Model theory for multiple-valued logics

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1 Many-valued logics

1.1 Paradoxes of vague terms and fuzzy approach

Human description of the world provides us with remarkable situations of a special kind. Consider the following form of the *sorites* paradox (paradox of heap): "A heap consisting of just one grain of sand is small. If you add one grain to small heap, it remains small. Therefore, each heap is small."

To see clearly that there is nothing wrong in the inference of the paradox (in a classical logic approach), let us formalize: let $S(x)$ stand for "a heap with x grains of sand is small". Then the assumptions axioms are: $S(1)$ and $S(x) \Rightarrow S(x+1)$. Now, let there be a heap consisting of n grains of sand and let us prove that the heap is small, i.e. we want to prove $S(n)$. This can be done by repeated application of *modus ponens*. Indeed, from $S(1)$ and $S(1) \Rightarrow S(2)$, we get $S(2)$ which together with $S(2) \Rightarrow S(3)$ gives S(3) etc. Thus we obtain $S(n)$.

Let us consider another example. Let $A \approx B$ denote the fact that the colors A and B are not much different. Intuitively clear is that if A is not much different from B and B is not much different from C, then A is not much different from C, i.e. $A \approx B$ and $B \approx C$ implies $A \approx C$ (transitivity of $\approx$). Now, take two colors A and B. There is no doubt that we can find a series $B_1, \ldots, B_n$ of colors such that $A \approx B_1$, $B_1 \approx B_2, \ldots, B_n \approx B$, i.e. a chain of colors starting with A and ending with B such that the successive colors are not much different. By transitivity, we get $A \approx B$, i.e. any two colors are not much different.

Looking closer at the above paradoxes we find a common feature: they are formulated in natural language and contain terms which are referred to as vague ("small", "not much different"). When analyzing paradoxes, terms used to formulate them should have exact meaning. And there is a problem with vague terms because they have no exact meaning.

The discussion leads us to an old tradition in science, namely to the *ideal of precision* and to the *principle of bivalence*. Statements that are to be useful and scientific statements in particular have been traditionally required to be precise (the more, the better). Thus, data were required to be as precise as possible in order to be considered serious, valuable. On the other hand, if the data were not precise, they were considered unreliable and somewhat suspicious. This view is called *ideal of precision* and is prevailed in philosophy and science for centuries (and still has a strong influence up to now). The principle of bivalence says that any statement is either true or false. Thus, given a number x, either "x in odd" is true or "x is odd" is false; given a heap h, either "h is small" is true or "h is small" is false.

Lofti A. Zadeh recognize vagueness as immanent to human description of the world. He proposed a natural formalism, so-called *fuzzy sets*, for dealing with vagueness. Zadeh's motivations came from systems engineering. In his paper [21] he summarized the arguments:

More often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership. For example, the class of animals clearly includes dogs, horses, birds, etc. as its members, and clearly excludes such objects as rocks, fluids, plants etc. However, such objects as starfish, bacteria etc. have an ambiguous status with respect to the class of animals. The same kind of ambiguity arises in the case of a number such as 10 in relation to the "class" of all real numbers which are much greater than 1.

Clearly, the "class of all real numbers which are much greater than 1", or "the class of beautiful *women", or "the class of tall men", do not constitute classes or sets in the usual mathematical sense of these terms. Yet, the fact remains that such imprecisely defined "classes" play an important role in human thinking, particularly in the domains of pattern recognition, communication of information, and abstraction.*

Zadeh came up with the concept of a *fuzzy set*:

A fuzzy set (class) A in X is characterized by a membership (characteristic) function $f_A(X)$ which associates with each point in X a real number in the interval $[0, 1]$, with the value of $f_A(x)$ at x representing the "grade of membership" of x in A. Thus, the nearer the value of $f_A(x)$ to unity, the higher the grade of membership of x in A. When A is a set in the ordinary sense of the term, its membership function can take on only two values 0 and 1, with $f_A(x) = 1$ or 0 according as x does or does not belong to A. Thus, in this case $f_A(x)$ reduces to the familiar characteristic function of a set A.

Zadeh's message points out two important facts. First, collections of objects encountered in human description of the world are, more often than not, nonsharp in that there is a gradual transition between being a member and not being a member of the collection. Second, classical mathematics does not have means for description of such collections in a natural way. To overcome this difficulty, Zadeh proposed the concept of a fuzzy set as a formal model of a collection with gradual transition from non-membership to membership. The main idea is that, in addition to "fully belong" (truth value 1) and "fully not belong" (truth value 0), there can also be other cases (other truth values) of belonging of an element to a fuzzy set; being a member of a fuzzy set is thus a graded property. This is in contrary with the principle of bivalence stating that a property either applies to an element or not. A fuzzy set is specified by a rule assigning to each element of a given universe set its membership degree, i.e. the degree to which the element belongs to the fuzzy set. Formally, fuzzy sets are just those rules: a fuzzy set A in a given universe X is a mapping of X into L, where L is a suitable set of membership degrees; $A(x)$, the element of L assigned to x by A, is called the membership degree of x in A. $A(x)$ can be thought as the truth degree (truth value) of the proposition "x belongs to A". Note that in the above quoted paper, Zadeh distinguishes a fuzzy set A from its membership function f_A (the membership degree of x in A is then $f_A(x)$, not $A(x)$); some authors proceed this way, we will not. Assigning (possibly intermediate) truth degrees to elementary propositions like "an element belongs to a fuzzy set" is a particular case of a more general one. In general, propositions may be composed out of elementary propositions using logical connectives and quantifiers and possible other connectives. In this way, one obtains propositions like "for all x and y, if x is big and if x and y do not differ much, then y is also big" containing elementary propositions that concern possibly non-sharp collections or relationship.

Intuitively, truth degrees can be compared, and 0 (degree corresponding to full falsity) and 1 (degree corresponding to full truth) are the least and the greatest truth degrees, respectively. Thus, L is a partially ordered set and bounded by 0 and 1. The most appealing set of truth degrees is the interval $[0, 1]$ (or some of its subsets containing 0 and 1) with its natural ordering.

The approach to modeling where acknowledges are gradual transitions from non-being a member to being a member of a collection, from not having to having a relationship, from not being true to being true, etc., and that uses the concept of a truth degree with the understanding that there are intermediate truth degrees (between 0 and 1) is called *fuzzy approach* or, perhaps more succinctly, *graded truth approach*.

Fuzzy approach seems to be most natural in modeling of human reasoning and human description of the outer world. For example, consider the term *tall man*. Using fuzzy approach, the collection of tall men can be modeled by a fuzzy set A to which a man 5 feet tall belongs with membership degree 0 (is not tall at all), a man 6 feet tall belongs with membership degree 0.8 and a man 6 feet 5 inches tall belongs with membership degree 1 (is absolutely tall). If we were to model the collection of tall men by an ordinary set B, one would have to pick up a particular height x (say 6 feet) and define B to be the set of those men whose height is at least x. This yields problems: First, why just x? Second and more important, a man whose height is just a little bit under 6 feet is not considered tall while a man who is 6 feet tall is - this is very counterintuitive.

Using graded truth approach, the *sorites* paradox can be resolved. Being a small heap is clearly a graded property and thus, $S(x)$ can take more truth values that just 0 or 1, heaps with less grains are "more small" than heaps with more grains. Take the real interval $[0, 1]$ for the set of truth values. Consider the axioms ($S(1)$ and $S(x) \Rightarrow S(x+1)$). There is no doubt that a heap consisting of one grain is small, i.e. we can accept $S(1)$ with the truth degree 1. On the other hand, it is intuitively appealing that $S(x) \Rightarrow S(x+1)$ should be accepted with a high truth degree which is, however, (strictly) less than 1, say 0.999 (the exact value does not matter for the argument). A direct generalization of *modus ponens* gives the following: if the truth degree of φ is (at least) a and the truth degree of $\varphi \Rightarrow \psi$ is (at least) b, then the truth degree of ψ is (at least) the truth degree of $\varphi\&(\varphi \Rightarrow \psi)$. Then if $\otimes$ defined by $a \otimes b = \max(0, a+b-1)$ is the operation corresponding to the connective $\&$ (we will see later that this is a reasonable choice, but we will also see that there are other reasonable choices), the truth degree of $\varphi\&(\varphi \Rightarrow \psi)$ is (at least) $a \otimes b$.

But now, there is no paradox any more. Indeed, let $\parallel \varphi \parallel$ denote the truth degree of φ. Then from $\parallel S(1) \parallel = 1$ and $\parallel S(1) \Rightarrow S(2) \parallel = 0.999$ we get $\parallel S(2) \parallel \geq 0.999$ which together with $\parallel S(2) \Rightarrow S(3) \parallel = 0.999$ gives $\parallel S(3) \parallel \geq 0.999^2$ which ... together with $\parallel S(n-1) \Rightarrow S(n) \parallel = 0.999$ gives $\parallel S(n) \parallel \geq 0.999^{n-1}$ (the power is taken w.r.t $\otimes$, so 0.999^3 is $0.999 \otimes 0.999 \otimes 0.999$). So, for example, we can deduce that the truth degree of "a heap with 11 grains of sand is small" is (at least) $0.999^{10} = 0.99$, that the truth degree of "a heap with 201 grains of sand is small" is (at least) $0.999^{200} = 0.8$ etc (one may change 0.999 to get intuitively more acceptable results).

The graded truth approach also helps to resolve the paradox "any two colors are not much different". The point to be rejected is the assumption that the relation "not much different" is a bivalent one (i.e. two colors either are much different or are not much different). We should rather construct the relation $\approx$ as admitting graded truth which is quite a natural assumption. For example, take colors A, B and C with wave-lengths x, y and z, respectively, and suppose, moreover, $x < y < z$ and $\|x - y\| < \|y - z\|$. Then A is closer to B than B is to C, and thus the truth value of $A \approx B$ is greater that the truth value of $B \approx C$. Starting from the (intuitively acceptable) assumption that the formula "if $A \approx B$ and $B \approx C$ then $A \approx C$" (transitivity of $\approx$) should be true and taking again $[0, 1]$ as the set of truth grades and product the operation corresponding to conjunction (we stress again that it is only one particular choice of many others possible), graded approach to $\approx$ translates the assumption into the condition $\parallel A \approx B \parallel \cdot \parallel B \approx C \parallel \leq \parallel A \approx C \parallel$ ($\parallel X \parallel$ denotes the truth value of X). Thus, if, for instance, the truth values of "A and B are not much different" and "B and C are not much different" are 0.8 and 0.5, respectively, then the restriction imposed on A and C by "if $A \approx B$ and $B \approx C$ then $A \approx C$" is the truth degree of "A and C are not much different" is at least $\parallel A \approx B \parallel \cdot \parallel B \approx C \parallel = 0.8 \cdot 0.5 = 0.4$ which is intuitively satisfactory.

As demonstrated, replacing bivalence with graded truth approach extends the scope of modeling capabilities of logic beyond the original limits forced by bivalence. Ignoring fuzzy approach in applications does not necessarily have to lead to a "disaster" like in the case of the paradoxes. It may, however, imply a significant loss compared to the situation when fuzzy approach is used. In this sense, the paradoxes are extreme examples showing the usefulness of fuzzy approach.

1.2 Fuzzy logic: formal foundations of fuzzy approach

Multiple-valued logic and fuzzy logic are the main tools to deal with vague knowledge modelized by means of fuzzy sets. These logics allow to handle propositions involving vague predicates. In multiple-valued logics, we assume that all the informations are complete, while in the fuzzy logic we assume that available information is imprecise or vague. In this thesis we will deal with multiple-valued logic, but sometimes we will not make the difference between multiple-valued logic and fuzzy logic, i.e. we will just consider them tools for dealing with vague predicates.

These days, the term *fuzzy logic* is used basically in two ways. In the broad sense, fuzzy logic is understood to cover any kind of methods and applications inspired by fuzzy approach. In the narrow sense, fuzzy logic refers to logical calculi that admit propositions to take intermediate truth values (between 0 and 1) which are interpreted as truth degrees; these calculi aim at formalization of reasoning in the presence of vagueness.

The main concern of fuzzy logic in the broad sense is to provide tools enabling us to deal with and utilize fuzziness. Inspiration for this is the kind of approximate reasoning people perform when reasoning with "fuzzy data" in everyday life.

Formal rules of reasoning are the subject of logic. If we allow fuzziness, i.e. more truth degrees, we get to fuzzy logic (in the narrow sense). This is the way fuzzy logic in the broad and narrow senses are connected: fuzzy logic in the narrow sense offers foundations for fuzzy logic in the broad sense.

Among other things, fuzzy logic in the narrow sense tells us: how to express formally propositions formulated in natural language (mathematically: what is the language of fuzzy logic, what are formulas), how to formally describe the (part of a) real world the propositions refer to (mathematically: what are structures in which formulas are evaluated), how to interpret natural language connectives (like "...or...", "...and...", "if...then...") and quantifiers (like "for all...", "for some...") that apply to propositions with possibly intermediate truth values (mathematically: what is the semantics of connectives and quantifiers in fuzzy setting), how to evaluate truth degrees of propositions, how propositions follow from other propositions that are possibly valid to only a certain degree and how to formalize approximate inference (that is: what is deduction, provability etc. in fuzzy logic), what are the properties of approximate inference (that is: what about classical properties like syntactic-semantical completeness etc).

1.3 Basic systems of Many-Valued Logics

If one looks systematically for many-valued logics which have been designed for quite different applications, one finds four main types of systems:

- the Gödel logics $\mathrm{G_k}$ from [11]
- the Łukasiewicz logics $\mathrm{Ł_k}$ as explained in [16]
- the Product logic $\prod$ studied in [14]
- the Post logics $\mathrm{P_m}$, for $2 \leq m \in \mathbb{N}$, from [18]

The first two types of many-valued logics each offer a uniformly defined family of systems which differ in their sets of truth degrees and comprise finitely valued logics for each one of the truth degree sets together with an infinite valued system with truth degree set, which formally is indicated by choosing $k \in \{n \in \mathbb{N} \mid n \geq 2\} \cup \infty$. For the fourth type an infinite valued version is lacking.

In their original presentations, these logics look rather different, regarding their propositional parts. For the first order extensions, however, there is a unique strategy: one adds a universal and an existential quantifier such that quantified formulas get, respectively, as their truth degrees the infimum and the supremum of all the particular cases in the range of the quantifiers.

1.3.1 The Gödel logics

The simplest one of these logics are the *Gödel logics* $\mathrm{G_k}$ which have a conjunction $\wedge$ and a disjunction $\vee$ defined by the minimum and the maximum, respectively, of the truth degrees of the constituents:

$$u \wedge v = \min(u, v) \qquad u \vee v = \max(u, v) \tag{1}$$

These Gödel logics have also a negation $\sim$ and an implication $\rightarrow_G$ defined by the truth degree functions:

$$\sim u = \begin{cases} 1, & \text{if } u = 0; \\ 0, & \text{if } u > 0. \end{cases} \qquad u \rightarrow_G v = \begin{cases} 1, & \text{if } u \leq v; \\ v, & \text{if } u > v. \end{cases}$$

This systems differ in their truth degree sets: for each $2 \leq k \leq \infty$, the truth degree set of $\mathrm{G_k}$ is $W_k = \{\frac{m}{k-1} \mid 0 \leq m \leq k-1\}$.

1.3.2 The Łukasiewicz logics

The *Łukasiewicz logics* $\mathrm{L_k}$, again with $2 \leq k \leq \infty$, have originally been designed in [16] with only two primitive connectives, an implication $\rightarrow_L$ and a negation $\neg$ characterized by the truth degree functions:

$$\neg u = 1 - u \qquad u \rightarrow_L v = \min\{1, 1 - u + v\}$$

The systems differ in their truth degree sets: for each $2 \leq k \leq \infty$ the truth degree set of $\mathrm{L_k}$ is $W_k = \{\frac{m}{k-1} \mid 0 \leq m \leq k-1\}$.

However, it is possible to define further connectives from these primitive ones. With

$$\varphi \& \psi =_{df} \neg(\varphi \rightarrow_L \neg\psi) \qquad \varphi \underline{\vee} \psi =_{df} \neg\varphi \rightarrow_L \psi$$

one gets a (strong) conjunction and a (strong) disjunction with truth degree functions

$$u\&v = \max\{u + v - 1, 0\} \qquad u\underline{\vee}v = \min\{u + v, 1\} \tag{2}$$

usually called the *Łukasiewicz* (arithmetical) *conjunction* and the *Łukasiewicz* (arithmetical) *disjunction*. It should be mentioned that these connectives are linked together via a De Morgan law using the standard negation of this system:

$$\neg(u\&v) = \neg u\underline{\vee}\neg v.$$

With the additional definitions

$$\varphi \wedge \psi =_{df} \varphi\&(\varphi \to_L \psi) \qquad \varphi \vee \psi =_{df} (\varphi \to_L \psi) \to_L \psi$$

one gets another (weak) conjunction $\wedge$ with the truth degree function $\min$ and a further (weak) disjunction $\vee$ with $\max$ as truth degree function, i.e. one has the conjunction and the disjunction of the Gödel logics also available.

1.3.3 The Product logic

The *product logic* $\prod$ has a fundamental conjunction $\odot$ with the ordinary product of reals as its truth degree function, as well as an implication $\to_{\prod}$ with truth degree function

$$u \to_{\prod} v = \begin{cases} 1, & \text{if } u \leq v; \\ \frac{v}{u}, & \text{if } u > v. \end{cases}$$

Additionally it has a truth degree constant $\overline{0}$ to denote the truth degree zero.

In this context, a negation and a further conjunction are defined as

$$\sim \varphi =_{df} \varphi \to_{\prod} \overline{0} \qquad \varphi \wedge \psi =_{df} \varphi \odot (\varphi \to_{\prod} \psi).$$

Routine calculations show that both connectives coincide with the corresponding ones of the infinite valued Gödel logic G_∞. And also the disjunction $\vee$ of this Gödel logic becomes available via the definition

$$\varphi \vee \psi =_{df} ((\varphi \to_{\prod} \psi) \to_{\prod} \psi) \wedge ((\psi \to_{\prod} \varphi) \to_{\prod} \varphi).$$

There is, however, no natural way to combine with this (infinite valued) product logic a whole family of finite valued systems by simply restricting the set of truth degrees to some W_m as in the previous two cases: besides W_2 no such set is closed under the ordinary product, and for W_2 the product coincides with the minimum operation, where $W_m = \{\frac{k}{m-1} \mid 0 \leq k \leq m - 1\}$.

1.3.4 The Post logics

The Post system $\mathrm{P_m}$, for $m \geq 2$, has truth degree set $W_m = \{\frac{k}{m-1} \mid 0 \leq k \leq m-1\}$. These propositional systems have been originally formulated uniformly in negation and disjunction as basic connectives with the following truth degree functions:

$$\sim u = \begin{cases} 1, & \text{if } u = 0; \\ u - \frac{1}{m-1}, & \text{if } u \neq 0 \end{cases} \qquad u \vee v = \max\{u, v\}.$$

Contrary to the previous systems, the definition of the negation here does not seem to be given in a uniform way independent of the number of truth degrees. However, it is always just a cyclic permutation of all truth degrees (in their natural order).

1.4 Graded truth and structures of truth values

From the previous discussion it is clear that the structure of the set of truth values deserves special attention. The aim of this subsection is to show how certain natural logical assumptions reflect themselves in corresponding properties of the structure of truth values.

The graded truth approach directly leads to the assumption that the set L of truth values is partially ordered (we denote the partial order by $\leq$) with 0 and 1 being the least and the greatest element, respectively. For every two truth values a and b there is a truth value greater than both a and b (one can take 1). Moreover, one may require that there is the least truth value which is greater than both a and b. In this way we come to the requirement of existence of suprema (and dually of infima) of two-element subsets in L. Let $\{\varphi_i \mid i \in I\}$ be a set of propositions. A generalization from the classical case of two-valued logic leads to the assumption that the truth value of "there exists $i \in I$ such that φ_i" is the supremum of the truth values of φ, i.e. $\|$ "there exists $i \in I$ such that φ_i" $\| = \bigvee_{i \in I} \| \varphi_i \|$ ($\| \varphi \|$ is the truth value of φ). Therefore, if one wants to evaluate such existential (and dually, universal) propositions, suprema (and infima) of arbitrary subsets of L should exist. In this way one comes to the assumption that L should be a complete lattice.

We now get to the question of operations on L which model logical connectives. The general principle to which we adhere says that they should extend the classical operations in that restricting the operations to "classical truth values" 0 and 1, they coincide with classical operations. Furthermore, as in the classical case we want the logic to be truth functional, i.e. the truth value of a compound formula depends only on the truth values of its parts. We start by conjunction (we denote it by $\&$). Denote the operation which corresponds to $\&$ by $\otimes$, i.e. $\otimes$ is a binary operation on L. Since $\otimes$ should extend the operation corresponding to classical conjunction, we require $1 \otimes 1 = 1$, $1 \otimes 0 = 0 \times 1 = 0 \times 0 = 0$. If we want the truth value of $\varphi \& \psi$ to be the same as the truth value of $\psi \& \varphi$, the truth functionality leads to the requirement of commutativity of $\otimes$. Similarly, if the truth values of $\varphi \& (\psi \& \chi)$ and $(\varphi \& \psi) \& \chi$ are to be the same, truth functionality leads to the associativity of $\otimes$. In this way we came to the assumption that $(L, \otimes, 1)$ is a commutative monoid. Furthermore, it is intuitively appealing to require that $\otimes$ is nondecreasing, i.e. that $a_1 \leq a_2$ and $b_1 \leq b_2$ imply $a_1 \otimes b_1 \leq a_2 \otimes b_2$ (the bigger the truth degrees of two propositions the bigger the truth degree of their conjunction).

Let us turn to implication. In classical logic, conjunction and implication play an important role in the formulation of an inference rule called *modus ponens*. *Modus ponens* says that if φ is valid and

$\varphi \Rightarrow \psi$ is valid ($\Rightarrow$ denote "implies") then (we may infer that) ψ is valid. Let us now reformulate a little bit using validity degree. Let us say that φ is valid in degree 1 if φ is valid and that φ is valid in degree 0 if φ is not valid (we have no other degrees than 0 and 1 in bivalent logic). A moment's reflection shows that an equivalent formulation of *modus ponens* is the following one: if φ is valid in degree at least a, and $\varphi \Rightarrow \psi$ is valid in degree at least b ($a, b \in \{0, 1\}$), then ψ is valid in degree at least $a \otimes_2 b$ ($\otimes_2$ denote the operation on $\{0, 1\}$ that corresponds to classical conjunction). This rule is sound in the following sense: if φ and ψ are formulas such that $a \leq\| \varphi \|$ and $b \leq\| \varphi \Rightarrow \psi \|$, then $a \otimes_2 b \leq\| \psi \|$. This means that when using *modus ponens* it cannot happen that "we infer more than is actually true", i.e. it cannot happen that the actual truth degree of ψ is less than the degree inferred by *modus ponens*. Therefore, the role of $\otimes_2$ in *modus ponens* is to get the lower estimation of the validity of the inferred formula ψ from the lower estimations of validities of φ and $\varphi \Rightarrow \psi$. It is easy to see that $\otimes_2$ gives the highest possible estimation under the condition that the inference rule still be sound.

This way of looking at *modus ponens* is suitable for generalization to graded-truth case. Knowing that $a \leq\| \varphi \|$ and $b \leq\| \varphi \Rightarrow \psi \|$ ($a, b \in L$) we want to use $\otimes$ to (1) obtain the lower estimation of validity of ψ. Moreover, we want (2) to obtain the highest possible lower estimation such that the rule still be sound. Denote by $\rightarrow$ the binary operation on L which corresponds to $\Rightarrow$. Condition (1) translates then to: $a \leq\| \varphi \|$ and $b \leq\| \varphi \Rightarrow \psi \|$ implies $a \otimes b \leq\| \psi \|$, i.e., by truth-functionality, $a \leq\| \varphi \|$ and $b \leq\| \varphi \|\rightarrow\| \psi \|$ implies $a \otimes b \leq\| \psi \|$. Putting $a =\| \varphi \|$ and denoting $c =\| \psi \|$, we obtain a special case of this implication, i.e. $b \leq a \rightarrow c$ implies $a \otimes b \leq c$. Condition (2): we want the inference of *modus ponens* to be as powerful as possible. From $a =\| \varphi \|$ and $\| \varphi \Rightarrow \psi \|$ we get a lower estimation $a \otimes \| \varphi \Rightarrow \psi \|$ of $c =\| \psi \|$, i.e. $a \otimes \| \varphi \Rightarrow \psi \|\leq c$. The rule is the more powerful the bigger the estimated value $a \otimes \| \varphi \Rightarrow \psi \|$ (of course, under the condition that the rule is still sound, i.e. it really gets the lower estimation of $\| \psi \|$). Since a is given, $a \otimes \| \varphi \Rightarrow \psi \|$ depends on $\| \varphi \Rightarrow \psi \|$. Since $\otimes$ is nondecreasing, bigger $\| \varphi \Rightarrow \psi \|$ leads to bigger (or at least the same) $a \otimes \| \varphi \Rightarrow \psi \|$. Now, as $\| \varphi \Rightarrow \psi \|=\| \varphi \|\rightarrow\| \psi \|= a \rightarrow c$, the requirement that *modus ponens* be as powerful as possible yields that $a \rightarrow c$ should be the largest possible value that leads to lower estimation of c. That is, we want that whenever $a \otimes b \leq c$ (i.e. whenever b is a possible candidate for $\| \varphi \|\rightarrow\| \psi \|= a \rightarrow c$, b is considered a possible candidate since it leads to lower estimation) then $b \leq a \rightarrow c$ (i.e. then $a \rightarrow c$ is at least as good as b since it leads to at least as good lower estimation of c).

Putting the conditions which derived from (1) and (2) together we get

$$a \otimes b \leq c \quad \text{iff} \quad b \leq a \rightarrow c.$$

This condition (one may check that it is true for $\otimes_2$ and $\rightarrow_2$) will be called the *adjointness property*. We just saw the assumptions from which it was derived.

Algebraic structures which satisfy the above conditions will be called *residuated lattices*. The logical assumptions from which the algebraic conditions have been derived are relatively simple. Further logical requirements can be taken into account by adding appropriate algebraic conditions. For example, one may require idempotent conjunction (i.e. the truth value of $\varphi \& \varphi$ be the same as the truth value of φ) and this leads to an additional condition $x \otimes x = x$. In a similar way one can obtain MV-algebras (the algebras of Łukasiewicz logic) as a special case of residuated lattices etc.

Definition 1.1. A *residuated lattice* is an algebra $\mathbf{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ where

(i) $(L, \vee, \wedge, 0, 1)$ is a lattice with the least element 0 and the greatest element 1,

(ii) $(L, \otimes, 1)$ is a commutative monoid, i.e. $\otimes$ is associative, commutative, and the identity $x \otimes 1 = x$ holds,

(iii) the *adjointness property*, i.e.

$$x \leq y \to z \quad \text{iff} \quad x \otimes y \leq z$$

holds for each $x, y, z \in L$ ($\leq$ denotes the lattice ordering).

A residuated lattice is called *complete* if $(L, \vee, \wedge, 0, 1)$ is a complete lattice.

Definition 1.2. A *t-norm* is a binary operation on $[0, 1]$ which is associative, commutative, monotone and with 1 acting as its unit element, i.e. $\otimes$ is a mapping $\otimes : [0, 1] \times [0, 1] \to [0, 1]$ satisfying

· $(x \otimes y) \otimes z = x \otimes (y \otimes z)$

· $x \otimes y = y \otimes x$

· $y_1 \leq y_2$ implies $x \otimes y_1 \leq x \otimes y_2$

· $x \otimes 1 = x$

for any $x, y, y_1, y_2 \in [0, 1]$.

Remark 1.1. Let $\otimes$ be a left-continuous t-norm (i.e. $\lim_{n\to\infty}(a_n \otimes b) = (\lim_{n\to\infty} a_n) \otimes b$ for any increasing sequence $\{a_n \in [0, 1] \mid n = 1, 2, 3, \ldots\}$). Put

$$a \to b = \sup\{c \mid a \otimes c \leq b\}.$$

Then $([0, 1], \min, \max, \otimes, \to, 0, 1)$ is a residuated lattice (a complete one).

T-norms have been used in the context of probabilistic metric spaces [15]. At the same time they are considered natural candidates for truth degree functions of conjunction connectives. From such a t-norm one is able to derive (essentially) all the other truth degree functions for further connectives.

The minimum operation $u \wedge v$ from (1), the Łukasiewicz arithmetic conjunction $u \& v$ from (2) and the ordinary product are the best known examples of t-norms. They are also examples of continuous t-norms.

2 Institutions

The theory of institutions introduced by Goguen and Burstall in [12] formalizes the intuitive notion of logical system into a mathematical object, including syntax, semantics and the satisfaction between them. The original goal for introducing the notion of an institution was to provide an abstract, logic-independent framework for algebraic specifications of computer science systems. It is natural to develop a theory of specification formalism in a way that is as much as possible independent of the choice of the underlying system: this would not only bring a separation of different issues (details of a particular logic and general concepts) but it would also allow to apply the abstract results of the theory to a certain formalism well suited for a given task.

Since they were defined, institutions gained the position of major tool in the development of the theory of specification and it became standard in the field to express the logical system underlying a particular language or system in the language of the theory of institutions (see CASL[2], CafeOBJ[9]).

Besides its importance for algebraic specification, the theory of institutions provides an appropriate framework for the development of abstract model theory, a model theory which do not have an underlying logical system. Such a model theory based on institutions may be called 'institution-independent model theory'. A monograph dedicated to this topic is [8] and further motivations for developing a model theory without a commitment to any logical system can be found in [7].

Definition 2.1. An *institution* consists of

1. a category $\mathbb{S}ig$, whose objects are called *signatures*,
2. a functor $\mathrm{Sen} : \mathbb{S}ig \to \mathbb{S}et$, providing for each signature Σ a set whose elements are called (Σ-)*sentences*,
3. a functor $\mathrm{Mod} : \mathbb{S}ig \to \mathbb{C}at^{op}$, providing for each signature Σ a category whose objects are called (Σ-)*models* and whose arrows are called (Σ-)*homomorphisms*,
4. a relation $\models_\Sigma \subseteq |\mathrm{Mod}(\Sigma)| \times \mathrm{Sen}(\Sigma)$ for each $\Sigma \in |\mathbb{S}ig|$, called ($\Sigma$-)*satisfaction*,

such that for each morphism $\varphi : \Sigma \to \Sigma'$ in $\mathbb{S}ig$, the *satisfaction condition*

$$M' \models_{\Sigma'} \mathrm{Sen}(\varphi)(\rho) \text{ iff } \mathrm{Mod}(\varphi)(M') \models_\Sigma \rho$$

holds for all $M' \in |\mathrm{Mod}(\Sigma')|$ and $\rho \in \mathrm{Sen}(\Sigma)$.

Following the usual notational conventions, we sometimes let $(\_ \upharpoonright_\varphi)$ denote the reduct functor $\mathrm{Mod}(\varphi)$ and let φ denote the sentence translation $\mathrm{Sen}(\varphi)$. When $M = M' \upharpoonright_\varphi$, we say that M' is a *φ-expansion* of M and that M is the *φ-reduct* of M' (similarly for model homomorphisms).

In the rest of this section we will provide some of the best well known examples of institutions.

2.1 FOL - the institution of first order logic with equality

Signatures. Signatures are triples (S, F, P), where S is a set (of sorts), $F = \bigcup_{w \in S^*, s \in S} F_{w \to s}$ is the set of operation symbols, organized by their arity $w \in S^*$ and their rank $s \in S$, and $P = \bigcup_{w \in S^*} P_w$ is the set of predicate symbols, also organized by arity. If $w = \lambda$, an element of $F_{w \to s}$ is called a constant symbol or a constant.

A signature morphism $\varphi : (S, F, P) \to (S', F', P')$ is a triplet $\varphi = (\varphi^{st}, \varphi^{op}, \varphi^{rel})$, where $\varphi^{st} : S \to S'$, $\varphi^{op} : F \to F'$, $\varphi^{rel} : P \to P'$ such that $\varphi^{op}(F_{w \to s}) \subseteq F'_{\varphi^{st}(w) \to \varphi^{st}(s)}$ and $\varphi^{rel}(P_w) \subseteq P'_{\varphi^{st}(w)}$, for all $w \in S^*$ and $s \in S$.

Models. A model M of a signature (S, F, P) interprets sorts as sets, operation symbols as functions such that if $\sigma \in F_{w \to s}$, $M_\sigma : M_w \to M_s$ and predicate symbols $\pi \in P_w$ as subsets $M_\pi \subseteq M_w$, where if $w = w_1 \dots w_n$, then $M_w = M_1 \times \dots \times M_n$.

A model homomorphism $h : M \to N$ is an S-sorted function $\{h_s : M_s \to N_s \mid s \in S\}$ that preserves both operation and predicate symbols: $h_s(M_\sigma(m_1, \dots, m_n)) = N_\sigma(h_{s_1}(m_1), \dots, h_{s_n}(m_n))$, for any operation symbol $\sigma \in F_{s_1 \dots s_n \to s}$ and any $m_i \in M_{s_i}$, $i = 1, \dots, n$ and $h_w(M_\pi) \subseteq N_\pi$, for any predicate symbol $\pi \in P_w$.

For each signature morphism $\varphi : \Sigma \to \Sigma'$, the functor $\mathrm{Mod}(\varphi) = (\_ \upharpoonright_\varphi)$ assigns to each Σ'-model M' a Σ-model M such that $M_x = M'_{\varphi(x)}$, where x stands for each sort, operation symbol or predicate symbol, and to each Σ'-model homomorphism $h' : M' \to N'$ the model homomorphism $h' \upharpoonright_\varphi : M' \upharpoonright_\varphi \to N' \upharpoonright_\varphi$ defined by $(h' \upharpoonright_\varphi)_s = h'_{\varphi(s)}$.

Sentences. Given a signature (S, F, P), we define the F-terms inductively: each $\sigma \in F_{\to s}$ is a term of sort s and for each $\sigma \in F_{w \to s}$, $\sigma(t_1, \dots, t_n)$ is a term of sort s if t_i are terms of sort s_i. The atomic formulas are either of the form $t = t'$ (equational), where t, t' are terms of the same sort or $\pi(t_1, \dots, t_n)$ (relational), where t_i is a term of sort s_i. The set of (S, F, P)-sentences is the least set that contains the atoms and is closed under Boolean connectives and quantification. A universal quantified sentence by a finite set of variables X is of the form $(\forall X)\rho$, where ρ is a $(S, F \uplus X, P)$-sentence and we added to the signature the variables X as new constants (the existential quantified sentences can be treated similarly).

The sentence translation along a signature morphism $\varphi : (S, F, P) \to (S', F', P')$ is defined inductively on the structure of sentences by replacing the symbols from (S, F, P) with their corresponding symbols by φ in (S', F', P'). The only thing that requires attention is that when translating a constant symbol of sort s, it becomes a constant symbol of sort $\varphi(s)$.

Satisfaction. Each term $t = \sigma(t_1, \dots, t_n)$ is interpreted in a model M as $M_\sigma(M_{t_1}, \dots, M_{t_n})$. The satisfaction relation between models and sentences is defined inductively on the structure of sentences. For a fixed signature (S, F, P):

- $M \models t = t'$ if $M_t = M_{t'}$;
- $M \models \pi(t_1, \dots, t_n)$ if $(M_{t_1}, \dots, M_{t_n}) \in M_\pi$;
- $M \models \neg\rho$ if $M \not\models \rho$;
- $M \models \rho_1 \wedge \rho_2$ if $M \models \rho_1$ and $M \models \rho_2$, and similarly for all Boolean connectives;

- $M \models (\forall X)\rho$ if $M' \models \rho$ for each expansion M' of M along the signature inclusion $(S, F, P) \hookrightarrow (S, F \uplus X, P)$ (i.e. M is the reduct of M'), where the signature $(S, F \uplus X, P)$ is obtained from (S, F, P) by adding the finite set X of variables as new constants;
- $M \models (\exists X)\rho$ if $M \models \neg(\forall X)\neg\rho$.

One can show that the satisfaction condition holds, which defines completely the institution of the first order logic with equality.

2.2 HCL - the institution of Horn clauses logic

An *universal Horn sentence* for a FOL signature (S, F, P) is a universal quantified conditional atomic sentence of the form $(\forall X)H \Rightarrow c$, where H is a finite conjunction of (relational or equational) atoms and c is a (relational or equational) atom, and $H \Rightarrow c$ is the implication of c by H. In the tradition of logic programming, universal Horn sentences are known as *Horn clauses*. Thus HCL has the same signatures and models as FOL, but only universal Horn sentences as sentences.

2.3 The institution of presentation $\mathrm{I}^{\mathrm{pres}}$ over a base institution I

Given an institution $\mathrm{I} = (\mathbb{S}ig, \mathrm{Sen}, \mathrm{Mod}, \models)$, a presentation is a pair (Σ, E), with $\Sigma \in |\mathbb{S}ig|$ and $E \subseteq \mathrm{Sen}(\Sigma)$. A presentation morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ is a signature morphism $\varphi : \Sigma \to \Sigma'$ such that $E' \models \varphi(E)$ (where we denote by $\models$ the relation of semantical consequence between sets of sentences - for any two sets of sentences $E, E' \in \mathrm{Sen}(\Sigma)$, $E \models E'$ if and only if any Σ-model M of E is also a model of E'). For a presentation (Σ, E), we let $\mathrm{Mod}(\Sigma, E)$ denote the category of all Σ-models M such that $M \models_\Sigma E$.

We define the institution of the presentations I^{pres} over the base institution as follows:

- $\mathbb{S}ig^p$ is the category $\mathbb{P}res$ of the presentations of I
- for each presentation (Σ, E), $\mathrm{Mod}^p(\Sigma, E) = \mathrm{Mod}(\Sigma, E)$
- for each presentation (Σ, E), $\mathrm{Sen}^p(\Sigma, E) = \mathrm{Sen}(\Sigma)$
- $M \models^p_{(\Sigma,E)} \rho$ if and only if $M \models_\Sigma \rho$, for each (Σ, E)-model M and each (Σ, E)-sentence ρ.

3 The institutions of multiple-valued logics with equality

In this section we will extend the institutions for multiple-valued logics developed in [1] by introducing function symbols in the signatures and also by considering equational atoms. As in [1], the sentences will be pairs, but in our case, we will consider only one value from the truth algebra, instead of a set of values. The satisfaction relation in this approach is an inequality instead of an equality as in [1].

As in [1], we will organize multiple-valued logics as families of institutions, each one being indexed by a class of truth-value algebras.

3.1 Truth-value algebras

Let us consider TV a signature in FOL with only one sort V (for the set of truth-values) and function and relation symbols. Let EQ be a set of axioms that the models of the signature TV must fulfill.

Therefore, a *truth-value algebra* L will be an object of the category of models of the presentation $(\mathrm{TV}, \mathrm{EQ})$ in FOL.

We present an example of a presentation $(\mathrm{TV}, \mathrm{EQ})$ for residuated lattices.

Example 3.1. Let $\mathrm{TV} = (S, F, P)$ be a signature, where $S = \{V\}$, $F_{\to V} = \{0, 1\}$, $F_{VV \to V} = \{\vee, \wedge, \otimes, \to\}$ and $P_{VV} = \{\leq\}$.

Let EQ be the following set of axioms:

bounded lattice:

$x \vee y = y \vee x$

$x \wedge y = y \wedge x$

$x \vee (y \vee z) = (x \vee y) \vee z$

$x \wedge (y \wedge z) = (x \wedge y) \wedge z$

$x \wedge (x \vee y) = x$

$x \vee (x \wedge y) = x$

$x \vee x = x$

$x \wedge x = x$

$x \wedge 1 = x$

$x \vee 0 = x$

commutative monoid:

$x \otimes (y \otimes z) = (x \otimes y) \otimes z$

$x \otimes y = y \otimes x$

$x \otimes 1 = x$

order relation

$x \leq x$

$((x \leq y)\ and\ (y \leq z))\ implies\ x \leq z$

$(x \leq y)\ or\ (y \leq x)$

adjointness property

$(x \leq y \to z)\ implies\ (x \otimes y \leq z)$

$(x \otimes y \leq z)\ implies\ (x \leq y \to z)$

In order to be able to introduce equational atoms on the institutions of multiple-valued logics, we need to make some assumptions regarding the presentation $(\mathrm{TV}, \mathrm{EQ})$ that gives the truth-value algebras. Therefore, we consider that in every presentation $(\mathrm{TV}, \mathrm{EQ})$ we have symbols and equations that models the usual infimum in a lattice $\wedge$, the lattice order $\leq$, 1 (the greatest value) and 0 (the smallest value). We also consider that every truth algebra has arbitrary suprema and infima.

We recall the following notions from [3].

Let L be a truth-value algebra and M a nonempty set. An *L-set* in M is a mapping $A : M \to L$. The set M is called the *universe* and $A(x)$ is called the *degree of membership* of x in A. L-sets are also called *fuzzy sets*.

If $M = M_1 \times \ldots \times M_n$, then an L-set in M is called an (n-ary) *L-relation* (or *fuzzy relation*) on M.

For every truth-value algebra L and any set M, we define an *L-equivalence* on M as a binary fuzzy relation $\approx: M \times M \to L$ satisfying the conditions:

· $m \approx m = 1$, for any $m \in M$ (reflexivity);

· $m \approx m' = m' \approx m$, for any $m, m' \in M$ (symmetry);

· $(m \approx m') \wedge (m' \approx m'') \leq (m \approx m'')$, for any $m, m', m'' \in M$ (transitivity).

An L-equivalence is an *L-equality* if it also satisfies the condition: if $m \approx m' = 1$, then $m = m'$, for any $m, m' \in M$.

If $\approx_i$ is an L-equivalence on M_i ($i = 1, \ldots, n$), an L-relation R on $M_1 \times \ldots \times M_n$ is said to be *compatible* with $\approx_1, \ldots, \approx_n$ if

$$(x_1 \approx_1 y_1) \wedge \ldots \wedge (x_n \approx_n y_n) \wedge R(x_1, \ldots, x_n) \leq R(y_1, \ldots, y_n),$$

for any $x_i, y_i \in X_i$, $i = 1, \ldots, n$.

If $\approx_i$ and $\approx$ are L-equivalences on M_i ($i = 1, \ldots, n$) and M, respectively, a function $f : M_1 \times \ldots \times M_n \to M$ is said to be *compatible* with $\approx_1, \ldots, \approx_n, \approx$ if

$$(x_1 \approx_1 y_1) \wedge \ldots \wedge (x_n \approx_n y_n) \leq (f(x_1, \ldots, x_n) \approx f(y_1, \ldots, y_n)),$$

for any $x_i, y_i \in X_i$, $i = 1, \ldots, n$.

From now on, L will stand for an object in the category of models of the presentation $(\mathrm{TV}, \mathrm{EQ})$. By abuse of notation, we will denote the carrier of L also by L.

We assume that we have a set C of logical connective symbols, along with their arity and a function con from C to the set of operations symbols of the signature TV, that maintains the arities for all logical connective symbols.

3.2 $\mathrm{MVL(L)}$ - the institution of multiple-valued logic

For each truth-value algebra L of $\mathrm{Mod}(\mathrm{TV}, \mathrm{EQ})$ and each set C of logical connective symbols, we are going to describe the L-Multiple-Valued Logic institution $\mathrm{MVL_C(L)} = (\mathbb{S}ig^L, \mathrm{Sen}^L, \mathrm{Mod}^L, \models^L)$. In the rest of this thesis we will not concern with the set of logical connective symbols, therefore we will simply denote the L-Multiple-Valued Logic institution by $\mathrm{MVL(L)} = (\mathbb{S}ig^L, \mathrm{Sen}^L, \mathrm{Mod}^L, \models^L)$.

3.3 Signatures

The category of signatures of $\mathrm{MVL(L)}$ is just the category of signatures of FOL.

Remark 3.1. All the properties known for $\mathbb{S}ig^{\mathrm{FOL}}$ hold also for $\mathbb{S}ig^L$.

3.4 Models

Given a signature (S, F, R), an (S, F, R)-*model* M consists of:

- each sort $s \in S$ is interpreted as a pair $(M_s, \approx_s^M)$, where M_s is a set and $\approx_s^M$ is an L-equality on M_s;

- each operation symbol $f \in F_{s_1 \ldots s_n \to s}$ is interpreted as a function $M_f : M_{s_1 \ldots s_n} \to M_s$ compatible with $\approx_{s_1}^M, \ldots, \approx_{s_n}^M, \approx_s^M$, i.e.

$$(x_1 \approx_{s_1}^M y_1) \wedge \ldots \wedge (x_n \approx_{s_n}^M y_n) \leq (M_f(x_1, \ldots, x_n) \approx_s^M M_f(y_1, \ldots, y_n)),$$

  for each $x_i, y_i \in M_{s_i}$;

- each relation symbol $r \in R_{s_1 \ldots s_n}$ is interpreted as a fuzzy relation $M_r : M_{s_1 \ldots s_n} \to L$ compatible with $\approx_{s_1}^M, \ldots, \approx_{s_n}^M$, i.e.

$$(x_1 \approx_{s_1}^M y_1) \wedge \ldots \wedge (x_n \approx_{s_n}^M y_n) \wedge M_r(x_1, \ldots, x_n) \leq M_r(y_1, \ldots, y_n),$$

  for each $x_i, y_i \in M_{s_i}$.

Given two (S, F, R)-models M and M', an (S, F, R)-*model homomorphism* $h : M \to M'$ is an indexed family of functions $\{h_s : M_s \to M'_s\}_{s \in S}$ such that:

- $(m \approx_s^M m') \leq (h_s(m) \approx_s^{M'} h_s(m'))$, for any $s \in S$ and $m, m' \in M_s$;

- $h_s(M_f(m)) = M'_f(h_w(m))$, for any $f \in F_{w \to s}$ and $m \in M_w$;

- $M_r(m) \leq M'_r(h_w(m))$, for any $r \in R_w$ and $m \in M_w$.

Fact 3.1. For any signature (S, F, R), the (S, F, R)-models and (S, F, R)-model homomorphisms form a category, $\mathrm{Mod}^{\mathrm{L}}(S, F, R)$, where the composition of homomorphisms is made component-wise as many sorted functions.

For any signature morphism $\varphi : (S, F, R) \to (S', F', R')$ and any (S', F', R')-model M', we define $\mathrm{Mod}^{\mathrm{L}}(\varphi)(M') = M' \upharpoonright_\varphi$ by:

- for any $s \in S$, $(M' \upharpoonright_\varphi)_s = M'_{\varphi^{st}(s)}$ and $\approx_s^{M' \upharpoonright_\varphi} = \approx_{\varphi^{st}(s)}^{M'}$;

- for each operation symbol $f \in F_{w \to s}$, $(M' \upharpoonright_\varphi)_f = M'_{\varphi^{op}_{w \to s}(f)}$;

- for each relation symbol $r \in R_w$, $(M' \upharpoonright_\varphi)_r = M'_{\varphi^{rel}_w(r)}$.

Each (S', F', R')-model homomorphism $h' : M' \to N'$ is mapped into $\mathrm{Mod}^{\mathrm{L}}(\varphi)(h') = h' \upharpoonright_\varphi$, where $h' \upharpoonright_\varphi : M' \upharpoonright_\varphi \to N' \upharpoonright_\varphi$ is defined by $(h' \upharpoonright_\varphi)_s = h'_{\varphi^{st}(s)}$, for each sort $s \in S$.

Fact 3.2. For each signature morphism $\varphi : (S, F, R) \to (S', F', R')$, $\mathrm{Mod}^{\mathrm{L}}(\varphi) : \mathrm{Mod}^{\mathrm{L}}(S', F', R') \to \mathrm{Mod}^{\mathrm{L}}(S, F, R)$ is a functor. Moreover, $\mathrm{Mod}^{\mathrm{L}} : \mathbb{S}ig^L \to \mathbb{C}at^{op}$ is a functor.

3.5 Sentences

Let (S, F, R) be a signature. An F-term of sort s is a syntactic structure $f(t_1, \ldots, t_n)$, where $f \in F_{s_1 \ldots s_n \to s}$ is an operation symbol and $t_1, \ldots, t_n$ are F-terms of sorts $s_1, \ldots, s_n$. Let us denote by T_F the set of all F-terms.

For any signature morphism $\varphi : (S, F, R) \to (S', F', R')$, we define $\varphi^{tm} : T_F \to T_{F'}$ by

$$\varphi^{tm}(f(t_1, \ldots, t_n)) = \varphi^{op}(f)(\varphi^{tm}(t_1), \ldots, \varphi^{tm}(t_n)).$$

First, let us define the set $Sen(S, F, R)$ as being the least set containing the followings:

- $t = t'$ is in $Sen(S, F, R)$, for any F-terms t and t' of the same sort;
- $r(t_1, \ldots, t_n)$ is in $Sen(S, F, R)$, for any $r \in R_w$ and $(t_1, \ldots, t_n) \in (T_F)_w$;
- $c(\rho_1, \ldots, \rho_n)$ is in $Sen(S, F, R)$, for any logical connective $c \in C$ with arity n and $\rho_1, \ldots, \rho_n$ from $Sen(S, F, R)$
- $(\forall X)\rho$, $(\exists X)\rho$ are in $Sen(S, F, R)$, for any ρ from $Sen(S, F \uplus X, R)$. The signature $(S, F \uplus X, R)$ is obtained from the signature (S, F, R) by adding the variables from X as new constant symbols.

For any signature morphism $\varphi : (S, F, R) \to (S', F', R')$, we define $Sen(\varphi) : Sen(S, F, R) \to Sen(S', F', R')$ by:

- $Sen(\varphi)(t = t') = (\varphi^{tm}(t) = \varphi^{tm}(t'))$;
- $Sen(\varphi)(r(t)) = \varphi^{rl}(r)(\varphi^{tm}(t))$;
- $Sen(\varphi)(c(\rho_1, \ldots, \rho_n)) = c(Sen(\varphi)(\rho_1), \ldots, Sen(\varphi)(\rho_n))$
- $Sen(\varphi)((\forall X)\rho) = (\forall X^{\varphi})Sen(\varphi')(\rho)$
- $Sen(\varphi)((\exists X)\rho) = (\exists X^{\varphi})Sen(\varphi')(\rho)$

where $X^{\varphi} = \{(x : \varphi^{st}(s)) \mid (x : s) \in X\}$ and $\varphi' : (S, F \uplus X, R) \to (S', F' \uplus X^{\varphi}, R')$ extends φ canonically.

We will often denote $Sen(\varphi)(\rho)$ by $\varphi(\rho)$.

Now we define the functor $\mathrm{Sen}^{\mathrm{L}} : \mathbb{S}ig^L \to \mathbb{S}et$ by:

- $\mathrm{Sen}^{\mathrm{L}}(S, F, R) = Sen(S, F, R) \times L$, for any signature (S, F, R) from Sig^L;

- $\mathrm{Sen}^{\mathrm{L}}(\varphi) = (Sen(\varphi), Id)$, for any signature morphism $\varphi : (S, F, R) \to (S', F', R')$, where Id is the identity function on L.

Also, we will often denote $\mathrm{Sen}^{\mathrm{L}}(\varphi)(\rho)$ by $\varphi(\rho)$.

Remark 3.2. For a signature (S, F, R), the *equational atoms* are of the form $[t = t', x]$, where t, t' are F-terms of the same sort and $x \in L$, and the *relational atoms* are of the form $[r(t), x]$, where $r \in R_w$, $t \in (T_F)_w$ and $x \in L$.

3.6 Satisfaction

Let (S, F, R) be a signature. Let us first notice that any F-term t is interpreted in any (S, F, R)-model M as

$$M_{f(t_1,\ldots,t_n)} = M_f(M_{t_1}, \ldots, M_{t_n}).$$

Fact 3.3. Let $h : M \to N$ be a model homomorphism. Then $h(M_t) = N_t$, for any term t.

Proof:
We prove this by induction on the structure of terms: $h(M_{f(t_1,\ldots,t_n)}) = h(M_f(M_{t_1}, \ldots, M_{t_n})) = N_f(h(M_{t_1}), \ldots, h(M_{t_n})) = N_f(N_{t_1}, \ldots, N_{t_n}) = N_{f(t_1,\ldots,t_n)}$. □

We define the *satisfaction degree* $d : |\mathrm{Mod}^{\mathrm{L}}(S, F, R)| \times Sen(S, F, R) \to L$ by:

- $d(M, t = t') = (M_t \approx^M M_{t'})$;
- $d(M, r(t)) = (M_r(M_t))$;
- $d(M, c(\rho_1, \ldots, \rho_n)) = con(c)(d(M, \rho_1), \ldots, d(M, \rho_n))$;
- $d(M, (\forall X)\rho) = \bigwedge_{M'\upharpoonright_{(S,F,R)}=M} d(M', \rho)$;
- $d(M, (\exists X)\rho) = \bigvee_{M'\upharpoonright_{(S,F,R)}=M} d(M', \rho)$.

The satisfaction relation between an (S, F, R)-model M and an (S, F, R)-sentence ρ in MVL(L) is defined by:

$$M \models^L [\rho, x] \text{ if and only if } d(M, \rho) \geq x.$$

3.7 The satisfaction condition

Proposition 3.1. For any signature morphism $\varphi : (S, F, R) \to (S', F', R')$, any (S', F', R')-model M' and any (S, F, R)-sentence $[\rho, x]$ in $\mathrm{MVL(L)}$,

$$M' \restriction_\varphi \models^L [\rho, x] \text{ iff } M' \models^L \mathrm{Sen}^{\mathrm{L}}(\varphi)([\rho, x]).$$

Proof:

First, notice that for any (S, F, R)-term t we have $(M'\restriction_\varphi)_t = M'_{\varphi^{tm}(t)}$:

$(M'\restriction_\varphi)_{f(t_1,\ldots,t_n)} = (M'\restriction_\varphi)_f((M'\restriction_\varphi)_{t_1}, \ldots, (M'\restriction_\varphi)_{t_n}) = M'_{\varphi^{op}(f)}(M'_{\varphi^{tm}(t_1)}, \ldots, M'_{\varphi^{tm}(t_n)})$
$= M'_{\varphi^{op}(f)(\varphi^{tm}(t_1),\ldots,\varphi^{tm}(t_n))} = M'_{\varphi^{tm}(f(t_1,\ldots,t_n))}$.

We can easily show by induction on the structure of sentences that $d(M'\restriction_\varphi, \rho) = d(M', \varphi(\rho))$:

· $d(M'\restriction_\varphi, t = t') = ((M'\restriction_\varphi)_t \approx_s^{M'\restriction_\varphi} (M'\restriction_\varphi)_{t'}) = (M'_{\varphi^{tm}(t)} \approx_{\varphi^{st}(s)}^{M'} M'_{\varphi^{tm}(t')})$
$= d(M', \varphi^{tm}(t) = \varphi^{tm}(t')) = d(M', \varphi(t = t'))$

· $d(M'\restriction_\varphi, r(t)) = (M'\restriction_\varphi)_r((M'\restriction_\varphi)_t) = M'_{\varphi^{rl}(r)}(M'_{\varphi^{tm}(t)}) = d(M', \varphi^{rl}(r)(\varphi^{tm}(t)))$
$= d(M', \varphi(r(t)))$

· $d(M'\restriction_\varphi, c(\rho_1, \ldots, \rho_n)) = con(c)(d(M'\restriction_\varphi, \rho_1), \ldots, d(M'\restriction_\varphi, \rho_n))$
$= con(c)(d(M', \varphi(\rho_1)), \ldots, d(M', \varphi(\rho_n))) = d(M', c(\varphi(\rho_1), \ldots, \varphi(\rho_n))) =$
$d(M', \varphi(c(\rho_1, \ldots, \rho_n)))$

· In the case of quantified sentences, the conclusion follows by noticing that there is a canonical bijection between the expansions N of $M'\restriction_\varphi$ to $(S, F \uplus X, R)$ and the expansions M'' of M' to $(S', F' \uplus X^\varphi, R')$ given by $N = M''\restriction_{\varphi'}$.

$$\begin{tikzcd}
(S, F, R) \arrow[r, "\varphi"] \arrow[d] & (S', F', R') \arrow[d] \\
(S, F \uplus X, R) \arrow[r, "\varphi'"'] & (S', F' \uplus X^\varphi, R')
\end{tikzcd}$$

Therefore, for the universal quantifier we have:

$$\begin{aligned}
d(M'\restriction_{\varphi}, (\forall X)\rho) &= \bigwedge_{N\restriction_{(S,F,R)}=M'\restriction_{\varphi}} d(N,\rho) \\
&= \bigwedge_{\substack{M''\restriction_{(S',F',R')}=M' \\ (M''\restriction_{\varphi'})\restriction_{(S,F,R)}=M'\restriction_{\varphi}}} d(M''\restriction_{\varphi'},\rho) \\
&= \bigwedge_{\substack{M''\restriction_{(S',F',R')}=M' \\ (M''\restriction_{\varphi'})\restriction_{(S,F,R)}=M'\restriction_{\varphi}}} d(M'',\varphi'(\rho)) \\
&= \bigwedge_{\substack{M''\restriction_{(S',F',R')}=M' \\ (M''\restriction_{(S',F',R')})\restriction_{\varphi}=M'\restriction_{\varphi}}} d(M'',\varphi'(\rho)) \\
&= \bigwedge_{M''\restriction_{(S',F',R')}=M'} d(M'',\varphi'(\rho)) \\
&= d(M',(\forall X^{\varphi})\varphi'(\rho)) = d(M',\varphi((\forall X)\rho)).
\end{aligned}$$

The case of the existential quantifier can be treated similarly.

Now we can finish the proof:

$M'\restriction_{\varphi}\models^{L} [\rho, x]$ iff $d(M'\restriction_{\varphi},\rho)\geq x$ iff $d(M',\varphi(\rho))\geq x$ iff $M'\models^{L} [\varphi(\rho),x]$. $\square$

4 Morphisms and comorphisms

4.1 Institution comorphisms

The embedding relationship between institutions is formalized by the concept of institution comorphism [13].

Definition 4.1. Given two institutions, $I = (\mathbb{S}ig, \mathrm{Sen}, \mathrm{Mod}, \models)$ and $I' = (\mathbb{S}ig', \mathrm{Sen}', \mathrm{Mod}', \models')$, an *institution comorphism* $(\Phi, \alpha, \beta) : I \to I'$ consists of:

1) a functor $\Phi : \mathbb{S}ig \to \mathbb{S}ig'$,

2) a natural transformation $\alpha : \mathrm{Sen} \Rightarrow \Phi; \mathrm{Sen}'$, and

3) a natural transformation $\beta : \Phi^{op}; \mathrm{Mod}' \Rightarrow \mathrm{Mod}$

such that the following *satisfaction condition* holds:

$$M' \models'_{\Phi(\Sigma)} \alpha_\Sigma(e) \text{ if and only if } \beta_\Sigma(M') \models_\Sigma e,$$

for any signature $\Sigma \in |\mathbb{S}ig|$, for any $\Phi(\Sigma)$-model M' and any Σ-sentence e.

Let us consider the subinstitution FOL* of FOL which restricts the sentences only to those without negation. Thus FOL* has the same sentences and models as FOL. Let us consider $(\mathrm{TV}, \mathrm{EQ})$ a presentation such that $\mathrm{Mod}(TV, EQ)$ is the class of all Gödel-algebras (the algebras corresponding to Gödel logics). Let L be a model for this presentation. In this settings we can define the comorphism:

$$(\Phi, \alpha, \beta) : \mathrm{FOL}^* \to \mathrm{MVL(L)}$$

1) $\Phi : \mathbb{S}ig^{\mathrm{FOL}^*} \to \mathbb{S}ig^L$ is the identity functor, i.e. $\Phi((S, F, R)) = (S, F, R)$.

2) Let (S, F, R) be a signature in FOL^*. Because Φ is the identity functor, we must define $\alpha_{(S,F,R)}$: $\mathrm{Sen}^{\mathrm{FOL}^*}(S, F, R) \to \mathrm{Sen}^L(S, F, R)$.

   We define $\alpha_{(S,F,R)}(\rho) = [\rho, 1]$, for any $\rho \in \mathrm{Sen}^{\mathrm{FOL}^*}(\mathrm{S}, \mathrm{F}, \mathrm{R})$. It is easy to observe that the following diagram is commutative:

$$\begin{tikzcd}
(S,F,R) \arrow[d, "\varphi"'] & \mathrm{Sen}^{\mathrm{FOL}^*}(S,F,R) \arrow[r, "\alpha_{(S,F,R)}"] \arrow[d, "\mathrm{Sen}^{\mathrm{FOL}^*}(\varphi)"'] & \mathrm{Sen}^{\mathrm{L}}(S,F,R) \arrow[d, "\mathrm{Sen}^{\mathrm{L}}(\varphi)"] \\
(S',F',R') & \mathrm{Sen}^{\mathrm{FOL}^*}(S',F',R') \arrow[r, "\alpha_{(S',F',R')}"'] & \mathrm{Sen}^{\mathrm{L}}(S',F',R')
\end{tikzcd}$$

3) Let (S, F, R) be a signature in FOL^*. We must define $\beta_{(S,F,R)}$: $\mathrm{Mod}^L(S, F, R) \to \mathrm{Mod}^{\mathrm{FOL}^*}(S, F, R)$.

We define $\beta_{(S,F,R)}((\{M_s, \approx_s\}_{s\in S}, \{M_f\}_{f\in F}, \{M_r\}_{r\in R})) = (\{M_s\}_{s\in S}, \{M_f\}_{f\in F}, \{\overline{M_r}\}_{r\in R})$, where $\overline{M_r} = \{m \in M_w \mid M_r(m) = 1\}$, for any $r \in R$.

We can easily check that the following diagram commutes:

$$\begin{tikzcd}
(S,F,R) \arrow[d, "\varphi"] & \mathrm{Mod}^{\mathrm{L}}(S,F,R) \arrow[r, "\beta_{(S,F,R)}"] & \mathrm{Mod}^{\mathrm{FOL}^*}(S,F,R) \\
(S',F',R') & \mathrm{Mod}^{\mathrm{L}}(S',F',R') \arrow[u, "(_\restriction_{\varphi})^{\mathrm{L}}"] \arrow[r, "\beta_{(S',F',R')}"'] & \mathrm{Mod}^{\mathrm{FOL}^*}(S',F',R') \arrow[u, "(_\restriction_{\varphi})^{\mathrm{FOL}^*}"']
\end{tikzcd}$$

We must also prove the satisfaction condition. Let (S,F,R) be a FOL^*-signature, M' an (S,F,R)-model in $\mathrm{MVL(L)}$ and $\rho \in \mathrm{Sen}^{\mathrm{FOL}^*}(S,F,R)$. We must check that

$$\beta_{(S,F,R)}(M') \models^{\mathrm{FOL}^*} \rho \text{ iff } M' \models^{L} \alpha_{(S,F,R)}(\rho).$$

The proof is by induction on the structure of ρ:

- Equational atoms:
  $\beta_{(S,F,R)}(M') \models^{\mathrm{FOL}^*} t = t'$ iff $\beta_{(S,F,R)}(M')_t = \beta_{(S,F,R)}(M')_{t'}$ iff $M'_t = M'_{t'}$ iff $(M'_t \approx^{M'} M'_{t'}) = 1$ iff $M' \models^L [t = t', 1]$ iff $M' \models^L \alpha_{(S,F,R)}(t = t')$.

- Relational atoms:
  $\beta_{(S,F,R)}(M') \models^{\mathrm{FOL}^*} r(t)$ iff $\beta_{(S,F,R)}(M')_t \in \beta_{(S,F,R)}(M')_r$ iff $M'_r(M'_t) = 1$ iff $M' \models^L [r(t), 1]$ iff $M' \models^L \alpha_{(S,F,R)}(r(t))$.

- Conjunction:
  $\beta_{(S,F,R)}(M') \models^{\mathrm{FOL}^*} \rho_1 \wedge \rho_2$ iff $\beta_{(S,F,R)}(M') \models^{\mathrm{FOL}^*} \rho_i$, $i = \overline{1,2}$ iff $M' \models^L \alpha_{(S,F,R)}(\rho_i)$, $i = \overline{1,2}$ iff $M' \models^L [\rho_i, 1]$, $i = \overline{1,2}$ iff $d(M', \rho_i) = 1$, $i = \overline{1,2}$ iff $d(M', \rho_1) \wedge d(M', \rho_2) = 1$ iff $d(M', \rho_1 \wedge \rho_2) \geq 1$ iff $M' \models^L [\rho_1 \wedge \rho_2, 1]$ iff $M' \models \alpha_{(S,F,R)}(\rho_1 \wedge \rho_2)$.

- Disjunction:
  $\beta_{(S,F,R)}(M') \models^{\mathrm{FOL}^*} \rho_1 \vee \rho_2$ iff $\beta_{(S,F,R)}(M') \models^{\mathrm{FOL}^*} \rho_1$ or $\beta_{(S,F,R)}(M') \models^{\mathrm{FOL}^*} \rho_2$ iff $M' \models^L [\rho_1, 1]$ or $M' \models^L [\rho_2, 1]$ iff $d(M', \rho_1) = 1$ or $d(M', \rho_2) = 1$ iff $d(M', \rho_1) \vee d(M', \rho_2) = 1$ iff $d(M', \rho_1 \vee \rho_2) \geq 1$ iff $M' \models^L [\rho_1 \vee \rho_2, 1]$.

- Implication:
  Suppose $\beta_{(S,F,R)}(M') \models^{\mathrm{FOL}^*} \rho_1 \to \rho_2$. Suppose $\beta_{(S,F,R)}(M') \models^{\mathrm{FOL}^*} \rho_1$. Then $\beta_{(S,F,R)}(M') \models^{\mathrm{FOL}^*} \rho_2$. From induction hypothesis we get that $M' \models^L [\rho_i, 1]$, $i = \overline{1,2}$, i.e. $d(M', \rho_1) = d(M', \rho_2) = 1$. Therefore $d(M', \rho_1) \to d(M', \rho_2) = 1$, thus $M' \models \alpha_{(S,F,R)}(\rho_1 \to \rho_2)$.

  Suppose $M' \models \alpha_{(S,F,R)}(\rho_1 \to \rho_2)$, thus $d(M', \rho_1) \to d(M', \rho_2) = 1$. Therefore $d(M', \rho_1) \leq d(M', \rho_2)$ from the definition of $\to$ in a Gödel-algebra. Suppose $\beta_{(S,F,R)}(M') \models^{\mathrm{FOL}^*} \rho_1$. Thus

$M' \models^L [\rho_1, 1]$, i.e. $d(M', \rho_1) = 1$. Therefore $d(M', \rho_2) = 1$. Thus $M' \models^L [\rho_2, 1]$ and from the induction hypothesis it follows that $\beta_{(S,F,R)}(M') \models^{\mathrm{FOL}^*} \rho_2$.

- Universal quantified formulas:

  Let us first notice that N' is an expansion of M' to $(S, F \uplus X, R)$ in MVL(L) if and only if $\beta_{(S,F\uplus X,R)}(N')$ is an expansion of $\beta_{(S,F,R)}(M')$ to $(S, F \uplus X, R)$ in FOL* and that for any expansion N of $\beta_{(S,F,R)}(M')$ to $(S, F \uplus X, R)$ in FOL*, there exists an expansion N' of M' to $(S, F \uplus X, R)$ such that $\beta_{(S,F\uplus X,R)}(N') = N$.

  We have the following equivalences:

  $\beta_{(S,F,R)}(M') \models^{\mathrm{FOL}^*} (\forall X)\rho$ iff $M'' \models^{\mathrm{FOL}^*} \rho$, for any M'' expansion of $\beta_{S,F,R}(M')$ to $(S, F \uplus X, R)$

  iff $\beta_{(S,F\uplus X,R)}(N') \models^{\mathrm{FOL}^*} \rho$, for any $\beta_{(S,F\uplus X,R)}(N')$ expansion of $\beta_{S,F,R}(M')$ to $(S, F \uplus X, R)$

  iff $N' \models^L \alpha_{(S,F\uplus X,R)}(\rho)$, for any N' expansion of M' to $(S, F \uplus X, R)$ in MVL(L)

  iff $d(N', \rho) = 1$, for any N' expansion of M' to $(S, F \uplus X, R)$ in MVL(L)

  iff $\bigwedge_{\substack{N' \in \mathrm{Mod}(S,F\uplus X,R) \\ N'\upharpoonright_{(S,F,R)} = M'}} d(N', \rho) = 1$ iff $M' \models^L \alpha_{(S,F,R)}((\forall X)\rho)$.

Remark 4.1.

1) We cannot define a comorphism $(\Phi, \alpha, \beta) : \mathrm{FOL} \to \mathrm{MVL(L)}$, because the satisfaction conditions does not hold for sentences from FOL of the form $\neg\rho$.

   Let (S, F, R) be a FOL signature, M' an (S, F, R)-model in MVL(L) and $\neg\rho \in \mathrm{Sen}^{\mathrm{FOL}}(S, F, R)$.

   Suppose that $\beta_{(S,F,R)}(M') \models^{\mathrm{FOL}} \neg\rho$. Then $\beta_{(S,F,R)}(M') \not\models^{\mathrm{FOL}} \rho$. From the induction hypothesis we get $M' \not\models^L \alpha_{(S,F,R)}(\rho)$, equivalent with $M' \not\models^L [\rho, 1]$, i.e. $d(M', \rho) < 1$.

   In order to prove that $M' \models^L \alpha_{(S,F,R)}(\neg\rho)$, it is enough to show that $\neg d(M', \rho) = 1$. In a Gödel algebra, $\neg d(M', \rho) = 1$ if $d(M', \rho) = 0$, but we know only that $d(M', \rho) < 1$. Notice that we have the same situation in all the algebras corresponding to the many-valued logics presented in subsection 1.3.

   But the other implication of the satisfaction condition holds. Suppose $M' \models^L \alpha_{(S,F,R)}(\neg\rho)$. Then $\neg d(M', \rho) = 1$, thus $d(M', \rho) = 0 < 1$. Therefore $M' \not\models^L [\rho, 1]$. By the induction hypothesis we get $\beta_{(S,F,R)}(M') \not\models^{\mathrm{FOL}} \rho$, therefore $\beta_{(S,F,R)}(M') \models^{\mathrm{FOL}} \neg\rho$.

2) If we consider $L = \{0, 1\}$, then the satisfaction condition holds also for sentences of the form $\neg\rho$ from FOL, because from $d(M', \rho) < 1$ we can infer that $d(M', \rho) = 0$. Thus we can define a comorphism

   $$(\Phi, \alpha, \beta) : \mathrm{FOL} \to \mathrm{MVL(L)}$$

   which establish the relationship between FOL and MVL(L).

4.2 Institution morphisms

Structure preserving mappings from a more complex to a simpler institution can be formalized by the general concept of institution morphism [12].

Definition 4.2. Given two institutions, $I' = (\mathbb{S}ig', \mathrm{Sen}', \mathrm{Mod}', \models')$ and $I = (\mathbb{S}ig, \mathrm{Sen}, \mathrm{Mod}, \models)$, an *institution morphism* $(\Phi, \alpha, \beta) : I' \to I$ consists of:

1) a functor $\Phi : \mathbb{S}ig' \to \mathbb{S}ig$, called the *signature functor*,

2) a natural transformation $\alpha : \Phi; \mathrm{Sen} \Rightarrow \mathrm{Sen}'$, called the *sentence transformation*,

3) a natural transformation $\beta : \mathrm{Mod}' \Rightarrow \Phi^{op}; \mathrm{Mod}$ called the *model transformation*,

such that the following *satisfaction condition* holds:

$$M' \models'_{\Sigma'} \alpha_{\Sigma'}(e) \text{ if and only if } \beta_{\Sigma'}(M') \models_{\Phi(\Sigma')} e,$$

for any signature $\Sigma' \in |\mathbb{S}ig'|$, for any Σ'-model M' and any $\Phi(\Sigma')$-sentence e.

By adjoint relationship, we can also define a morphism

$$(\Phi', \alpha', \beta') : \mathrm{MVL(L)} \to \mathrm{FOL}^*$$

where $\Phi' : \mathbb{S}ig^L \to \mathbb{S}ig^{\mathrm{FOL}^*}$ is the identity functor, and α' and β' are defined in the same way as α and β from the comorphism $(\Phi, \alpha, \beta) : \mathrm{FOL}^* \to \mathrm{MVL(L)}$. We observe that the satisfaction condition can be treated similarly.

5 Model-theoretic properties

In this section we will investigate different model-theoretic properties developed at institutional level for multiple-valued logics. This is one of the main motivations for the present work, i.e. to show how one can obtain model-theory results for multiple-valued logics using the mechanism of institutions.

For the rest of this section, we fix L to be a truth-value algebra.

5.1 Signature (co)-limits

Because $\mathrm{MVL(L)}$ signatures are exactly FOL signatures and since the category of FOL signatures has small (co)-limits [8], it follows that the category of $\mathrm{MVL(L)}$ signatures has small (co)-limits.

Corollary 5.1. The category of $\mathrm{MVL(L)}$ signatures has small (co)-limits.

5.2 Initial model for signatures

Proposition 5.1. For any signature in $\mathrm{MVL(L)}$ there exists an initial model.

Proof:

Let (S, F, R) a signature in $\mathrm{MVL(L)}$. We define the (S, F, R)-model $0_{(S,F,R)}$ by:

- for each $s \in S$, $(0_{(S,F,R)})_s = (T_F)_s$ and $t \approx^{0_{(S,F,R)}} t' = \begin{cases} 1, & \text{if } t = t' \\ 0, & \text{otherwise} \end{cases}$
- for each $f \in F_{w \to s}$, $(0_{(S,F,R)})_f(t_1, \dots, t_n) = f(t_1, \dots, t_n)$
- for each $r \in R_w$, $(0_{(S,F,R)})_r(t_1, \dots, t_n) = 0$

For any (S, F, R)-model M, we define the map $h : 0_{(S,F,R)} \to M$ by

$$h(t) = M_t.$$

We check that h is an (S, F, R)-model homomorphism:

- Let $t, t' \in (T_F)_s$. If $t = t'$, then $t \approx^{0_{(S,F,R)}} t' = 1$ and $h_s(t) \approx^M h_s(t') = 1$. If $t \neq t'$, then $t \approx^{0_{(S,F,R)}} t' = 0$ and $0 \leq x$, for any $x \in L$. Therefore, $t \approx^{0_{(S,F,R)}} t' \leq h_s(t) \approx^M h_s(t')$.
- $h(f(t)) = M_f(h(t))$ by definition.
- $(0_{(S,F,R)})_r(t) = 0$, therefore $(0_{(S,F,R)})_r(t) \leq M_r(h(t))$.

Hence, $0_{(S,F,R)}$ is the initial model for the signature (S, F, R). $\square$

5.3 Model amalgamation

Model amalgamation is the institutional property which is required by almost all institution-independent model theoretic properties [20], [10].

Definition 5.1. In any institution, a commuting square of signatures

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\ \Sigma_2 \arrow[r, "\theta_2"'] & \Sigma' \end{tikzcd}$$

is an *amalgamation square* if for each Σ_1-model M_1 and each Σ_2-model M_2 such that $M_1\upharpoonright_{\varphi_1}= M_2\upharpoonright_{\varphi_2}$, there exists an unique Σ'-model M', called *the amalgamation of* M_1 *and* M_2, such that $M'\upharpoonright_{\theta_i}= M_i$, $i \in \{1, 2\}$.

If M' is not unique, we say that the square is a *weak amalgamation square*.

Definition 5.2. An institution has *model amalgamation* if every pushout of signatures is an amalgamation square.

Proposition 5.2. $\mathrm{MVL(L)}$ has model amalgamation.

Proof:

Let us consider the following pushout of signatures in $\mathrm{MVL(L)}$:

$$\begin{tikzcd} (S, F, R) \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & (S_1, F_1, R_1) \arrow[d, "\theta_1"] \\ (S_2, F_2, R_2) \arrow[r, "\theta_2"'] & (S', F', R') \end{tikzcd}$$

Let M_i be an (S_i, F_i, R_i)-model, $i = 1, 2$, such that $M_1\upharpoonright_{\varphi_1}= M_2\upharpoonright_{\varphi_2}$.

Let us notice that we can associate to any signature (S, F, R) from $\mathrm{MVL(L)}$ a signature from FOL, $\overline{(S, F, R)} = (S \cup \{l\}, \overline{F} \cup \overline{R}, \emptyset)$, where $\overline{F}_{w\to s} = F_{w\to s}$, for any $w \in S^*$, $s \in S$, $\overline{F}_{ss\to l} = \{\approx\}$, for any $s \in S$, $\overline{F}_{u\to t} = \emptyset$, otherwise, and $\overline{R}_{w\to l} = R_w$, for any $w \in S^*$, $\overline{R}_{u\to t} = \emptyset$, otherwise.

Moreover, to any (S, F, R)-model M in $\mathrm{MVL(L)}$ we can associate an $\overline{(S, F, R)}$-model in FOL $\overline{M}$, where:

- $\overline{M}_s = M_s$
- $\overline{M}_l = L$
- $\overline{M}_f = M_f$, for any $f \in \overline{F}_{w\to s}$, $w \in S^*$, $s \in S$

- $\overline{M}_r = M_r$, for any $r \in \overline{R}_{w \to l}$, $w \in S^*$
- $\overline{M}_{\approx} = \approx_s^M$, for any $\approx \in \overline{F}_{ss \to l}$, $s \in S$.

Every signature morphism $\varphi : (S, F, R) \to (S', F', R')$ in MVL(L) can be extended to a signature morphism $\overline{\varphi} : \overline{(S, F, R)} \to \overline{(S', F', R')}$ in FOL by:

- $\overline{\varphi}^{st}(s) = \varphi^{st}(s)$, for any $s \in S$,
- $\overline{\varphi}^{st}(l) = l$,
- $\overline{\varphi}^{op}(f) = \varphi^{op}(f)$, for any $f \in \overline{F}_{w \to s}$, $w \in S^*, s \in S$,
- $\overline{\varphi}^{op}(r) = \varphi^{rel}(r)$, for any $r \in \overline{R}_{w \to l}$, $w \in S^*$,
- $\overline{\varphi}^{op}(\approx_{ss \to l}) = \approx_{\varphi^{st}(s)\varphi^{st}(s) \to l}$, for any $s \in S$.

If $\varphi : (S, F, R) \to (S', F', R')$ is a signature morphism in MVL(L) and M' is an (S', F', R')-model, we can easily prove that $\overline{M' \upharpoonright_{\varphi}} = \overline{M'} \upharpoonright_{\overline{\varphi}}$.

Therefore we have a pushout of signatures in FOL:

$$\begin{tikzcd}
\overline{(S, F, R)} \arrow[r, "\overline{\varphi_1}"] \arrow[d, "\overline{\varphi_2}"'] & \overline{(S_1, F_1, R_1)} \arrow[d, "\overline{\theta_1}"] \\
\overline{(S_2, F_2, R_2)} \arrow[r, "\overline{\theta_2}"'] & \overline{(S', F', R')}
\end{tikzcd}$$

and $\overline{M_i}$ model of $\overline{(S_i, F_i, R_i)}$, $i = 1, 2$, such that $\overline{M_1} \upharpoonright_{\overline{\varphi_1}} = \overline{M_2} \upharpoonright_{\overline{\varphi_2}}$.

Since FOL has model amalgamation [20], it follows that there exists a unique $\overline{(S', F', R')}$-model M' such that $M' \upharpoonright_{\overline{\theta_i}} = \overline{M_i}$, $i = 1, 2$. We notice that $L = (\overline{M_i})_l = (M' \upharpoonright_{\overline{\theta_i}})_l = M'_{\overline{\theta_i}^{st}(l)} = M'_l$. Therefore, we can define an (S', F', R')-model in MVL(L) N by $N_s = M'_s$, $\approx_s^N = M'_{\approx_{ss \to l}}$, $N_f = M'_f$ and $N_r = M'_r$.

By the construction of M' and using the fact that $(M_i)_f$ and $(M_i)_r$ are compatible with $\approx^{M_i}$, we can prove that N_r and N_f are compatible with $\approx^N$. Thus, N is a proper (S', F', R')-model in MVL(L). Moreover $\overline{N} = M'$.

From $M' \upharpoonright_{\overline{\theta_i}} = \overline{M_i}$, it follows immediately that $N \upharpoonright_{\theta_i} = M_i$, $i = 1, 2$. Since M' was unique with the property $M' \upharpoonright_{\overline{\theta_i}} = \overline{M_i}$, $i = 1, 2$, we can easily prove that N is unique with the property $N \upharpoonright_{\theta_i} = M_i$, $i = 1, 2$. $\square$

An institution has J-*model amalgamation* for a category J when all co-limits of all diagrams $J \to \mathbb{S}ig$ have model amalgamation.

Proposition 5.3. FOL has J-model amalgamation for all small categories J.

Corollary 5.2. MVL(L) has J-model amalgamation for all small categories J.

Proof:

The proof can be treated in a similar way as the proof of Proposition 5.2 and using the fact that FOL has J-model amalgamation. □

5.4 Elementary diagrams

The institution-independent method of diagrams used here was developed in [5].

An institution $I = (\mathbb{S}ig, \mathrm{Sen}, \mathrm{Mod}, \models)$ has *elementary diagrams* if for each signature Σ and each Σ-model M, there exists a signature morphism $\iota_\Sigma(M) : \Sigma \to \Sigma_M$, "functorial" in Σ and M, and a set E_M of Σ_M-sentences such that $\mathrm{Mod}(\Sigma_M, E_M)$ and the comma category $M/\mathrm{Mod}(\Sigma)$ are naturally isomorphic, i.e. the following diagram commutes by the isomorphism $i_{\Sigma,M}$ "natural" in Σ and M:

$$\begin{tikzcd} \mathrm{Mod}(\Sigma_M, E_M) \arrow[r, "i_{\Sigma,M}"] \arrow[dr, "\mathrm{Mod}(\iota_\Sigma(M))"'] & (M/\mathrm{Mod}(\Sigma)) \arrow[d, "\text{forgetful}"] \\ & \mathrm{Mod}(\Sigma) \end{tikzcd}$$

The signature morphism $\iota_\Sigma(M) : \Sigma \to \Sigma_M$ is called the *elementary extension of* Σ *via* M and the set E_M of Σ_M-sentences is called the *elementary diagram* of the model M. For each model homomorphism $h : M \to N$, let N_h denote $i^{-1}_{\Sigma,M}(h)$.

The "functoriality" of ι means that for each signature morphism $\varphi : \Sigma \to \Sigma'$ and each Σ-model homomorphism $h : M \to M'\restriction_\varphi$, there exists a presentation morphism $\iota_\varphi(h) : (\Sigma_M, E_M) \to (\Sigma'_{M'}, E_{M'})$ such that

$$\begin{tikzcd} \Sigma \arrow[r, "\iota_\Sigma(M)"] \arrow[d, "\varphi"'] & \Sigma_M \arrow[d, "\iota_\varphi(h)"] \\ \Sigma' \arrow[r, "\iota_{\Sigma'}(M')"'] & \Sigma'_{M'} \end{tikzcd}$$

commutes and $\iota_\varphi(h); \iota_{\varphi'}(h') = \iota_{\varphi;\varphi'}(h; h'\restriction_\varphi)$ and $\iota_{1_\Sigma}(1_M) = 1_{\Sigma_M}$.

The "naturality" of i means that for each signature morphism $\varphi : \Sigma \to \Sigma'$ and each Σ-model homomorphism $h : M \to M'\restriction_\varphi$ the following diagram commutes:

$$\begin{tikzcd} \mathrm{Mod}(\Sigma_M, E_M) \arrow[r, "i_{\Sigma,M}"] & M/\mathrm{Mod}(\Sigma) \\ \mathrm{Mod}(\Sigma'_{M'}, E_{M'}) \arrow[r, "i_{\Sigma',M'}"'] \arrow[u, "\mathrm{Mod}(\iota_\varphi(h))"] & M'/\mathrm{Mod}(\Sigma') \arrow[u, "h/\mathrm{Mod}(\varphi) = h; (\_)\restriction_\varphi"'] \end{tikzcd}$$

Let $\Sigma = (S, F, R)$ be an MVL(L) signature and M a Σ-model. We define $\Sigma_M = (S, F_M, R)$, where

· $(F_M)_{w\to s} = F_{w\to s}$;

· $(F_M)_{\to s} = F_{\to s} \cup M_s$ (the elements of M are added to the signature as constant symbols).

Let us consider $\iota_\Sigma(M) : \Sigma \to \Sigma_M$ the signature inclusion.

Let M_M be the Σ_M-model such that $(M_M)\restriction_\Sigma = M$ and that interprets all the elements of M as themselves, i.e. $(M_M)_m = m$, for any $m \in M$. Notice that $\approx^{M_M} = \approx^M$.

Let E_M be the set of all atoms (either equational or relational) satisfied by M_M.

Proposition 5.4. E_M is the elementary diagram of the model M.

Proof:

We show first that there exists an isomorphism of categories i,

$$i_{\Sigma,M} : \mathrm{Mod}^L(\Sigma_M, E_M) \to (M/\mathrm{Mod}^L(\Sigma))$$

· The isomorphism $i_{\Sigma,M}$ maps each Σ_M-model N satisfying E_M to the Σ-model homomorphism $h_N : M \to N\restriction_\Sigma$ such that $h_N(m) = N_m$, for any $m \in M$.

Let us check that h_N is indeed a Σ-model homomorphism:

– For each $m, m' \in M$, $[m = m', m \approx^M m'] \in E_M$, because $M_M \models^L [m = m', m \approx^M m']$ iff $(M_M)_m \approx^M (M_M)_{m'} \geq m \approx^M m'$ iff $m \approx^M m' \geq m \approx^M m'$. Therefore $N \models^L [m = m', m \approx^M m']$ which means that $N_m \approx^N N_{m'} \geq m \approx^M m'$, therefore we have $m \approx^M m' \leq h_N(m) \approx^N h_N(m')$.

– For each operation symbol $f \in F_{w\to s}$ and for each $m \in M_w$, $[M_f(m) = f(m), 1] \in E_M$, therefore $N \models^L [M_f(m) = f(m), 1]$, equivalent to $N_{M_f(m)} \approx^N N_f(N_m) \geq 1$. Because $\approx^N$ is an L-equality, we get that $N_{M_f(m)} = N_f(N_m)$. By definition of h_N, the last equality is equivalent to $h_N(M_f(m)) = N_f(h_N(m))$.

– For each relation symbol $r \in R_w$ and for each $m \in M_w$, $[r(m), M_r(m)] \in E_M$, therefore $N \models^L [r(m), M_r(m)]$, i.e. $N_r(N_m) \geq M_r(m)$. By definition of h_N we get $N_r(h_N(m)) \geq M_r(m)$.

If $g : N \to N'$ is an Σ_M-model homomorphism, then $i_{\Sigma,M}$ maps g into $g\restriction_\Sigma$. We show that $i_{\Sigma,M}(g)$ is an arrow in the category $M/\mathrm{Mod}^L(\Sigma)$, i.e. $h_N; g\restriction_\Sigma = h_{N'}$.

$$\begin{tikzcd} M \arrow[r, "h_N"] \arrow[d, "h_{N'}"'] & N\restriction_\Sigma \arrow[dl, "g\restriction_\Sigma"] \\ N'\restriction_\Sigma & \end{tikzcd}$$

Let $m \in M$. Then $(g\restriction_\Sigma)(h_N(m)) = (g\restriction_\Sigma)(N_m) = g(N_m) = N'_m = h_{N'}(m)$.

· The inverse isomorphism $i^{-1}_{\Sigma,M}$ maps any Σ-model homomorphism $h : M \to N$ to the Σ_M-model $i^{-1}_{\Sigma,M}(h) = N_h$, where $N_h\restriction_\Sigma = N$ and $(N_h)_m = h(m)$, for any $m \in M$. We consider $\approx^{N_h} = \approx^N$.

First, let us notice that h is also an Σ_M-model homomorphism $M_M \to N_h$.

We must check that $N_h \models^L E_M$:

– Let $[t = t', x] \in E_M$. From $M_M \models^L E_M$, we get that $(M_M)_t \approx^{M_M} (M_M)_{t'} \geq x$. Because h is also an Σ_M-model homomorphism, it follows that $x \leq (M_M)_t \approx^{M_M} (M_M)_{t'} \leq h((M_M)_t) \approx^{N_h} h((M_M)_{t'})$. Hence $(N_h)_t \approx^{N_h} (N_h)_{t'} \geq x$, thus $N_h \models^L [t = t', x]$.

– Let $[r(t), x] \in E_M$. Thus $(M_M)_r((M_M)_t) \geq x$. Since h is an Σ_M-model homomorphism, it follows that $x \leq (M_M)_r((M_M)_t) \leq (N_h)_r(h((M_M)_t)) = (N_h)_r((N_h)_t)$, thus $N_h \models^L [r(t), x]$.

If g is an arrow in $M/\mathrm{Mod}^{\mathrm{L}}(\Sigma)$, $g : (M \xrightarrow{h_1} N) \to (M \xrightarrow{h_2} N')$, let $i^{-1}_{\Sigma,M} = g'$, where $g' : N_{h_1} \to N_{h_2}$ such that $g'(n) = g(n)$, for any $n \in N_{h_1}$.

$$\begin{tikzcd} M \arrow[r, "h_1"] \arrow[d, "h_2"'] & N \arrow[dl, "g"] \\ N' & \end{tikzcd}$$

We must check whether g' is an Σ_M-model homomorphism. By using the fact that g is a Σ-model homomorphism, we must only show that $g'((N_{h_1})_m) = (N_{h_2})_m$, for any $m \in M$. But $g'((N_{h_1})_m) = g(h_1(m)) = h_2(m) = (N_{h_2})_m$.

It is easy to see that $i_{\Sigma,M}$ is an isomorphism of categories.

"Functoriality" of ι:

Let $\Sigma' = (S', F', R')$ be an MVL(L) signature and $\varphi : \Sigma \to \Sigma'$ be a signature morphism. Let M' be a Σ'-model and $h : M \to M'\restriction_\varphi$ a Σ-model homomorphism. We define $\iota_\varphi(h) : \Sigma_M \to \Sigma'_{M'}$ by

– for each sort $s \in S$, $\iota_\varphi(h)(s) = \varphi(s)$;

– for each constant symbol $f \in F_{\to s}$, $\iota_\varphi(h)(f) = \varphi(f)$, and for each constant symbol $m \in M_s$, $\iota_\varphi(h)(m) = h(m)$;

– for each operation symbol $f \in F_{w \to s}$, $\iota_\varphi(h)(f) = \varphi(f)$;

– for each relation symbol $r \in R_w$, $\iota_\varphi(h)(r) = \varphi(r)$.

$\iota_\varphi(h)$ is an MVL(L)-signature morphism.

We can easily observe that the following diagram commutes:

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi"] \arrow[d, "\iota_\Sigma(M)"'] & \Sigma' \arrow[d, "\iota_{\Sigma'}(M')"] \\ \Sigma_M \arrow[r, "\iota_\varphi(h)"'] & \Sigma'_{M'} \end{tikzcd}$$

Let $M \xrightarrow{h} M'\restriction_\varphi$ and $M' \xrightarrow{h'} M''\restriction_{\varphi'}$.

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi"] \arrow[d, "\iota_\Sigma(M)"'] & \Sigma' \arrow[r, "\varphi'"] \arrow[d, "\iota_{\Sigma'}(M')"] & \Sigma'' \arrow[d, "\iota_\Sigma(M'')"] \\ \Sigma_M \arrow[r, "\iota_\varphi(h)"'] & \Sigma'_{M'} \arrow[r, "\iota_{\varphi'}(h')"'] & \Sigma''_{M''} \end{tikzcd}$$

It follows immediately that $\iota_\varphi(h); \iota_{\varphi'}(h') = \iota_{\varphi;\varphi'}(h; h'\restriction_\varphi)$ and $\iota_{1_\Sigma}(1_M) = 1_{\Sigma_M}$.

We only have to check that $\iota_\varphi(h)$ is a presentation morphism. We will prove that $M_M \models^L [\rho, x]$ implies $M'_{M'} \models^L \mathrm{Sen}^L(\iota_\varphi(h))[\rho, x]$, where $\rho \in \{t = t', r(t) \mid t, t' \text{ terms }\}$. By the satisfaction condition, this is equivalent to $M_M \models^L [\rho, x]$ implies $N \models^L [\rho, x]$, where we denote $N = (M'_{M'})\restriction_{\iota_\varphi(h)}$.

We prove that $N_t = h((M_M)_t)$, for any term t, by structural induction on terms:

– if $t \in F_{\to s}$, then $N_t = (M'_{M'})_{\iota_\varphi(h)(t)} = (M'_{M'})_{\varphi(t)} = M'_{\varphi(t)} = (M'\restriction_\varphi)_t = h(M_t) = h((M_M)_t)$.

– if $t = m \in M_s$, then $N_m = (M'_{M'})_{\iota_\varphi(h)(m)} = (M'_{M'})_{h(m)} = h(m) = h((M_M)_m)$.

– if $t = f(t_1, \ldots, t_n)$, then $N_{f(t_1,\ldots,t_n)} = N_f(N_{t_1}, \ldots, N_{t_n}) =$
$(M'_{M'})_{\iota_\varphi(h)(f)}(h((M_M)_{t_1}), \ldots, h((M_M)_{t_n})) = (M'_{M'})_{\varphi(f)}(h((M_M)_{t_1}), \ldots, h((M_M)_{t_n})) =$
$M'_{\varphi(f)}(h((M_M)_{t_1}), \ldots, h((M_M)_{t_n})) = (M'\restriction_\varphi)_f(h((M_M)_{t_1}), \ldots, h((M_M)_{t_n})) =$
$h(M_f((M_M)_{t_1}, \ldots, (M_M)_{t_n})) = h((M_M)_f((M_M)_{t_1}, \ldots, (M_M)_{t_n})) = h((M_M)_{f(t_1,\ldots,t_n)})$.

If $M_M \models^L [t = t', x]$, we have that $(M_M)_t \approx^{M_M} (M_M)_{t'} \geq x$. But h is also an Σ_M-model homomorphism, therefore $x \leq (M_M)_t \approx^{M_M} (M_M)_{t'} \leq h((M_M)_t) \approx^N h((M_M)_{t'}) = N_t \approx^N N_{t'}$, hence $N \models^L [t = t', x]$.

If $M_M \models^L [r(t), x]$, then $(M_M)_r((M_M)_t) \geq x$. Using the fact that h is also an Σ_M-model homomorphism, we get that $(M_M)_r((M_M)_t) \leq N_r(h(M_M)_t)$, therefore $x \leq N_r(N_t)$, i.e. $N \models^L [r(t), x]$.

”Naturality” of i:

Let $\Sigma' = (S', F', R')$ be an MVL(L) signature and $\varphi : \Sigma \to \Sigma'$ be a signature morphism. Let M' be a Σ'-model and $h : M \to M'\restriction_\varphi$ be a Σ-model homomorphism. We must check if the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Mod}^L(\Sigma_M, E_M) \arrow[r, "i_{\Sigma,M}"] & M/\mathrm{Mod}^L(\Sigma) \\
\mathrm{Mod}^L(\Sigma'_{M'}, E_{M'}) \arrow[u, "\mathrm{Mod}^L(\iota_\varphi(h))"] \arrow[r, "i_{\Sigma',M'}"'] & M'/\mathrm{Mod}^L(\Sigma') \arrow[u, "h;(_)\restriction_\varphi"']
\end{tikzcd}$$

– Let N' be a $\Sigma'_{M'}$-model such that $N' \models^L E_{M'}$. We have to verify if $h; (h_{N'}\restriction_\varphi) = h_{N'\restriction_{\iota_\varphi(h)}}$. Notice that $h; (h_{N'}\restriction_\varphi) : M \to N'\restriction_{\varphi;\iota_{\Sigma'}(M')}$ and $h_{N'\restriction_{\iota_\varphi(h)}} : M \to N'\restriction_{\iota_\Sigma(M);\iota_\varphi(h)}$. By the functoriality condition, we have that the homomorphisms have the same domain and codomain.

Let $m \in M_s$. Then $(h; (h_{N'}\restriction_\varphi))(m) = (h_{N'}\restriction_\varphi)(h(m)) = h_{N'}(h(m)) = N'_{h(m)} = N'_{\iota_\varphi(h)(m)} = (N'\restriction_{\iota_\varphi(h)})_m = h_{N'\restriction_{\iota_\varphi(h)}}(m)$.

– Let $g : N'_1 \to N'_2$ be a $(\Sigma'_{M'}, E_{M'})$-model homomorphism. We know that $h; i_{\Sigma',M'}(g)\restriction_\varphi = h; (g\restriction_{\iota_{\Sigma'}(M')})\restriction_\varphi$. On the other hand $i_{\Sigma,M}(g\restriction_{\iota_\varphi(h)}) = (g\restriction_{\iota_\varphi(h)})\restriction_{\iota_\Sigma(M)}$. By using the functoriality condition we obtain the conclusion.

$\square$

5.5 Co-limits of models

In the presence of diagrams and model amalgamation, co-limits of models can be obtained from corresponding co-limits of signatures. This is an important consequence of the existence of diagrams because in the actual institutions co-limits of models are much more difficult to establish than co-limits of signatures.

Let (S, F, R) be a signature in MVL(L), Γ a set of (S, F, R)-atomic sentences (equational and relational) and M an (S, F, R)-model.

We define M_Γ by:

- $(M_\Gamma)_s = \{[m] \mid m \in M_s\}$, where $[m] = \{m' \in M_s \mid h(m) = h(m'),$ for any $h : M \to N \models \Gamma\}$, for $m \in M_s$
- $[m] \approx^{M_\Gamma} [m'] = \bigwedge_{h:M\to N\models\Gamma} h(m) \approx^N h(m')$
- $(M_\Gamma)_f([m_1], \ldots, [m_n]) = [M_f(m_1, \ldots, m_n)]$
- $(M_\Gamma)_r([m_1], \ldots, [m_n]) = \bigwedge_{h:M\to N\models\Gamma} Nr(h(m_1), \ldots, h(m_n))$.

Fact 5.1. The definition of M_Γ is correct.

Proof:

Let $m_1 \in [m]$ and $m'_1 \in [m']$. Therefore $h(m) = h(m_1)$ and $h(m') = h(m'_1)$, for any $h : M \to N \models \Gamma$. Thus $[m] \approx^{M_\Gamma} [m'] = \bigwedge_{h:M\to N\models\Gamma}(h(m) \approx^N h(m')) = \bigwedge_{h:M\to N\models\Gamma}(h(m_1) \approx^N h(m'_1))$.

Let $f \in F_{s_1\ldots s_n\to s}$, $[m_i] \in (M_\Gamma)_{s_i}$ and $m'_i \in [m_i]$, for any $i \in \{1, \ldots, n\}$. Thus $h(m_i) = h(m'_i)$, for any $h : M \to N \models \Gamma$. In order to show that $(M_\Gamma)_f([m_1], \ldots, [m_n]) = [M_f(m'_1, \ldots, m'_n)]$, it is enough to show that $[M_f(m_1, \ldots, m_n)] = [M_f(m'_1, \ldots, m'_n)]$. Let $m \in [M_f(m_1, \ldots, m_n)]$. Therefore $h(m) = h(M_f(m_1, \ldots, m_n))$, for any homomorphism $h : M \to N \models \Gamma$. We have $h(m) = h(M_f(m_1, \ldots, m_n)) = N_f(h(m_1), \ldots, h(m_n)) = N_f(h(m'_1), \ldots, h(m'_n)) = h(M_f(m'_1, \ldots, m'_n))$, for any homorphism $h : M \to N \models \Gamma$. Thus $m \in [M_f(m'_1, \ldots, m'_n)]$.

Let $r \in R_{s_1\ldots s_n}$, $[m_i] \in (M_\Gamma)_{s_i}$ and $m'_i \in [m_i]$, for any $i \in \{1, \ldots, n\}$. Thus $h(m_i) = h(m'_i)$, for any $h : M \to N \models \Gamma$. We have $(M_\Gamma)_r([m_1], \ldots, [m_n]) = \bigwedge_{h:M\to N\models\Gamma} N_r(h(m_1), \ldots, h(m_n)) = \bigwedge_{h:M\to N\models\Gamma} N_r(h(m'_1), \ldots, h(m'_n))$. □

Fact 5.2. M_Γ is an (S, F, R)-model.

Proof:

We prove the followings:

- $\approx^{M_\Gamma}$ is an L-equality:

We have $[m] \approx^{M_\Gamma} [m] = \bigwedge_{h:M\to N\models\Gamma}(h(m) \approx^N h(m)) = 1$. Obviously, $[m] \approx^{M_\Gamma} [m'] = [m'] \approx^{M_\Gamma} [m]$. Since $\approx^N$ is an L-equality for any $h : M \to N \models \Gamma$ we have $([m] \approx^{M_\Gamma} [m']) \wedge ([m'] \approx^{M_\Gamma}$

$[m''])=\bigwedge_{h:M\to N\models\Gamma}(h(m)\approx^N h(m'))\wedge\bigwedge_{h:M\to N\models\Gamma}(h(m')\approx^N h(m''))=\bigwedge_{h:M\to N\models\Gamma}((h(m)\approx^N h(m'))\wedge(h(m')\approx^N h(m'')))\leq\bigwedge_{h:M\to N\models\Gamma}(h(m)\approx^N h(m''))=[m]\approx^{M_\Gamma}[m']$. Finally, suppose $[m]\approx^{M_\Gamma}[m']=1$. It follows that $h(m)=h(m')$, for any homomorphism $h:M\to N\models\Gamma$. Suppose $n\in[m]$. Thus $h(m)=h(n)$, for any $h:M\to N\models\Gamma$, therefore we obtain $h(m')=h(n)$, for any $h:M\to N\models\Gamma$. Hence $n\in[m']$.

· $(M_\Gamma)_f$ compatible with $\approx^{M_\Gamma}$:

Let $h:M\to N\models\Gamma$ be a homomorphism. Then $(h(m_1)\approx^N h(m_1'))\wedge\ldots\wedge(h(m_n)\approx^N h(m_n'))\leq(N_f(h(m_1),\ldots,h(m_n))\approx^N N_f(h(m_1'),\ldots,h(m_n')))=(h(M_f(m_1,\ldots,m_n))\approx^N h(M_f(m_1',\ldots,m_n')))$. Since h is arbitrary, we obtain that $([m_1]\approx^{M_\Gamma}[m_1'])\wedge\ldots\wedge([m_n]\approx^{M_\Gamma}[m_n'])\leq((M_\Gamma)_f([m_1],\ldots,[m_n])\approx^\Gamma(M_\Gamma)_f([m_1'],\ldots,[m_n']))$.

· $(M_\Gamma)_r$ compatible with $\approx^{M_\Gamma}$:

Let $h:M\to N\models\Gamma$ be a homomorphism. Since N_r is compatible with $\approx^N$, we have $(h(m)\approx^N h(m'))\wedge N_r(h(m))\leq N_r(h(m'))$. As h is arbitrary, it follows that $([m]\approx^{M_\Gamma}[m'])\wedge(M_\Gamma)_r([m])\leq(M_\Gamma)_r([m'])$.

□

We define $q_\Gamma:M\to M_\Gamma$ by $q_\Gamma(m)=[m]$.

Fact 5.3. q_Γ is a surjective (S,F,R)-model homomorphism.

Proposition 5.5. $(M_\Gamma)_t=[M_t]$, for any term t of (S,F,R).

Proof:

We prove this by induction on the structure of terms: $(M_\Gamma)_{f(t_1,\ldots,t_n)}=(M_\Gamma)_f((M_\Gamma)_{t_1},\ldots,(M_\Gamma)_{t_n})=(M_\Gamma)_f([M_{t_1}],\ldots,[M_{t_n}])=[M_f(M_{t_1},\ldots,M_{t_n})]=[M_{f(t_1,\ldots,t_n)}]$.

□

Proposition 5.6. For each (S,F,R)-model homomorphism $h:M\to N$ such that $N\models\Gamma$, there exists a unique model homomorphism $h_\Gamma:M_\Gamma\to N$ such that $q_\Gamma;h_\Gamma=h$.

$$\begin{tikzcd} M \arrow[rr, "q_\Gamma"] \arrow[dr, "h"'] & & M_\Gamma \arrow[dl, "h_\Gamma"] \\ & N\models\Gamma & \end{tikzcd}$$

Proof:

We define $h_\Gamma:M_\Gamma\to N$ by $h_\Gamma([m])=h(m)$.

Let $m'\in[m]$. Since $h:M\to N\models\Gamma$, it follows that $h(m)=h(m')$. Thus $h_\Gamma([m])=h(m)=h(m')=h_\Gamma([m'])$. Therefore the definition of h is correct.

We have the followings relations:

- $[m] \approx^{M_\Gamma} [m'] = \bigwedge_{g:M\to N\models\Gamma}(g(m) \approx^N g(m')) \leq h(m) \approx^N h(m')$
- $h_\Gamma((M_\Gamma)_f([m])) = h_\Gamma([M_f(m)]) = h(M_f(m)) = N_f(h(m)) = N_f(h_\Gamma([m]))$
- $(M_\Gamma)_r([m]) = \bigwedge_{g:M\to N\models\Gamma} N_r(g(m)) \leq N_r(h(m)) = N_r(h_\Gamma([m]))$.

Therefore h_Γ is an (S, F, R)-model homomorphism.

The uniqueness of h_Γ follows from the fact that q_Γ is surjective. $\square$

Proposition 5.7. $M_\Gamma \models \Gamma$.

Proof:

Let $C \in \Gamma$. We have two cases:

1) $C = [t = t', x]$

Let $h : M \to N \models \Gamma$. Therefore $N \models [t = t', x]$, i.e. $N_t \approx^N N_{t'} \geq x$. Since $h(M_t) = N_t$ and $h(M_{t'}) = N_{t'}$, it follows that $h(M_t) \approx^N h(M_{t'}) \geq x$.

Since h was arbitrary chosen, we obtain $\bigwedge_{h:M\to N\models\Gamma} h(M_t) \approx^N h(M_{t'}) \geq x$.

Because $\bigwedge_{h:M\to N\models\Gamma}(h(M_t) \approx^N h(M_{t'})) = ([M_t] \approx^{M_\Gamma} [M_{t'}]) = ((M_\Gamma)_t \approx^{M_\Gamma} (M_\Gamma)_{t'})$, it follows that $M_\Gamma \models [t = t', x]$.

2) $C = [r(t), x]$

Let $h : M \to N \models \Gamma$. Thus $N \models [r(t), x]$, i.e. $N_r(N_t) \geq x$, equivalent to $N_r(h(M_t)) \geq x$.

Since $h : M \to N \models \Gamma$ was arbitrarily chosen, it follows $\bigwedge_{h:M\to N\models\Gamma} N_r(h(M_t)) \geq x$.

Because $\bigwedge_{h:M\to N\models\Gamma} N_r(h(M_t)) = (M_\Gamma)_r([M_t]) = (M_\Gamma)_r((M_\Gamma)_t)$, we get $M_\Gamma \models [r(t), x]$.

$\square$

Proposition 5.8. For any set Γ of atomic sentences, the model $0_\Gamma = (0_{(S,F,R)})_\Gamma$ is the initial Γ-model, i.e. initial model in $\mathrm{Mod}((S, F, R), \Gamma)$.

Proof:

By Proposition 5.7, it follows that $0_\Gamma \models \Gamma$.

Let N be a model in $\mathrm{Mod}((S, F, R), \Gamma)$. There exists a unique model-homomorphism $\alpha_N : 0_{(S,F,R)} \to N$.

By Proposition 5.6, there exists a unique arrow $h_N : 0_\Gamma \to N$ such that $q_\Gamma; h_N = \alpha_N$.

$$\begin{tikzcd}
0_{(S,F,R)} \arrow[rr, "q_\Gamma"] \arrow[dr, "\alpha_N"'] & & 0_\Gamma \arrow[dl, "h_N"] \\
& N \models \Gamma &
\end{tikzcd}$$

Suppose there exists another arrow $g : 0_\Gamma \to N$. Since α_N is the unique arrow from $0_{(S,F,R)}$ to N, we obtain $q_\Gamma; g = \alpha_N$. Thus $q_\Gamma; g = q_\Gamma; h_N$. Since q_Γ is surjective, it follows that $g = h_N$. □

We recall the following result from [5]:

Theorem 5.1. Consider an institution with diagrams and initial models of presentations. If the category $\mathbb{S}ig$ has J-co-limits and the institution has J-model amalgamation then, for each signature Σ, the category of Σ-models has J-co-limits.

Corollary 5.3. The category of models of any $\mathrm{MVL(L)}$ signature has small co-limits.

Proof:

Since the elementary diagrams of $\mathrm{MVL(L)}$ consists only of atomic sentences, let us consider the sub-institution $\mathrm{AMVL(L)}$ of $\mathrm{MVL(L)}$ which restricts the sentences only to atoms. Obviously, $\mathrm{AMVL(L)}$ inherits the $\mathrm{MVL(L)}$ diagrams and has initial models for all its presentations by Proposition 5.8. By Corollary 5.2, $\mathrm{MVL(L)}$ has J-model amalgamation for all small categories J. The category of signature has small co-limits (Corollary 5.1), therefore by Theorem 5.1, the category of models of any signature has small co-limits. □

5.6 Limits of models

In this subsection we show that $\mathrm{MVL(L)}$ has finite limits of models by a direct construction of models.

Proposition 5.9. $\mathrm{MVL(L)}$ has products of models.

Proof:

Let (S, F, R) be an $\mathrm{MVL(L)}$ signature and let M^1 and M^2 be (S, F, R)-models. We define M, the product of M^1 and M^2, by:

- $M_s = M^1_s \times M^2_s$;
- $\approx^M: M \times M \to L$, $(m^1, m^2) \approx^M (n^1, n^2) = (m^1 \approx^{M_1} n^1) \wedge (m^2 \approx^{M_2} n^2)$;
- $M_f : M_{s_1} \times \dots M_{s_n} \to M$, $M_f(m) = (\ M^1_f(m^1_1, \dots, m^1_n)\ ,\ M^2_f(m^2_1, \dots, m^2_n)\ )$, where $f \in F_{s_1 \dots s_n \to s}$ and $m = ((m^1_1, m^2_1), \dots, (m^1_n, m^2_n))$;
- $M_r : M_{s_1} \times \dots M_{s_n} \to L$, $M_r(m) = M^1_r(m^1_1, \dots, m^1_n) \wedge M^2_r(m^2_1, \dots, m^2_n)$, where $r \in R_{s_1 \dots s_n}$ and $m = ((m^1_1, m^2_1), \dots, (m^1_n, m^2_n))$.

M is an (S, F, R)-model:

– using the fact that $\approx^{M^1}$ and $\approx^{M^2}$ are L-equalities, we can show immediately that $\approx^M$ is an L-equality.

– M_f is compatible with $\approx^M$:

$((m_1^1, m_1^2) \approx^M (l_1^1, l_1^2)) \wedge \ldots \wedge ((m_n^1, m_n^2) \approx^M (l_n^1, l_n^2)) = (m_1^1 \approx^{M_1} l_1^1) \wedge (m_1^2 \approx^{M_2} l_1^2) \wedge \ldots \wedge (m_n^1 \approx^{M_1} l_n^1) \wedge (m_n^2 \approx^{M_2} l_n^2) \leq (M_f^1(m_1^1, \ldots, m_n^1) \approx^{M_1} M_f^1(l_1^1, \ldots, l_n^1)) \wedge (M_f^2(m_1^2, \ldots, m_n^2) \approx^{M_2} M_f^2(l_1^2, \ldots, l_n^2)) = (M_f^1(m_1^1, \ldots, m_n^1), M_f^2(m_1^2, \ldots, m_n^2)) \approx^M (M_f^1(l_1^1, \ldots, l_n^1), M_f^2(l_1^2, \ldots, l_n^2)) = M_f((m_1^1, m_1^2), \ldots, (m_n^1, m_n^2)) \approx^M M_f((l_1^1, l_1^2), \ldots, (l_n^1, l_n^2))$.

– M_r is compatible with $\approx^M$:

$((m_1^1, m_1^2) \approx^M (l_1^1, l_1^2)) \wedge \ldots \wedge ((m_n^1, m_n^2) \approx^M (l_n^1, l_n^2)) \wedge M_r((m_1^1, m_1^2), \ldots, (m_n^1, m_n^2)) = (m_1^1 \approx^{M_1} l_1^1) \wedge (m_1^2 \approx^{M_2} l_1^2) \wedge \ldots \wedge (m_n^1 \approx^{M_1} l_n^1) \wedge (m_n^2 \approx^{M_2} l_n^2) \wedge M_r^1(m_1^1, \ldots, m_n^1) \wedge M_r^2(m_1^2, \ldots, m_n^2) \leq M_r^1(l_1^1, \ldots, l_n^1) \wedge M_r^2(l_1^2, \ldots, l_n^2) = M_r((l_1^1, l_1^2), \ldots, (l_n^1, l_n^2))$.

We define $p_j : M \to M_j$ by $p_j(m^1, m^2) = m^j$, for any $j = \overline{1, 2}$. p_j is a model homomorphism:

– $(m^1, m^2) \approx^M (n^1, n^2) = (m^1 \approx^{M^1} n^1) \wedge (m^2 \approx^{M^2} n^2) \leq (m^j \approx^{M^j} n^j) = p_j(m^1, m^2) \approx^{M^j} p_j(n^1, n^2)$;

– $p_j(M_f(m)) = p_j(M_f^1(m_1^1, \ldots, m_n^1), M_f^2(m_1^2, \ldots, m_n^2)) = M_f^j(m_1^j, \ldots, m_n^j) = M_f^j(p_j(m))$, where $m = ((m_1^1, m_1^2), \ldots, (m_n^1, m_n^2))$;

– $M_r(m) = M_r^1(m_1^1, \ldots, m_n^1) \wedge M_r^2(m_1^2, \ldots, m_n^2) \leq M_r^j(m_1^j, \ldots, m_n^j) = M_r^j(p_j(m))$, where $m = ((m_1^1, m_1^2), \ldots, (m_n^1, m_n^2))$.

Let N be an (S, F, R)-model and let $h_j : N \to M^j$, $j = \overline{1, 2}$. We define $h : N \to M$ by $h(n) = (h_1(n), h_2(n))$, for any $n \in N$. We can immediately show that h is a model homomorphism and that $h; p_j = h_j$, for any $j = \overline{1, 2}$. Moreover, h is unique with the property $h; p_j = h_j$. Therefore M is indeed the product of M^1 and M^2.

□

Since L has arbitrary infima, we can generalize the construction from the proof of Proposition 5.9 for any finite set of models.

Proposition 5.10. MVL(L) has equalizers of models.

Proof:

Let (S, F, R) be a signature and $M \overset{h}{\underset{g}{\rightrightarrows}} N$ be model homomorphisms. Let us consider $X = \{x \in M \mid h(x) = g(x)\}$. We define:

· $x \approx^X y = x \approx^M y$, for any $x, y \in X$;

· $X_f(x) = M_f(x)$, for any $x \in X$;

· $X_r(x) = M_r(x)$, for any $x \in X$.

It is obvious that X is an (S, F, R)-model.

Let $h' : N' \to M$ be an (S, F, R)-model homomorphism such that $h'; h = h'; g$. Since $h(h'(n')) = g(h'(n'))$, for any $n' \in N'$, it follows that $h'(n') \in X$. Thus we can consider the model homomorphism $h' : N' \to X$. □

We recall the following result from [17]:

Proposition 5.11. A category has all finite limits if and only if it has finite products and equalizers.

Corollary 5.4. $\mathrm{MVL(L)}$ has all finite limits of models.

5.7 Initial models for presentations

Definition 5.3. $< I, E >$ is an *inclusion system* for a category $\mathbb{C}$ if I and E are two sub-categories with $|I| = |E| = |\mathbb{C}|$ such that

1. I is a partial order, and
2. every arrow f in $\mathbb{C}$ can be factored uniquely as $f = e_f; i_f$ with $e_f \in E$ and $i_f \in I$.

In any inclusion system $< I, E >$ for a category $\mathbb{C}$, the arrows of I are called *abstract inclusions* and the arrows of E are called *abstract surjections*.

The abstract surjections of some inclusion systems need not necessarily be surjective in the ordinary set-theoretic sense. Consider for example the trivial inclusion system for $\mathbb{S}et$ where each function is an abstract surjection and the abstract inclusions are just the identities. An inclusion system $< I, E >$ is *epic* when all the abstract surjections are epis.

Definition 5.4. In any category $\mathbb{C}$ with an inclusion system $< I, E >$, an object B is a E*-quotient representation* of A if there exists an abstract surjection $A \to B$. An E*-quotient* of A is an isomorphism class of E-quotient represention.

Definition 5.5. [19] If E is a class of morphisms from a category $\mathbb{C}$, then $\mathbb{C}$ is E*-co-well-powered* if for any object B and for any class of morphisms from E having domain B, let us say $\{e_j \mid j \in J\}$, there is a subset M of J such that for any j in J there is some $m \in M$ such that e_j and e_m are isomorphic, i.e. there is an isomorphism α with $e_j = e_m; \alpha$.

Definition 5.6. An inclusion system $< I, E >$ for a category $\mathbb{C}$ is *co-well-powered* if the category $\mathbb{C}$ is E-co-well powered, i.e. the class of E-quotients of each object is a set.

Definition 5.7. In any category $\mathbb{C}$ with an inclusion system $< I, E >$, we say that an object A is an I-*subobject* of another object B if there exists an abstract inclusion $A \to B \in I$.

Definition 5.8. In any category $\mathbb{C}$ with an inclusion system $< I, E >$, an object A of $\mathbb{C}$ is I-reachable if and only if it has no I-subobjects which are different from A.

Fact 5.4. In any category $\mathbb{C}$ with a given inclusion system and which has an initial object $0_{\mathbb{C}}$

– each object A is reachable if and only if the unique arrow $0_{\mathbb{C}} \to A$ is an abstract surjection, and

– each object has exactly one reachable subobject.

Inclusion system for $\mathrm{MVL(L)}$ **models**

Let (S, F, R) be a signature in $\mathrm{MVL(L)}$ and let $h : M \to M'$ be an (S, F, R)-homomorphism. We will factor the homomorphism h by:

$$\begin{tikzcd} M \arrow[rr, "h"] \arrow[dr, "e"'] & & M' \\ & h(M) \arrow[ur, "i"'] & \end{tikzcd}$$

where we define $e(m) = h(m)$, for any $m \in M$, and $i(m') = m'$, for any $m' \in h(M)$.

Notice that $h(M)$ is defined by:

- $h(M)_s = h_s(M_s)$
- $\approx^{h(M)} = \approx^{M'}$
- $h(M)_f = M'_f$
- $h(M)_r = M'_r$

Let $f \in F_{w \to s}$ and let $m' \in h(M)_w$. Thus there exists $n \in M_w$ such that $h(n) = m'$. We have $h(M)_f(m') = M'_f(m') = M'_f(h(n)) = h(M_f(n))$. Therefore there exists $M_f(n) \in M_s$ such that $h(M)_f(m') = h(M_f(n))$, i.e. $h(M)_f(m') \in h(M)_s$. Since M'_f and M'_r are compatible with $\approx^{M'}$, it follows that $h(M)_f$ and $h(M)_r$ are compatible with $\approx^{h(M)}$. In conclusion, $h(M)$ is an (S, F, R)-model.

It is obvious that e is a surjection. We check if e and i are model homomorphisms:

- Since $\approx^{h(M)} = \approx^{M'}$, it follows that $m'_1 \approx^{h(M)} m'_2 \leq i(m'_1) \approx^{M'} i(m'_2)$, for any $m'_1, m'_2 \in h(M)$. Let $m_1, m_2 \in M$. We have $(e(m_1) \approx^{h(M)} e(m_2)) = (h(m_1) \approx^{h(M)} h(m_2)) = (h(m_1) \approx^{M'} h(m_2)) \geq m_1 \approx^M m_2$.

- Let $f \in F_{w \to s}$. For any $m' \in h(M)_w$, $i(h(M)_f(m')) = h(M)_f(m') = M'_f(m') = M'_f(i(m'))$. For any $m \in M_w$, we have $e(M_f(m)) = h(M_f(m)) = M'_f(h(m)) = h(M)_f(h(m)) = h(M)_f(e(m))$.

- Let $r \in R_w$. Since $h(M)_r = M'_r$, it follows that $h(M)_r(m') \leq M'_r(i(m'))$, for any $m' \in h(M)$. Let $m \in M$. We have $M_r(m) \leq M'_r(h(m)) = h(M)_r(h(m)) = h(M)_r(e(m))$.

Definition 5.9. For any $\mathrm{MVL(L)}$ signature (S, F, R) and any (S, F, R)-model homomorphism $M \to N$ which is a set inclusion for each $s \in S$ such that

- $(m \approx^M m') = (m \approx^N m')$, for any $m, m' \in M$, and
- $M_r(m) = N_r(m)$, for any $m \in M$,

we say that M is a *submodel* of N.

Fact 5.5. Let (S, F, R) be an $\mathrm{MVL(L)}$ signature and let $M \to N$ be an (S, F, R)-submodel. Then $M_f(m) = N_f(m)$, for any $m \in M$ and $M_t = N_t$, for any (S, F, R)-term t.

Fact 5.6. For any $\mathrm{MVL(L)}$ signature (S, F, R), the category of (S, F, R)-models admits the following inclusion system:

abstract surjection	abstract inclusions
surjective homomorphisms	submodels

Fact 5.7. Let (S, F, R) be a signature in $\mathrm{MVL(L)}$. The above inclusion system for the category of (S, F, R)-models is co-well-powered.

Proof:

Let E be the class of all surjective (S, F, R)-homomorphisms. We must show that the category of (S, F, R)-models is E-co-well-powered.

Let M be an (S, F, R)-model and let $X = \{e \mid e : M \to N$, e surjective homomorphism $\}$ be a class of arrows from E having domain M.

Let us denote by $[e] = \{e' \in X \mid e' : M \to N'$ and there exists an isomorphism $g : N \to N'$, g homomorphism, such that $e; g = e'\}$, for any $e : M \to N \in X$. We will show that $\{[e] | \, e \in X\}$ is a set.

For every $[e]$, where $e : M \to N \in X$, we can define an equivalence relation $\sim_e$ on M by $m \sim_e m'$ if and only if $e(m) = e(m')$. Let $e' \in [e]$, $e' : M \to N'$ and let $m, m' \in M$ such that $e(m) = e(m')$. We have $e'(m) = (e; g)(m) = (e; g)(m') = e'(m')$, where $g : N \to N'$ is isomorphism such that $e; g = e'$. Thus, the definition of $\sim_e$ is correct. We can easily observe that for any $f \in F_{w \to s}$, if $m \sim_e m'$, for any $m, m' \in M_w$, then $M_f(m) \sim_e M_f(m')$.

If $e' \in [e]$, then $\sim_e = \sim_{e'}$. Therefore we cannot have more classes $[e]$, where $e \in X$, than equivalence relations on M. Since the equivalence relations on a set form a set, it follows that $\{[e] |\ e \in X\}$ is a set.

Finally, we can find a subset Y of the class X by choosing one surjective homomorphism e' from each $[e]$ (an representative arrow from each class). Obviously, for each arrow e from X we can find an arrow e' in Y which is isomorphic with e.

□

Definition 5.10. In any category $\mathbb{C}$ with a given inclusion system and with small products, a class of objects of $\mathbb{C}$ closed under isomorphisms is a *quasi-variety* if it is closed under small products and subobjects.

The existence of initial models of quasi-varieties can be obtained at the very general level of abstract categories with inclusion systems.

Proposition 5.12. Consider a category $\mathbb{C}$ with an initial object $0_{\mathbb{C}}$, small products, and with a co-well-powered epic inclusion system. Each quasi-variety Q of $\mathbb{C}$ has a reachable initial object.

Corollary 5.5. For any $\mathrm{MVL(L)}$ signature (S, F, R), any set of (S, F, R)-sentences of the form $[(\forall X)t = t', x]$ or $[(\forall X)r(t), x]$ has an initial model.

Proof:

Let (S, F, R) be an $\mathrm{MVL(L)}$-signature and let Γ be a set of (S, F, R)-sentences of the form $[(\forall X)t = t', x]$ or $[(\forall X)r(t), x]$.

Since by Proposition 5.1 there exists the initial (S, F, R)-model, the category of (S, F, R)-models admits a co-well-powered epic inclusion system and has small products, by Proposition 5.12 it is enough to show that Γ^* is a quasi-variety, where $\Gamma^* = \{A \in |\mathrm{Mod}(S, F, R)| \mid A \models \Gamma\}$.

· Closed under small products

Let $(A_i)_{i\in I}$ be a family of (S, F, R)-models satisfying Γ. We have to prove that the product $\prod_{i\in I} A_i$ satisfies each sentence ρ of Γ.

Let $[(\forall X)t = t', x]$ in Γ. Let A' be any $(S, F \uplus X, R)$-expansion of $\prod_{i\in I} A_i$. Each projection $p_i : \prod_{i\in I} A_i \to A_i$ lifts uniquely to $p'_i : A' \to A'_i$ and $(p'_i)_{i\in I}$ is a product cone. Moreover $A' = \prod_{i\in I} A'_i$.

By definition of $\prod_{i\in I} A'_i$, we have $A'_t \approx^{A'} A'_{t'} = \bigwedge_{i\in I} p'_i(A'_t) \approx^{A'_i} p'_i(A'_{t'})$. Since $A_i \models [(\forall X)t = t', x]$, we get $(A'_i)_t \approx^{A'_i} (A'_i)_{t'} \geq x$, for every $(S, F \uplus X, R)$-expansion A'_i of A_i. From $p_i(A'_t) = (A'_i)_t$, it follows that $\bigwedge_{i\in I} p'_i(A'_t) \approx^{A'_i} p'_i(A'_{t'}) \geq x$, thus $d(A', t = t') \geq x$. Therefore $\prod_{i\in I} A_i \models [(\forall X)t = t', x]$.

Let $[(\forall X)r(t), x] \in \Gamma$. We can prove in a similar way that $\prod_{i\in I} A_i \models [(\forall X)r(t), x]$, since for any $(S, F \uplus X, R)$-expansion A' of $\prod_{i\in I} A_i$, $A'_r(A'_t) = \bigwedge_{i\in I} (A'_i)_r(p'_i(A'_t))$. From $A_i \models [(\forall X)r(t), x]$, we get $(A'_i)_r((A'_i)_t) \geq x$, thus $\bigwedge_{i\in I} (A'_i)_r(p'_i(A'_t)) \geq x$.

· Closed under subobjects

Let $B \hookrightarrow A$ be a submodel of A, where A satisfies Γ.

Let $[(\forall X)t = t', x] \in \Gamma$. Let B' be an $(S, F \uplus X, R)$-expansion of B and let A' be the $(S, F \uplus X, R)$-expansion of A such that $A'_x = B'_x$, for each $x \in X$. Therefore $B' \hookrightarrow A'$ is a submodel for $(S, F \uplus X, R)$. From $A \models \Gamma$, it follows that $A'_t \approx^{A'} A'_{t'} \geq x$. Since B' is a submodel, we have $\approx^{B'} = \approx^{A'}$ and $B'_u = A'_u$, for any term u. Thus $B'_t \approx^{B'} B'_{t'} \geq x$. As B' was chosen arbitrarily, it follows that $B \models [(\forall X)t = t', x]$.

The case $[(\forall X)r(t), x] \in \Gamma$ can be treated similarly by noticing that $B'_r = A'_r$, for any submodel $B' \hookrightarrow A'$.

□

5.8 Epic basic sentences

In any institution, a set E of Σ-sentences is *basic* [4] if there exists a Σ-models M_E such that for each Σ-model M,

$$M \models_\Sigma E \text{ if and only if there exists a model homomorphism } M_E \to M.$$

For a basic set E of sentences, when for each model $M \models E$ the model homomorphism $M_E \to M$ is unique, we say that E is *epic basic*.

Proposition 5.13. Each set of atomic sentences from $\mathrm{MVL(L)}$ is epic basic.

Proof:

Let (S, F, R) be a signature in $\mathrm{MVL(L)}$ and let E be a set of atomic (S, F, R)-sentences.

We consider M_E to be 0_E, the initial model for E (Proposition 5.8).

Let M be an (S, F, R)-model. If $M \models E$, obviously, there exits a unique model homomorphism $0_E \to M$.

Suppose that there exists an unique model homomorphism $h_M : 0_E \to M$ and let us prove that $M \models E$. Let $\rho \in E$. We have two cases:

- $\rho = [t = t', x]$

  Since $0_E \models E$, it follows that $(0_E)_t \approx^{0_E} (0_E)_{t'} \geq x$. Since h_M is a model homomorphism, $a \approx^{0_E} a' \leq h_M(a) \approx^M h_M(a')$, for any $a, a' \in 0_E$. Therefore $x \leq (0_E)_t \approx^{0_E} (0_E)_{t'} \leq h_M((0_E)_t) \approx^M h_M((0_E)_{t'}) = M_t \approx^M M_{t'}$. Thus $M \models [t = t', x]$.

- $\rho = [r(t), x]$

  From $0_E \models [r(t), x]$, we obtain $(0_E)_r((0_E)_t) \geq x$. Since h_M is a model homomorphism, $(0_E)_r(a) \leq M_r(h_M(a))$, for any $a \in 0_E$. Therefore $x \leq (0_E)_r((0_E)_t) \leq M_r(h_M((0_E)_t)) = M_r(M_t)$. Thus $M \models [r(t), x]$.

□

5.9 Representable signature morphisms

The institutional notion of *representable* signature morphisms is an abstract concept meant to capture the phenomena of quantification over (sets of) first-order variables. The notion starts from the fact that semantics of quantification in first-order-like logics can be given in terms of signature extensions: $M \models_{(S,F,P)} (\forall X)\rho$ if and only if $M' \models_{(S,F\uplus X,P)} \rho$, for each $(S, F \uplus X, P)$-expansion M' of M (and similarly for existential quantification). Thus, in order to reach first-order quantification institutionally, one needs to define somehow what "injective signature morphism that only adds constant symbols" (such as $(S, F, P) \hookrightarrow (S, F \uplus X, P)$) means.

In any institution, a signature morphism $\chi : \Sigma \to \Sigma'$ is *quasi-representable* [4] when for each Σ'-model M', the canonical functor determined by the reduct functor $\mathrm{Mod}(\chi)$ is an isomorphism:

$$M'/\mathrm{Mod}(\Sigma') \cong (M'\upharpoonright_\chi)/\mathrm{Mod}(\Sigma).$$

This means that each Σ-model homomorphism $h : M'\upharpoonright_\chi \to N$ admits a unique χ-expansion $h' : M' \to N'$.

A signature morphism $\chi : \Sigma \to \Sigma'$ is *representable* if and only if there exists a Σ-model M_χ (called the *representation of* χ) and an isomorphism i_χ of categories such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Mod}(\Sigma') \arrow[r, "i_\chi"] \arrow[dr, "\mathrm{Mod}(\chi)"'] & (M_\chi/\mathrm{Mod}(\Sigma)) \arrow[d, "\text{forgetful}"] \\
 & \mathrm{Mod}(\Sigma)
\end{tikzcd}$$

Fact 5.8. A signature morphism $\chi : \Sigma \to \Sigma'$ is representable if and only if it is quasi-representable and $\mathrm{Mod}(\Sigma')$ has initial model.

Proposition 5.14. In $\mathrm{MVL(L)}$, all signature extension with constant symbols are quasi-representable.

Proof:

Let $\varphi : (S, F, R) \to (S, F \uplus X, R)$ be a signature extension with constant symbols and let M' be an $(S, F \uplus X, R)$-model. We denote $M = M'\upharpoonright_\varphi$.

We must define an isomorphism of categories between $M'/\mathrm{Mod}(S, F \uplus X, R)$ and $M/\mathrm{Mod}(S, F, R)$.

Let $i_{M',(S,F,R)} : M'/\mathrm{Mod}(S, F \uplus X, R) \to M/\mathrm{Mod}(S, F, R)$ be the following functor:

- for any object $h' : M' \to N'$, we define $i_{M',(S,F,R)} = h'\upharpoonright_\varphi$;
- for any arrow $g' : (h'_1 : M' \to N'_1) \to (h'_2 : M' \to N'_2)$, we define $i_{M',(S,F,R)} = g'\upharpoonright_\varphi$.

Let $i^{-1}_{M',(S,F,R)} : M/\mathrm{Mod}(S, F, R) \to M'/\mathrm{Mod}(S, F \uplus X, R)$ be the following functor:

- for any object $h : M \to N$, we define $i^{-1}_{M',(S,F,R)} = M' \xrightarrow{h'} N'$, where $N' \upharpoonright_\varphi = N$ and $N'_x = h(M'_x)$, for any $x \in X$ and $h'(m') = h(m')$, for any $m' \in M'$. We notice that h' is a model homomorphism between M' and N' because h is a model homomorphism and the definition of N' on the constants from X.

- for any arrow $g : (h_1 : M \to N_1) \to (h_2 : M \to N_2)$, we define $i^{-1}_{M',(S,F,R)}(g) = g' : (h'_1 : M' \to N'_1) \to (h'_2 : M' \to N'_2)$ by $g'(n') = g(n)$, for any $n' \in N'_1$.

  It is sufficient to prove the homomorphism condition for the constants symbols from X:

  $g'((N'_1)_x) = g'(h_1(M'_x)) = g(h_1(M'_x)) = h_2(M'_x) = (N'_2)_x$.

It is easy to check that $i_{M',(S,F,R)}$ is an isomorphism of categories. □

Corollary 5.6. In $\mathrm{MVL(L)}$, all signature extensions with constant symbols are representable.

Proof:

We know that $\mathrm{MVL(L)}$ has initial models of signatures and signature extensions with constant symbols are quasi-representable, therefore they are representable. □

Given a set X of new constants for an $\mathrm{MVL(L)}$ signature (S, F, R), the representation of the signature inclusion $(S, F, R) \to (S, F \uplus X, R)$ is given by the model of the $(F \uplus X)$-terms $T_F(X)$, which is the free (S, F, R)-model over X. This is due to the fact that $(S, F \uplus X, R)$-models M are in canonical bijection with evaluations of variables from X to the carrier sets of M. By the freeness property of $T_F(X)$, these evaluations are in canonical bijection with (S, F, R)-model homomorphisms $T_F(X) \to M$.

5.10 System of proof rules for $\mathrm{AMVL(L)}$

For this subsection, we suppose that L is finite and any two elements from L are comparable. We recall that $\mathrm{AMVL(L)}$ is the sub-institution of $\mathrm{MVL(L)}$ which restricts the sentences only to atoms.

Definition 5.11. A *system of proof rules* $(\mathbb{S}ig, \mathrm{Sen}, \mathrm{Rl}, h, c)$ consists of

- a category of signatures $\mathbb{S}ig$,

- a sentence functor $\mathrm{Sen} : \mathbb{S}ig \to \mathbb{S}et$,

- a proof rule functor $\mathrm{Rl} : \mathbb{S}ig \to \mathbb{S}et$ and

- two natural transformations $h, c : \mathrm{Rl} \Rightarrow \mathrm{Sen}; \mathcal{P}$, where $\mathcal{P} : \mathbb{S}et \to \mathbb{S}et$ is the power-set function.

For each signature Σ, $\mathrm{Rl}(\Sigma)$ gives the set of the Σ-proof rules, $h_\Sigma : \mathrm{Rl}(\Sigma) \to \mathcal{P}(\mathrm{Sen}(\Sigma))$ gives the hypotheses of the rules, and $c_\Sigma : \mathrm{Rl}(\Sigma) \to \mathcal{P}(\mathrm{Sen}(\Sigma))$ gives the conclusion. A Σ-rule r can be therefore written as $h_\Sigma(r) \xrightarrow{r} c_\Sigma(r)$. The functoriality of Rl and the naturality of the hypotheses h and of the conclusions c say that the translation of rules along signature morphisms is coherent with the translation of the sentences. When there is no danger of confusion, we identify a system of proof rules $(\mathbb{S}ig, \mathrm{Sen}, \mathrm{Rl}, h, c)$ with Rl.

For each set Γ of Σ-sentences, we define $\overline{\Gamma} = \bigcup_{n \in N} \Gamma^n$, where:

$$\begin{array}{rcl} \Gamma^0 & = & \Gamma, \\ \Gamma^{n+1} & = & \Gamma^n \cup \{e \mid \dfrac{E_1}{E_2} \in \mathrm{Rl}(\Sigma), E_1 \subseteq \Gamma^n, e \in E_2\}. \end{array}$$

We say that Γ infer e and we denote $\Gamma \vdash_\Sigma e$ if $e \in \overline{\Gamma}$.

Notice that $\vdash = (\vdash_\Sigma)_{\Sigma \in |\mathbb{S}ig|}$ has the following properties:

1. *reflexivity:* $\{e\} \vdash_\Sigma e$, for each $e \in \mathrm{Sen}(\Sigma)$,
2. *transitivity:* if $E \vdash_\Sigma e'$, for each $e' \in E'$, and $E \cup E' \vdash_\Sigma e$, then $E \vdash_\Sigma e$, for each $E, E' \subseteq \mathrm{Sen}(\Sigma)$ and each $e \in \mathrm{Sen}(\Sigma)$;
3. *monotonicity:* if $E \subseteq E'$ and $E \vdash_\Sigma e$, then $E' \vdash_\Sigma e$;
4. *translation:* if $E \vdash_\Sigma e'$ and $\varphi : \Sigma \to \Sigma'$, then $\mathrm{Sen}(\varphi)(E) \vdash_{\Sigma'} \mathrm{Sen}(\varphi)(e)$.

When there is no danger of confusion we omit the subscript of $\vdash_\Sigma$. We say that $(\mathbb{S}ig, \mathrm{Sen}, \vdash)$ is the *entailment system freely generated* by the system of proof rules $(\mathbb{S}ig, \mathrm{Sen}, \mathrm{Rl}, h, c)$.

A system of proof rules $(\mathbb{S}ig, \mathrm{Sen}, \mathrm{Rl}, h, c)$ is *sound* (*complete*) for an institution $I = (\mathbb{S}ig, \mathrm{Sen}, \mathrm{Mod}, \models)$ whenever the entailment system freely generated by Rl, $(\mathbb{S}ig, \mathrm{Sen}, \vdash)$, is sound (complete) for I, i.e. $\vdash \subseteq \models$ ($\models \subseteq \vdash$).

The following result is proved in [6].

Proposition 5.15. A system of proof rules $(\mathbb{S}ig, \mathrm{Sen}, \mathrm{Rl}, h, c)$ is sound for an institution $I = (\mathbb{S}ig, \mathrm{Sen}, \mathrm{Mod}, \models)$ if for each signature Σ, $\mathrm{Rl}(\Sigma) \subseteq \models_\Sigma$.

The system of proof rules for $\mathrm{AMVL(L)}$ is given by the following set of rules for any $\mathrm{AMVL(L)}$ signature (S, F, R):

(R) $\emptyset \vdash [t = t, 1]$, where $t \in T_F$;

(S) $[t = t', x] \vdash [t' = t, x]$, where $t, t' \in T_F$, $x \in L$;

(T) $\{[t = t', x], [t' = t'', y]\} \vdash [t = t'', x \wedge y]$, where $t, t', t'' \in T_F$, $x, y \in L$;

(EQF) $\emptyset \vdash [t = t', 0]$, where $t, t' \in T_F$;

(RELF) $\emptyset \vdash [r(t), 0]$, where $t \in T_F$, $r \in R$;

(DESC-EQ) $[t = t', x] \vdash [t = t', y]$, where $t, t' \in T_F$, $x, y \in L$ such that $x \geq y$;

(DESC-REL) $[r(t), x] \vdash [r(t), y]$, where $t \in T_F$, $r \in R$, $x, y \in L$ such that $x \geq y$;

(OP) $\{[t_i = t'_i, x_i] \mid 1 \leq i \leq n\} \vdash [f(t_1,\ldots,t_n) = f(t'_1,\ldots,t'_n), x_1 \wedge \ldots \wedge x_n]$, where $f \in F$;

(REL) $\{[t_i = t'_i, x_i] \mid 1 \leq i \leq n\} \cup \{[r(t_1,\ldots,t_n), y]\} \vdash [r(t'_1,\ldots,t'_n), x_1 \wedge \ldots \wedge x_n \wedge y]$, where $r \in R$.

Proposition 5.16. AMVL(L) with the above system of proof rules is sound.

Proof:

By Proposition 5.15, it is enough to show that for each MVL(L)-signature (S, F, R), $\mathrm{Rl}(S, F, R) \subseteq \models_{(S,F,R)}$.

Let (S, F, R) be an MVL(L)-signature.

(R) Let M be an (S, F, R)-model. Since $\approx^M$ is an L-equality, it follows that $M_t \approx^M M_t = 1$, thus $M \models [t = t, 1]$.

(S) Let M be an (S, F, R)-model such that $M \models [t = t', x]$. Therefore $M_t \approx^M M_{t'} \geq x$. Since $\approx^M$ is an L-equality, we have $M_{t'} \approx^M M_t = M_t \approx^M M_{t'} \geq x$, thus $M \models [t' = t, x]$.

(T) Let M be an (S, F, R)-model such that $M \models [t = t', x]$ and $M \models [t' = t'', y]$. Therefore $M_t \approx^M M_{t'} \geq x$ and $M_{t'} \approx^M M_{t''} \geq y$. Since $x \geq x \wedge y$ and $y \geq x \wedge y$, it follows that $M_t \approx^M M_{t'} \geq x \wedge y$ and $M_{t'} \approx^M M_{t''} \geq x \wedge y$. We obtain $(M_t \approx^M M_{t'}) \wedge (M_{t'} \approx^M M_{t''}) \geq (x \wedge y) \wedge (x \wedge y) = x \wedge y$. Since $\approx^M$ is an L-equality, we have $(M_t \approx^M M_{t'}) \wedge (M_{t'} \approx^M M_{t''}) \leq M_t \approx^M M_{t''}$, therefore $M_t \approx^M M_{t''} \geq x \wedge y$. Thus $M \models [t = t'', x \wedge y]$.

(EQF) Let M be an (S, F, R)-model. It is obvious that $M_t \approx^M M_{t'} \geq 0$, since $0 \leq x$, for any $x \in L$. Thus $M \models [t = t', 0]$.

(RELF) Let M be an (S, F, R)-model. As $0 \leq x$, for any $x \in L$, it follows that $M_r(M_t) \geq 0$, hence $M \models [r(t), 0]$.

(DESC-EQ) Let M be an (S, F, R)-model such that $M \models [t = t', x]$. Therefore $M_t \approx^M M_{t'} \geq x \geq y$, thus $M \models [t = t', y]$.

(DESC-REL) Let M be an (S, F, R)-model such that $M \models [r(t), x]$. We have $M_r(M_t) \geq x \geq y$, therefore $M \models [r(t), y]$.

(OP) Let M an (S,F,R)-model such that $M \models [t_i = t'_i, x_i]$, for all $1 \le i \le n$. Thus $M_{t_i} \approx^M M_{t'_i} \ge x_i$, therefore we have $x_1 \wedge \ldots \wedge x_n \le (M_{t_1} \approx^M M_{t'_1}) \wedge \ldots \wedge (M_{t_n} \approx^M M_{t'_n}) \le (M_f(M_{t_1}, \ldots, M_{t_n}) \approx^M M_f(M_{t'_1}, \ldots, M_{t'_n}))$, for any $f \in F$, since M_f is compatible with $\approx^M$. By the definition of the satisfaction relation, we obtain that $M \models [f(t_1, \ldots, t_n) = f(t'_1, \ldots, t'_n), x_1 \wedge \ldots \wedge x_n]$.

(REL) Let M be an (S,F,R)-model such that $M \models [t_i = t'_i, x_i]$, for all $1 \le i \le n$, and $M \models [r(t_1, \ldots, t_n), y]$. Therefore we have $M_{t_i} \approx^M M_{t'_i} \ge x_i$, for all $1 \le i \le n$, and $M_r(M_{t_1}, \ldots, M_{t_n}) \ge y$. We obtain $x_1 \wedge \ldots \wedge x_n \wedge y \le (M_{t_1} \approx^M M_{t'_1}) \wedge \ldots \wedge (M_{t_n} \approx^M M_{t'_n}) \wedge M_r(M_{t_1}, \ldots, M_{t_n})$. Since M_r is compatible with $\approx^M$, we have $(M_{t_1} \approx^M M_{t'_1}) \wedge \ldots \wedge (M_{t_n} \approx^M M_{t'_n}) \wedge M_r(M_{t_1}, \ldots, M_{t_n}) \le M_r(M_{t'_1}, \ldots, M_{t'_n})$, thus $M_r(M_{t'_1}, \ldots, M_{t'_n}) \ge x_1 \wedge \ldots \wedge x_n \wedge y$. Hence $M \models [r(t'_1, \ldots, t'_n), x_1 \wedge \ldots \wedge x_n \wedge y]$.

$\square$

In order to prove that AMVL(L) with the above system of proof rules is also complete, let us first define, for any signature (S,F,R) and any set E of (S,F,R)-sentences from AMVL(L), P_E by:

- $(P_E)_s = \{[t] \mid t \in (T_F)_s\}$, where $[t] = \{t' \in (T_F)_s \mid E \vdash [t = t', 1]\}$;
- $[t] \approx^{P_E} [t'] = \bigvee\{y \in L \mid E \vdash [t = t', y]\}$, for any $[t], [t'] \in P_E$;
- $(P_E)_f([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$, for any $f \in F_{s_1 \ldots s_n \to s}$ and $[t_i] \in (P_E)_{s_i}$;
- $(P_E)_r([t_1], \ldots, [t_n]) = \bigvee\{y \in L \mid E \vdash [r(t_1, \ldots, t_n, y)]\}$, for any $r \in R_{s_1 \ldots s_n}$ and $[t_i] \in (P_E)_{s_i}$.

Fact 5.9. The definition of P_E is correct.

Proof:

The proof follows by simple calculation.

- $[t] \approx^{P_E} [t'] = \bigvee\{y \mid E \vdash [t = t', y]\}$

  Let $t_1 \in [t]$ and $t'_1 \in [t']$. Then $E \vdash [t = t_1, 1]$ and $E \vdash [t' = t'_1, 1]$. Suppose $E \vdash [t = t', y]$. From $E \vdash [t = t_1, 1]$, using (S), we obtain $E \vdash [t_1 = t, 1]$. We can apply (T) and we obtain $E \vdash [t_1 = t', y]$. We apply again (T) and it follows that $E \vdash [t_1 = t'_1, y]$.

- $(P_E)_f([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$

  Let $t'_i \in [t_i]$, for all $1 \le i \le n$. Then $E \vdash [t_i = t'_i, 1]$, for all $1 \le i \le n$. By rule (OP) we get $E \vdash [f(t_1, \ldots, t_n) = f(t'_1, \ldots, t'_n), 1]$, thus $f(t'_1, \ldots, t'_n) \in [f(t_1, \ldots, t_n)]$.

- $(P_E)_r([t_1], \ldots, [t_n]) = \bigvee\{y \mid E \vdash [r(t_1, \ldots, t_n), y]\}$

  Let $t'_i \in [t_i]$, for all $1 \le i \le n$. Then $E \vdash [t_i = t'_i, 1]$, for all $1 \le i \le n$. Suppose $E \vdash [r(t_1, \ldots, t_n), y]$. By rule (REL) we get $E \vdash [r(t'_1, \ldots, t'_n), y]$.

$\square$

Proposition 5.17. P_E is an (S, F, R)-model in $\mathrm{AMVL(L)}$.

Proof:

For P_E to be an (S, F, R)-model we must check the followings:

– $\approx^{P_E}$ is an L-equality:

By rule (R), we obtain $E \vdash [t = t, 1]$, thus $[t] \approx^{P_E} [t] = 1$.

Using rule (S), we can immediately show that $[t] \approx^{P_E} [t'] = [t'] \approx^{P_E} [t]$.

Let $[t] \approx^{P_E} [t'] = z = \bigvee\{y \mid E \vdash [t = t', y]\}$ and $[t'] \approx^{P_E} [t''] = z' = \bigvee\{y \mid E \vdash [t' = t'', y]\}$. Since any two elements from L are comparable, we have that $E \vdash [t = t', z]$ and $E \vdash [t' = t'', z']$. By rule (T) we get $E \vdash [t = t'', z \wedge z']$. Thus $[t] \approx^{P_E} [t''] = \bigvee\{y \mid E \vdash [t = t'', y]\} \geq z \wedge z' = [t] \approx^{P_E} [t'] \wedge [t'] \approx^{P_E} [t'']$.

Suppose $[t] \approx^{P_E} [t'] = 1$. Thus $E \vdash [t = t', 1]$. Let $t_1 \in [t]$. Then $E \vdash [t = t_1, 1]$. By rule (S) we obtain $E \vdash [t_1 = t, 1]$ and by rule (T) we get $E \vdash [t_1 = t', 1]$. We apply (S) again and we get $E \vdash [t' = t_1, 1]$. Thus $t_1 \in [t']$. Similarly, we can prove that if $t_1 \in [t']$, then $t_1 \in [t]$. Therefore $[t] = [t']$.

– $(P_E)_f$ is compatible with $\approx^{P_E}$:

Let $[t_i] \approx^{P_E} [t'_i] = z_i$, where $z_i = \bigvee\{y \mid E \vdash [t_i = t'_i, y]\}$, for any $1 \leq i \leq n$. Since in L any two elements are comparable, it follows that $E \vdash [t_i = t'_i, z_i]$, for any $1 \leq i \leq n$. Using rule (OP) we get that $E \vdash [f(t_1, \ldots, t_n) = f(t'_1, \ldots, t'_n), z_1 \wedge \ldots \wedge z_n]$. We have the followings $((P_E)_f([t_1], \ldots, [t_n]) \approx^{P_E} (P_E)_f([t'_1], \ldots, [t'_n])) = ([f(t_1, \ldots, t_n)] \approx^{P_E} [f(t'_1, \ldots, t'_n)]) = \bigvee\{y \mid E \vdash [f(t_1, \ldots, t_n) = f(t'_1, \ldots, t'_n), y]\} \geq z_1 \wedge \ldots \wedge z_n = ([t_1] \approx^{P_E} [t'_1]) \wedge \ldots \wedge ([t_n] \approx^{P_E} [t'_n])$.

– $(P_E)_r$ is compatible with $\approx^{P_E}$:

Let $[t_i] \approx^{P_E} [t'_i] = z_i$, where $z_i = \bigvee\{y \mid E \vdash [t_i = t'_i, y]\}$, for any $1 \leq i \leq n$. Thus $E \vdash [t_i = t'_i, z_i]$, for any $1 \leq i \leq n$. Suppose $(P_E)_r([t_1], \ldots, [t_n]) = z = \bigvee\{y \mid E \vdash [r(t_1, \ldots, t_n), y]\}$. Since in L any two elements are comparable, we have $E \vdash [r(t_1, \ldots, t_n), z]$. We apply rule (REL) and we obtain $E \vdash [r(t'_1, \ldots, t'_n), z_1 \wedge \ldots \wedge z_n \wedge z]$. We have $(P_E)_r([t'_1], \ldots, [t'_n]) = \bigvee\{y \mid E \vdash [r(t'_1, \ldots, t'_n), y]\} \geq z_1 \wedge \ldots \wedge z_n \wedge z = ([t_1] \approx^{P_E} [t'_1]) \wedge \ldots \wedge ([t_n] \approx^{P_E}][t'_n]) \wedge (P_E)_r([t_1], \ldots, [t_n])$.

□

Fact 5.10. $(P_E)_t = [t]$, for any (S, F, R)-term t.

Proof:

We proof by induction on the structure of terms:

$(P_E)_{f(t_1,\ldots,t_n)} = (P_E)_f((P_E)_{t_1}, \ldots, (P_E)_{t_n}) = (P_E)_f([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$. □

Proposition 5.18. $P_E \models E$

Proof:

Let $[t = t', x] \in E$. Therefore $E \vdash [t = t', x]$. Since $(P_E)_t \approx^{P_E} (P_E)_{t'} = \bigvee\{y \mid E \vdash [t = t', y]\}$, we obtain that $(P_E)_t \approx^{P_E} (P_E)_{t'} \geq x$, thus $P_E \models [t = t', x]$.

Let $[r(t), x] \in E$. It follows that $E \vdash [r(t), x]$. By definition, $(P_E)_r((P_E)_t) = \bigvee\{y \mid E \vdash [r(t), y]\}$, therefore $(P_E)_r((P_E)_t) \geq x$, thus $P_E \models [r(t), x]$.

□

Proposition 5.19. If $P_E \models e$, then $E \vdash e$, for any (S, F, R)-sentence e in $\mathrm{AMVL(L)}$.

Proof:

Let $e = [t = t', x]$. Since $P_E \models [t = t', x]$, we have $(P_E)_t \approx^{P_E} (P_E)_{t'} \geq x$. From $(P_E)_t \approx^{P_E} (P_E)_{t'} = z = \bigvee\{y \mid E \vdash [t = t', y]\}$, we get $E \vdash [t = t', z]$, since any two elements from L are comparable. Since $z \geq x$, using rule (DESC-EQ) we obtain $E \vdash [t = t', x]$.

Let $e = [r(t_1, \ldots, t_n), x]$. From $P_E \models [r(t_1, \ldots, t_n), x]$, we get $(P_E)_r((P_E)_{t_1}, \ldots, (P_E)_{t_n}) \geq x$. Since $(P_E)_r((P_E)_{t_1}, \ldots, (P_E)_{t_n}) = z = \bigvee\{y \mid E \vdash [r(t_1, \ldots, t_n), y]\}$ and any two elements of L are comparable, we obtain $E \vdash [r(t_1, \ldots, t_n), z]$. We have that $z \geq x$, thus we can apply rule (DESC-REL) and we obtain $E \vdash [r(t_1, \ldots, t_n), x]$. □

Proposition 5.20. $\mathrm{AMVL(L)}$ with the above proof rules is complete.

Proof:

Let E be a set of atomic sentences for any signature (S, F, R). Suppose $E \models e$, where e is an atomic sentence for (S, F, R). By Proposition 5.18 we obtain that $P_E \models e$. Finally, by Proposition 5.19, we get $E \vdash e$.

□

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