, $j \leq -1$, vanish. Thus for such u_n's we have that

$$(Au_n)_j = b_2^{-2}(\Delta_d u_n)_j, \ j \geq 1.$$

Using the fact that any $\lambda \in [-4b_2^{-2}, 0]$ belongs to $\sigma(b_2^{-2}\Delta_d)$ we can construct sequences $(u_{n,j})_{j\geq 1}$ such that $\|u_n\|_{l^2(\mathbb{Z}^*)} \leq 1$ and $\|(A-\lambda I)u_n\|_{l^2(\mathbb{Z}^*)} \to 0$. This implies that $\lambda \in \sigma(A)$ and the proof is finished. $\square$

Before computing the resolvent $(A-\lambda I)^{-1}$ we need some results for difference equations.

Lemma 3.2.1. *For any $\lambda \in \mathbb{C} \setminus [-4, 0]$ and $g \in l^2(\mathbb{Z}^*)$, any solution $f \in l^2(\mathbb{Z}^*)$ of*

$$\Delta_d f(j) - \lambda f(j) = g(j), \quad j \neq 0$$

with $f(0)$ prescribed is given by

$$f(j) = \alpha r^{|j|} + \frac{1}{2r - 2 - \lambda} \sum_{k\in\mathbb{Z}^*} r^{|j-k|} g(k) \tag{3.3}$$

where α is determined by $f(0)$ and r is the unique solution with $|r| < 1$ of

$$r^2 - 2r + 1 = \lambda r.$$

Moreover

$$f(j) = f(0)r^{|j|} + \frac{1}{r - r^{-1}} \sum_k (r^{|j-k|} - r^{|j|+|k|}) g(k), \quad j \neq 0.$$

Proof. Let us consider the case when $j \geq 1$, the other case $j \leq -1$ can be treated similarly. Writing the equation satisfied by f we obtain that

$$f(j+1) - (2+\lambda)f(j) + f(j-1) = g(j), \quad j \geq 1.$$

This is an inhomogeneous difference equation whose solutions are written as the sum between a particular solution and the general solution for the homogeneous difference equation

$$f(j+1) - (2+\lambda)f(j) + f(j-1) = 0, \quad j \geq 1.$$

Let us denote by r_1 and r_2, $|r_1| \leq |r_2|$, the two solutions of the second order equation

$$r^2 - (2+\lambda)r + 1 = 0.$$

Since $2+\lambda \in \mathbb{C}\setminus[-2,2]$ we have that r_1 and r_2 belong to $\mathbb{C}\setminus\mathbb{R}$ and more than that $|r_1|<1<|r_2|$. Thus we obtain that

$$f(j)=\alpha r_1^j+\beta r_2^j+\frac{1}{2r-2-\lambda}\sum_{k\in\mathbb{Z}^*} r_1^{|j-k|}g(k). \tag{3.4}$$

Since f is an $l^2(\mathbb{Z}^+)$ function we should have $\beta=0$. Then formula (3.3) holds. The last identity is obtained by putting $j=0$ in (3.4) and using that $2r-2-\lambda=r-r^{-1}$. $\square$

As an application of the previous Lemma we have the following result.

Lemma 3.2.2. *Set $Z_1=\mathbb{Z}\cap(-\infty,-1]$ and $Z_2=\mathbb{Z}\cap[1,\infty)$. For any $\lambda\in\mathbb{C}\setminus[-4\max\{b_1^{-2},b_2^{-2}\},0]$ and $g\in l^2(\mathbb{Z}^*)$, any solution $f\in l^2(\mathbb{Z})$ of*

$$b_s^{-2}\Delta_d f(j)-\lambda f(j)=g(j),\quad j\in Z_s,$$

with $f(0)$ prescribed is given by

$$f(j)=\alpha_s r_s^{|j|}+\frac{b_s^2}{2r_s-2-\lambda b_s^2}\sum_{k\in Z_s} r_s^{|j-k|}g(k),\quad j\in Z_s, s\in\{1,2\} \tag{3.5}$$

where for $s\in\{1,2\}$, constant α_s is determined by $f(0)$ and r_s is the unique solution with $|r_s|<1$ of

$$r_s^2-2r_s+1=\lambda r_s b_s^2.$$

Moreover

$$f(j)=f(0)r_s^{|j|}+\frac{b_s^2}{r_s-r_s^{-1}}\sum_{k\in Z_s}(r_s^{|j-k|}-r_s^{|j|+|k|})g(k), j\in Z_s. \tag{3.6}$$

The proof of this lemma consists in just applying Lemma 3.2.1 to the difference equations in Z_1 and Z_2.

Lemma 3.2.3. *Let $\lambda\in\mathbb{C}\setminus[-4\max\{b_1^{-2},b_2^{-2}\},0]$. For any $g\in l^2(\mathbb{Z}^*)$ there exists a unique solution $f\in l^2(\mathbb{Z}^*)$ of the equation $(A-\lambda I)f=g$. Moreover, it is given by the following formula*

$$\begin{aligned} f(j)=&\frac{-r_s^{|j|}}{b_2^{-2}(1-r_2)+b_1^{-2}(1-r_1)}\Big[\sum_{k\in Z_1} r_1^{|k|}g(k)+\sum_{k\in Z_2} r_2^{|k|}g(k)\Big] \\ &+\frac{b_s^2}{r_s-r_s^{-1}}\sum_{k\in Z_s}(r_s^{|j-k|}-r_s^{|j|+|k|})g(k),\quad j\in Z_s, \end{aligned} \tag{3.7}$$

where for $s\in\{1,2\}$, $r_s=r_s(\lambda)$ is the unique solution with $|r_s|<1$ of the equation

$$r_s^2-2r_s+1=\lambda b_s^2 r_s.$$

Proof. Any solution of $(A-\lambda I)f=g$ satisfies

$$\begin{cases} \Delta_d f(j) - b_s^2\lambda f(j) = b_s^2 g(j), \quad j\in Z_s, \\ b_1^{-2}(f(-1)-f(0)) = b_2^{-2}(f(0)-f(1)), \end{cases}$$

where $f(0)$ is artificially introduced in order to write the system in a convenient form that permits us to apply Lemma 3.2.2.

Using (3.6) we obtain

$$f(-1) = f(0)r_1 - b_1^2\sum_{k\in Z_2} r_1^{|k|}g(k)$$

and

$$f(1) = f(0)r_2 - b_2^2\sum_{k\in Z_2} r_2^{|k|}g(k).$$

The coupling condition gives us that

$$f(0) = \frac{-1}{b_1^{-2}(1-r_1)+b_2^{-2}(1-r_2)}\sum_{s=1,2,\ k\in Z_s} r_s^{|k|}g(k).$$

Introducing this formula in (3.6) we obtain the explicit formula of the resolvent. □

3.2.2 Limiting absorption principle

In this subsection we write a limiting absorption principle. From Lemma 3.2.3 we know that for any $\lambda\in\mathbb{C}\backslash[-4\max\{b_1^{-2},b_2^{-2}\},0]$ and $\varphi\in l^2(\mathbb{Z}^*)$ there exists $R(\lambda)\varphi=(A-\lambda)^{-1}\varphi\in l^2(\mathbb{Z}^*)$ and it is given by

$$\begin{aligned}(R(\lambda)\varphi)(j) = {} & \frac{-r_s^{|j|}}{b_2^{-2}(1-r_2)+b_1^{-2}(1-r_1)}\Big[\sum_{k\in I_1} r_1^{|k|}\varphi(k) + \sum_{k\in I_2} r_2^{|k|}\varphi(k)\Big] \\ & + \frac{b_s^2}{r_s-r_s^{-1}}\sum_{k\in I_s}(r_s^{|j-k|}-r_s^{|j|+|k|})\varphi(k), \quad j\in Z_s,\end{aligned} \tag{3.8}$$

where $r_s=r_s(\lambda)$, $s\in\{1,2\}$, is the unique solution with $|r_s|<1$ of the equation

$$r_s^2 - 2r_s + 1 = \lambda b_s^2 r_s.$$

Let us now consider $I=[-4\max\{b_1^{-2},b_2^{-2}\},0]$. As we proved in Theorem 3.2.1 we have that $\sigma(A)=I$. For any $\omega\in I$ and $\epsilon\geq 0$ let us denote by $r_{s,\epsilon}^{\pm}$ the unique solution with modulus less than one of

$$r^2-2r+1=(\omega\pm i\epsilon)b_s^2 r.$$

Denoting $r^+_{s,\epsilon} = \exp(z^+_{s,\epsilon})$ with $z^+_{s,\epsilon} = a^+_{s,\epsilon} + i\tilde{a}^+_{s,\epsilon}$, $a^+_{s,\epsilon} < 0$ and $\tilde{a}^+_{s,\epsilon} \in [-\pi, \pi]$ we obtain by taking the imaginary part in the equation satisfied by $r^+_{s,\epsilon}$ that

$$(\exp(a^+_{s,\epsilon}) - \exp(-a^+_{s,\epsilon}))\sin(\tilde{a}^+_{s,\epsilon}) = \epsilon b_s^2.$$

Thus $\tilde{a}^+_{s,\epsilon} \in [-\pi, 0]$. A similar result holds for $r^-_{s,\epsilon}$, $\tilde{a}^-_{s,\epsilon} \in [0, \pi]$.

Let us set $r^\pm_s = \lim_{\epsilon \downarrow 0} r^\pm_{s,\epsilon}$. Using the sign of the imaginary part of $r^\pm_{s,\epsilon}$ we obtain that $r^\pm_s$ are the solutions with $\operatorname{Im}(r^+_s) \leq 0 \leq \operatorname{Im}(r^-_s)$ of the equation

$$r^2 - 2r + 1 = \omega b_s^2 r.$$

Also, using that $r^-_{s,\epsilon} = \overline{r^+_{s,\epsilon}}$ we obtain $r^-_s = \overline{r^+_s}$.

For any $\omega \in J = I \setminus \{-4b_1^{-2}, -4b_2^{-2}, 0\}$ and $\varphi \in l^1(\mathbb{Z}^*)$ let us set

$$\begin{aligned}(R^\pm(\omega)\varphi)(j) =& \frac{-(r^\pm_s)^{|j|}}{b_2^{-2}(1 - r^\pm_2) + b_1^{-2}(1 - r^\pm_1)}\Big[\sum_{k \in I_1}(r^\pm_1)^{|k|}\varphi(k) + \sum_{k \in I_2}(r^\pm_2)^{|k|}\varphi(k)\Big] \\ &+ \frac{b_s^2}{r^\pm_s - (r^\pm_s)^{-1}}\sum_{k \in I_s}((r^\pm_s)^{|j-k|} - (r^\pm_s)^{|j|+|k|})\varphi(k), \quad j \in Z_s.\end{aligned}$$

We will prove that $R^\pm(\omega)$ are well defined as bounded operators from $l^1(\mathbb{Z}^*)$ to $l^\infty(\mathbb{Z}^*)$. We point out that we cannot define $R^\pm(\omega)$ for $\omega \in \{-4b_1^{-2}, -4b_2^{-2}, 0\}$ since for $\omega = 0$ we have $r_1 = r_2 = 1$ and for $\omega = 4b_s^{-2}, s \in \{1, 2\}$, we have $r_s = -1$. We also emphasize that $R^-(\omega)\varphi = \overline{R^+(\omega)\overline{\varphi}}$. This is a consequence of the fact that for any $\omega \in I$, $r^-_s(\omega) = \overline{r^+_s}(\omega)$. Formally, the above operator equals $R(\omega \pm i\epsilon)$ with $\epsilon = 0$. We point out that as operators on $l^2(\mathbb{Z}^*)$, $R(\omega \pm i\epsilon)$ are defined for any $\omega \in I$ but only if $\epsilon \neq 0$.

Lemma 3.2.4. *For any $\varphi \in l^1(\mathbb{Z}^*)$ operator $\exp(itA)$ satisfies*

$$e^{itA}\varphi = \frac{1}{2i\pi}\int_I e^{it\omega}[R^+(\omega) - R^-(\omega)]\varphi \, d\omega. \tag{3.9}$$

Proof. To clarify the ideas behind the proof we divide it in several steps.

Step 1. Let I_1 be a bounded interval such that $I \subset I_1$. There exists a constant

$$C(\omega) = \frac{1}{|\omega|^{1/2}} + \frac{1}{|\omega b_1^2 + 4|^{1/2}} + \frac{1}{|\omega b_2^2 + 4|^{1/2}} \in L^1(I_1) \tag{3.10}$$

such that for all $\omega \in I_1 \setminus \{-4b_1^{-2}, -4b_2^{-2}, 0\}$ the following inequality

$$|(R(\omega \pm i\epsilon)\varphi)(n)| \lesssim C(\omega)\|\varphi\|_{l^1(\mathbb{Z}^*)}, \text{ for all } \varphi \in l^1(\mathbb{Z}^*) \text{ and } n \in \mathbb{Z}^*,$$

holds uniformly on small enough ϵ.

Step 2. For any $\omega \in J$, $R^{\pm}(\omega)$ are bounded operators from $l^1(\mathbb{Z}^*)$ to $l^\infty(\mathbb{Z}^*)$ and

$$\|R^{\pm}(\omega)\|_{l^1(\mathbb{Z}^*)-l^\infty(\mathbb{Z}^*)} \lesssim C(\omega).$$

Step 3. For any $\omega \in J$, $\varphi \in l^1(\mathbb{Z}^*)$ and $n \in \mathbb{Z}^*$ the following holds

$$\lim_{\epsilon \downarrow 0}(R(\omega \pm i\epsilon)\varphi)(n) = (R^{\pm}(\omega)\varphi)(n).$$

Step 4. For any $\varphi \in l^1(\mathbb{Z}^*)$ and $n \in \mathbb{Z}^*$ we have

$$\lim_{\epsilon \downarrow 0} \int_I e^{it\omega}(R(\omega \pm i\epsilon)\varphi)(n)d\omega = \int_I e^{it\omega}(R^{\pm}(\omega)\varphi)(n)d\omega.$$

Step 5. For any $\varphi \in l^1(\mathbb{Z}^*)$

$$e^{itA}\varphi = \frac{1}{2i\pi}\int_I e^{it\omega}[R^+(\omega) - R^-(\omega)]\varphi d\omega.$$

Proof of Step 1. Observe that for any $\omega \in \mathbb{R}$ and $\epsilon > 0$ we have

$$|(R(\omega\pm i\epsilon)\varphi)(n)| \lesssim \|\varphi\|_{l^1(\mathbb{Z}^*)}\Big(\frac{1}{|b_2^{-2}(1-r_{2,\epsilon}^{\pm}) + b_1^{-2}(1-r_{1,\epsilon}^{\pm})|} + \frac{1}{|r_{1,\epsilon}^{\pm} - (r_{1,\epsilon}^{\pm})^{-1}|} + \frac{1}{|r_{2,\epsilon}^{\pm} - (r_{2,\epsilon}^{\pm})^{-1}|}\Big).$$

Solution $r_{s,\epsilon}^{\pm}$ of equation $r^2 - 2r + 1 = (\omega \pm i\epsilon)b_s^2 r$ satisfies

$$\frac{1}{|r_{s,\epsilon}^{\pm}|} - |r_{s,\epsilon}^{\pm}| \le \left| r_{s,\epsilon}^{\pm} - \frac{1}{r_{s,\epsilon}^{\pm}}\right| = b_s|\omega \pm i\epsilon|^{1/2}.$$

Then for all $\omega \in I_1$ and ϵ small enough we have

$$|r_{s,\epsilon}^{\pm}| \ge \frac{2}{b_s|\omega \pm i\epsilon|^{1/2} + (b_s^2|\omega \pm i\epsilon| + 4)^{1/2}} \ge C > 0$$

and

$$|r_{s,\epsilon}^{\pm}| \le \frac{1}{|r_{s,\epsilon}^{\pm}|} + \left| r_{s,\epsilon}^{\pm} - \frac{1}{r_{s,\epsilon}^{\pm}}\right| \le C_1 < \infty.$$

Thus for any $\omega \in I_1$ we have

$$\frac{1}{|r_{s,\epsilon}^{\pm} - (r_{s,\epsilon}^{\pm})^{-1}|} \lesssim \frac{1}{|1 - r_{s,\epsilon}^{\pm}||1 + r_{s,\epsilon}^{\pm}|} \lesssim \frac{1}{|1 - r_{s,\epsilon}^{\pm}|} + \frac{1}{|1 + r_{s,\epsilon}^{\pm}|}.$$

Using the equation satisfied by $r_{s,\epsilon}^{\pm}$ we find that

$$|1 - r_{s,\epsilon}^{\pm}| = b_s|\omega \pm i\epsilon|^{1/2}|r_{s,\epsilon}^{\pm}| \gtrsim |\omega \pm i\epsilon|^{1/2} \ge |\omega|^{1/2}$$

and

$$|1 + r_{s,\epsilon}^{\pm}| = |(\omega \pm i\epsilon)b_s^2 + 4|^{1/2}|r_{s,\epsilon}^{\pm}| \gtrsim |(\omega \pm i\epsilon)b_s^2 + 4|^{1/2} \ge |\omega b_s^2 + 4|^{1/2}.$$

Putting together the above estimates for the roots $r^{\pm}_{s,\epsilon}$ we find that for all $\omega \in I_1$ and ϵ small enough the following holds

$$\frac{1}{|r^{\pm}_{1,\epsilon} - (r^{\pm}_{1,\epsilon})^{-1}|} + \frac{1}{|r^{\pm}_{2,\epsilon} - (r^{\pm}_{2,\epsilon})^{-1}|} \lesssim \frac{1}{|\omega|^{1/2}} + \frac{1}{|\omega b_1^2 + 4|^{1/2}} + \frac{1}{|\omega b_2^2 + 4|^{1/2}}.$$

We now prove that

$$\frac{1}{|b_2^{-2}(1 - r^{\pm}_{2,\epsilon}) + b_1^{-2}(1 - r^{\pm}_{1,\epsilon})|} \lesssim \frac{1}{|\omega|^{1/2}}.$$

We recall that the sign of the imaginary parts of $r^{\pm}_{1,\epsilon}$ and $r^{\pm}_{2,\epsilon}$ is the same. Also, since $|r^{\pm}_{s,\epsilon}| < 1$, the real parts of $1 - r^{\pm}_{1,\epsilon}$ and $1 - r^{\pm}_{2,\epsilon}$ are positive. These properties of the roots imply that

$$|b_2^{-2}(1 - r^{\pm}_{2,\epsilon}) + b_1^{-2}(1 - r^{\pm}_{1,\epsilon})| \geq b_2^{-2}|1 - r^{\pm}_{2,\epsilon}| + b_1^{-2}|1 - r^{\pm}_{1,\epsilon}| \gtrsim |\omega|^{1/2}.$$

Putting together the above results we obtain that Step 1 is satisfied with $C(\omega)$ given by (3.10) uniformly on all $\epsilon > 0$ sufficiently small.

Step 2 follows as Step 1 by putting $\epsilon = 0$ and replacing $r^{\pm}_{s,\epsilon}$ with $r^{\pm}_{s}$.

Proof of Step 3. We write

$$R(\omega \pm i\epsilon)\varphi(n) = \sum_{k \in \mathbb{Z}^*} R(\omega \pm i\epsilon, n, k)\varphi(k),$$

where $R(\omega \pm i\epsilon, n, k)$ collects all the coefficients in front of $\varphi(k)$ in formula (3.7).

Using that, for any $\omega \in J$, $r^{\pm}_{s,\epsilon}(\omega) \to r^{\pm}_{s}(\omega)$ we obtain that $R(\omega \pm i\epsilon, n, k)\varphi(k) \to R^{\pm}(\omega, n, k)\varphi(k)$. Since for any $\omega \in J$ and ϵ small enough we have the uniform bound

$$|R(\omega \pm i\epsilon, n, k)\varphi(k)| \leq C(\omega)|\varphi(k)|, \forall k \in \mathbb{Z}^*,$$

we can apply Lebesgue's dominated convergence theorem to conclude that

$$\sum_{k \in \mathbb{Z}^*} R(\omega \pm i\epsilon, n, k)\varphi(k) \to \sum_{k \in \mathbb{Z}^*} R^{\pm}(\omega, n, k)\varphi(k),$$

which proves Step 3.

Step 4 follows by Lebesgue's dominated convergence theorem since we have the pointwise convergence in Step 3 and the uniform bound in Step 1.

Proof of Step 5. Applying Cauchy's formula we obtain that

$$e^{itA} = \frac{1}{2i\pi} \int_{\Gamma} e^{it\omega} R(\omega) d\omega$$

for any curve Γ that rounds the spectrum of operator A. For small parameter ϵ we choose in the above formula path Γ_ϵ to be the following rectangle

$$\begin{aligned}\Gamma_\epsilon =& \{\omega \pm i\epsilon, \omega \in [-4\max\{b_1^{-2}, b_2^{-2}\} - \epsilon, \epsilon]\} \\ &\cup \{-4\max\{b_1^{-2}, b_2^{-2}\} - \epsilon + i\eta, \eta \in [-\epsilon, \epsilon]\} \cup \{\epsilon + i\eta, \eta \in [-\epsilon, \epsilon]\}.\end{aligned}$$

Using the estimates for $R(\lambda)$, $\lambda \in \Gamma_\epsilon$ obtained in Step 1 and the convergence in Step 4 we obtain that for any $\varphi \in l^1(\mathbb{Z}^*)$ the following holds:

$$e^{itA}\varphi = \frac{1}{2\pi i}\int_I e^{it\omega}(R^+(\omega) - R^-(\omega))\varphi d\omega.$$

The proof is now complete. □

3.2.3 Proof of the main result

We now prove the main result of this paper.

Proof of Theorem 3.0.6. For any $\varphi \in l^1(\mathbb{Z}^*)$ Lemma 3.2.4 gives us that

$$(e^{itA}\varphi)(n) = \frac{1}{2\pi i}\int_I e^{it\omega}(R^+(\omega) - R^-(\omega))\varphi(n)ds,\ n \in \mathbb{Z}^*,$$

where $I = [-4\max\{b_1^{-2}, b_2^{-2}\}, 0]$. Using the fact that $R^-(\omega)\varphi = \overline{R^+(\omega)\overline{\varphi}}$ we obtain

$$(e^{itA}\varphi)(n) = \frac{1}{\pi}\int_I e^{it\omega}((\operatorname{Im} R^+)(\omega)\varphi)(n)d\omega,\ \ n \in \mathbb{Z}^*,$$

where $\operatorname{Im} R^+$ is given by

$$\begin{aligned}
(\operatorname{Im} R^+)(\omega)\varphi(j) =& \frac{(R^+(\omega)\varphi)(j) - (R^-(\omega)\varphi)(j)}{2i} \\
=& \sum_{k\in Z_1} \varphi(k) \operatorname{Im} \frac{-(r_s^+)^{|j|}(r_1^+)^{|k|}}{b_2^{-2}(1-r_2^+) + b_1^{-2}(1-r_1^+)} \\
&+ \sum_{k\in Z_2} \varphi(k) \operatorname{Im} \frac{-(r_s^+)^{|j|}(r_2^+)^{|k|}}{b_2^{-2}(1-r_2^+) + b_1^{-2}(1-r_1^+)} \\
&+ \sum_{k\in Z_s} \varphi(k) \operatorname{Im} \frac{b_s^2}{r_s^+ - (r_s^+)^{-1}}((r_s^+)^{|j-k|} - (r_s^+)^{|j|+|k|}), \quad j \in Z_s
\end{aligned}$$

and for $s \in \{1, 2\}$, r_s^+ is the root of $r^2 - 2r + 1 = \omega b_s^2 r$ with the imaginary part nonpositive.

In order to prove (3.15) it is sufficient to show the existence of a constant $C = C(b_1, b_2)$ such that

$$\sum_{k\in Z_1} |\varphi(k)| \Big| \int_I e^{it\omega} \operatorname{Im} \frac{(r_s^+)^{|j|}(r_1^+)^{|k|}}{b_2^{-2}(1-r_2^+) + b_1^{-2}(1-r_1^+)} d\omega \Big| \le C(|t|+1)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z}^*)},\ \forall j \in \mathbb{Z}^*, \tag{3.11}$$

and

$$\sum_{k\in Z_s} |\varphi(k)| \Big| \int_I e^{it\omega} \operatorname{Im} \frac{(r_s^+)^{|j-k|}}{r_s^+ - (r_s^+)^{-1}} d\omega \Big| \le C(|t|+1)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z}^*)},\ \forall j \in \mathbb{Z}^*. \tag{3.12}$$

The estimates for the other two terms occurring in the representation of $\operatorname{Im} R^+(\omega)$ are similar.

Step I. Proof of (3.12)**.** We prove that

$$\sup_{j\in\mathbb{Z}}\Big|\int_I e^{it\omega}\operatorname{Im}\frac{(r_s^+)^{|j|}}{r_s^+-(r_s^+)^{-1}}d\omega\Big|\le C(b_1,b_2)(|t|+1)^{-1/3},\quad \forall\ t\in\mathbb{R}. \tag{3.13}$$

We split I as $I=I_1\cup I_2$ where $I_1=[-4\max\{b_1^{-2},b_2^{-2}\},4b_s^{-2}]$ and $I_2=[4b_s^{-2},0]$. If $\omega\in I_1$, the following equation

$$r+\frac{1}{r}=2+\omega b_s^2$$

has real roots and then

$$\int_I e^{it\omega}\operatorname{Im}\frac{(r_s^+)^{|j|}}{r_s^+-(r_s^+)^{-1}}d\omega=0.$$

When $\omega\in I_2$, root r_s of equation $r_s+\frac{1}{r_s}=2+\omega b_s^2$ has the form $r_s=e^{-i\theta},\theta\in[0,\pi]$.

Using the change of variables $\omega=b_s^{-2}(2\cos\theta-2)$ we get

$$\begin{aligned}\int_{I_2} e^{it\omega}\operatorname{Im}\frac{(r_s^+)^{|j|}}{r_s^+-(r_s^+)^{-1}}d\omega=&\,2b_s^{-2}\int_0^\pi e^{itb_s^{-2}(2\cos\theta-2)}\operatorname{Im}\frac{e^{-i|j|\theta}}{e^{-i\theta}-e^{i\theta}}\sin\theta d\theta\\ =&-2b_s^{-2}\int_0^\pi e^{itb_s^{-2}(2\cos\theta-2)}\operatorname{Im}\frac{e^{-i|j|\theta}}{2i\sin\theta}\sin\theta d\theta\\ =&\,b_s^{-2}\int_0^\pi e^{itb_s^{-2}(2\cos\theta-2)}\operatorname{Re}e^{-i|j|\theta}d\theta\\ =&\,\frac{b_s^{-2}}{2}\int_0^\pi e^{itb_s^{-2}(2\cos\theta-2)}(e^{i|j|\theta}+e^{-i|j|\theta})d\theta.\end{aligned}$$

Van der Corput's Lemma applied to the phase function $\phi(\theta)=(2\cos\theta-2)b_s^{-2}+j\theta/t$ shows that

$$\Big|\int_0^\pi e^{it(2\cos\theta-2)b_s^{-2}}e^{ij\theta}d\theta\Big|\le C(b_s)(|t|+1)^{-3},\ \forall\ t\in\mathbb{R},\forall j\in\mathbb{Z} \tag{3.14}$$

The proof of (3.12) is now finished.

Step II. Proof of (3.11)**.** It is sufficient to prove that

$$\sup_{j,k\in\mathbb{N}}\Big|\int_I e^{it\omega}\frac{(r_1^+)^j(r_2^+)^k}{b_2^{-2}(1-r_2^+)+b_1^{-2}(1-r_1^+)}d\omega\Big|\le C(b_1,b_2)(|t|+1)^{-1/3},\ \forall t\in\mathbb{R}.$$

To fix the ideas let us assume that $b_2\le b_1$. We split interval I as follows $I=I_1\cup I_2$ where $I_1=[-4b_2^{-2},-4b_1^{-2}]$ and $I_2=[-4b_1^{-2},0]$. We remark that on I_1, $r_1^+\in\mathbb{R}$ and $r_2^+\in\mathbb{C}\setminus\mathbb{R}$. On I_2 both r_1^+ and r_2^+ belong to $\mathbb{C}\setminus\mathbb{R}$. We prove that

$$\sup_{j,k\in\mathbb{N}}\Big|\int_{I_1} e^{it\omega}\frac{(r_1^+)^j(r_2^+)^k}{b_2^{-2}(1-r_2^+)+b_1^{-2}(1-r_1^+)}d\omega\Big|\le C(b_1,b_2)(|t|+1)^{-1/3} \tag{3.15}$$

and

$$\sup_{j,k\in\mathbb{N}}\Big|\int_{I_2} e^{it\omega}\frac{(r_1^+)^j(r_2^+)^k}{b_2^{-2}(1-r_2^+)+b_1^{-2}(1-r_1^+)}d\omega\Big| \leq C(b_1,b_2)(|t|+1)^{-1/3}. \tag{3.16}$$

Let us set $h(\omega)=b_2^{-2}(1-r_2^+(\omega))+b_1^{-2}(1-r_1^+(\omega))$ Using the same arguments as in the proof of Lemma 3.2.4 we get that $|h(\omega)|\geq C(b_1,b_2)|\omega|^{1/2}$. Then, on I_1, $|h(\omega)|\geq c>0$. Moreover $|h'(\omega)|\leq c_2<\infty$. Using integration by parts we obtain that

$$\begin{aligned}
&\Big|\int_{I_1} e^{it\omega}\frac{(r_1^+)^j(r_2^+)^k}{b_2^{-2}(1-r_2^+)+b_1^{-2}(1-r_1^+)}d\omega\Big| \\
&\qquad\leq \sup_{x\in I_1}\Big|\int_{-4b_2^{-2}}^{x} e^{it\omega}(r_1^+)^j(r_2^+)^k d\omega\Big|\Big(\|1/h\|_{L^\infty(I_1)}+\|(1/h)'\|_{L^1(I_1)}\Big)\\
&\qquad\leq C(b_1,b_2)\sup_{x\in I_1}\Big|\int_{-4b_2^{-2}}^{x} e^{it\omega}(r_1^+)^j(r_2^+)^k d\omega\Big|.
\end{aligned}$$

A similar argument shows that

$$\Big|\int_{-4b_2^{-2}}^{x} e^{it\omega}(r_1^+)^j(r_2^+)^k d\omega\Big| \leq \sup_{y\leq x}\Big|\int_{-4b_2^{-2}}^{y} e^{it\omega}(r_2^+)^k d\omega\Big|\Big(\|(r_1^+)^j\|_{L^\infty(I_1)}+\|((r_1^+)^j)'\|_{L^\infty(I_1)}\Big).$$

Observe that for $\omega\in I_1$, $r_1^+(\omega)$ given by

$$r_1^+(\omega)=\frac{2+b_1^2\omega-\sqrt{(2+b_1^2\omega)^2-4}}{2}$$

is a decreasing function. Thus

$$\|((r_1^+)^j)'\|_{L^1(I_1)}\leq \|(r_1^+)^j\|_{L^\infty(I_1)}\leq 1, \quad \forall j\in\mathbb{N}.$$

The proof of (3.15) is now reduced to the following estimate:

$$\sup_{y\in I_1}\Big|\int_{-4b_2^{-2}}^{y} e^{it\omega}(r_2^+(\omega))^k d\omega\Big| \leq C(b_1,b_2)(|t|+1)^{-1/3}, \forall k\in\mathbb{N}, t\in\mathbb{R}.$$

Making the change of variables $\omega=b_2^{-2}(2\cos\theta-2)$ and applying Van der Corput's Lemma as in the final step of Step I we obtain that

$$\Big|\int_{-4b_2^{-2}}^{y} e^{it\omega}(r_2^+(\omega))^k d\omega\Big| = 2b_2^{-2}\Big|\int_{2\arcsin(b_2^2/y)}^{\pi} e^{itb_2^2(2\cos\theta-2)}e^{-ik\theta}\sin\theta d\omega\Big|\leq C(b_2)(|t|+1)^{-1/3}.$$

We now prove (3.16). We first make the change of variables $\omega=b_1^{-2}(2\cos\theta-2)$. Thus

$$\int_{I_2} e^{it\omega}\frac{(r_1^+)^j(r_2^+)^k}{b_2^{-2}(1-r_2^+)+b_1^{-2}(1-r_1^+)}d\omega = 2b_1^{-2}\int_0^{\pi} e^{itb_1^{-2}(2\cos\theta-2)}e^{-ij\theta}e^{-2ik\arcsin(b_2b_1^{-1}\sin\frac{\theta}{2})}\frac{\sin\theta}{h(\theta)}d\theta,$$

where $h(\theta)=b_2^{-2}(1-r_2^+(\theta))+b_1^{-2}(1-r_1^+(\theta))$, $r_1^+(\theta)=e^{-i\theta}$ and $r_2^+(\theta)=e^{-2i\arcsin(b_2b_1^{-1}\sin\frac{\theta}{2})}$.

Using that far from $\theta = 0$ function h satisfies $|h(\theta)| > 0$ we choose a small parameter ϵ and split our integral as follows:

$$\begin{aligned}\int_0^\pi e^{itb_1^{-2}(2\cos\theta-2)} e^{-ij\theta} e^{-2ik\arcsin(b_2 b_1^{-1}\sin\frac{\theta}{2})}\frac{\sin\theta}{h(\theta)}d\theta &= T_1 + T_2\\ &= \int_0^\epsilon e^{itb_1^{-2}(2\cos\theta-2)} e^{-ij\theta} e^{-2ik\arcsin(b_2 b_1^{-1}\sin\frac{\theta}{2})}\frac{\sin\theta}{h(\theta)}d\theta\\ &\quad+ \int_\epsilon^\pi e^{itb_1^{-2}(2\cos\theta-2)} e^{-ij\theta} e^{-2ik\arcsin(b_2 b_1^{-1}\sin\frac{\theta}{2})}\frac{\sin\theta}{h(\theta)}d\theta.\end{aligned}$$

Observe that on interval $[0, \epsilon]$

$$\Big\|\frac{\sin\theta}{h(\theta)}\Big\|_{L^\infty(0,\epsilon)} + \Big\|\frac{d}{d\theta}(\frac{\sin\theta}{h(\theta)})\Big\|_{L^1(0,\epsilon)} \le M < \infty$$

and on interval $[\epsilon, \pi]$

$$\Big\|\frac{1}{h(\theta)}\Big\|_{L^\infty(\epsilon,\pi)} + \Big\|\frac{d}{d\theta}(\frac{1}{h(\theta)})\Big\|_{L^1(\epsilon,\pi)} \le M < \infty.$$

Then we have the following estimates for T_1 and T_2

$$|T_1| \le M \sup_{x\in[0,\epsilon]}\Big|\int_0^x e^{itb_1^{-2}(2\cos\theta-2)} e^{-ij\theta} e^{-2ik\arcsin(b_2 b_1^{-1}\sin\frac{\theta}{2})}d\theta\Big|$$

and

$$|T_2| \le M \sup_{x\in[\epsilon,\pi]}\Big|\int_x^\pi e^{itb_1^{-2}(2\cos\theta-2)} e^{-ij\theta} e^{-2ik\arcsin(b_2 b_1^{-1}\sin\frac{\theta}{2})}\sin\theta d\theta\Big|.$$

We now apply the following lemma that we prove later.

Lemma 3.2.5. *Let $a \in (0, 1]$ and $0 \le \delta \le \pi$. There exists $C(a, \delta)$ such that for all real numbers y, z and t*

$$\Big|\int_\delta^\pi e^{it(2\cos\theta+2z\arcsin(a\sin\frac{\theta}{2}))} e^{iy\theta}\sin\theta d\theta\Big| \le C(a,\delta)(|t|+1)^{-1/3} \tag{3.17}$$

and if $\delta > 0$

$$\Big|\int_0^{\pi-\delta} e^{it(2\cos\theta+2z\arcsin(a\sin\frac{\theta}{2}))} e^{iy\theta}d\theta\Big| \le C(a,\delta)(|t|+1)^{-1/3}. \tag{3.18}$$

We obtain that

$$|T_1| \le MC(a,\epsilon)(|t|+1)^{-1/3}$$

and

$$|T_2| \le MC(a,\epsilon)(|t|+1)^{-1/3}.$$

The proof of Theorem 3.0.6 is now finished. □

Proof of Lemma 3.2.5. Since the integrals in (3.17) and (3.18) are on bounded intervals it is sufficient to prove that, for t large enough, each of the integrals is bounded by $|t|^{-1/3}$. In the case of (3.17) we will consider the case $\delta = 0$ since the proof for $\delta > 0$ is similar.

Let us denote by ψ either the function $\chi_{(0,\pi-\delta)}$ or $\sin\theta$. We set

$$p(\theta) = 2\cos\theta + 2z \arcsin(a \sin\frac{\theta}{2}),\ \ \theta \in [0, \pi].$$

Using the Maple software we obtain that

$$\min_{\theta\in[0,\pi]}[(p''(\theta))^2 + (p'''(\theta))^2] \geq \min\Big\{4 + \frac{z^2a^2(a^2-1)^2}{16}, \frac{a^2}{4(1-a^2)}\big(z - \frac{4\sqrt{1-a^2}}{a}\big)^2\Big\}.$$

If z is such that $|z - \frac{4\sqrt{1-a^2}}{a}| \geq \epsilon > 0$ then Van der Corput's lemma applied to the phase function $p(\theta) + y\theta/t$ guarantees that

$$\Big|\int_0^{\pi} e^{itp(\theta)}e^{iy\theta}\psi(\theta)d\theta\Big| \leq C(a,\epsilon)(|t|+1)^{-1/3}.$$

Assume now that $|z - \frac{4\sqrt{1-a^2}}{a}| < \epsilon$ with ϵ small enough that we will specify later. Let us write

$$z = \frac{4\sqrt{1-a^2}}{a} + b$$

with b a small parameter such that $|b| < \epsilon$. With this notation $p(\theta) = p_b(\theta) = q(\theta) + br(\theta)$ where

$$q(\theta) = 2\cos(\theta) + \frac{8\sqrt{1-a^2}}{a}\arcsin(a\sin\frac{\theta}{2})$$

and

$$r(\theta) = 2\arcsin(a\sin\frac{\theta}{2}).$$

Solving system $(q''(\theta), q'''(\theta)) = (0,0)$ with Maple software we obtain that it has a unique solution $\theta = \pi$. Thus for any $\delta < \pi$ there exists a positive constant $c(a,\delta)$ such that

$$|q''(\theta)| + |q'''(\theta)| \geq c(a,\delta), \quad \forall\ \theta \in [0, \pi-\delta].$$

It implies the existence of an $\epsilon = \epsilon(a,\delta)$ such that for all $|b| \leq \epsilon$

$$|p_b''(\theta)| + |p_b'''(\theta)| \geq c(a,\delta) - |b| \sup_{x\in[0,\pi]}(|r''| + |r'''|) \geq \frac{c(a,\delta)}{2}, \quad \forall\ \theta \in [0, \pi-\delta].$$

Hence, Van der Corput's Lemma applied to the phase function $p_b(\theta) + y\theta/t$ guarantees that

$$\Big|\int_0^{\pi-\delta} e^{itp_b(\theta)}e^{iy\theta}\psi(\theta)d\theta\Big| \leq C(a,\delta)(|t|+1)^{-1/3}, \quad \forall |b| < \epsilon, \forall\ t, y \in \mathbb{R}.$$

The proof of (3.18) is finished.

To prove estimate (3.17) it remains to show that we can choose $\delta(a)$ small enough such that for all $|b| < \epsilon$

$$|I_b(t)| := \Big| \int_{\pi-\delta(a)}^{\pi} e^{itp_b(\theta)} e^{iy\theta} \sin(\theta) d\theta \Big| \le C(a)(|t|+1)^{-1/3}, \quad \forall y, t \in \mathbb{R}. \tag{3.19}$$

The Taylor expansions of q and r near $\theta = \pi$ are as follows

$$q(\theta) = \frac{-2a + 8\sqrt{1-a^2}\arcsin(a)}{a} - \frac{1}{16}\frac{(2a^2-1)(\theta-\pi)^4}{-1+a^2} - \frac{1}{384}\frac{(4a^2-1)(\theta-\pi)^6}{(-1+a^2)^2} + O((\theta-\pi)^8),$$

and

$$r(\theta) = 2\arcsin(a) - \frac{1}{4}\frac{a}{\sqrt{1-a^2}}(\theta-\pi)^2 + \frac{1}{192}\frac{a(2a^2+1)}{(1-a^2)^{3/2}}(\theta-\pi)^4 + O\left((\theta-\pi)^6\right).$$

Also the second derivatives of q and r satisfy

$$q''(\theta) = -\frac{3}{4}\frac{(2a^2-1)(\theta-\pi)^2}{-1+a^2} + O(|\theta-\pi|^4) \quad \text{as } \theta \sim \pi,$$

and

$$r''(\theta) = -\frac{1}{2}\frac{a}{\sqrt{1-a^2}} + O(\theta-\pi)^2 \quad \text{as } \theta \sim \pi.$$

Observe that for $a \neq 1/\sqrt{2}$, the second derivative of q behaves as $(\theta-\pi)^2$ near $\theta = \pi$. Otherwise it behaves as $(\theta-\pi)^4$ near the same point. Since the proof of (3.19) is quite different in the two cases we will treat then separately.

In the sequel $\delta(a)$ is chosen such that we can compare q and r with their Taylor expressions near $\theta = \pi$.

Case 1. $a \neq 1/\sqrt{2}$. The main idea is to split the interval $[\pi-\delta(a), \pi]$ in three intervals where we can compare $|\theta-\pi|$ with $|b|^{1/2}$ and decide which of them dominates the other:

$$[\pi-\delta(a),\pi] = [\pi-\delta(a), \pi-\alpha_2|b|^{1/2}] \cup [\pi-\alpha_2|b|^{1/2}, \pi-\alpha_1|b|^{1/2}] \cup [\pi-\alpha_1|b|^{1/2}, \pi],$$

where $\alpha_1 << 1 << \alpha_2$ are independent of b but depend on the parameter a. More precisely the parameters α_1 and α_2 are chosen in terms of the first two coefficients of the Taylor expansion of functions q and r near $\theta = \pi$.

Let us consider the interval $[\pi-\delta(a), \pi-\alpha_2|b|^{1/2}]$ with α_2 large enough. In this interval $|\theta-\pi|$ dominates $|b|^{1/2}$ and we apply Lemma 1.3.1. We check the hypotheses of this lemma. In this interval the first derivative of p_b is of the same order as $|\theta-\pi|^3$:

$$|p_b'(\theta)| \ge |q'(\theta)| - |b||r'(\theta)| \ge C_1|\theta-\pi|(|\theta-\pi|^2 - C_2|b|) \ge C_3|\theta-\pi|^3$$

and

$$|p_b'(\theta)| \le |q'(\theta)| + |b||r'(\theta)| \ge C_4|\theta-\pi|(|\theta-\pi|^2 + C_5|b|) \ge C_6|\theta-\pi|^3.$$

Also, the second derivative satisfies:

$$|p_b''(\theta)| \geq |q''(\theta)| - |b||r''(\theta)| \geq C_7(|\theta-\pi|^2 - C_8|b|) \geq C_9|\theta-\pi|^2$$

and

$$|p_b''(\theta)| \leq |q''(\theta)| + |b||r''(\theta)| \geq C_{10}(|\theta-\pi|^2 + C_{11}|b|) \geq C_{12}|\theta-\pi|^2.$$

We emphasize that all the above constants are independent of b. Observe that on the considered interval $|p_b''| \gtrsim |b|$. If we try to apply Van der Corput's Lemma with $k=2$ we obtain

$$\Big|\int_{\pi-\delta(a)}^{\pi-\alpha_2|b|^{1/2}} e^{itp_b(\theta)}e^{iy\theta}\sin(\theta)d\theta\Big| \leq (|tb|)^{-1/2} \max_{[\pi-\delta(a),\pi-\alpha_2|b|^{1/2}]} |\sin\theta| \leq C(\delta(a))|tb|^{-1/2},$$

an estimate that is not uniform in the parameter b.

However, using Lemma 1.3.1 we obtain the existence of a constant C depending on all the constants $C_i, i=1,...,12$ but independent of the parameter b, such that

$$\begin{aligned}
\Big|\int_{\pi-\delta(a)}^{\pi-\alpha_2|b|^{1/2}} e^{itp_b(\theta)}e^{iy\theta}\sin(\theta)d\theta\Big| &= \Big|\int_{\pi-\delta(a)}^{\pi-\alpha_2|b|^{1/2}} e^{itp_b(\theta)}e^{iy\theta}|p_b''(\theta)|^{1/2}\frac{\sin(\theta)}{|p_b''(\theta)|^{1/2}}d\theta\Big| \qquad (3.20)\\
\leq C|t|^{-1/2}\Big(\max_{[\pi-\delta(a),\pi-\alpha_1|b|^{1/2}]}\frac{|\sin(\theta)|}{|p_b''(\theta)|^{1/2}} &+ \int_{\pi-\delta(a)}^{\pi-\alpha_2|b|^{1/2}}\Big|\Big(\frac{\sin(\theta)}{|p_b''(\theta)|^{1/2}}\Big)'(\theta)\Big|d\theta\Big)\\
\leq C|t|^{-1/2}\max_{[\pi-\delta(a),\pi-\alpha_2|b|^{1/2}]}\frac{|\sin(\theta)|}{|p_b''(\theta)|^{1/2}} &\lesssim C|t|^{-1/2}\max_{[\pi-\delta(a),\pi-\alpha_2|b|^{1/2}]}\frac{|\sin(\theta)|}{|\theta-\pi|} \lesssim C|t|^{-1/2}.
\end{aligned}$$

On the interval $[\pi-\alpha_2|b|^{1/2}, \pi-\alpha_1|b|^{1/2}]$ the third derivative of p_b satisfies:

$$|p'''(\theta)| \simeq |\theta-\pi||C(a)+b| \simeq |b|^{1/2},$$

since $C(a) \neq 0$ in the case $a \neq 1/\sqrt{2}$. Applying Van der Corput's Lemma with $k=3$ we get

$$\Big|\int_{\pi-\alpha_2|b|^{1/2}}^{\pi-\alpha_1|b|^{1/2}} e^{itp_b(\theta)}e^{iy\theta}\sin(\theta)d\theta\Big| \lesssim (|tb|^{1/2})^{-1/3} \max_{\theta\in[\pi-\alpha_2|b|^{1/2},\pi-\alpha_1|b|^{1/2}]} |\sin\theta| \lesssim |t|^{-1/3}. \qquad (3.21)$$

On interval $[\pi-\alpha_1|b|^{1/2}, \pi]$ with α_1 small enough, the term $|br''(\theta)|$ dominates $|q''(\theta)|$. The the behavior of $p_b''(\theta)$ is given by $|br''(\theta)|$:

$$|p_b''(\theta)| \geq |br''(\theta)| - |q''(\theta)| \geq C_1(|b| - C_2|\theta-\pi|^2) \geq C_3|b|,$$

for some positive constants C_1 and C_2 independent of the parameter b. Applying Van der Corput's Lemma with $k=2$ we get

$$\Big|\int_{\pi-\alpha_1|b|^{1/2}}^{\pi} e^{itp_b(\theta)}e^{iy\theta}\sin(\theta)d\theta\Big| \lesssim (|tb|)^{-1/2} \max_{\theta\in[\pi-\alpha_1|b|^{1/2},\pi]} |\sin\theta| \lesssim |t|^{-1/2}. \qquad (3.22)$$

Using (3.20), (3.21) and (3.22) we obtain that (3.19) holds uniformly for all $|b| < \epsilon, y$ and t real numbers.

Case 2. $a = 1/\sqrt{2}$. In this case the Taylor expansion of function q at $\theta = \pi$ is given by

$$q(\theta) = \frac{-2a + 8\sqrt{1-a^2}\arcsin(a)}{a} - \frac{1}{384}\frac{(4a^2-1)(\theta-\pi)^6}{(-1+a^2)^2} + O(|\theta-\pi|^8).$$

We split the interval $[\pi - \delta(a), \pi]$ as follows:

$$\begin{aligned}[\pi - \delta(a), \pi] =& [\pi - \delta(a), \pi - \alpha_3|b|^{1/4}] \cup [\pi - \alpha_3|b|^{1/4}, \pi - \alpha_2|b|^{1/4}] \\ & \cup [\pi - \alpha_2|b|^{1/4}, \pi - \alpha_1|b|^{1/2}] \cup [\pi - \alpha_1|b|^{1/2}, \pi],\end{aligned}$$

where $\alpha_2 << 1 << \alpha_3$ and all $\alpha_1, \alpha_2, \alpha_3$ are independent of b.

On the first interval $[\pi - \delta(a), \pi - \alpha_3|b|^{1/4}]$ we apply Lemma 1.3.2. We have to check that the first third derivatives behave as powers of $|\theta - \pi|$ in this interval. Observe that

$$|p_b'(\theta)| \geq C_1|\theta-\pi|(|\theta-\pi|^4 - C_2|b|) \geq C_3|\theta-\pi|^5$$

and

$$|p_b'(\theta)| \leq C_4|\theta-\pi|(|\theta-\pi|^4 + C_5|b|) \geq C_6|\theta-\pi|^5.$$

In a similar manner

$$C_7|\theta-\pi|^4 \leq |p_b''(\theta)| \leq C_8|\theta-\pi|^4.$$

Also the third derivative satisfies

$$|p_b'''(\theta)| \geq C_9|\theta-\pi|(|\theta-\pi|^2 - C_{10}|b|) \geq C_{11}|\theta-\pi|^3$$

and

$$|p_b'''(\theta)| \leq C_{12}|\theta-\pi|(|\theta-\pi|^2 + C_{13}|b|) \geq C_{14}|\theta-\pi|^3.$$

We now apply Lemma 1.3.2 taking into account that all the above constants are independent of b and we obtain

$$\begin{aligned}\Big|\int_{\pi-\delta(a)}^{\pi-\alpha_3|b|^{1/4}} e^{itp_b(\theta)}e^{iy\theta}\sin\theta d\theta\Big| &= \Big|\int_{\pi-\delta(a)}^{\pi-\alpha_3|b|^{1/4}} e^{itp_b(\theta)}e^{iy\theta}|p_b'''(\theta)|^{1/3}\frac{\sin\theta}{|p_b'''(\theta)|^{1/3}}d\theta\Big| \qquad (3.23)\\ &\lesssim |t|^{-1/3}\Big(\max_{[\pi-\delta(a),\pi-\alpha_3|b|^{1/4}]}\frac{|\sin\theta|}{|p_b'''(\theta)|^{1/3}} + \int_{\pi-\delta(a)}^{\pi-\alpha_3|b|^{1/4}}\Big|\Big(\frac{\sin\theta}{|p_b'''(\theta)|^{1/3}}\Big)'\Big|d\theta\Big)\\ &\lesssim |t|^{-1/3}\max_{[\pi-\delta(a),\pi-\alpha_3|b|^{1/4}]}\frac{|\sin\theta|}{|p_b'''(\theta)|^{1/3}}\\ &\lesssim |t|^{-1/3}\max_{[\pi-\delta(a),\pi-\alpha_3|b|^{1/4}]}\frac{|\sin\theta|}{|\theta-\pi|} \leq C|t|^{-1/3}.\end{aligned}$$

In the case of the interval $[\pi - \alpha_3|b|^{1/4}, \pi - \alpha_2|b|^{1/4}]$ we apply Van der Corput's Lemma with $k = 3$ and use that

$$|p_b'''(\theta)| \geq C_1|\theta - \pi|(|\theta - \pi|^2 - C_2|b|) \geq C_1|\theta - \pi|(\alpha_2^2|b|^{1/2} - C_2|b|) \geq C_3|b|^{1/4+1/2}.$$

Then

$$\Big|\int_{\pi-\alpha_3|b|^{1/4}}^{\pi-\alpha_2|b|^{1/4}} e^{itp_b(\theta)}e^{iy\theta}\sin\theta\Big| \leq (|t||b|^{3/4})^{-1/3} \max_{[\pi-\alpha_3|b|^{1/4},\pi-\alpha_2|b|^{1/4}]} |\sin\theta| \leq C|t|^{-1/3}. \tag{3.24}$$

Let us now consider the integral on the interval $[\pi - \alpha_2|b|^{1/4}, \pi - \alpha_1|b|^{1/2}]$. Observe that in this case

$$\begin{aligned}\Big|\int_{\pi-\alpha_2|b|^{1/4}}^{\pi-\alpha_1|b|^{1/2}} e^{itp_b(\theta)}e^{iy\theta}\sin\theta d\theta\Big| &\leq \int_{\pi-\alpha_2|b|^{1/4}}^{\pi-\alpha_1|b|^{1/2}} |\sin\theta|d\theta \leq \int_{\alpha_1|b|^{1/2}}^{\alpha_2|b|^{1/4}} |\sin\theta|d\theta \\ &\leq \int_{\alpha_1|b|^{1/2}}^{\alpha_2|b|^{1/4}} \theta d\theta \leq C|b|^{1/2} \leq C|t|^{-1/3},\end{aligned} \tag{3.25}$$

as long as $|b| \leq |t|^{-2/3}$.

We now consider the case $|b| \geq |t|^{-2/3}$ and prove that a similar estimate can be obtained. Observe that on the considered interval the second derivative of p_b satisfies

$$|p_b''(\theta)| \geq |b||r''(\theta)| - |q''(\theta)| \geq C_1(|b| - C_2|\theta - \pi|^4) \geq C_1(|b| - C_2(\alpha_2|b|^{1/4})^4) \geq C_3|b|.$$

Thus, Van der Corput's Lemma with $k = 2$ gives us

$$\begin{aligned}\Big|\int_{\pi-\alpha_2|b|^{1/4}}^{\pi-\alpha_1|b|^{1/2}} e^{itp_b}e^{iy\theta}\sin\theta d\theta\Big| &\lesssim (|tb|)^{-1/2} \max_{\theta\in[\pi-\alpha_2|b|^{1/4},\pi-\alpha_1|b|^{1/2}]} |\sin\theta| \leq (|tb|)^{-1/2}|b|^{1/4} \\ &\leq |t|^{-1/2}|b|^{-1/4} \leq |t|^{-1/2}|t|^{1/6} = |t|^{-1/3}.\end{aligned} \tag{3.26}$$

On the last interval $[\pi-\alpha_1|b|^{1/2}, \pi]$ the term $|br''(\theta)|$ dominates $|q''(\theta)|$. Then the behavior of $p_b''(\theta)$ in the considered interval is given by $|br''(\theta)|$:

$$|p_b''(\theta)| \geq |br''(\theta)| - |q''(\theta)| \geq C_1(|b| - C_2|\theta - \pi|^4) \geq C_3|b|.$$

Thus

$$\Big|\int_{\pi-\alpha_1|b|^{1/2}}^{\pi} e^{itp_b(\theta)}e^{iy\theta}\sin(\theta)d\theta\Big| \lesssim (|tb|)^{-1/2} \max_{\theta\in[\pi-\alpha_1|b|^{1/2},\pi]} |\sin\theta| \lesssim |t|^{-1/2}. \tag{3.27}$$

Using the previous estimates (3.23), (3.24), (3.25), (3.26) and (3.27) we obtain that estimate (3.19) also holds in the case $a = 1/\sqrt{2}$.

The proof of Lemma 3.2.5 is now finished. $\square$

In the case of system (3.8) the proof of Theorem 3.1.2 follows the lines of the proof of Theorem 3.0.6 by taking into account the representation formula for the resolvent of the operator A given by (3.9).

Lemma 3.2.6. *Let $\lambda \in \mathbb{C} \setminus [-4\max\{b_1^{-2}, b_2^{-2}\}, 0]$ and A given by* (3.9)*. For any $g \in l^2(\mathbb{Z}^*)$ there exists a unique solution $f \in l^2(\mathbb{Z}^*)$ of the equation $(A-\lambda I)f = g$. Moreover, it is given by the following formula*

$$f(j) = \frac{-r_s^{|j|}}{b_1^{-2}(r_1^{-1}-r_1)+b_2^{-2}(r_2^{-1}-r_2)}\Big[g(0)+\sum_{k\in Z_1} r_1^{|k|}g(k)+\sum_{k\in Z_2} r_2^{|k|}g(k)\Big] \tag{3.28}$$
$$+\frac{b_s^2}{r_s-r_s^{-1}}\sum_{k\in Z_s}(r_s^{|j-k|}-r_s^{|j|+|k|})g(k), \quad j\in Z_s,$$

where for $s\in\{1,2\}$, $r_s = r_s(\lambda)$ is the unique solution with $|r_s|<1$ of the equation

$$r_s^2 - 2r_s + 1 = \lambda b_s^2 r_s.$$

We leave the complete details of the proof of Theorem 3.1.2 to the reader.

3.3 Open problems

In this article we have analyzed the dispersive properties of the solutions of a system consisting in coupling two discrete Schrödinger equations. However we do not cover the case when more discrete equations are coupled. The main difficulty is to write in an accurate and clean way the resolvent of the linear operator occurring in the system. Once this case will be understood then we can treat discrete Schödinger equations on trees similar to those considered in [4] in the continuous case.

There is another question which arises from this paper. Suppose that we have a system $iU_t + AU = 0$ with an initial datum at $t = 0$, where A is an symmetric operator with a finite number of diagonals not identically vanishing. Under which assumptions on the operator A does solution U decay and how we can characterize the decay property in terms of the properties of A? When A is a diagonal operator we can use Fourier's analysis tools but in the case of a non-diagonal operator this is not useful.

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