Some Modern Problems in Nonlinear Elliptic Partial Differential Equations

Dissertation

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Chapter 1

The Vazquez Maximum Principle and Applications

1.1 Introduction

In this first chapter we study a deep maximum principle for elliptic equations due to Professor Juan Luis Vazquez, together with some of its applications. This maximum principle is a result very useful in dealing with nonlinearities of type u^p. It uses a condition very close to that of Keller-Osserman type(see chapter 3) and it is an important improvement of the classical maximum principle.
The proof has a very elementary character, using essentially techniques of ordinary differential equations and the method of sub- and supersolutions. It also has many applications, and we will present here a theorem of Diaz and Saa, which is in fact an adaptation for the p-Laplace operator of the famous result of Brezis-Oswald.
We state here for convenience the main results of the method of sub- and supersolutions.
Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and consider a Caratheodory function $f(x,u) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ such that f is of class C^1 with respect to u. Consider the general problem:

$$-\Delta u = f(x,u) \quad in \quad \Omega$$

$$u = 0 \quad on \quad \partial\Omega$$

We look for classical solutions, i.e. solutions u which belongs to $C^2(\Omega) \cap C\overline{\Omega}$.

Definition 1. *A function $\underline{U}$ is said to be a subsolution of the problem above if*

$$-\Delta \underline{U} \le f(x, \underline{U}) \quad in \quad \Omega$$

$$\underline{U} \le 0 \quad on \quad \partial\Omega$$

A similar definition, by reversing the signs, holds for the notion of supersolution.

The main result of the method of sub- and supersolutions is the following:

Theorem 1. *Let $\underline{U}$ (respectively $\overline{U}$) be a subsolution (respectively a supersolution) to the problem above, such that $\underline{U} \leq \overline{U}$ in Ω. Then the following hold:*
(i) there exists a solution u such that $\underline{U} \leq u \leq \overline{U}$ in Ω;
(ii) there exists a minimal and a maximal solution $\underline{u}$ and $\overline{u}$ of our problem with respect to the interval $[\underline{U}, \overline{U}]$.

We will not prove this result. For a proof and for many extensions and applications the reader may consult [Rad].
The structure of this chapter is the following: we start with proving the version of the Vazquez maximum principle for the Laplacian and for the p-Laplacian, in order to show how a proof for the Laplace operator can be adapted for the p-Laplace. In section 1.3 we derive the theorem of Diaz and Saa and we present the slight simplifications we can do in the proof for the case of the usual Laplace operator.

1.2 The Vazquez Maximum Principle

In this section we state and prove the Vazquez maximum principle in its both variants, for the Laplace and for the p-Laplace operator. This result has appeared in the paper [Va84]

Theorem 2. *(The Vazquez Maximum Principle) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, where $N \geq 3$. Let u be a real function on Ω such that $u \in C^2(\Omega) \cap C(\overline{\Omega})$, $u \geq 0$ in Ω and*

$$\Delta u \leq f(u) \quad in \quad \Omega \tag{1.1}$$

where $f : [0, \infty) \to \mathbb{R}$ is a continuous and nondecreasing function such that $f(0) = 0$ and it satisfies the following integral condition:

$$\int_0^1 \frac{dt}{\sqrt{F(t)}} = +\infty \tag{1.2}$$

where $F(t) := \int_0^t f(s)ds$.
Then the following alternative holds: either $u > 0$ in Ω or $u \equiv 0$ in Ω.

Remark. *Typical examples of functions satisfying the condition (1.2) are $f(t) = t^p$ with $p \geq 1$ or $f(t) = t^p log(1 + t)$ for $p \geq 2$.*

For the proof we use the following

Lemma 1. *We consider the 1-dimensional Dirichlet problem*

$$v'' = K_1 v' + K_2 f(v) \quad in \quad (0, r_1) \tag{1.3}$$

$$v(0) = 0, \quad v(r_1) = v_1 > 0 \tag{1.4}$$

where K_1, K_2 and r_1 are positive real numbers. If $f(0) = 0$ and f is increasing, then the problem has a unique solution. If moreover f satisfies the condition (1.2), then $v'(0) > 0$ and $0 < v < v_1$ in $(0, r_1)$, where v is the solution of (1.3)-(1.4).

Proof. We obviously see that $\underline{v} \equiv 0$ is a subsolution and $\overline{v} \equiv C$ is a supersolution to (1.3)-(1.4), provided that C is very large. By using theorem 1 we deduce that problem (1.3)-(1.4) has at least a solution.
To prove uniqueness, suppose that v_0 and v_2 are two different solutions and let $v := v_0 - v_2$. Then v satisfies

$$v^{''} = K_1 v^{'} + K_2(f(v_0) - f(v_2)) \ \ in \ \ (0, r_1) \tag{1.5}$$

$$v(0) = v(r_1) = 0 \tag{1.6}$$

We prove that $v_0 \le v_2$. If not, then there exists a point $x_0 \in (0, r_1)$ such that $v(x_0) = \sup\limits_{x \in (0,r_1)} v(x) > 0$. Hence $v^{'}(x_0) = 0$ and $v^{''}(x_0) \le 0$. By replacing in (1.5) we obtain $0 \ge v^{''}(x_0) = K_1 v^{'}(x_0) + K_2(f(v_0(x_0)) - f(v_2(x_0))) > 0$, which is a contradiction. Hence $v_0 \le v_2$ and by reversing the roles we get also that $v_2 \le v_0$. The uniqueness follows.
Assume now that f satisfies (1.2) and let v be a solution of (1.5)-(1.6). Let $r_0 := \sup\{0 \le r \le r_1 : v(r) = 0\}$. Then $0 \le r_0 < r_1$ and $v(r_0) = 0$. We prove in the following lines that necessary $r_0 = 0$, which will end the proof.
We argue by contradiction and suppose that $r_0 > 0$. Then $v^{'}(r_0) = 0$ and $v^{'} \le 0$ on (r_0, r_1). Hence $v : [r_0, r_1] \to [0, v(r_1)]$ is a bijection(since v can not have local maxima in $(0, r_1)$, fact which derives from the equation). We multiply (1.5) by $v^{'}$ and we integrate. By setting $w := (v^{'})^2$ we obtain:

$$w^{'} = 2K_1 w + 2K_2(F(v))^{'} \tag{1.7}$$

We multiply again (1.7) by $e^{-2K_1 r}$ and we integrate in (r_0, r). We have:

$$e^{-2K_1 r} w(r) - e^{-2K_1 r_0} w(r_0) = 2K_2 \int_{r_0}^{r} e^{-2K_1 s}(F(v(s)))^{'} ds \tag{1.8}$$

Since $w(r_0) = 0$, it follows that $2K_2 \int_{r_0}^{r} e^{-2K_1 s}(F(v(s)))^{'} ds = e^{-2K_1 r} w(r) \le 2K_2 \int_{r_0}^{r} e^{-2K_1 r_0}(F(v(s)))^{'} ds = 2K_2 e^{-2K_1 r_0}(F(v(r)) - F(v(r_0)))$, hence $\frac{v^{'}(r)}{\sqrt{F(v(r))}} \le 2K_2 e^{-2K_1}(r - r_0) \le C$, for any $r \in (0, r_1)$. Since $v : [r_0, r_1] \to [0, v_1]$ is a bijection, by integrating and changing the variable we obtain:

$$\int_0^{v_1} \frac{dt}{\sqrt{F(t)}} \le C(r_1 - r_0) < +\infty$$

which is a contradiction to the condition (1.2) supposed for f. □

We denote for convenience the solution v in this general lemma by $v(r; K_1, K_2, r_1, v_1)$ in order to emphase its dependence on the parameters. We can now prove the Vazquez maximum principle in its form for the Laplace operator.

Proof. Let $\Omega_0 := \{x \in \Omega : u(x) = 0\}$, where $u \ge 0$ in Ω. Let x_1 be a point in $\Omega \setminus \Omega_0$ such that $d(x_1, \Omega_0) < d(x_1, \partial\Omega)$. Suppose that Ω_0 is a proper subset of Ω. Set $R := d(x_1, \Omega_0)$ and x_0 be a point in $\Omega_0 \cap \overline{B(x_1, R)}$. Then $u(x_0) = 0$. Also set $\omega := \{x \in \Omega : \frac{1}{2}R < |x - x_1| < R\}$.
We apply lemma 1 for the parameters $r_1 = \frac{R}{2}$, $v_1 = \inf\{u(x) : |x - x_1| = \frac{R}{2}\}$, $K_1 = \frac{2(N-1)}{R}$,

$K_2 = 1$. We can define $\overline{u}(x) := v(R - |x - x_1|; K_1, K_2, r_1, v_1)$, where v is given by lemma 1. Then $\overline{u}(R) = v(0) = 0$ and $\overline{u}(\frac{R}{2}) = v(\frac{R}{2})$. From the equation we deduce:

$$\Delta\overline{u} = \overline{u}^{''} - \frac{N-1}{r}\overline{u}^{'} \geq v^{''} - \frac{2(N-1)}{R}v^{'} = f(v) \tag{1.9}$$

in ω. In conclusion, using the function v given by lemma 1 we obtained $\overline{u}$. We prove that $\overline{u} \leq u$ in ω. We see that the equality holds on $\partial\omega$ and we will prove in fact that for any $\varepsilon > 0$ we have

$$\overline{u}(x) \leq u(x) + \varepsilon(1 + |x^2|)^{-\frac{1}{2}} \quad in \quad \omega \tag{1.10}$$

Suppose (1.10) is not true; hence there exists $y_0 \in \omega$ such that $\overline{u}(x) - u(x) - \varepsilon(1+|x|^2)^{-\frac{1}{2}}$ attains its maximum in y_0 and this maximum is positive. We obtain that $0 \geq \Delta(\overline{u}(x) - u(x) - \varepsilon(1+|x|^2)^{-\frac{1}{2}})|_{x=y_0}$ and by a straightforward calculation(using also (1.9)) we derive a contradiction for $N \geq 3$. By passing to the limit as $\varepsilon \to 0$ we obtain that $\overline{u} \leq u$ in ω.

Remark. *We see that the above comparison does not hold for $N = 2$. Instead of this, we can compare directly in the same way as before $\overline{u}$ and u and obtain the same result provided that f is supposed increasing instead of nondecreasing. Hence the principle is valid also for $N = 2$ in this stronger hypothesis on f.*

Hence $\overline{u}(x_0) = u(x_0) = 0$, where $x_0 \in \partial\omega$, and since $u \geq 0$ in Ω, $\nabla u(x_0) = 0$, hence $\frac{\partial u}{\partial n}(x_0) = 0$. On the other hand we compute the normal derivative of $u|_\omega$ in x_0. We have

$$\begin{aligned}
\frac{\partial u}{\partial n}(x_0) &= \lim_{t\to 0}\frac{u(x_0 - t(x - x_0))}{t} \\
&\geq \lim_{t\to 0}\frac{\overline{u}(x_0 - t(x - x_0))}{t} \\
&= \lim_{t\to 0}\frac{v(tR)}{t} \\
&= Rv^{'}(0) > 0
\end{aligned}$$

which is a contradiction with $\frac{\partial u}{\partial n}(x_0) = 0$. It follows that our assumption that Ω_0 is a proper subset of Ω is false, hence either $\Omega_0 = \Omega$ (and in this case $u \equiv 0$ in Ω) or $\Omega_0 = \varnothing$ (and in this case $u > 0$ in Ω). $\square$

We are going now to prove the Vazquez maximum principle for the p-Laplace operator, by showing the modifications we have to make in the above proof. For the beginning, let us define the p-Laplace operator, as $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$, where $p > 1$. We will need also the expression of the p-Laplacian in polar coordinates; this is

$$\Delta_p u = (|u^{'}(r)|^{p-2}u^{'}(r))^{'} + |u^{'}(r)|^{p-2}u^{'}(r)\frac{N-1}{r} \tag{1.11}$$

where $u = u(r) = u(|x|)$. We remark that the usual Laplace operator is a particular case of the p-Laplacian for $p = 2$. Based on these, we can formulate:

Lemma 2. *Consider the 1-dimensional Dirichlet problem:*

$$(|v^{'}|^{p-2}v^{'})^{'} = K_1|v^{'}|^{p-2}v^{'} + K_2 f(v), \quad in \quad (0, r_1) \tag{1.12}$$

$$v(0) = 0, v(r_1) = v_1 > 0 \tag{1.13}$$

where K_1, K_2 and r_1 are positive real numbers and $f(0) = 0$, f is increasing.
Then the problem (1.12)-(1.13) has a unique solution. If, in addition, f satisfies the condition

$$\int\limits_{0+} \frac{dt}{F(t)^{1/p}} = +\infty \tag{1.14}$$

where $F(t) := \int\limits_0^t f(s)ds$, then $v'(0) = 0$ and $0 < v < v_1$ in $(0, r_1)$.

We will prove completely this lemma because, even if the idea is the same as in the proof of lemma 1, there are some technical differences that are not always obvious.

Proof. We easily see, as before, that $\underline{v} \equiv 0$ is a subsolution and $\overline{v} \equiv C$ is a supersolution to (1.12)-(1.13); by theorem 1 it follows that (1.12)-(1.13) has at least a solution.
To prove uniqueness, suppose that v_0 and v_2 are two solutions of (1.12)-(1.13). We prove that $v_0 \leq v_2$ in $(0, r_1)$. Suppose not; then there exists a point $x_0 \in (0, r_1)$ such that $v_0(x_0) - v_2(x_0) = \sup\{v_0(x) - v_2(x) : x \in (0, r_1)\} > 0$. Hence $v_0'(x_0) = v_2'(x_0)$ and $v_0''(x_0) \leq v_2''(x_0)$.
By subtracting the equations (1.12) for v_0 and v_2 one has:

$$(|v_0'|^{p-2}v_0')'(x_0) - (|v_2'|^{p-2}v_2')'(x_0) = K_2(f(v_0(x_0)) - f(v_2(x_0))) > 0 \tag{1.15}$$

since the term with coefficient K_1 is the same in both expressions and it cancels. Since $v_0''(x_0) \leq v_2''(x_0)$ and $v_0'(x_0) = v_2'(x_0)$, one can easily check that the left hand side is nonpositive, contradiction. Hence $v_0 \leq v_2$ in $(0, r_1)$. By changing v_0 by v_2 uniqueness follows.
Suppose now that f satisfies (1.14). Let v be the solution of (1.12)-(1.13) and let $r_0 := \sup\{0 \leq r \leq r_1 : v(r) = 0\}$. Then $v(r_0) = 0$ and $r_0 < r_1$. We will prove that $r_0 = 0$.
Suppose that $r_0 > 0$. Since v has no local maxima in $(0, r_1)$, it follows that $v \equiv 0$ in $(0, r_0)$ and $v'(r_0) = 0$. We also remark that $v' > 0$ in (r_0, r_1), hence $v : [r_0, r_1] \to [0, v_1]$ is a bijection.
We multiply in both sides of (1.12) by v' and we integrate. We have:

$$w' = \frac{K_1 p}{p-1} w + \frac{K_2 p}{p-1} (F(v))' \tag{1.16}$$

where, as in lemma 1, we denote by $w := |v'|^p$. We multiply again in both sides of (1.16) by $e^{-\frac{K_1 p}{p-1} r}$.
We obtain

$$(w' - \frac{K_1 p}{p-1} w) e^{-\frac{K_1 p}{p-1} r} = \frac{K_2 p}{p-1} e^{-\frac{K_1 p}{p-1} r} (F(v))'$$

or, equivalently, after integration on (r_0, r)

$$w(r) e^{-\frac{K_1 p}{p-1} r} = \frac{K_2 p}{p-1} \int\limits_{r_0}^{r} e^{-\frac{K_1 p}{p-1} s} F(v(s))' \leq \frac{K_2 p}{p-1} e^{-\frac{K_1 p}{p-1} r_0} F(v(r)) \tag{1.17}$$

It follows that

$$\frac{w(r)}{F(v(r))} \leq \frac{K_2 p}{p-1} e^{-\frac{K_1 p}{p-1}(r_0 - r)} \leq C$$

for some constant $C > 0$. Hence

$$\frac{|v'(r)|}{F(v(r))^{1/p}} \leq C^{1/p}$$

Since $v : [r_0, r_1] \to [0, v_1]$ is a bijection, by integrating in the last equality and changing the variable, we obtain:

$$|\int_0^{v_1} \frac{dt}{F(t)^{\frac{1}{p}}}| \leq C(r_1 - r_0)$$

which contradicts (1.14). Hence $r_0 = 0$ and $v^{'}(r) > 0, \quad \forall r \in (0, r_1)$ and we are done. $\square$

We state now the Vazquez maximum principle for the p-Laplacian:

Theorem 3. *(The Vazquez Maximum Principle for the p-Laplacian) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a function satisfying $u \geq 0$ in Ω and*

$$\Delta_p u \leq f(u) \quad in \quad \Omega \tag{1.18}$$

where $f : [0, \infty) \to \mathbb{R}$ is a continuous, increasing function such that $f(0) = 0$ and it satisfies (1.14). Then the following alternative holds: either $u \equiv 0$ in Ω or $u > 0$ in Ω.

We will skip the proof of this theorem, since it follows the same lines as the proof of theorem 2. The only technical difference is when choosing the parameters in order to apply lemma 2. In this case, the correct parameters are: $r_1 = \frac{R(p-1)}{p}$, $v_1 = \inf\{u(x) : |x - x_1| = \frac{R(p-1)}{p}\}$, $K_1 = \frac{p(N-1)}{R(p-1)}$ and $K_2 = 1$. The interior set ω must be taken as it follows: $\omega := \{x \in \Omega : \frac{R(p-1)}{p} < |x - x_1| < R\}$. The rest of the proof is similar to that of theorem 2.

1.3 A Theorem of Diaz and Saa

In this section we prove a theorem of J.I.Diaz and J.E.Saa(see [DS87]) which extends to the p-Laplace case another famous theorem, that of Brezis and Oswald, published in [BO86].
We deal in what follows with the following problem:

$$-\Delta_p u = f(x, u), \quad in \quad \Omega \tag{1.19}$$

$$u \geq 0, \quad u = 0 \quad on \quad \partial\Omega \tag{1.20}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^N$ and $f : \Omega \times [0, \infty) \to \mathbb{R}$ is a function satisfying the following hypothesis:
(A) $r \to f(x, r)$ is continuous in $[0, \infty)$ for a.e. $x \in \Omega$ and $x \to f(x, r)$ is in $L^\infty(\Omega)$;
(B) The application $r \to \frac{f(x,r)}{r^{p-1}}$ is decreasing on $(0, \infty)$ for a.e. $x \in \Omega$;
(C) There exists a positive constant C such that $f(x, r) \leq C(1 + r^{p-1})$ for all $r \geq 0$ and for a.e $x \in \Omega$.
We look for solutions $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We remark that (A) implies in this case that $f(x, u(x)) \in L^\infty(\Omega)$, hence, using the well-known Sobolev regularity estimates, we derive that u is in the space $W_{loc}^{2,2}(\Omega)$. We also impose the condition that u is not identically 0.
We define the following quantities:

$$a_0(x) := \lim_{r \to 0} \frac{f(x, r)}{r^{p-1}}, \quad a_\infty(x) = \lim_{r \to 0} \frac{f(x, r)}{r^{p-1}} \tag{1.21}$$

and

$$\lambda_1(-\Delta_p v - a|v|^{p-2}v) := \inf\{\int_\Omega |\nabla v|^p dx - \int_{\{v\neq 0\}} a|v|^p dx : v \in W_0^{1,p}(\Omega), \|v\|_{L^p} = 1\} \tag{1.22}$$

where a is a real function on Ω.
Using these we state:

Theorem 4. *(Diaz-Saa) The problem (1.19)-(1.20) has at most one solution. It has a solution if and only if we have:*

$$\lambda_1(-\Delta_p v - a_0|v|^{p-2}v) < 0 < \lambda_1(\Delta_p v - a_\infty|v|^{p-2}v) \tag{1.23}$$

Before starting the proof, we have to introduce three technical lemmas,very useful for the general treatment of the uniqueness in the p-Laplace case. The main difficulty of the uniqueness part, that these lemmas are solving, is to prove the monotonicity of the rather complicated operator

$$w \to (-\Delta_p w^{\frac{1}{p}}) w^{\frac{1-p}{p}} \tag{1.24}$$

We will show at the end of this section how the proof can be simplified for $p = 2$, where we need only one of these lemmas.

Lemma 3. *If u is a solution of problem (1.19)-(1.20), with $u > 0$ in Ω, then $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$.*

Proof. Since $u \in L^\infty(\Omega)$, it follows that $u(x) \leq \|u\|_\infty$, hence

$$\frac{f(x, u(x))}{u(x)^{p-1}} \geq \frac{f(x, \|u\|_\infty)}{\|u\|_\infty^{p-1}} \geq -M$$

or equivalently $f(x, u(x)) \geq -Mu(x)^{p-1}$. Hence $\Delta_p(u) \leq Mu^{p-1}$ and we apply the Vazquez maximum principle for the p-Laplacian(theorem 3). Since we impose that u is not identically 0, it follows that $u > 0$ in Ω. From the maximum principle we derive also the conclusion. □

Lemma 4. *Let $J : L^1(\Omega) \to (-\infty, \infty]$ be the following functional:*

$$J(w) = \frac{1}{p}\int_\Omega |\nabla w^{\frac{1}{p}}|^p dx \tag{1.25}$$

for $w \geq 0$ and $w^{\frac{1}{p}} \in W_0^{1,p}(\Omega)$. We define $J(w)$ to be ∞ in the rest. Then J is not identically infinite and it is convex.

Proof. Let w_1, $w_2 \in L^1(\Omega)$ be two functions such that $w_1^{\frac{1}{p}}$, $w_2^{\frac{1}{p}} \in W^{1,p}(\Omega)$, $w_i \geq 0$ in Ω for $i = 1, 2$ and $w_1 = w_2$ on $\partial\Omega$. Let $w_3 := tw_1 + (1-t)w_2$, where $t \in (0, 1)$ is fixed and let $z_i = w_i^{\frac{1}{p}}$, for $i = 1, 2, 3$. The convexity of J expresses as:

$$\int_\Omega |\nabla w_3^{\frac{1}{p}}|^p dx \leq t\int_\Omega |\nabla w_1^{\frac{1}{p}}|^p dx + (1-t)\int_\Omega |\nabla w_2^{\frac{1}{p}}|^p dx$$

or, equivalently,

$$\int_\Omega (|\nabla z_3|^p - t|\nabla z_1|^p - (1-t)|\nabla z_2|^p) dx \leq 0 \tag{1.26}$$

To prove this, we can write:

$$\begin{aligned}
|\nabla w_3| &= |\nabla z_3^p| = p z_3^{p-1}|\nabla z_3| \\
&= p(tz_1^p + (1-t)z_2^p)^{\frac{p-1}{p}}|\nabla z_3| \\
&= p(tz_1^p + (1-t)z_2^p)^{\frac{p-1}{p}}|\nabla (tz_1^p + (1-t)z_2^p)^{\frac{1}{p}}| \\
&= p(tz_1^p + (1-t)z_2^p)^{\frac{p-1}{p}}\frac{1}{p}(tz_1^p + (1-t)z_2^p)^{-\frac{p-1}{p}}|t\nabla z_1^p + (1-t)\nabla z_2^p| \\
&= p|tz_1^{p-1}\nabla z_1 + (1-t)z_2^{p-1}\nabla z_2| \\
&\leq p(tz_1^p + (1-t)z_2^p)^{\frac{p-1}{p}}(t|\nabla z_1|^p + (1-t)|\nabla z_2|^p)^{\frac{1}{p}}
\end{aligned}$$

where for the inequality we have applied the Holder inequality with the exponents $\frac{p-1}{p}$ and $\frac{1}{p}$. Hence $|\nabla z_3|^p \leq t|\nabla z_1|^p + (1-t)|\nabla z_2|^p$. Then (1.26) follows by integration. $\square$

Lemma 5. *For $i = 1, 2$ let $w_i \in L^\infty(\Omega)$ be two functions such that $w_i \geq 0$ a.e. in Ω, $w_i^{\frac{1}{p}} \in W^{1,p}(\Omega)$ and $\Delta_p w_i^{\frac{1}{p}} \in L^\infty(\Omega)$. We suppose also that:*
(i) $w_1 = w_2$ on $\partial\Omega$;
(ii) $\frac{w_1}{w_2}$ and $\frac{w_2}{w_1}$ belongs to $L^\infty(\Omega)$
Then we have:

$$\int_\Omega (-\frac{\Delta_p w_1^{\frac{1}{p}}}{w_1^{\frac{p-1}{p}}} + \frac{\Delta_p w_2^{\frac{1}{p}}}{w_2^{\frac{p-1}{p}}})(w_1 - w_2)dx \geq 0 \tag{1.27}$$

This lemma shows what we understand rigourously by the monotonicity of the operator (1.24). After proving it, the uniqueness follows immediately, as we shall see.

Proof. The idea is to compute the Gateaux differential of the functional J considered above in the direction $w := w_1 - w_2$ and to use the convexity of J. We have:

$$J'(w_1; w) = \frac{1}{p}\lim_{h\to 0}\frac{1}{h}\int_\Omega (|\nabla (w_1 + hw)^{\frac{1}{p}}|^p - |\nabla w_1^{\frac{1}{p}}|^p)dx \tag{1.28}$$

We use a limited Taylor developpment. We have

$$(w_1 + hw)^{\frac{1}{p}} = w_1^{\frac{1}{p}}(1 + h\frac{w}{w_1})^{\frac{1}{p}} = w_1^{\frac{1}{p}}(1 + \frac{1}{p}h\frac{w}{w_1} + o(h))$$

hence

$$\nabla((w_1 + hw)^{\frac{1}{p}}) = \nabla w_1^{\frac{1}{p}} + \frac{1}{p}h\nabla\frac{w}{w_1^{\frac{p-1}{p}}} + o(h)$$

When deriving the last equation we have used essentially the condition $\frac{w_1}{w_2} \in L^\infty(\Omega)$. By using again the limited Taylor developpment one obtains:

$$|\nabla((w_1 + hw)^{\frac{1}{p}})|^p = |\nabla w_1^{\frac{1}{p}}|^p + \frac{h}{p}|\nabla w_1^{\frac{1}{p}}|^{p-1}\nabla\frac{w}{w_1^{\frac{p-1}{p}}} + o(h)$$

Then

$$\begin{aligned}
\int_\Omega \lim_{h\to 0}\frac{1}{h}(|\nabla((w_1 + hw)^{\frac{1}{p}})|^p - |\nabla w_1^{\frac{1}{p}}|^p)dx &= \frac{1}{p}\int_\Omega |\nabla w_1^{\frac{1}{p}}|^{p-2}\nabla w_1^{\frac{1}{p}}\nabla\frac{w}{w_1^{\frac{p-1}{p}}} \\
&= \frac{1}{p}\int_\Omega -\frac{\Delta_p w_1^{\frac{1}{p}}}{w_1^{\frac{p-1}{p}}} w dx
\end{aligned} \tag{1.29}$$

By Lebesgue's theorem, we can commute the limit and the integral in (1.29) and we obtain:

$$J^{'}(w_1; w) = \frac{1}{p}\int\limits_{\Omega} \frac{-\Delta_p w_1^{\frac{1}{p}}}{w_1^{\frac{p-1}{p}}} w dx \tag{1.30}$$

Hence the expression in (1.27) is $J^{'}(w_1; w) - J^{'}(w_2; w)$, which is nonnegative by the convexity of J. $\square$

Now we can pass to the proof of the main theorem. We start with the uniqueness part, which follows easily from these lemmas.

Proof. (Uniqueness) Let w be an arbitrary solution of problem (1.19)-(1.20). Then $w \in L^{\infty}(\Omega)$ and $f(x, w(x)) \leq C(1+\|w\|_{\infty}^{p-1})$. On the other hand $f(x, w(x)) \geq -M\|w\|_{\infty}^{p-1}$, hence $\Delta_p w = -f(x, w) \in L^{\infty}(\Omega)$.

Consider now two different solutions w_1, w_2 of (1.19)-(1.20). From lemma 3 and using by example the l'Hopital rule, we deduce that $\frac{w_2}{w_1}$ and $\frac{w_1}{w_2}$ are in $L^{\infty}(\Omega)$ and we can apply lemma 5. We have

$$0 \leq \int\limits_{\Omega} (-\frac{\Delta_p w_1}{w_1^{p-1}} + \frac{-\Delta_p w_2}{w_2^{p-1}})(w_1^p - w_2^p) < 0$$

which is a contradiction. $\square$

Proof. (Existence) We start with the "if part". Suppose that there exists a solution u of problem (1.19)-(1.20). Then, it follows from the definition that:

$$\lambda_1(-\Delta_p v - a_0|v|^{p-2}v) \leq \frac{1}{\int\limits_{\Omega} |u|^p dx}(\int\limits_{\Omega} |\nabla u|^p dx - \int\limits_{\Omega} a_0|u|^p dx) \tag{1.31}$$

On the other hand,

$$-\Delta_p u \cdot u = f(x, u)u = \frac{f(x, u)}{u^{p-1}} u^p$$

,hence, by integrating,

$$\int\limits_{\Omega} -div(|\nabla u|^{p-2}\nabla u)udx = \int\limits_{\Omega} \frac{f(x, u)}{u^{p-1}} u^p dx < \int\limits_{\Omega} a_0 u^p dx$$

It follows immediately that $\lambda_1(-\Delta_p v - a_0|v|^{p-2}v) < 0$.

To prove that $\lambda_1(-\Delta_p v - a_{\infty}|v|^{p-2}v) > 0$, we first set $a(x) := \frac{f(x,1+\|u\|_{\infty})}{(1+\|u\|_{\infty})^{p-1}}$, hence $a(x) \geq a_{\infty}(x)$, for a.e. $x \in \Omega$ and $\frac{f(x,u(x))}{u(x)^{p-1}} > a(x)$. We derive that

$$\lambda_1(-\Delta_p v - a_{\infty}|v|^{p-2}v) > \lambda_1(-\Delta_p v - a|v|^{p-2}v) =: \mu \tag{1.32}$$

We prove next that $\mu \geq 0$.

From the variational definition of λ_1 we deduce that there exists an "eigenfunction" Ψ solving the following problem:

$$-\Delta_p \Psi - a\Psi^{p-1} = \mu\Psi^{p-1} \quad in \quad \Omega \tag{1.33}$$

$$\Psi > 0 \quad in \quad \Omega, \quad \Psi = 0 \quad on \quad \partial\Omega \tag{1.34}$$

We see that also $k\Psi$ is a solution for the problem above, for any $k > 0$. Set $\Omega_k := \{x \in \Omega : k\Psi(x) > u(x)\}$. Obviously $\Omega = \cup_{k>0}\Omega_k$. We apply lemma 5 and we get:

$$\begin{aligned} 0 &\leq \int_{\Omega_k} (-\frac{\Delta_p u}{u^{p-1}} + \frac{\Delta_p(k\Psi)}{(k\Psi)^{p-1}})(u^p - (k\Psi^p)) \\ &= \int_{\Omega_k} (\frac{f(x,u)}{u^{p-1}} - (a(x) + \mu)(u^p - (k\Psi^p)))dx \end{aligned}$$

It follows that $\frac{f(x,u)}{u^{p-1}} < a(x) + \mu$. But $\frac{f(x,u)}{u^{p-1}} > a(x)$, hence $\mu > 0$.

Conversely, let us assume that (1.23) holds and we will prove that (1.19)-(1.20) has a solution. The proof uses only classical arguments in nonlinear analysis, as a variational technique followed by an interesting and not at all trivial bootstrap argument. First of all, we can generalize more the problem by replacing condition (B) on f by the following:

(B') For all $\delta > 0$, there exists $C_\delta \geq 0$ such that $f(x,r) \geq -C_\delta r^{p-1}$, $\forall r \in [0,\delta]$.

In view of this new condition, we define

$$a_0(x) := \liminf_{u\to 0} \frac{f(x,u)}{u^{p-1}}, \quad a_\infty(x) := \limsup_{u\to 0} \frac{f(x,u)}{u^{p-1}} \tag{1.35}$$

hence $a_0(x) \geq -C$ and $a_\infty(x) \leq C$. We renote by $f(x,u)$ the function equal to the old $f(x,u)$ for $u \geq 0$ and equal to $f(x,0)$ for $u < 0$. We introduce the energy functional:

$$E(u) := \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega F(x,u)dx, \tag{1.36}$$

defined for $u \in W_0^{1,p}(\Omega)$, where as usual $F(x,u) = \int_0^u f(x,s)ds$. From (C) we have that $|F(x,u)| \leq \frac{C}{p}(p|u| + |u|^p)$, hence E is well-defined on $W_0^{1,p}(\Omega)$. We prove the following three properties for the energy functional:

(a) $\lim_{\|u\|\to\infty} E(u) = +\infty$ (i.e. E is coercive);

(b) E is weakly lower semicontinuous in $W_0^{1,p}(\Omega)$;

(c) there exists $\phi \in W_0^{1,p}(\Omega)$ such that $E(\phi) < 0$.

We start our program by proving (b), which is easy. Let $(u_n)_n$ be a sequence converging weakly in $W_0^{1,p}(\Omega)$ to a function u. Then, by the lower semicontinuity of the norm, $\|u\|^p_{W_0^{1,p}(\Omega)} \leq \liminf_{n\to\infty} \|u_n\|^p_{W_0^{1,p}(\Omega)}$. By extracting a subsequence if needed, we may suppose that $u_n \to u$ in $L^p(\Omega)$ and a.e. Also by taking a larger constant, we may suppose that $F(x,u_n) \leq C(1+u_n^p)$. By Fatou's lemma, we obtain

$$\limsup_{n\to\infty} \int_\Omega F(x,u_n)dx \leq \int_\Omega F(x,u)dx$$

hence $E(u) \leq \liminf_{n\to\infty} E(u_n)$.

We next prove assumption (c). We know that $\lambda_1(-\Delta_p v - a_0|v|^{p-2}v) < 0$, hence there exists a function $\phi \in W_0^{1,p}$ such that

$$\int_\Omega |\nabla\phi|^p dx < \int_{\{\phi\neq 0\}} a_0|\phi|^p dx \tag{1.37}$$

In (1.37) we may suppose without loss of generality that $\phi \geq 0$ and $\phi \in L^\infty(\Omega)$. For the first assumption, we replace ϕ by ϕ^+, and one can check that in this way the inequality (1.37) is conserved.

For the second assumption, set as before $\Omega_k := \{x \in \Omega : \phi(x) \le k\}$, for any $k \in \mathbb{N}$. Then obviously $\Omega = \cup_k \Omega_k$ and $\Omega_k \subset \Omega_{k+1}$, $\forall k \in \mathbb{N}$. It follows that

$$\int_\Omega |\nabla \phi|^p dx = \lim_{k\to\infty} \int_{\Omega_k} |\nabla \phi|^p dx$$

We deduce from (1.37) that there exists a $k \in \mathbb{N}$ such that the same inequality (1.37) holds with the integrals taken on Ω_k. Fix such a k and set $\tilde\phi(x) := \phi(x)$ if $\phi(x) \le k$ and $\tilde\phi(x) = 0$ if $\phi(x) > k$. Hence $\tilde\phi \in L^\infty(\Omega)$ and

$$\int_\Omega |\nabla \tilde\phi|^p dx = \int_{\Omega_k} |\nabla \phi|^p dx < \int_{\Omega_k \cap \{\phi \ne 0\}} a_0 |\phi|^p dx = \int_{\{\tilde\phi \ne 0\}} a_0 |\phi|^p dx$$

In this way we may replace ϕ by $\tilde\phi \in L^\infty(\Omega)$. Hence we take from the beginning $\phi \in W_0^{1,p}(\Omega) \cup L^\infty(\Omega)$ and $\phi \ge 0$.

Then, it follows from the definition of a_0 that $\liminf\limits_{u\to 0} \frac{F(x,u)}{u^p} \ge \frac{1}{p} a_0(x)$. We take in this inequality $u = \varepsilon\phi$, with $\varepsilon > 0$ and ϕ as above. Then $\liminf\limits_{\varepsilon\to 0} \frac{F(x,\varepsilon\phi(x))}{\varepsilon^p} \ge \frac{a_0(x)\phi(x)^p}{p}$ a.e in $\{\phi \ne 0\}$. By integration an use of (1.37) we have

$$\frac{1}{p}\int_\Omega |\nabla \phi|^p dx < \int_\Omega \frac{F(x,\varepsilon\phi(x))}{\varepsilon^p} dx$$

for $\varepsilon > 0$ small. Hence $E(\varepsilon\phi) < 0$.

Finally, we prove (a). Suppose that there exists a sequence $(u_n)_n \subset W_0^{1,p}(\Omega)$ and there exists $C > 0$ such that $\|u_n\|_{W_0^{1,p}(\Omega)} \to \infty$ and $E(u_n) \le C$. Then

$$\frac{1}{p}\int_\Omega |\nabla u_n|^p dx \le \int_\Omega F(x,u_n)dx + C \le C(1 + \|u_n\|^p_{L^p(\Omega)})$$

Set $u_n := t_n v_n$, where $t_n = \|u_n\|_{L^p(\Omega)} \to \infty$ and $v_n = \frac{u_n}{\|u_n\|_{L^p(\Omega)}}$. Then $\frac{1}{p}\int_\Omega |\nabla v_n|^p dx \le C(1 + \frac{1}{t_n^p})$, hence $(v_n)_n$ is bounded in $W_0^{1,p}(\Omega)$. By reflexivity of $W_0^{1,p}(\Omega)$ (see [Br83]) we may suppose(by subtracting a subsequence) that $v_n \to v$ weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and a.e. in Ω. We deduce that $\|v\|_{L^p(\Omega)} = 1$.

By passing to the limit we obtain:

$$\frac{1}{p}\int_\Omega |\nabla v|^p dx \le \liminf_{n\to\infty} \int_\Omega \frac{F(x,t_n v_n)}{t_n^p} dx \le \limsup_{n\to\infty} \int_\Omega \frac{F(x,t_n v_n)}{t_n^p} dx \tag{1.38}$$

On the other hand, we have:

$$\begin{aligned}
\int_\Omega \frac{F(x,t_n v_n)}{t_n^p} dx &= \int_{\{v_n \le 0\}} \frac{F(x,t_n v_n)}{t_n^p} dx + \int_{\{v_n > 0\}} \frac{F(x,t_n v_n)}{t_n^p} dx \\
&= \int_{\{v_n \le 0\}} \frac{F(x,t_n v_n)}{t_n^p} dx + \int_{\{v \le 0\}} \frac{F(x,t_n v_n^+)}{t_n^p} dx + \int_{\{v > 0\}} \frac{F(x,t_n v_n^+)}{t_n^p} dx \\
&= I_1 + I_2 + I_3
\end{aligned}$$

Trying to estimate I_1, I_2 and I_3 we obtain :

$$I_1 = \int_{\{v_n \le 0\}} \frac{F(x,t_n v_n)}{t_n^p} dx \le \frac{C}{t_n^p} \int_{\{v_n \le 0\}} t_n |v_n| dx \le \frac{\overline{C}}{t_n} \to 0$$

$$I_2 = \int\limits_{\{v\le 0\}} \frac{F(x,t_n v_n^+)}{t_n^p}dx \le \frac{C}{t_n^p}\int\limits_{\{v\le 0\}} ((t_n v_n^+)^p + 1)dx \to 0$$

From the definition of a_∞ it follows that $\limsup\limits_{v\to\infty} \frac{F(x,v)}{v^p} \le \frac{1}{p}a_\infty(x)$, hence $\limsup\limits_{n\to\infty} \frac{F(x,t_n v_n^+)}{t_n^p} \le \frac{a_\infty(x)v_n^p}{p}$ for a.e. $x \in \{v > 0\}$. By Fatou's lemma, $\limsup\limits_{n\to\infty} I_3 \le \frac{1}{p}\int\limits_{\{v>0\}} a_\infty v^p dx$.
By putting these estimates together it follows that

$$\int\limits_\Omega |\nabla v|^p dx \le \int\limits_\Omega a_\infty v^p dx \tag{1.39}$$

and

$$\int\limits_\Omega |\nabla v|^p dx - \int\limits_\Omega a_\infty v^p dx \ge \alpha \|v\|^p_{L^p(\Omega)} \tag{1.40}$$

where $\alpha := \lambda_1(-\Delta_p v - a_\infty |v|^{p-2}v) > 0$. But (1.39) and (1.40) imply that $v = 0$, which is contradictory to $\|v\|_{L^p(\Omega)} = 1$.
Using the properties (a),(b),(c) we derive the existence result. Let $m := \inf\limits_{u\in W_0^{1,p}(\Omega)} E(u)$. By (a) and (c), $-\infty < m < 0$. Let $(u_n)_n$ be a minimizing sequence, i.e. $E(u_n) \to m$. Hence $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$ and we may suppose there exists $u \in W_0^{1,p}(\Omega)$ such that $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and a.e. in Ω. Then, by (b), $E(u) \le \liminf\limits_{n\to\infty} E(u_n) = m$, hence $E(u) = m$. By replacing u by u^+, the energy decreases, hence $E(u^+) = m$. It follows that u^+ is a nonnegative solution of (1.19)-(1.20).
We prove that $u \in L^\infty(\Omega)$. For any $k \ge 1$, let $f^k(x,u) := f(x,0)$ for $u \le 0$ and $f^k(x,u) := max\{f(x,u), -ku^{p-1}\}$ for $u > 0$. We remark easily that f^k satisfies (A),(B),(C) for any k provided that f satisfies. Hence there exists a unique solution u_k of

$$-\Delta_p u_k = f^k(x,u_k) \ \ in \ \ \Omega \tag{1.41}$$

$$u_k = 0 \ \ on \ \ \partial\Omega, \ \ u_k > 0 \ \ in \ \ \Omega \tag{1.42}$$

Then $u_k \in W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ (if $N > p$), hence $f^k(x,u_k) \in L^{p^*}(\Omega)$. From the classical L^p estimates (see [GT02], chapter 9) it follows that $u_k \in W^{2,p^*}(\Omega) \subset L^q(\Omega)$, where $q = \frac{Np^*}{N-2p^*}$, if $N > 2p^*$. We replace $p^* = \frac{Np}{N-p}$ and we obtain $q = \frac{Np}{N-3p}$, in the case that $N > 3p$. In all the contrary cases to these inequalities, the space is $L^\infty(\Omega)$. In the case that $N > 3p$ we continue our bootstrap technique. The next space will be $L^r(\Omega)$ with $r = \frac{Np}{N-5p}$ if $N > 5p$ or $L^\infty(\Omega)$ if $N \le 5p$. We continue and after a finite number of steps(N being fixed) we will arrive to the situation that $N < kp$ for k sufficiently large. It follows that (see again [GT02], chapter 7) $u_k \in L^\infty(\Omega)$, for any $k \in \mathbb{N}$.
But

$$-\Delta_p u_k = f^k(x,u_k) \ge f(x,u_k) \ \ in \ \ \Omega \tag{1.43}$$

and $u_k = 0$ on $\partial\Omega$, hence u_k is a supersolution to (1.19)-(1.20). We see that 0 is a subsolution to (1.19)-(1.20), hence the unique solution u of the problem (1.19)-(1.20) satisfies $0 \le u \le u_k$. But we have just proved that $u_k \in L^\infty(\Omega)$, hence $u \in L^\infty(\Omega)$ and the proof ends. □

To end this section, we will show how the proof can be simplified in the case $p = 2$. This corresponds to the usual Laplacian and was proved before the Diaz-Saa theorem, by Brezis and Oswald(see [BO86]).

We will prove just the uniqueness of the solution, since this is the main point where the simplification is indeed important.

Proof. (uniqueness for $p = 2$) Let u_1, u_2 be two solutions of the problem (1.19)-(1.20). From the previous lemmas, we will use only lemma 3. From the result of this lemma and the usual l'Hopital rule we derive that

$$\frac{u_1}{u_2} \in L^\infty(\Omega), \quad \frac{u_1^2}{u_2} \in H_0^1(\Omega) \tag{1.44}$$

From the equation we derive that $\frac{-\Delta u_1}{u_1} = \frac{f(x,u_1)}{u_1}$. We multiply in both sides by $u_1^2 - u_2^2$ and we integrate. By applying also the Green formula, we obtain:

$$\begin{aligned}\int_\Omega \frac{f(x,u_1)}{u_1}(u_1^2 - u_2^2)dx &= -\int_\Omega \Delta u_1 (u_1 - \frac{u_2^2}{u_1})dx \\ &= \int_\Omega \nabla u_1 (\nabla u_1 - \nabla \frac{u_2^2}{u_1})dx \\ &= \int_\Omega |\nabla u_1|^2 dx - \int_\Omega \nabla u_1 (\frac{2u_2}{u_1}\nabla u_2 - \frac{u_2^2}{u_1^2}\nabla u_1)dx \\ &= \int_\Omega (|\nabla u_1|^2 - 2\frac{u_2}{u_1}\nabla u_1 \nabla u_2 + \frac{u_2^2}{u_1^2}|\nabla u_1|^2)dx\end{aligned}$$

We obtain a similar result by reversing the roles of u_1 and u_2. By summing these equalities, one has:

$$\int_\Omega (\frac{f(x,u_1)}{u_1}) - \frac{f(x,u_2)}{u_2})(u_1^2 - u_2^2)dx = \int_\Omega (|\nabla u_1 - \frac{u_1}{u_2}\nabla u_2|^2 - |\nabla u_2 - \frac{u_2}{u_1}\nabla u_1|^2) \tag{1.45}$$

It follows that the left hand side is nonnegative, and, since $\frac{f(x,u)}{u}$ is decreasing, we deduce that $u_1 = u_2$ in Ω. $\square$

The rest of the proof follow essentially the same lines as the general one. We remark that the essential tool that allow us to do such simplification in the proof is the Green formula. This is a fact that often happens when passing from the Laplacian to the p-Laplacian.

Open question Establish a similar result without monotonicity of f, using possibly a stronger integral condition instead of this. For example, try to make an analogue of the facts in chapter 3, where the strong Keller-Osserman condition allows us to treat very general nonlinearities.

References

1.[BO86]-H.Brezis, L.Oswald-Remarks on sublinear elliptic equations, Nonl. Anal. TMA, 10(1986), no.1, 55-64;

2.[DS87]-J.I.Diaz, J.E.Saa-Existence et unicite des solutions positives pour certaines equations elliptiques quasilineaires, C.R. Acad. Sci. Paris, t.305, Serie I,1987, 521-524;

3.[GT02]-D.Gilbarg, N.Trudinger-Elliptic partial differential equations of second order, Springer-Verlag, Berlin, 2002;

4.[Rad] -V.Radulescu-Treatment methods for nonlinear elliptic equations, Lecture Notes, University

of Craiova, 2005;
5.[Va84]-J.L.Vazquez-A strong maximum principle for some quasilinear elliptic equations, Appl. Math. and Optimization, 12(1984), no.3, 191-202.

Chapter 2

A Result for a Singular Neumann Problem in a Ball

2.1 Introduction

In this chapter we are concerned in studying the existence of radial solutions for the following singular elliptic Neumann problem:

$$-\Delta u + u^{-\nu} = h(x) + f(u), \quad in \quad B \tag{2.1}$$

$$\frac{\partial u}{\partial n} = 0, \quad on \quad \partial B \tag{2.2}$$

where B is the unit ball in $\mathbb{R}^N$, with $N \geq 2$, $\nu > 1$ and $h \in C(\overline{B})$ is a radially symmetric function.
We introduce the following hypothesis on function f:
(i)$f \in C^1([0, \infty))$, $f \geq 0$, $f(0) = 0$;
(ii)f is a nondecreasing function in $[0, \infty)$;
(iii) f is asymptotically sublinear, i.e. $\lim\limits_{s \to \infty} \frac{f(s)}{s} = 0$.
Typical examples for the nonlinearity f are $f(x) = arctg(x)$ or $f(x) = \frac{x}{1+x}$ for f bounded or $f(x) = log(1 + x)$, $f(x) = x^\alpha$ with $\alpha < 1$ or $f(x) = x^\alpha log(1 + x)$ for $\alpha < 1$.
We say that a function $u \in C^2(\overline{B} \setminus \{0\}) \cap C(\overline{B})$ is a solution of problem (2.1)-(2.2) if it solves the problem distributionally and it satisfies $u > 0$ in $\overline{B} \setminus \{0\}$. We will see in the next section that this definition of solution will appear as natural.
Our starting point for this study is the paper of M. del Pino and G. Hernandez where a similar problem is considered, but without the nonlinearity f. That's why the ideas and methods we use follow closely those in the above mentioned paper.
The main techniques used for proving existence for the problem (2.1)-(2.2) are the following:
A: We perturb the equation (2.1) in order to avoid the singularity. This is a change that one usually do when treating problems with singular terms. More precisely, we consider the family of approximating problems:

$$-\Delta u + (max\{u, s\})^{-\nu} = h(x) + f(u), \quad in \quad B \tag{2.3}$$

$$\frac{\partial u}{\partial n} = 0, \quad on \ \ \partial B \tag{2.4}$$

for small $u > 0$.
B: We try to obtain apriori bounds from below for the solutions, uniformly with respect to $s > 0$. We will see at this step that our change in the equation (2.3) is not sufficient, and we have to introduce another perturbation, useful for technical reasons:

$$-\Delta u + (max\{u, s\})^{-\nu} = h(x) + f(u) + \varepsilon\delta_0, \quad in \ \ B \tag{2.5}$$

where we will look for solutions which are positive in B. Here $\varepsilon > 0$ is small and δ_0 denotes the Dirac mass supported at the origin. We will see that this problem, in appearance more difficult than (2.3)-(2.4), posesses a uniform lower estimate for its solutions. For this, we will need to pass through the step of proving an upper estimate first.
C: The final argument uses a passage to the limit in the apriori bounds of B and a theorem of saddle-point type.
The chapter is divided in five sections. In section 2.2 we prove some basic and general results. Section 2.3 contains some preliminaries, regarding some theorems of saddle-point type which will be needed to conclude. In section 2.4 we prove(very technically) some apriori bounds which will be important for the existence theorems. Finally, in the last section we prove the main result and some other completions.

2.2 General results

In this section we prove several technical lemmas that we will use further in the text. Here appears essential to have the asymptotic sublinearity condition (iii). The following lemma shows that our definition of solution is natural.

Lemma 6. *If u is a solution of (2.1)-(2.2) and $u \geq 0$, then $u > 0$ in $\overline{B} \setminus \{0\}$.*

Proof. We prove that if there exists some $\overline{r} \in (0,1)$ such that $u(\overline{r}) > 0$, then $u(r) > 0, \forall \ \ 0 < r < \overline{r}$.
Set $F(t) = \int\limits_0^t f(s)ds$.
We write the equation in radial form:

$$u'' + \frac{N-1}{r}u' - u^{-\nu} + h + f(u) = 0 \tag{2.6}$$

By multiplying by r^{N-1} we have:

$$(r^{N-1}u')' = (u^{-\nu} - h - f(u))r^{N-1}$$

We multiply again by $r^{N-1}u'$ and we arrive to the following inequality:

$$\frac{d}{ds}(\frac{|s^{N-1}u'|^2}{2}) \geq -\frac{d}{ds}(\frac{u^{-\nu+1}}{-\nu+1}r^{2N-2} + \|h\|_\infty u + F(u)) \tag{2.7}$$

on $[r, \overline{r}]$, where $0 < r < \overline{r}$. By fixing $s \in (r, \overline{r})$ and integrating on $[s, \overline{r}]$ we find:

$$r^{2(N-1)}\frac{u^{1-\nu}(s)}{\nu-1} + \frac{|s^{N-1}u'(s)|^2}{2} + \|h\|_\infty u(s) + F(u(s)) \leq r^{2(N-1)}\frac{u^{1-\nu}(\overline{r})}{\nu-1} + \frac{|\overline{r}^{N-1}u'(\overline{r})|^2}{2} + \|h\|_\infty u(\overline{r}) + F(u(\overline{r})) \tag{2.8}$$

We may suppose(with no loss of generality) that $u > 0$ on $(r, \overline{r}]$ and that u is nondecreasing on $[r, \overline{r}]$. By passing to the limit in (2.9) as $s \to r$, we conclude that

$$r^{2N-2}\frac{u^{-\nu+1}(r)}{\nu - 1} \leq C(\overline{r}) < \infty$$

hence $u(r) > 0$.
In a similar way one can prove that $u(r) > 0$ for $\overline{r} \leq r \leq 1$. □

This lemma produces an interesting question: is it possible that u vanishes at the origin? For $N = 1$ the answer is negative. If we multiply by $u^{'}$ in the equation and integrate on $[s, 1]$, where $s \in (0, 1)$, we find:

$$\frac{1}{2}u^{'}(s)^2 + \frac{u^{-\nu+1}(s)}{\nu - 1} + F(u(s)) + \|h\|_\infty u(s) \leq \frac{1}{2}u^{'}(1)^2 + \frac{u^{-\nu+1}(1)}{\nu - 1} + F(u(1)) + \|h\|_\infty u(1) \quad (2.9)$$

If $u(0) = 0$, then, by making $s > 0$ very small, the left hand side goes to infinity (since $-\nu + 1 < 0$), contradicting the inequality (2.10).
For $N \geq 2$ things are very different and it may happen that $u(0) = 0$, as the following example, taken from the paper [DPH96], shows:
Let θ be a smooth function on $[0, 1]$ such that $\theta \equiv 1$ on $[0, \frac{1}{3}]$ and $\theta \equiv 0$ on $[\frac{2}{3}, 1]$, and $0 \leq \theta \leq 1$ on $[0, 1]$. Consider

$$u(r) = c\theta(r)r^{\frac{2}{\nu+1}} + 1 - \theta(r)$$

with some constant c. By a rather long, but straightforward calculation, it follows that u solves such an equation with an appropriate value of c and some function h. Here $f \equiv 0$. But obviously $u(0) = 0$. The main technical difference between the cases $N = 1$ and $N \geq 2$ is here the appearance of the term $\frac{N-1}{r}u^{'}$ in the radial form of the Laplacian.

Lemma 7. *Let u be a function in $C^2[a, b]$, where $[a, b] \subset [0, 1]$, such that:*

$$u^{''} + \frac{N-1}{r}u^{'} - u^{-\nu} + m + f(u) \geq 0 \quad (2.10)$$

where $m > 0$ and f is assimptotically sublinear (in the sense that $\lim\limits_{t\to\infty} \frac{f(t)}{t} = 0$). Suppose also that $u(a) \geq \mu > 0$, $u^{'}(a) = 0$ and $u^{'} \leq 0$ on $(a, b]$. Then there exists some constant $\theta = \theta(m, \nu, \mu)$ such that $u(b) > \theta > 0$.

Proof. By multiplying by $u^{'} \leq 0$, we have:

$$\frac{d}{dr}(\frac{u^{'2}}{2}) + \frac{u^{-\nu+1}}{\nu - 1} + mu + F(u)) \leq -\frac{N-1}{2}(u^{'})^2 \leq 0$$

hence

$$\frac{u^{'2}(r)}{2} + \frac{u(r)^{-\nu+1}}{\nu - 1} + mu(r) + F(u(r)) \leq \frac{u(a)^{-\nu+1}}{\nu - 1} + mu(a) + F(u(a))$$

or, equivalently

$$u^{'}(r) \leq \sqrt{2\int\limits_{u(r)}^{u(a)}(m - s^{-\nu} + f(s))ds} \quad (2.11)$$

By integrating (2.12) with respect to r on $[a,b]$ and by changing the variable, we obtain:

$$\int_{u(a)}^{u(b)} \frac{dt}{\sqrt{2\int_t^{u(a)}(m-s^{-\nu}+f(s))ds}} \leq 1 \tag{2.12}$$

Suppose $u(b)<\theta<\mu\leq u(a)$. Then, by nondecreasingness, we have $\int_{\mu}^{u(a)} \frac{dt}{\sqrt{2\int_t^{u(a)}(m-s^{-\nu}+f(s))ds}} \leq 1$.
Since $u(a)\geq\mu>\theta$, there exists a constant C such that $u(a)\geq\frac{C}{\theta^{\nu-1}}$.
We prove next that

$$\lim_{\alpha\to\infty}\int_{\mu}^{\alpha} \frac{dt}{\sqrt{2\int_t^{\alpha}(m-s^{-\nu}+f(s))ds}} = \infty \tag{2.13}$$

For this, we remark first that $m-s^{-\nu}+f(s)\leq m-\alpha^{-\nu}+f(\alpha)$ for $s\in(\mu,\alpha]$, hence

$$\sqrt{\frac{\alpha-\mu}{2(m-\alpha^{-\nu}+f(\alpha))}} = \int_{\mu}^{\alpha}\frac{dt}{\sqrt{2(\alpha-t)(m-\alpha^{-\nu}+f(\alpha))}} \leq \int_{\mu}^{\alpha}\frac{dt}{\sqrt{2\int_t^{\alpha}(m-s^{-\nu}+f(s))ds}} \tag{2.14}$$

But the first term in (2.15) is unbounded as $\alpha\to\infty$, hence (2.14) is proved. It follows from here that $u(a)$ must have an upper bound. Since $u(a)\geq\frac{C}{\theta^{\nu-1}}$, θ must have a lower bound (depending only on the parameters m,μ,ν) and we are done. $\square$

Remark. *In the last proof we have used in an essential way the asymptotic sublinearity of f.*

Lemma 8. *Assume f,h as before, with $\|h\|_\infty\leq m$. Then there exists a number $\theta=\theta(m,\nu)>0$ such that if u is a solution of*

$$u'' + \frac{N-1}{r}u' = u^{-\nu} - h(r) - f(u) \quad in \quad (0,1) \tag{2.15}$$

$$u'(1) = 0 \tag{2.16}$$

for which there are numbers $a\leq\rho<1$ with $u(\rho)<\theta$, $u'(a)=0$ and $u'(\rho)\geq 0$, then u is nondecreasing on $[a,\rho]$.

Proof. By way of contradiction, suppose that u is not nondecreasing on $[a,\rho]$. Then for any $\theta>0$ there is some $b_1\in(a,\rho)$ such that $u(b_1)<\theta$ and $u'(b_1)<0$. Fix some $\theta>0$. Let a_1 be the first point at the left of b_1 such that $u'(a_1)=0$. It follows easily that u is decreasing on $[a_1,b_1]$.
We multiply in both sides of the equation by r^{N-1} and we obtain:

$$(r^{N-1}u')' = (u^{-\nu}-h(r)-f(u))r^{N-1} > (u(a_1)^{-\nu}-h(r)-f(u))r^{N-1}$$

hence

$$\begin{aligned} 0 &> b_1^{N-1}u'(b_1) \\ &> \frac{1}{N}(b_1^N-a_1^N)(u(a_1)^{-\nu}-m) - \int_{a_1}^{b_1} f(u(r))r^{N-1}dr \\ &> \frac{1}{N}(b_1^N-a_1^N)(u(a_1)^{-\nu}-m-f(u(a_1))) \end{aligned}$$

It follows that $u(a_1)^{-\nu} < m + f(u(a_1))$. But the equation $x^{-\nu} = m + f(x)$ has only one solution $c_0 \in (0, \infty)$ (since the left hand side is decreasing and the right hand side is increasing) and $u(a_1) > c_0 = c_0(m, \nu)$.
The final step is applying the lemma 7 for the interval $[a_1, b_1]$ and the parameters m, ν and $\mu = c_0(m, \nu)$. It follows that $u(b_1) > \theta$, which yields to a contradiction. $\square$

We will need one more general result concerning the differential inequality

$$-(u^{''} + \frac{N-1}{r}u^{'}) \leq m + f(u) \tag{2.17}$$

Lemma 9. *Assume that u is a function of class $C^2([\delta, 1])$ for some $0 \leq \delta < 1$ and satisfies (2.18), where $m > 0$. Assume also that $u^{'}(1) = u^{'}(\delta) = 0$. Then, for any $\delta \leq r \leq 1$, we have:*
(i) $\sup\limits_{\delta \leq r \leq 1} u \leq \inf\limits_{\delta \leq r \leq 1} u + (mc + cf(\sup\limits_{\delta \leq r \leq 1} u))(1 + r^{2-N})$, *if* $N \geq 3$ *;*
(ii) $\sup\limits_{\delta \leq r \leq 1} u \leq \inf\limits_{\delta \leq r \leq 1} u + (mc + cf(\sup\limits_{\delta \leq r \leq 1} u))(1 - log(r))$, *if* $N = 2$.

Proof. We perform te usual multiplication by r^{N-1} on both sides of the inequality and we integrate on $[\delta, r]$. We obtain:

$$-\int_\delta^r (s^{N-1}u^{'}(s))^{'} ds \leq \int_\delta^r (m + f(u(s)))s^{N-1} ds$$

hence

$$-r^{N-1}u^{'}(r) \leq \frac{m}{N}(r^N - \delta^N) + \int_\delta^r f(u(s))s^{N-1} ds$$

By dividing the last equation by r^{N-1} and integrating again it follows that:

$$\begin{aligned} u(r) &\leq u(1) + \frac{m}{2N}(1 - r^2) + \int_r^1 \frac{1}{t^{N-1}} \int_\delta^t f(u(s))s^{N-1} ds dt \\ &\leq u(1) + \frac{m}{2N}(1 - r^2) + \int_r^1 \int_\delta^t f(u(s)) ds dt \end{aligned}$$

By taking into account that $r \in [0, 1]$, we can further obtain that $u(r) \leq u(1) + \frac{m}{2N}(1 - r^2) + f(\sup\limits_{\delta \leq r \leq 1} u)$, for any $r \in [\delta, 1]$. By passing to the supremum with respect to r we have:

$$\sup_{\delta \leq r \leq 1} u \leq u(1) + mc(N) + f(\sup_{\delta \leq r \leq 1} u) \tag{2.18}$$

On the other hand, we fix for the moment $r \in [\delta, 1]$ and we integrate on $[r, 1]$. We suppose $N \geq 3$. We have:

$$-\int_r^1 (s^{N-1}u^{'}(s))^{'} ds \leq \int_r^1 (m + f(u(s)))s^{N-1} ds$$

or

$$u^{'}(r) \leq \frac{m}{N}(1 - r^N)\frac{1}{r^{N-1}} + \frac{1}{r^{N-1}} \int_r^1 f(u(s))s^{N-1} ds$$

By integrating again on $[r, 1]$ it follows that

$$\begin{aligned} u(1) - u(r) &\leq m \int_r^1 \frac{1 - s^N}{N s^{N-1}} ds + \int_r^1 \frac{1}{t^{N-1}} \int_t^1 f(u(s)) s^{N-1} ds dt \\ &\leq m \int_r^1 \frac{1 - s^N}{N s^{N-1}} ds + f(\sup_{\delta \leq r \leq 1} u) \int_r^1 \frac{1 - s^N}{N s^{N-1}} ds \\ &\leq (m + f(\sup_{\delta \leq r \leq 1} u)) c(N) r^{2-N} \end{aligned}$$

We pass to the infimum with respect to r and we obtain:

$$u(1) \leq \inf_{\delta \leq r \leq 1} u + c(N)(m + f(\sup_{\delta \leq r \leq 1} u)) r^{2-N} \tag{2.19}$$

We introduce this estimate for $u(1)$ in (2.19) and we arrive to the desired inequality.
For $N = 2$ the calculations are very similar, the only difference is that at the final integration we obtain in (2.20) $-log(r)$ instead of r^{2-N}. □

All the results of this section, in spite of their elementary character, will play a key role in the derivation of apriori bounds for the perturbed singular problem.

2.3 A Saddle-Point Type Theorem

This section is devoted to presenting and proving explicitely a saddle-point theorem of Rabinowitz which will be used for deriving existence in the last section. For this we introduce the following:

Definition 2. *Let X be a real Banach space and $F : X \to \mathbb{R}$ be a C^1 functional on X. We say that F satisfies the Palais-Smale condition in the point $c \in \mathbb{R}$ if for any sequence $(u_n)_n$ of elements of X such that $(F(u_n))_n$ converges to c and $\|F'(u_n)\|_{X^*} \to 0$, then $(u_n)_n$ contains a convergent subsequence.*
We say that F satisfies the global Palais-Smale condition if for any sequence $(u_n)_n$ in X such that $\sup_{n \in \mathbb{N}} |F(u_n)|$ is finite and $\|F'(u_n)\|_{X^*} \to 0$, then $(u_n)_n$ contains a convergent subsequence.

We start by stating a famous result of Ambrosetti and Rabinowitz:

Theorem 5. *(The Mountain-Pass Theorem)*
Let X be a real Banach space and $F \in C^1(X, \mathbb{R})$ be a functional on X. Let K be a compact metric space and K^* be a nontrivial subset of K. We fix a map $p^* \in C(K^*, X)$. Set $\mathcal{P} := \{p \in C(K, X) : p = p^* \text{ on } K^*\}$ (such extensions exists as a consequence of the theorem of Dugundji). Let

$$c := \inf_{p \in \mathcal{P}} \sup_{t \in K} F(p(t))$$

If $c > \sup_{t \in K^*} F(p^*(t))$ and F satisfies the Palais-Smale condition in the point c, then c is a critical value of F.

We omit the proof of this theorem. The original proof of Ambrosetti and Rabinowitz appears in [AR73]; another proof, based on a different idea, can be found in the paper [BN91].
The next theorem is an important generalisation of the Mountain-Pass theorem, which gives an idea of where we can find the critical point of F which realises the critical value c.

Theorem 6. *(The Ghoussoub-Preiss Theorem)*
Let X be a real Banach space and F be a functional of class C^1 on X. Assume that there exists a closed subset $\Sigma \subset X$ such that:
(i) $p^*(K^*) \cap \Sigma = \varnothing$;
(ii) $p(K) \cap \Sigma \neq \varnothing$;
(iii) $F \geq c$ on Σ,
where the notations are similar to those of the Mountain-Pass theorem(in this case we say that Σ is a linking between K and K^*).
Then there exists an "almost critical" sequence $(u_n)_n \subset X$ such that $F(u_n) \to c$, $\|F^{'}(u_n)\|_{X^*} \to 0$ and $d(u_n, \Sigma) \to 0$. If moreover F satisfies the Palais-Smale condition in c, then there exists a critical point $u \in \Sigma$ of F such that $F(u) = c$.

The proof of this theorem is based on the following famous result in nonlinear analysis:

Lemma 10. *(The Ekeland Variational Principle)*
Let (M, d) be a complete metric space and $\Psi : M \to (-\infty, \infty]$ be a lower semicontinuous function. Assume that Ψ is bounded from below and that Ψ is not identically infinite. Then for any $\varepsilon > 0$ and for any point $z_0 \in M$, there exists $z \in M$ such that:

$$\Psi(z) \leq \Psi(z_0) - \varepsilon d(z, z_0) \tag{2.20}$$

$$\Psi(x) \geq \Psi(z) - \varepsilon d(x, z), \forall x \in M \tag{2.21}$$

We omit the proof of this lemma, which can be found in the original paper [Ek74] and in many books in nonlinear analysis, by example [Rad]. Also in the last reference one can find many variants and many applications of the Ekeland variational principle.
We need another technical result in functional analysis, which is usually known as the **pseudogradient lemma**.

Lemma 11. *Let K be a compact metric space and X be a real Banach space. Let $f : K \to X^*$ be a continuous mapping. Then, for any $\varepsilon > 0$, there exists a locally Lipschitz continuous function $v : K \to X$ such that $\|v(t)\| \leq 1, \quad \forall t \in K$ and*

$$\langle f(t), v(t)\rangle \geq \|f(t)\|_{X^*} - \varepsilon, \quad \forall t \in K \tag{2.22}$$

Proof. Fix $\varepsilon > 0$. For any $t_0 \in K$, we have that $\|f(t_0)\| = \sup_{\|w\|<1} \langle f(t_0), w\rangle$. Hence there exists some $w \in X$ such that $\|w\| < 1$ and $\langle f(t_0), w\rangle > \|f(t_0)\|_{X^*} - \varepsilon$. As f is continuous in t_0, it follows that there exists a neighborhood V_{t_0} of t_0 such that for all $t \in V_{t_0}$ we have

$$\langle f(t), w\rangle \geq \|f(t)\|_{X^*} - \varepsilon$$

We have established a mapping $K \ni t_0 \to V_{t_0} \in \mathcal{V}(t_0)$. Since K is compact, there are $t_1, ..., t_n \in K$ such that $K \subset \cup_{i=1}^n V_{t_i}$, and for every t_i we associate $w_i \in X$.
We construct a partition of unity. Let $\rho_i(t) := d(t, V_{t_i}^c)$ and

$$\Psi_i(t) := \frac{\rho_i(t)}{\sum\limits_{j=1}^{n} \rho_j(t)}, \quad t \in K, \quad i = \overline{1,n}$$

We finally construct $v(t) := \sum\limits_{i=1}^{n} \Psi_i(t) w_i$. One can easily check, by a straightforward argument, that this function v is the one we look for. □

Now we are in position to prove the Ghoussoub-Preiss theorem. The proof follows the ideas of the original paper [GP89]

Proof. From (i) we derive that $d(p^*(K^*), \Sigma) > 0$, hence there exists ε such that $0 < \varepsilon < d(p^*(K^*))$. Consider $p_0 \in \mathcal{P}$ such that $\sup\limits_{t \in K} F(p_0(t)) < c + \frac{1}{2}\varepsilon^2$. Let $K_0 := \{t \in K : d(p_0(t), \Sigma) \geq \varepsilon\}$. Obviously, $K^* \subset K_0$. Set also $\mathcal{P}_0 := \{p \in C(K, X) : p = p_0 \;\; on \;\; K_0\}$.
We make a small perturbation of the functional F by replacing it by $G := F + \eta$, where $\eta(u) := \varepsilon max\{0, \varepsilon - d(u, \Sigma)\}$. We remark that $\eta(u) \leq \varepsilon^2$ with equality if and only if $u \in \Sigma$. This change is important in the proof in order to "identify" the elements of Σ. Define also $c_0 := \inf\limits_{p \in \mathcal{P}_0} \sup\limits_{t \in K} G(p(t))$.
From (iii) we obtain that $c + \varepsilon^2 \leq F_{|\Sigma} + \varepsilon^2$, hence $c + \varepsilon^2 \leq c_0 \leq c + \frac{3}{2}\varepsilon^2$.
Define $\Psi(t) := \sup\limits_{t \in K} G(p(t))$, hence $c_0 = \inf\limits_{p \in \mathcal{P}_0} \Psi(p)$. On the space $\mathcal{P}_0$ we introduce the natural metric $d(p, q) := \sup\limits_{t \in K} \|p(t) - q(t)\|$, and $(\mathcal{P}_0, d)$ become a complete metric space. We apply lemma 10 for the function Ψ, which is obviously continuous and bounded below(by $c + \varepsilon^2$), and for $(M, d) = (\mathcal{P}_0, d)$. Then there exists $p \in \mathcal{P}_0$ such that

$$\Psi(p) - \Psi(q) + \varepsilon d(p, q) \geq 0, \quad \forall q \in \mathcal{P}_0 \tag{2.23}$$

$$\Psi(p) \leq \Psi(p_0) - \varepsilon d(p, p_0) \tag{2.24}$$

It follows that

$$c + \varepsilon^2 \leq c_0 \leq \Psi(p) \leq \Psi(p_0) - \varepsilon d(p, p_0)$$

hence $d(p, p_0) \leq \frac{\varepsilon}{2}$. Define $B(p) := \{t \in K : G(p(t)) = \Psi(p)\}$; one can easily see that $B(p) \subset K \setminus K_0$. Indeed, if there exists $t \in B(p) \cap K_0$, then $G(p(t)) = \Psi(p) \geq c + \varepsilon^2$, but on the other hand $d(p(t), p_0(t)) \geq \varepsilon$, which contradicts the previous estimate.
Now it is the moment to use lemma 11 for the compact metric space K and for the function F' in order to obtain a locally Lipschitz continuous function $w : K \to X$ such that $\|w(t)\| \leq 1$ for all $t \in K$ and

$$\left\langle F'(p(t)), w(t) \right\rangle \geq \|F'(p(t))\| - \varepsilon, \quad \forall t \in K \tag{2.25}$$

Since $B(p)$ and K_0 are two compact disjoint sets, there exists a continuous mapping $\alpha : K \to [0, 1]$ such that $\alpha = 1$ in a neighborhood of $B(p)$ and $\alpha = 0$ on K_0. For any $h > 0$ we introduce in (2.24) the mapping $q_h(t) := p(t) - h\alpha(t)w(t)$, which is in $\mathcal{P}_0$ since $\alpha = 0$ on K_0. It follows that

$$\Psi(q_h) - \Psi(p) + \varepsilon d(p, q_h) \geq 0 \tag{2.26}$$

On the other hand, by compactness one can find some $t_h \in K$ such that $\Psi(q_h) = G(q_h(t_h))$. Since K is compact, we may suppose that(passing to a subsequence if needed) $t_h \to t_0$ for some $t_0 \in K$. One obviously see that t_0 belongs to $B(p)$. It follows, from the construction of α, that $\alpha(t_h) = 1$ for $h > 0$ small enough.
We perform a limited Taylor developpment in (2.27). For this we remark that

$$F(p(t_h) - hw(t_h)) = F(p(t_h)) - h\left\langle F^{'}(p(t_h)), w(t_h)\right\rangle + o(h)$$

Using this and the fact that $\Psi(p) \geq F(p(t_h)) + \varepsilon\eta(t_h)$, we obtain that $\left\langle F^{'}(p(t_h)), w(t_h)\right\rangle \leq o(1)$. We replace this in (2.26) and we get that $\|F^{'}(p(t_h))\| \leq \varepsilon + o(1)$ for all sufficiently small h, hence $\|F^{'}(p(t_0))\| \leq \varepsilon$. On the other hand, $G = F + O(\varepsilon^2)$, hence $F(p(t_0)) = c + O(\varepsilon^2)$.
We also estimate

$$d(p(t), \Sigma) \leq d(p, p_0) + d(p_0(t), \Sigma) \leq d(p_0(t), \Sigma) + \frac{\varepsilon}{2}$$

But $t_0 \in B(p)$, hence t_0 is not in K_0 and $d(p_0(t_0), \Sigma) \leq \varepsilon$. It follows that $d(p(t_0), \Sigma) \leq \frac{3\varepsilon}{2}$.
By taking succesively $\varepsilon_n = \frac{1}{n}$ and t_n the corresponding value of t_0 for ε_n, we obtain that $u_n := p(t_n)$ is the desired sequence. Moreover, if we have the Palais-Smale condition in c, let u_0 be a limit point of the sequence $(u_n)_n$. Then $F(u_0) = c$, $F^{'}(u_0) = 0$ and, since $d(u_n, \Sigma)$ converges to 0, $u_0 \in \Sigma$. □

We arrive now to the main result of this section. This is a remarkable theorem due to P.H. Rabinowitz(see by example [Rab86]) which has many applications in nonlinear analysis and that we will use in order to derive existence for our singular problem.

Theorem 7. *(The Saddle-Point Theorem) Let X be a real Banach space and suppose that $X = X_1 \bigoplus X_2$, where X_1 and X_2 are closed subspaces of X with $dim X_2 < \infty$. Let $F \in C^1(X, \mathbb{R})$ be a functional on X. Assume that there exists $R > \rho \geq 0$ such that*

$$F(u) \geq \rho, \quad \forall u \in X_1 \tag{2.27}$$

and

$$F(u) \leq 0, \quad \forall u \in X_2, \quad \|u\| = R \tag{2.28}$$

If F satisfies the global Palais-Smale condition, then it admits a critical point u such that $F(u) \geq 0$.

Proof. The idea of the proof is that of trying to arrive to the conditions of the Ghoussoub-Preiss theorem. We mention here that this is not the original proof of P.H. Rabinowitz(see [Rab86] for it), since the Ghoussoub-Preiss theorem has appeared later.
For this goal we consider the sets $K := \{x \in X_2 : \|x\| \leq R\}$ and $K^* := \{x \in X_2 : \|x\| = R\}$ and the mapping $p^* : K^* \to X$, $p^*(x) = x$. Let $\mathcal{P}$ be the set of the extensions of p^* to K. We take in the Ghoussoub-Preiss theorem $\Sigma = X_1$. In what follows we check that the conditions in the Ghoussoub-Preiss theorem are satisfied for X_1.
Obviously, $K^* \subset X_2$ and does not contain 0, hence $p^*(K^*) \cap X_1 = K^* \cap X_1 = \varnothing$. For the condition (ii) we need to use a topological degree argument. Let $P : X \to X_2$ be the orthogonal projection. Then we may rewrite the condition (ii) as $\forall p \in \mathcal{P}\ \exists x \in K$ such that $p(x) \in X_1$, or equivalently $p(x)$ is not in X_2, or again $P \circ p(x) = 0$. But $P \circ p$ is the identity on K^*, hence $d(P \circ p, K, 0) = d(I, K, 0) = 1$. Here by $d(P, K, 0)$ we understand the Brouwer topological degree. From the existence property of

the degree we derive the existence of x such that $p \circ p(x) = 0$, hence (ii). From this we also obtain that $\sup\limits_{t \in K} F(p(t)) \geq \rho$ for any $p \in \mathcal{P}$, hence

$$c := \inf_{p \in \mathcal{P}} \sup_{t \in K} F(p(t)) \geq \rho$$

and we obtain the last condition.
We end the proof by applying the Ghoussoub-Preiss theorem. □

Remark. *In the Saddle-Point theorem the values ρ and 0 are not important. The theorem holds for any values r_1 and r_2 such that $r_1 > r_2$, $F(u) \geq r_1$ for $u \in X_1$ and $F(u) \leq r_2$ for $u \in X_2$ and $\|u\| = R$ for R big enough.*

2.4 Apriori Estimates for the Singular Problem

In this section, we consider the problem of finding apriori bounds for the radial solutions of the perturbed singular problem:

$$-\Delta u + (max\{u, s\})^{-\nu} = h(x) + f(u) + \varepsilon\delta_0, \quad in \quad B \tag{2.29}$$

$$\frac{\partial u}{\partial n} = 0 \quad on \quad \partial B \tag{2.30}$$

where $\varepsilon > 0$, $s > 0$, $\nu > 1$ and δ_0 is the Dirac mass concentrated in the origin. In this section we will assume that the nonlinearity f satisfies the conditions (i) and (ii) from the introduction. Our goal is to obtain a lower bound for the radial solutions, which will be decissive in proving existence. But, as we will see, for proving it we will need an upper bound for these solutions.
Let us denote by Φ_N the fundamental solution for the Laplacian in dimension N, i.e. $\Phi_N(r) = \frac{1}{\omega_N(N-2)} r^{N-2}$ for $N \geq 3$ and $\Phi_N(r) = -\frac{1}{2\pi} log(r)$ for $N = 2$. Here ω_N denotes the surface measure of the unit sphere in $\mathbb{R}^N$. We start with the upper bound:

Proposition 1. *Assume $h \in C(\overline{B})$ is radially symmetric and $\int\limits_B h dx > 0$. Then, there exists the numbers $\varepsilon_0 > 0$, $s_0 > 0$, $\beta > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $s \in (0, s_0)$ and for any radial solution of (2.30)-(2.31) for such s and ε we have:*

$$u(r) - \varepsilon\Phi_N(r) \leq \beta, \quad \forall r \in (0, 1] \tag{2.31}$$

Proof. We will prove the proposition only for $N \geq 3$, the other case is very similar. Set

$$v(r) := u(r) - \frac{\varepsilon}{\omega_N(N-2)} r^{2-N} \tag{2.32}$$

Then by a simple computation we have that

$$\Delta v \leq h + f(v + \varepsilon\Phi_N)$$

and by the same integration as in lemma 3, we obtain

$$\sup_B v \leq v(1) + c\|h\|_{L^\infty(B)} + f(\sup_B v) \tag{2.33}$$

If we suppose that we do not have a uniform bound for every v, then, by sublinearity of f, we have for some v that $\frac{1}{2}\sup\limits_B v \leq f(\sup\limits_B v)$, or

$$\sup_B v \leq 2(v(1) + c\|h\|_{L^\infty(B)}) \tag{2.34}$$

Hence one must only prove that $v(1)$ is uniformly bounded, or equivalently that $u(1)$ is. Suppose by contradiction that this is not true. Then there exists sequences $\varepsilon_n > 0$, $\sigma_n > 0$ which are decreasing to 0, and solutions u_n for (2.30)-(2.31) with parameters s_n and ε_n, such that $\lim\limits_{n\to\infty} u_n(1) \to \infty$. From lemma 9 we deduce that $\inf\limits_{\rho\leq r\leq 1} u_n(r) \to \infty$ as $n \to \infty$, for all $\rho > 0$.
By integration in the equation, we obtain:

$$\int_B (max\{u_n, s_n\})^{-\nu} dx = \int_B h dx + \varepsilon_n + \int_B f(u_n) dx \tag{2.35}$$

From here we deduce that for any $\delta > 0$, there exists n_δ such that $\inf\limits_B u_n < \delta$ for $n > n_\delta$. Otherwise, there would be some δ such that $u_n > \delta$ on B, for any $n \in \mathbb{N}$ and the fact that $\inf\limits_{\rho\leq r\leq 1} u(r) \to \infty$ would imply that the left-hand side of (2.36) will go to 0, which is a contradiction. Hence there exists a value $\delta_n \in (0, 1]$ such that $\delta_n \to 0$ as $n \to \infty$ and $u_n(\delta_n) = \inf\limits_B u_n$. We derive also that u_n is nondecreasing in $(0, \delta_n]$.
Fix a point $\theta > 0$ and consider

$$\gamma_n := \inf\{0 < \rho < 1 : u_n(r) \geq \theta, \quad \forall r \geq \rho\} \tag{2.36}$$

Since $\inf\limits_{\rho\leq r\leq 1} u(r) \to \infty$ for all $\rho > 0$, we must have $\lim\limits_{n\to\infty} \gamma_n = 0$ and from lemma 8 we derive that u_n is nondecreasing on $[\delta_n, \gamma_n]$.
We introduce the change of variable $r = t^{-\frac{1}{N-2}}$; denote by $\overline{f}(t) := f(t^{-\frac{1}{N-2}})$. Then, by a straightforward calculation, we deduce that $\overline{u_n}$ solves

$$\overline{u_n}'' = \frac{1}{(N-2)^2 t^\lambda}(max\{\overline{u_n}, s_n\}^{-\nu} - \tilde{h}(t) - f(\overline{u_n})) \tag{2.37}$$

where $\lambda = \frac{2(N-1)}{N-2}$.
In this setting, let $R_n := \delta_n^{2-N}$ and $S_n := \gamma_n^{2-N}$. Since $\delta_n \leq \gamma_n$, it follows that $S_n \leq R_n$ and $\overline{u}(R_n) = \inf\limits_{[1,\infty)} \overline{u_n}$.
We prove next the following facts:

$$\lim_{n\to\infty}(\overline{u_n}'(S_n + t) + \frac{1}{(N-2)^2}\int_1^{S_n+t} \frac{f(\overline{u_n}(s))}{s^\lambda} ds) = -\alpha \tag{2.38}$$

for all $t \in [0, \frac{\theta}{\alpha})$, and

$$\lim_{n\to\infty}(R_n - S_n) = \frac{\theta}{\alpha} \tag{2.39}$$

where $\alpha := \frac{1}{(N-2)^2}\int_1^\infty \frac{\overline{h}(t)}{t^\lambda} dt$.
To prove (2.39), we first integrate in the equation and obtain:

$$\overline{u_n}'(S_n) = \frac{1}{(N-2)^2}\int_1^{S_n}(max\{\overline{u_n}, s_n\}^{-\nu} - \overline{h} - f(\overline{u_n}))\frac{1}{t^\lambda} dt$$

hence

$$\lim_{n\to\infty}(\overline{u_n}'(S_n) + \frac{1}{(N-2)^2}\int_1^{S_n} \frac{f(\overline{u_n})(t)}{t^\lambda}dt) = -\alpha \tag{2.40}$$

Consider now the function

$$\phi_n(t) := \overline{u_n}(S_n+t) + \frac{1}{(N-2^2)}\int_0^t\int_1^{S_n+l} \frac{f(\overline{u_n}(s))}{s^\lambda}dsdl \tag{2.41}$$

By differentiation of ϕ_n we arrive to the left-hand side of (2.39) and we easily see that for small s_n, i.e. for large n, ϕ_n is a convex function. We express the result of (2.41) as $\lim_{n\to\infty}\phi_n'(0) = -\alpha$. Using also the convexity of ϕ_n, we have $\phi_n(t) - \phi_n(0) \geq t\phi_n'(0)$, or for n sufficiently large

$$\phi_n(t) \geq \theta - (\alpha+\varepsilon)t \geq \theta - \frac{\alpha+\varepsilon}{\alpha+2\varepsilon}\theta > 0$$

for $t \in [0, \frac{\theta}{\alpha+2\varepsilon})$. On the other hand, by direct computation we obtain

$$\phi_n''(t) = \frac{1}{(N-2)^2(S_n+t)^\lambda}(max\{\overline{u_n}(S_n+t), s_n\}^{-\nu} - \overline{h}) \tag{2.42}$$

If we prove that in (2.43) the term in brackets in the right-hand side is uniformly bounded, then $\phi_n''(t) \to 0$ uniformly for $t \in [0, \frac{\theta}{\alpha+2\varepsilon}]$ and (2.39) follows from the Lagrange theorem.
To prove that $max\{\overline{u_n}(S_n+t), s_n\}^{-\nu} - \overline{h}$ is uniformly bounded, it suffices to show that $\overline{u_n}(S_n+t) > C > 0$. But we know first that $\phi_n(t) \geq \frac{\varepsilon\theta}{\alpha+2\varepsilon} > 0$ for all $t \in [0, \frac{\theta}{\alpha+2\varepsilon}]$. Then we have:

$$\phi_n(t) = \overline{u_n}(S_n+t)(1 + \frac{1}{(N-2)^2\overline{u_n}(S_n+t)}\int_0^t\int_1^{S_n+l} \frac{f(\overline{u_n}(s))}{s^\lambda}dsdl) \tag{2.43}$$

and, since f is asimptotically sublinear and $\overline{u_n}(S_n+t) \to \infty$, we can write

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{\overline{u_n}(S_n+t)}\int_0^t\int_1^{S_n+l}\frac{f(\overline{u_n}(s))}{s^\lambda}dsdl &\leq \lim_{n\to\infty}\frac{1}{\overline{u_n}(S_n+t)}\int_0^t lf(\overline{u_n}(S_n+l))dl \\
&\leq \lim_{n\to\infty}\frac{t^2 f(\overline{u_n}(S_n+t))}{\overline{u_n}(S_n+t)} \\
&= 0
\end{aligned}$$

Hence there exists $C > 0$ such that $\overline{u_n}(S_n+t) > C > 0$ and (2.39) is proved.
To prove (2.40), suppose that there exists $\eta > 0$ such that $R_n \geq S_n + \frac{\theta}{\alpha} + 2\eta$. Since $\phi_n'(t) \to -\alpha$ uniformly on compact sets in $[0, \frac{\theta}{\alpha})$ and $\phi_n(0) = \theta$, it follows that $\phi_n(t) \to \theta - \alpha t$ uniformly on compact sets in $[0, \frac{\theta}{\alpha})$. On the other hand, since ϕ_n is decreasing and convex in $[0, \frac{\theta}{\alpha}+\eta)$, it follows that $\phi_n'(\frac{\theta}{\alpha}+\eta) \to 0$. Hence

$$\phi_n'(\frac{\theta}{\alpha}+\eta) - \phi_n'(\frac{\theta}{\alpha}-\eta) \to \alpha \tag{2.44}$$

as $n \to \infty$, and

$$\phi_n'(\frac{\theta}{\alpha}+3\eta) - \phi_n'(\frac{\theta}{\alpha}+\eta) \to 0 \tag{2.45}$$

as $n \to \infty$. On the other hand,

$$\phi_n^{'}(\frac{\theta}{\alpha}+\eta)-\phi_n^{'}(\frac{\theta}{\alpha}-\eta)=\overline{u_n}^{'}(S_n+\frac{\theta}{\alpha}+\eta)-\overline{u_n}^{'}(S_n+\frac{\theta}{\alpha}-\eta)+\frac{1}{(N-2)^2}\int\limits_{S_n+\frac{\theta}{\alpha}-\eta}^{S_n+\frac{\theta}{\alpha}+\eta}\frac{f(\overline{u_n}(s))}{s^{\lambda}}ds$$

$$=\int\limits_{\frac{\theta}{\alpha}-\eta}^{\frac{\theta}{\alpha}+\eta}(\overline{u_n}^{''}(S_n+s)+\frac{f(\overline{u_n}(S_n+s))}{(N-2)^2(S_n+s)^{\lambda}})ds$$

$$=\frac{1}{(N-2)^2}\int\limits_{\frac{\theta}{\alpha}-\eta}^{\frac{\theta}{\alpha}+\eta}\frac{max\{\overline{u_n}(S_n+s),s_n\}^{-\nu}}{(S_n+s)^{\lambda}}ds+o(1)$$

$$\leq\frac{1}{(N-2)^2}(1-\frac{2\eta}{S_n})^{-\lambda}\int\limits_{\frac{\theta}{\alpha}+\eta}^{\frac{\theta}{\alpha}+3\eta}\frac{max\{\overline{u_n}(S_n+t-2\eta),s_n\}^{-\nu}}{(S_n+t)^{\lambda}}dt$$

where the last inequality holds by a change of variable $s \to t-2\eta$. But $\overline{u_n}(S_n+t-2\eta) \geq \overline{u_n}(S_n+t)$, hence

$$\phi_n^{'}(\frac{\theta}{\alpha}+\eta)-\phi_n^{'}(\frac{\theta}{\alpha}-\eta)\leq C(\phi_n^{'}(\frac{\theta}{\alpha}+3\eta)-\phi_n^{'}(\frac{\theta}{\alpha}+\eta))\to 0$$

which is a contradiction with (2.45) and (2.46). It follows that $\lim\limits_{n\to\infty}(R_n-S_n)\leq\frac{\theta}{\alpha}$. But from the choices of R_n, S_n and from (2.39), we obtain that $\lim\limits_{n\to\infty}(R_n-S_n)\geq\frac{\theta}{\alpha+2\varepsilon}$, for all $\varepsilon>0$. We obtain (2.40).

Let $\eta>0$ and choose $\theta>0$ such that for any $\tau<\theta$ we have

$$(1-\eta)(max\{\tau,s_n\})^{-\nu}<(max\{\tau,s_n\})^{-\nu}-\overline{h}-f(\tau)<(1+\eta)(max\{\tau,s_n\})^{-\nu} \tag{2.46}$$

Since

$$\overline{u_n}^{''}(R_n+s)=\frac{1}{(N-2)^2(R_n+s)^{\lambda}}((max\{\overline{u_n}(R_n+s),s_n\})^{-\nu}-\overline{h}(t)-f(\overline{u_n}(R_n+s)))$$

we obtain from (2.47) for $s<0$ and $\overline{u_n}(R_n+s)<\theta$ that

$$\overline{u_n}^{''}(R_n+s)\geq\frac{1}{(N-2)^2R_n^{\lambda}}(1-\eta)(max\{\overline{u_n}(R_n+s),s_n\})^{-\nu} \tag{2.47}$$

In a similar way, for $s>0$ and $\overline{u_n}(R_n+s)<\theta$ we obtain

$$\overline{u_n}^{''}(R_n+s)\leq\frac{1}{(N-2)^2R_n^{\lambda}}(1+\eta)(max\{\overline{u_n}(R_n+s),s_n\})^{-\nu} \tag{2.48}$$

We multiply in both sides of (2.48) and by integration we obtain

$$\frac{1}{2}\overline{u_n}^{'}(R_n+s)^2\geq\frac{1}{(N-2)^2R_n^{\lambda}}(1-\eta)\int\limits_{\overline{u_n}(R_n)}^{\overline{u_n}(R_n+s)}(max\{\tau,s_n\})^{-\nu}d\tau$$

for all $s\in[-(R_n-S_n),0]$, or equivalently

$$\sqrt{1-\eta}\leq\frac{\overline{u_n}^{'}(R_n+s)}{\sqrt{\frac{2}{(N-2)^2R_n^{\lambda}}\int\limits_{\overline{u_n}(R_n)}^{\overline{u_n}(R_n+s)}(max\{\tau,s_n\})^{-\nu}d\tau}}$$

By integration again we find

$$\sqrt{1-\eta}(-s) \leq \int_{\overline{u_n}(R_n)}^{\overline{u_n}(R_n+s)} \Big(\frac{1}{(N-2)^2 R_n^\lambda} \int_{\overline{u_n}(R_n)}^{z} max\{\tau, s_n\}^{-\nu} d\tau\Big)^{-\frac{1}{2}} dz \tag{2.49}$$

Similarly, by multiplying in (2.49) and integrating we obtain

$$\overline{s}\sqrt{1+\eta} \geq \int_{\overline{u_n}(R_n)}^{\overline{u_n}(R_n+s)} \Big(\frac{1}{(N-2)^2 R_n^\lambda} \int_{\overline{u_n}(R_n)}^{z} max\{\tau, s_n\}^{-\nu} d\tau\Big)^{-\frac{1}{2}} dz \tag{2.50}$$

where $\overline{s} > 0$ is the point where $\overline{u_n}(R_n + s) = \overline{u_n}(R_n + \overline{s})$, for a fixed $s \in (-(R_n - S_n), 0)$. It follows that $\overline{s}\sqrt{1+\eta} \geq \sqrt{1-\eta}(-s)$ and from the monotonicity of $\overline{u_n}$ we have

$$\overline{u_n}(R_n + s) \geq \overline{u_n}(R_n - \mu s) \tag{2.51}$$

where we denoted $\mu := \sqrt{\frac{1-\eta}{1+\eta}}$.
Consider now the function

$$\psi_n(t) := \overline{u_n}(R_n + t) + \frac{1}{(N-2)^2} \int_0^t \int_{R_n}^{R_n+l} \frac{f(\overline{u_n}(s))}{s^\lambda} ds dl \tag{2.52}$$

which is convex and nondecreasing. Let b_n be such that $\overline{u_n}(R_n + b_n) = \theta$; b_n is unique since $\overline{u_n}$ is increasing on $[R_n, \infty)$. Then

$$\overline{u_n}(R_n + a) \leq \overline{u_n}(R_n - \frac{a}{\mu}) = \overline{u_n}(R_n - \frac{\theta}{2\alpha}) < \theta$$

hence $a < b_n$.
We have

$$\psi_n^{'}(+\infty) = \psi_n^{'}(b_n) + \int_{b_n}^{\infty} \psi_n^{''}(t) dt$$

hence

$$\psi_n^{'}(\infty) \geq \psi_n^{'}(a) + \frac{1}{(N-2)^2} \int_{b_n}^{\infty} \frac{max\{\overline{u_n}(R_n+\tau), s_n\}^{-\nu} - \overline{h}}{(R_n+\tau)^\lambda} d\tau = \psi_n^{'}(a) + o(1) \tag{2.53}$$

On the other hand,

$$\begin{aligned}
\psi_n^{'}(a) &= \frac{1}{(N-2)^2} \int_0^a \frac{max\{\overline{u_n}(R_n+\tau), s_n\}^{-\nu}}{(R_n+\tau)^\lambda} d\tau + o(1) \\
&= \frac{1}{(N-2)^2} \int_{-a}^0 \frac{max\{\overline{u_n}(R_n-\tau), s_n\}^{-\nu}}{(R_n-\tau)^\lambda} d\tau + o(1) \\
&\geq \frac{1}{(N-2)^2} \int_{-a}^0 \frac{max\{\overline{u_n}(R_n+\frac{\tau}{\mu}), s_n\}^{-\nu}}{(R_n-\tau)^\lambda} d\tau + o(1)
\end{aligned}$$

since $\overline{u_n}(R_n - \tau) \le \overline{u_n}(R_n + \frac{\tau}{\mu})$. We perform the change of variable $\tau \to \mu\tau$ and we obtain:

$$\begin{aligned}
\psi_n^{'}(a) &\ge \frac{1}{(N-2)^2} \int\limits_{-\frac{a}{\mu}}^{0} \frac{max\{\overline{u_n}(R_n+\tau), s_n\}^{-\nu}}{(R_n - \tau\mu)^\lambda} \mu d\tau + o(1) \\
&= \frac{1}{(N-2)^2} \int\limits_{-\frac{a}{\mu}}^{0} \frac{max\{\overline{u_n}(R_n+\tau), s_n\}^{-\nu}}{(R_n + \tau)^\lambda} \frac{(R_n+\tau)^\lambda}{(R_n - \tau\mu)^\lambda} \mu d\tau + o(1) \\
&\ge \frac{1}{(N-2)^2} (\frac{R_n - \frac{a}{\mu}}{R_n + a})^\lambda \mu \int\limits_{-\frac{a}{\mu}}^{0} \frac{max\{\overline{u_n}(R_n+\tau), s_n\}^{-\nu}}{(R_n + \tau)^\lambda} d\tau + o(1) \\
&= \mu (\frac{R_n - \frac{a}{\mu}}{R_n + a})^\lambda (\psi_n^{'}(-\frac{a}{\mu}) + o(1)) + o(1)
\end{aligned}$$

Since $-\frac{a}{\mu} \in (-(R_n - S_n), 0)$, it follows from (2.39) that $\lim\limits_{n\to\infty} \psi_n^{'}(-\frac{a}{\mu}) = -\alpha$. Hence $\psi_n^{'}(a) \ge \mu\alpha + o(1)$. On the other hand $\psi_n^{'}(+\infty) = \frac{1}{(N-2)\omega_N}\varepsilon_n \le \frac{1}{(N-2)\omega_N}\varepsilon_0$ or

$$\frac{\mu}{(N-2)\omega_N} \int\limits_B h dx = \mu\alpha \le \frac{1}{(N-2)\omega_N}\varepsilon_0 \tag{2.54}$$

But μ can be chosen arbitrarily close to 1, hence by choosing ε_0 small enough(for example $\varepsilon_0 < \frac{1}{2}\int\limits_B hdx$) we arrive to a contradiction □

Proposition 2. *(bound from below) There exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, there exists $s_0 > 0$, $\delta_\varepsilon > 0$ such that for any solution u of (2.30)-(2.31) for $s \in (0, s_0)$ and ε, one has $u(r) \ge \delta_\varepsilon$.*

Proof. We divide for simplicity this proof into three steps.
Step 1: As in proposition 1, for $N \ge 3$ suppose there exists a decreasing sequence $s_n \to 0$ and u_n solutions of (2.30)-(2.31) for $s = s_n$ such that $\inf\limits_B u_n \le o(1)$. Let $t = r^{2-N}$ as before; then $\overline{u_n}$ solves (2.38).
Consider, as in the proof of proposition 1, the function

$$\phi_n(t) := \overline{u_n}(t) + \int\limits_0^t \int\limits_1^l \frac{f(\overline{u_n}(s))}{s^\lambda} dsdl \tag{2.55}$$

and let R_n be a point such that $\phi_n(R_n) = \inf\limits_t \phi_n(t)$. It follows that

$$\phi_n^{'}(t) = \overline{u_n}^{'}(t) + \frac{1}{(N-2)^2} \int\limits_1^t \frac{f(\overline{u_n}(s))}{s^\lambda} ds = \frac{1}{(N-2)^2} \int\limits_1^t \frac{(max\{\overline{u_n}, s_n\}^{-\nu} - \overline{h})}{s^\lambda} ds \le k < \infty$$

hence $\phi_n^{'}(t)$ is uniformly bounded. Suppose that there exists a sequence $t_n \ge 1$ such that $\phi_n(t_n) \le$

$o(1)$. In this case $\overline{u_n}(t_n) \leq o(1)$. Fix a number $\delta > 0$. Then

$$\phi_n^{'}(\infty) - \phi_n^{'}(1) + \frac{1}{(N-2)^2}\int_1^\infty \frac{\overline{h}}{t^\lambda}dt \geq \frac{1}{(N-2)^2}\int_{t_n}^{t_n+\delta} \frac{max\{\overline{u_n}, s_n\}^{-\nu}}{t^\lambda}dt$$

$$\geq \frac{1}{(\delta+t_n)^\lambda(N-2)^2}\int_{t_n}^{t_n+\delta} max\{\overline{u_n}, s_n\}^{-\nu}dt$$

$$\geq \frac{1}{(\delta+t_n)^\lambda(N-2)^2}\int_{t_n}^{t_n+\delta} max\{\overline{u_n}(t_n)+k(t-t_n), s_n\}^{-\nu}dt$$

$$\geq \frac{1}{(\delta+t_n)^\lambda(N-2)^2}\frac{\delta}{max\{s_n, \overline{u_n}(t_n)+k\delta\}^{-\nu}}$$

for some $k > 0$, where the third inequality holds since $\overline{u_n}(t_1)-\overline{u_n}(t_2) \leq k(t_1-t_2)$ for t_1, t_2 sufficiently close. It follows that

$$\liminf_{n\to\infty}(t_n+\delta)^\lambda \geq \frac{C}{\delta^{\nu-1}} \tag{2.56}$$

for some $C > 0$. Since δ is arbitrary, it follows that $t_n \to \infty$. In particular, $u_n(1)$ and $\phi_n(1)$ remain bounded away from 0. Using this remark, we can define the numbers

$$S_n(\theta) := \sup\{R > 1 : \phi_n(t) \geq \theta, \quad \forall t \in [1, R]\} \tag{2.57}$$

A similar argument as in the last calculation shows that

$$C \geq \int_1^{S_n(\theta)} \frac{max\{\overline{u_n}, s_n\}^{-\nu}}{t^\lambda}dt = \int_1^{S_n(\theta)} \frac{dt}{\overline{u_n}^\nu t^\lambda} \geq \frac{1}{S_n(\theta)^\lambda}\int_1^{S_n(\theta)} \frac{dt}{(\theta+k(S_n(\theta)-t))^\nu}$$

hence

$$S_n(\theta) \geq \frac{C}{\theta^{\frac{\nu-1}{\lambda}}} \tag{2.58}$$

for all sufficiently small θ and for sufficiently large n.

Step 2: we prove the following estimate: given a number $z > 0$, we have for sufficiently small θ:

$$\limsup_{n\to\infty}|\phi_n^{'}(S_n(\theta))| < z \tag{2.59}$$

Fix a number $\theta_0 > 0$. By construction, ϕ_n is nonincreasing and convex on $[S_n(\theta_0), R_n]$. We have two cases:

Case 1: If $S_n(\theta_0) \to \infty$, we prove that $\phi_n^{'}(S_n(\theta_0)) \to 0$. If not, there exists $\alpha > 0$ such that $-\phi_n^{'}(S_n(\theta_0)) \geq \alpha > 0$. Set $v_n(t) := \phi_n(S_n(\theta_0) - t)$. Then $v_n(0) = \theta_0$ and by direct differentiation we obtain that

$$v_n^{''}(t) = \frac{1}{(N-2)^2(S_n(\theta_0)-t)^\lambda}(max\{\overline{u_n}, s_n\}^{-\nu} - \overline{h}(S_n(\theta_0)-t))$$

which converges to 0 uniformly on compact subsets in $[0, \infty)$. Hence $v_n(s) \geq \theta_0 + \alpha s + o(1)s^2$. We deduce that v_n and, consequently, ϕ_n, take arbitrarily large values. But this contradicts the results of proposition 1.

Case 2: If $S_n(\theta_0) \leq S_0 < \infty$, then ϕ_n is nonincreasing and convex on $(S_n(\theta_0), S_n(\theta))$ for some $\theta < \theta_0$. Hence, by convexity,

$$\phi_n(S_n(\theta_0)) - \phi_n(S_n(\theta)) \geq \phi_n^{'}(S_n(\theta))(S_n(\theta_0) - S_n(\theta))$$

or equivalently

$$\theta_0 - \theta \geq -\phi_n^{'}(S_n(\theta) - S_n(\theta_0))$$

It follows that, using (2.59)

$$-\phi_n^{'}(S_n(\theta)) \leq \frac{\theta_0 - \theta}{S_n(\theta) - S_n(\theta_0)} \leq \frac{\theta_0}{C\theta^{\frac{1-\nu}{\lambda}} + S_0} \leq z$$

for θ_0 small enough. The estimate (2.60) is proved.
Step 3: fix as before the numbers z, $\eta > 0$ and $\theta > 0$ such that (2.60) holds and for all $\tau < \theta$ and n large

$$(1-\eta)(max\{\tau, s_n\})^{-\nu} \leq (max\{\tau, s_n\})^{-\nu} + \overline{h}(t) - f(\tau) \leq (1+\eta)(max\{\tau, s_n\})^{-\nu} \tag{2.60}$$

Set also

$$T_n(\theta) := \inf\{R > R_n : \overline{u_n}(R) = \theta\} \tag{2.61}$$

By an integration similar to that leading to (2.50) and (2.51) in proposition 1, we arrive to the inequality

$$|\overline{u_n}^{'}(T_n(\theta))| \leq (\frac{1+\eta}{1-\eta})^{\frac{1}{2}} |\overline{u_n}^{'}(S_n(\theta))| \tag{2.62}$$

On the other hand, we can choose the parameter θ so small(for η fixed) such that

$$(\frac{1+\eta}{1-\eta})^{\frac{1}{2}} \int_1^{S_n(\theta)} \frac{f(\overline{u_n}(s))}{s^\lambda} ds \geq \int_1^{T_n(\theta)} \frac{f(\overline{u_n}(s))}{s^\lambda} ds$$

hence, by adding the two inequalities, we arrive to

$$(\frac{1+\eta}{1-\eta})^{\frac{1}{2}} \phi_n^{'}(S_n(\theta)) \geq \phi_n^{'}(T_n(\theta)) \tag{2.63}$$

On the other hand, we have:

$$\phi_n^{'}(+\infty) = \phi_n^{'}(T_n(\theta)) + \frac{1}{(N-2)^2} \int_{T_n(\theta)}^{\infty} (\overline{u_n}^{-\nu} - \overline{h}) \frac{1}{t^\lambda} dt = \phi_n^{'}(T_n(\theta)) + o(1)$$

But using the second step of this proof, we remark that $|\phi_n^{'}(S_n(\theta))| < z$, hence $|\phi_n^{'}(T_n(\theta))| < (\frac{1+\eta}{1-\eta})^{\frac{1}{2}} z$. It follows that

$$0 < \phi_n^{'}(\infty) < (\frac{1+\eta}{1-\eta})^{\frac{1}{2}} z \tag{2.64}$$

which holds for every $z > 0$(since (2.60) is proved for any z). But this is a contradiction for z sufficiently small, which proves the proposition. $\square$

2.5 The Main Existence Result

In this section we prove existence results, first for the perturbed problem (2.30)-(2.31). This is a necessary step in order to prove existence for our starting problem (2.1)-(2.2). Here is the place where we will use the variational technique of Rabinowitz described in section 2.3.

Theorem 8. *There exists a constant $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists at least one (radial) solution of the problem*

$$-\Delta u + u^{-\nu} = h + \varepsilon\delta_0 + f(u), \quad in \quad B \tag{2.65}$$

$$\frac{\partial u}{\partial n} = 0 \quad on \quad \partial B \tag{2.66}$$

Proof. By proposition 3, it is enough to prove that for any $s > 0$ small, (2.30)-(2.31) has a solution. Consider $\phi \in C^2(\overline{B} \setminus \{0\})$ radially symmetric with $\phi^{'}(1) = 0$ and $\psi := \phi - \Phi_N$, which is a function in $C^2(\overline{B})$ and $\Delta\psi \geq 0$ in $\overline{B}$. By changing u in $v + \varepsilon\phi$, the problem (2.30)-(2.31) is equivalent to

$$-\Delta v + max\{\varepsilon\phi(r) + v, s\}^{-\nu} = h_\varepsilon + f(\varepsilon\phi(r) + v) \quad in \quad B \tag{2.67}$$

$$\frac{\partial v}{\partial n} = 0 \quad on \quad \partial B \tag{2.68}$$

where $h_\varepsilon := h + \varepsilon\Delta\psi$.
We define the energy functional $J_s : H_r^1(B) \to \mathbb{R}$ by

$$J_s(v) := \frac{1}{2}\int\limits_B |\nabla v|^2 dx + \int\limits_B F_s(\varepsilon\phi + v)dx - \int\limits_B h_e v dx - \int\limits_B F(\varepsilon\phi + v)dx \tag{2.69}$$

where $F_s(z) := \int\limits_0^z (max\{\tau, s\})^{-\nu} d\tau$ and we extend f by 0 for $x < 0$. Here the notation $H_r^1(B)$ indicates the Sobolev space of radial functions on the unit ball.
The idea is to prove the Palais-Smale condition for J_s. Let $(v_n)_n$ be a Palais-Smale sequence for J_s, i.e. $(J_s(v_n))_n$ is bounded and $J_s^{'}(v_n) \to 0$. It suffices to prove that $(v_n)_n$ is bounded in L^1 norm.
The condition $J_s^{'}(v_n) \to 0$ in $H^1(B)$ implies

$$\int\limits_B |\nabla v_n|^2 dx + \int\limits_B max\{\varepsilon\phi + v_n, s\}^{-\nu} v_n dx = \int\limits_B h_\varepsilon v_n dx + \int\limits_B f(\varepsilon\phi + v_n)v_n dx + o(1)\|v_n\|_{H^1(B)} \tag{2.70}$$

In order to apply the Poincare inequality we need to decompose $v_n := w_n + \alpha_n$ where $\alpha_n := \int\limits_B v_n dx$ and $\int\limits_B w_n dx = 0$. We remark that, if the sequence $(v_n)_n$ is not bounded in $H^1(B)$, then, by extracting a suitable subsequence, $f(\varepsilon\phi + v_n) = o(1)v_n$, from the condition of asymptotic sublinearity.
We assume that $(v_n)_n$ is not bounded in $H^1(B)$. By considering the decomposition $v_n := w_n + \alpha_n$ and using also the previous remark about the behaviour of the f part in (2.71), we can separate the part with w_n and we derive:

$$\int\limits_B |\nabla w_n|^2 = \int\limits_B (h_\varepsilon - max\{\varepsilon\phi + v_n, s\}^{-\nu})w_n + o(1)\|w_n\|_{H^1(B)} + o(1)\|w_n\|^2_{H^1(B)} \tag{2.71}$$

Since $\int\limits_B w_n = 0$ we can use the Poincare inequality. We obtain that

$$(1 - o(1))\|w_n\|^2_{H^1(B)} \leq (C\|h\|_\infty + o(1))\|w_n\|_{H^1(B)}$$

hence $(w_n)_n$ is bounded in $H^1(B)$. It is still possible that $(\alpha_n)_n$ to be unbounded. In this case, on a subsequence, we have $\alpha_n \to \infty$ or $\alpha_n \to -\infty$. Since $J_s^{'}(v_n) \to 0$ and by separating the part with $(\alpha_n)_n$, one has

$$\int\limits_B (max\{\varepsilon\phi + v_n, s\}^{-\nu} - h_\varepsilon)dx = o(1) + \int\limits_B f(\varepsilon\phi + \alpha_n + w_n)dx \tag{2.72}$$

If $\alpha_n \to \infty$ on a subsequence, the boundedness of $(w_n)_n$ already proved implies that $\varepsilon\phi + v_n \to \infty$, hence the equality (2.73) reads

$$o(1) = \int_B h_\varepsilon dx + o(1) + \int_B f(\varepsilon\phi + \alpha_n + w_n)dx$$

which is a contradiction, since the right-hand side goes to a positive value as $n \to \infty$. If $\alpha_n \to -\infty$, then by boundedness of $(w_n)_n$ we have that $\varepsilon\phi + \alpha_n + w_n < 0$ for sufficiently large n, hence $f(\varepsilon\phi + v_n) = 0$. By taking limits as $n \to \infty$ in both sides of (2.73) we find

$$s^{-\nu} = \int_B h_\varepsilon dx + o(1) \tag{2.73}$$

which is a contradiction for a small $s > 0$.
Hence J_s satisfies the global Palais-Smale condition for $s \in (0, s_0)$ with $s_0 > 0$ small.
We consider as before the direct sum

$$H_r^1(B) = W \oplus Z$$

where $W := \{w \in H_r^1(B) : \int_B w dx = 0\}$ and Z is the space of constant functions. We compute

$$J_s(\alpha) = \int_B F_s(\varepsilon\phi + \alpha)dx - \alpha \int_B h_\varepsilon dx - \int_B F(\varepsilon\phi + \alpha)dx$$

hence

$$\frac{1}{\alpha} J_s(\alpha) = \int_B \frac{F_s(\varepsilon\phi + \alpha)}{\alpha} dx - \int_B h_\varepsilon dx - \int_B \frac{F(\varepsilon\phi + \alpha)}{\alpha} dx < 0$$

(in fact if f is not constant, the limit of $\frac{1}{\alpha} J_s(\alpha)$ is $-\infty$ as $\alpha \to \infty$), hence $\lim_{\alpha\to\infty} J_s(\alpha) = -\infty$. Hence on the 1-dimensional space Z the functional J_s is not bounded below. On the other hand,

$$\begin{aligned} J_s(w) &= \frac{1}{2}\int_B |\nabla w|^2 + \int_B F_s(\varepsilon\phi + w)dx - \int_B h_\varepsilon w dx - \int_B F(\varepsilon\phi + w)dx \\ &\geq \frac{1}{2}\int_B |\nabla w|^2 - C(s, \|h_\varepsilon\|_\infty)(\int_B w^2 dx)^{\frac{1}{2}} - \int_B F(\varepsilon\phi + w)dx \end{aligned}$$

But by the Poincare inequality, $\frac{1}{2}\int_B |\nabla w|^2 \geq c \int_B w^2 dx$ and, since f is asimptotically sublinear, the term with F has a growth weaker then w^2. It follows that J_s is bounded below on W.
We apply the saddle-point theorem of Rabinowitz(see theorem 7 and the remark after) and we conclude that for any $s > 0$ sufficiently small, the functional J_s has a critical point, which is a solution of problem (2.68)-(2.69). □

We end this section with the main existence result:

Theorem 9. *The problem (2.1)-(2.2) has at least a radial solution provided that* $\int_B h dx > 0$.

Proof. We know from theorem 8 that there exists a solution $u_\varepsilon > 0$ to the problem (2.68)-(2.69), for any $\varepsilon \in (0, \int_B h dx)$. Set as before $v_\varepsilon := u_\varepsilon - \varepsilon\Phi_N$, where Φ_N is the fundamental solution for the Laplacian. Then v_ε satisfies

$$\Delta v_\varepsilon = u_\varepsilon^{-\nu} - h - f(u_\varepsilon) \tag{2.74}$$

and v_ε is bounded uniformly for small $\varepsilon > 0$, as proposition 1 shows. We have two cases:
Case 1: Suppose that

$$\inf_B u_\varepsilon \geq c > 0 \tag{2.75}$$

Then $u_\varepsilon^{-\nu}$ is uniformly bounded, and by classical elliptic estimates(see [GT02], chapter 9) we may assume, passing to a subsequence if necessary, that $v_\varepsilon \to v$ in $C^1(\overline{B})$ as $\varepsilon \to 0$. Then $v > 0$ and v is a solution of (2.1)-(2.2).
Case 2: Assume that, by passing to a subsequence, we have

$$\inf_B u_\varepsilon \to 0 \tag{2.76}$$

as $\varepsilon \to 0$. Let δ_ε be such that

$$u_\varepsilon(\delta_\varepsilon) = \inf_B u_\varepsilon$$

and

$$\eta_\varepsilon := \inf\{0 < \alpha < 1 : u_\varepsilon(r) \geq \theta, \quad \forall \alpha \leq r \leq 1\}$$

From the arguments in the proof of proposition 1, we derive that $\lim_{\varepsilon\to 0} \delta_\varepsilon = 0$ and η_ε is well-defined for all small $\theta > 0$. Fix such a number $\theta > 0$. By the general results of section 2.2, u_ε is increasing on $(\delta_\varepsilon, \eta_\varepsilon)$.
We prove next the following inequality:

$$u_\varepsilon(r) \geq C(r - \delta_\varepsilon)^{\frac{2}{\nu+1}}, \quad \forall r \in [\delta_\varepsilon, 1] \tag{2.77}$$

for some $C > 0$.
By multiplying in both sides of the radial form of (2.68) by r^{N-1} and integrating we obtain

$$r^{N-1}u_\varepsilon^{'}(r) = \int_{\delta_\varepsilon}^{r} (\frac{1}{u_\varepsilon(s)^\nu} - h(s) - f(u_\varepsilon(s)))s^{N-1}ds \tag{2.78}$$

Since $f(0) = 0$ and f is continuous, we may choose $\theta > 0$ so small that

$$\frac{1}{\tau^\nu} - h(s) - f(\tau) \geq \frac{1}{2\tau^\nu}$$

for all $s > 0$ and $0 < \tau < \theta$. Then, from (2.78) we have for all $r \in [\delta_\varepsilon, \eta_\varepsilon]$

$$r^{N-1}u_\varepsilon^{'}(r) \geq \int_{\delta_\varepsilon}^{r} \frac{s^{N-1}}{2u_\varepsilon(s)^\nu}ds = \frac{1}{2Nu_\varepsilon(r)^\nu}(r^N - \delta_\varepsilon^N)$$

and by integrating again

$$u_\varepsilon(r)^{\nu+1} - u^\varepsilon(\delta_\varepsilon)^{\nu+1} \geq \frac{\nu+1}{2N}\int_{\delta_\varepsilon}^{r}(r - \delta_\varepsilon)dr \tag{2.79}$$

hence we have (2.77) for $r \in (\delta_\varepsilon, \eta_\varepsilon)$. Since for $r \in [\eta_\varepsilon, 1]$ we have $u_\varepsilon(r) \geq \theta$, (2.77) follows. In particular, $u_\varepsilon^{-\nu}$ remains bounded on each set $A_\alpha := \{x : \alpha < |x| < 1\}$. Since

$$\Delta u_\varepsilon = u_\varepsilon^{-\nu} - h - f(u_\varepsilon)$$

in A_α, by classical elliptic estimates we conclude the existence of a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ converging uniformly on compact sets of $\overline{B} \setminus \{0\}$ to a function $u \in C^2(\overline{B} \setminus \{0\})$ and u satisfies

$$\Delta u = u^{-\nu} - h - f(u), \quad in \quad \overline{B} \setminus \{0\} \tag{2.80}$$

$$\frac{\partial u}{\partial n} = 0 \quad on \quad \partial B, \quad u > 0 \quad in \quad \overline{B} \setminus \{0\} \tag{2.81}$$

and $u \in L^\infty(B)$ by proposition 1. Since $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$, by (2.77) we have

$$u(r) \geq cr^{\frac{2}{\nu+1}}, \quad \forall r \in (0, 1] \tag{2.82}$$

hence $u(r)^{-\nu} \leq cr^{1\frac{2\nu}{\nu+1}}$, which is an L^p function for p very large. We choose such a $p > \frac{N}{2}$ and it follows that $u^{-\nu} - h - f(u) \in L^p(B)$.
Consider now v the solution of

$$\Delta v = u^{-\nu} - h - f(u) \quad in \quad B \tag{2.83}$$

$$v = u \quad on \quad \partial B \tag{2.84}$$

Then by elliptic estimates and Sobolev inequalities $v \in W^{2,p}(B) \subset C(\overline{B})$. It follows that $w := v - u$ is bounded and it satisfies

$$\Delta w = 0 \quad in \quad \overline{B} \setminus \{0\}$$

$$w = 0 \quad on \quad \partial B$$

From standard harmonic function theory it follows that the singularity in 0 is removable and $w \equiv 0$. Hence u can be extended in 0 and becomes a solution of (2.1)-(2.2). □

2.6 Open Questions and Final Comments

The study of singular problems with Neumann boundary conditions appears to be a very difficult problem today. For example, in this chapter we have proved a very particular result and we have spent a lot of technical effort in doing this. Up to our knowledge there are very few papers written on this type of problem. That's why, by contradiction to the case of the Dirichlet boundary condition, which has been intensively studied in the last decades, the investigation for singular problems with Neumann boundary condition is very far from a huge development. That's why there are a lot of open questions.
The main difficulty concerning the case of the Neumann problem is that many of the main tools in elliptic equations doesn't hold. For example the method of sub- and supersolutions is not valid in this case, as the following simple example shows:
Consider the problem:

$$-\Delta u = f(u) \quad in \quad \Omega \tag{2.85}$$

$$\frac{\partial u}{\partial n} = g(x) \quad on \quad \partial\Omega \tag{2.86}$$

where Ω is a smooth, bounded domain in $\mathbb{R}^N$ and $f : \mathbb{R} \to \mathbb{R}$ is an increasing function. Consider $\underline{u}$ a subsolution and $\overline{u}$ a supersolution of (2.85)-(2.86)(where the adaptation of definition of sub- and supersolution for Neumann problems is obvious). Assume that $\overline{u} \geq \underline{u}$ and define $u := \overline{u} - \underline{u}$. Then

$$-\Delta u \geq f(\overline{u}) - f(\underline{u}) \geq 0 \quad in \quad \Omega$$

and $u \geq 0$ in Ω with $\frac{\partial u}{\partial n} \geq 0$ on $\partial\Omega$. By the classical maximum principle, we have either $u \equiv 0$ in Ω or $u > 0$ in Ω and $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$. But the second case is not possible, hence $u \equiv 0$ in Ω, which implies $\overline{u} \equiv \underline{u}$, which is contradictory. Hence the method of sub- and supersolutions does not hold always when dealing with Neumann boundary condition. Also, up to our knowledge, there are no works trying to apply a topological degree technique.

We will propose here several open problems:

Open problem 1 What happens if we consider for example $f(u) = u^p$ with $p > 1$, i.e for the superlinear case? But if we consider a function f such that $\lim_{t\to\infty} \frac{f(t)}{t} = a \in (0, \infty)$? The previous method does not seem to be applicable for these cases. There exists two papers treating these problems in the 1-dimensional case, which is much simpler, but which can be taken as a possible starting point for further investigation: these are [DPMM92] for the case of an asymptotically linear f and [DPM93] for a superlinear function.

Open problem 2 What if we introduce a gradient term in the equation? There were many interesting results in the study of nonlinear problems with gradient terms in the last years, but using Dirichlet conditions or mixed(Robyn) conditions. It will be interesting to develop a technique to treat singular Neumann problems with gradient terms, which up to now is not done.

Open problem 3 Our result presented here is very particular, since it holds only on balls and with a radial function h. It is an open problem to study this on a general domain $\Omega \subset \mathbb{R}^N$, where the ODE techniques doesn't work. An idea could be that of using the results obtained for balls and to use a homotopy between Ω and a ball inside Ω, together with a degree setting. The main difficulty is that in absence of a boundary condition for u itself, it is hard to show that the triples we consider are admissible for the degree.

Open problem 4 Is it true that any solution of a problem like (2.1)-(2.2) on a ball must be radially symmetric? Very recently it has been proved(see chapter 3) that any blow-up solution of a general semilinear elliptic equation on a ball is symmetric. But for problems with singular terms or with Neumann condition on the boundary this is still unknown.

There are many other open questions concerning for example uniqueness(in [DPH96] it is proved a uniqueness result for the case $f = 0$, but only for radial solutions) or asymptotic behaviour of solutions near the boundary or near the singularity point.

References

1.[AR73]-A.Ambrosetti, P.H.Rabinowitz-Dual variational methods in critical point theory and applications, J. Funct. Analysis, no. 14(1973), 349-381;

2.[BN91]-H. Brezis, L.Nirenberg-Remarks on finding critical points. Comm. Pure Appl. Math. 44(1991), no 8-9, 939-963;

3.[DPH96]-M.del Pino, G. Hernandez-Solvability of the Neumann problem in a ball for $-\Delta u + u^{-\nu} = h(|x|), \ \ \nu > 1$, Jour. of Diff. Equations 124(1996), 108-131;

4.[DPM93]-M.del Pino, R.Manasevich-Infinitely many T-periodic solutions for a problem arising in nonlinear elasticity, Jour. of Diff. Equations 103(1993), 260-277;

5.[DPMM92]-M.del Pino, R.Manasevich, A.Montero-T-periodic solutions for some second order differential equations with singularities, Proc. Roy. Soc. Edinburgh sect. A 120(1992), 231-243;

6.[Ek74]-I.Ekeland-On the variational principle, J. Math. Anal. Appl. 47(1974), 324-353;

7.[GP89]-N.Ghoussoub, D.Preiss-A general mountain pass principle for locating and classifying crit-

ical points, Ann. Inst. Poincare Ann. Non Lineaire, 6(1989), no.5, 321-330;
8.[GT02]-D.Gilbarg, N.Trudinger-Elliptic partial differential equations of second order, Springer-Verlag, Berlin, 2002;
9.[Rab86]-P.H.Rabinowitz-Minimax methods in critical point theory with applications to differential equations, CBMS Regional Conference Series in Mathematics, vol 65, AMS, Providence, 1986;
10.[Rad]-V.Radulescu-Treatment methods for nonlinear elliptic equations, Lecture Notes, University of Craiova, 2005;

Chapter 3

Blow-up Solutions for Nonlinear Elliptic Equations

3.1 Introduction and Hystorical Facts

In this chapter we develop the theory of existence, uniqueness, asymptotic analysis and qualitative theory of blow-up solutions for semilinear elliptic partial differential equations. This, together with the similar problem posed in the parabolic case, are very active fields of research in present and there are a lot of recent good result in this area. What we intent to do here is to realise a survey of the developpment of this problem, in order to show how the theory was constructed since 1956.
We deal mostly with the following general problem:

$$\Delta u = f(u) \quad in \quad \Omega \tag{3.1}$$

$$u(x) \to \infty \quad as \quad x \to \partial\Omega \tag{3.2}$$

where all the time Ω is a bounded and sufficiently regular domain. We will call, as usual, the function f the nonlinearity of the problem. In the last part we will study also other type of problems with blow-up, to see how the theory could be adapted in different cases.
Hystorically speaking, the discussion on these problems started in '50, with the papers of J.B. Keller(see [Ke56] and [Ke57]) and the paper of R.Osserman([Os57]). In the first of them, Keller showed how this problem arises from natural(physical) phenomena, more specifically an electrohydrodynamic model, which will be presented in the next section. The work of Keller is continued by himself with a mathematical paper in the next year, where he give a condition for existence of these solutions. He could do this at that time only for monotone increasing nonlinearities. Independently, R. Osserman obtained in the same year a similar result. After several years, the problem had been found as interesting also in geometry, by Loewner and Nirenberg([LN74]), which have found the nonlinearity $f(u) = u^{\frac{N+2}{N-2}}$ from a study of invariance of some PDEs under conformal and projective mapping.
The problem regained its attraction for researchers many years ago, when the developpment of functional analysis, nonlinear analysis and PDE provided new techniques for finding important results.

In this line we remark the most important ones, which are on one side the study of multiplicity of these solutions and the existence of different solutions(which has been done by McKenna,Reichel and their collaborators in a series of papers, see [LMK94], [MKRW97] si [AfR97]) and on the other side the elimination of the monotonicity condition on the nonlinearity(done first in [AfR97]).
A different kind of problem was the study of asymptotic behaviour of these solutions. This research was initiated by C. Bandle and M. Marcus in two papers([BM92] and [BM95]) under the hypothesis of increasingness of f. Much more recently, the problem has been studied in a very general context(see for example [DDGR06]). Also, the existence of boundary blow-up solutions for other types of PDEs was studied recently(by example for the logistic equation, see a series of papers of V. Radulescu and F. Carstea). Also recently(see [RaC06]) the same authors used an approach based on the theory of regular variation of Karamata.
The structure of this chapter is as follows: in the next section we present in detail the physical model that gave birth to the interest on these phenomena. Then we will study what existence and uniqueness results can be obtained by supposing that f is increasing. In section 3.4 we state and prove results of multiplicity of blow-up solutions, followed in section 3.5 by the presentation of a very new symmetry result for solutions. The next section is dedicated to the more recent developpment of the theory in absence of the monotonicity of f. Here we introduce several Keller-Osserman type conditions and we study their connection with existence of blow-up solutions. In the next section we are concerned with uniqueness results and boundary blow-up rates. Finally, we end this chapter with a section devoted for comments and open problems.

3.2 Physical Motivation: a Model in Electrohydrodynamics

In this section we present a model that can be considerated one of the most important applications and practically started the modern research of the problem. This was introduced by Keller in [Ke56]. By electrohydrodynamics we understand the study of the motion of a fluid under the influence of an electric field, by example in vacuum tubes(where the fluid is an electron gas) or in plasmas. In our particular case we consider a uniformly charged gas of mass M in a container Ω. We want to study the equilibrium of this gas. This state is achieved when the pressure force and the electrostatic forces are balanced by each other. This equality leads to the equilibrium law:

$$\nabla p = a\rho E \tag{3.3}$$

where p is the pressure, ρ is the density and a_ρ the charge density, E the electric vector field and $a := \frac{a_\rho}{\rho}$, which is a constant, since the gas is supposed uniformly charged.
On the other hand, the electric field is given by the charge, hence

$$div(E) = 4\pi a\rho \tag{3.4}$$

If the container surface $\partial\Omega$ is a perfect conductor, then E has no tangential component on this surface., hence ∇p is normal to the surface. From classical mechanics we know that any fluid is characterized by a state equation which writes: $p = p(\rho, T)$, where T is the temperature of the fluid. In our case, we assume that the temperature is constant, hence the state equation is:

$$p = p(\rho) \tag{3.5}$$

Since this function is an increasing one of variable ρ (which is obvious from physical reasons, i.e when the density increases, of course the pressure should increase, since the container remains unchanged) and goes to ∞ if also ρ does so, we may reverse the roles and express the density as a function of pressure: $\rho = \rho(p)$.
We eliminate from (3.3) and (3.4) the electric field E: $div(E) = \frac{1}{a} div(\rho^{-1}\nabla p)$, hence

$$div(\frac{1}{\rho}\nabla p) = 4\pi a^2 \rho \tag{3.6}$$

which, by the previous considerations, is an equations of variable p or ρ. But for our goal it is more convenient to introduce a new variable:

$$v := \int_{p_0}^{p} \frac{dp}{\rho(p)} \tag{3.7}$$

where p_0 is the initial pressure when $\rho = 0$. In this notation, the equation (3.6) becomes

$$\Delta v = 4\pi a^2 \rho(p(v)) \tag{3.8}$$

and the function in the right-hand side is an increasing function in v. In this way we obtain the equation (3.1). On the other hand, since ∇p is normal on $\partial\Omega$, it follows that the pressure(hence ρ and v) is constant on $\partial\Omega$. Let α be this constant value for v. Hence from the classical theory there exists a unique solution v of (3.8) with boundary condition $v = \alpha$, and from the classical maximum principle this solution increases with α. Also, $\Delta v = f(v) \geq 0$, hence v is subharmonic, and $v \leq \alpha$ inside the container. We obviously have

$$M = \int_{\Omega} \rho(s) ds \tag{3.9}$$

and one can prove that for any given mass M, there exists exactly one value of α such that (3.9) holds(for the proof see the appendix of [Ke56]).
As an example we consider the case of an ideal gas, whose state equation is

$$p = \frac{RT}{m}\rho \tag{3.10}$$

where R is the Rayleigh constant and m is the average mass of the molecules in the gas. By a simple computation, it follows that $v = \frac{RT}{m} log(p)$ and $f(v) = \frac{4\pi a^2 m}{RT} e^{\frac{m}{RT}v}$. We introduce in (3.8) the new variable $u := \frac{m}{RT}v + log(4\pi\frac{(am)^2}{(RT)^2})$ and we deduce that

$$\Delta u = e^u \tag{3.11}$$

This equation was introduced in this particular case of the ideal gas by Max von Laue(see [ML18]), who deduced using statistical mechanics that in equilibrium the density at any point is proportional to an exponential function(which has been described explicitely in terms of the electrostatic potential).
We will prove in the next section that if the condition

$$\int_{p(1)}^{\infty} \frac{1}{\rho(p)\sqrt{p - p_0}} dp < \infty \tag{3.12}$$

is satisfied, then there exists a bound function $g(R)$ such that for any point P in the container(i.e. $P \in \Omega$ in mathematical terms) we have

$$v(P) \leq g(R(P)) \tag{3.13}$$

where $R(P) = d(P, \partial\Omega)$. The condition (3.12) was deduced also by R. Osserman in [Os57] and it is called today the Keller-Osserman((KO) for short) condition.

From (3.13) we deduce that for all $P \in \Omega$, v, ρ and p tends to finite value in P as $M \to \infty$, for all gases satisfying (3.12). Also from (3.12), ρ goes to ∞ as P closes the surface of the container. Thus, we obtain the physical conclusion: as the mass of the gas increases to infinity, the density remain bounded at any interior point and becomes infinite on the surface. Hence most of the gas accumulates in a thin layer near the surface as the mass of the gas increases.

From the mathematical point of view, we arrive to the boundary condition (3.2) for the variable ρ or for v.

3.3 Existence and Uniqueness for Nondecreasing Nonlinearity

From now on we will renounce at the physical notation as in section 3.2. We consider a bounded domain $\Omega \subset \mathbb{R}^N$ with sufficiently smooth boundary. The idea is first to study some qualitative properties of the solution of (3.1) without any boundary condition and to get a uniform upper bound.

We start by supposing that f is a nonnegative real continuous function which is also nondecreasing and satisfies the Keller-Osserman condition

$$\int_0^\infty \frac{dt}{\sqrt{F(t)}} < +\infty \tag{3.14}$$

where $F(t) := \int_0^t f(s)ds$. The main result is:

Theorem 10. *(Keller, [Ke57]) Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be an arbitrary solution of (3.1) with f as above. Then there exists a decreasing function g such that $g(x) = g(d(x))$ for any $x \in \Omega$, where $d(x) := d(x, \partial\Omega)$, such that the following inequality holds:*

$$u(x) \leq g(d(x)), \quad \forall x \in \Omega \tag{3.15}$$

Moreover, g has the asymptotic properties:

$$\lim_{t\to 0} g(t) = \infty, \quad \lim_{t\to\infty} g(t) = -\infty \tag{3.16}$$

Proof. Since for every point $x \in \Omega$ there exists a ball $B(x, R) \subset \Omega$, it suffices to prove the theorem for $\Omega = B(0, R)$. Fix a solution u of (3.1) on this ball and fix a constant $\theta \in (0, 1)$. Consider first the Dirichlet problem:

$$\Delta v = h(v) \ \ in \ \ \Omega \tag{3.17}$$

$$v = \alpha \quad on \quad \partial\Omega \tag{3.18}$$

where $\alpha \geq \sup_{x \in \partial\Omega} u(x)$ and $h(v) := \theta f(v)$.

Then, since h is nondecreasing, there exists a unique solution of (3.17)-(3.18). Indeed, uniqueness follows easily from a standard comparison argument: if u_1 and u_2 are two solutions and we suppose that the sets $\Omega_1 := \{x \in \Omega : u_1(x) < u_2(x)\}$ and $\Omega_2 := \{x \in \Omega : u_2(x) < u_1(x)\}$ are both nonempty, then on $\partial\Omega_1$ and $\partial\Omega_2$ we have $u_1 = u_2$. On the other hand, we obtain:

$$\Delta(u_1 - u_2) = \theta(f(u_1) - f(u_2)) \tag{3.19}$$

But in Ω_1, the right-hand side is nonpositive and from the classical maximum principle it follows that $u_1(x) - u_2(x) \geq 0$ in Ω_1, which is a contradiction. Hence $\Omega_1 = \varnothing$ and similarly $\Omega_2 = \varnothing$, which means that $u_1 \equiv u_2$. Existence follows also from a standard sub- and supersolutions argument, which we omit here.

Let v_α be the solution of (3.17)-(3.18). From the maximum principle it follows that v_α is a nondecreasing function of α and we can define

$$g(x) := \lim_{\alpha \to \infty} v_\alpha(x) \tag{3.20}$$

By the theorem of Gidas-Ni-Nirenberg(see [GNN79]) it follows that v_α is a radial function and g is a radial function. We want to show that this g is the desired function. First of all, since $v_\alpha \geq u$ on $\partial\Omega$, and

$$\Delta(v_\alpha - u) = \theta(f(v_a) - f(u)) + (\theta - 1)(f(u))$$

by the same comparison argument as above, it follows that $v_\alpha \geq u$ in Ω or $u \leq g$ in Ω. On the other hand, since v_α is radially symmetric, we arrive to the ordinary differential equation(where $v = v_\alpha$):

$$v^{''} + \frac{N-1}{r} v^{'} = h(v) \quad in \quad (0, R) \tag{3.21}$$

$$v^{'}(0) = 0, \quad v(R) = \alpha \tag{3.22}$$

Denote also $v_{0,\alpha} := v_\alpha(0)$, which is uniquely determined by α. We multiply by r^{N-1} in both sides of (3.21) and we integrate to obtain:

$$v^{'}(r) = \frac{1}{r^{N-1}} \int_0^r s^{N-1} h(v(s)) ds \geq 0 \tag{3.23}$$

hence v is nondecreasing. Then

$$v^{'}(r) \leq \frac{1}{r^{N-1}} h(v(r)) \int_0^r s^{N-1} ds = \frac{r}{N} h(v(r))$$

It follows that

$$v^{''}(r) \geq h(v(r)) - \frac{N-1}{N} h(v(r)) = \frac{1}{N} h(v(r))$$

and

$$\frac{1}{N} h(v(r)) \leq v^{''}(r) \leq h(v(r)) \tag{3.24}$$

We multiply the last inequality by $v^{'}(r)$ and we integrate from 0 to r. By making a change of variable and taking the power $-\frac{1}{2}$ in both sides one obtains:

$$\int_{v_0}^{v} \frac{1}{\sqrt{\theta F(s)}} ds \leq r \leq \sqrt{N} \int_{v_0}^{v} \frac{1}{\sqrt{\theta F(s)}} ds \tag{3.25}$$

But from condition (3.14), the integrals are convergent as $v \to \infty$, hence for some value of r, denoted by $R(v_0)$, v becomes infinite and the inequality (3.25) still holds. Since $\lim\limits_{v_0 \to \infty} \int_{v_0}^{\infty} \frac{1}{\sqrt{\theta F(s)}} ds = 0$ and $\lim\limits_{v_0 \to -\infty} \int_{v_0}^{\infty} \frac{1}{\sqrt{\theta F(s)}} ds = \infty$, it follows that $R(v_0)$ has the same behaviour as a function of v_0. Also one can easily check that $R(v_0)$ is nonincreasing.

We finally define $g(r) := \inf\{v_0 : R(v_0) = r\}$, which is a sort of "inverse" of the function $R(v_0)$. It is easy to see that g is the desired function and it corresponds with the first definition. □

Corollary. *If f is as in theorem 10, the equation (3.1) posed in the whole $\mathbb{R}^N$ has no solutions.*

Proof. If there exists a solution $u \in C^2(\mathbb{R}^N)$, then by theorem 10, $u(x) \leq g(R)$, for any $x \in \mathbb{R}^N$ and for any $R > 0$. But from the conditions (3.16) on g, it follows that $u(x) \leq t$, $\forall t \in \mathbb{R}$, which is absurd. □

Now we derive the most interesting result for us:

Theorem 11. *If f is as above, then in any bounded domain $\Omega \subset \mathbb{R}^N$ there exists a solution of (3.1)-(3.2).*

Proof. We remark as before that for any α there exista a solution u_α of (3.1) with $u = \alpha$ on $\partial\Omega$ and from the maximum principle u_α is a nondecreasing function with respect to α. From theorem 10 we deduce that for any $x \in \Omega$ we have $u_\alpha(x) \leq g(d(x))$, hence are bounded above uniformly in α. Thus there exists $u(x) := \lim\limits_{\alpha \to \infty} u_\alpha(x) < \infty$, $\forall x \in \Omega$, and the limit u is still a solution of (3.1). But as $x \to \partial\Omega$, $u(x)$ becomes indefinitely large, since on $\partial\Omega$, $u_\alpha(x) = \alpha \to \infty$. Hence u is the desired solution. □

In the same paper of Keller there exists another complementary result which gives conditions in order to have solution in $\mathbb{R}^N$, and that we only state:

Theorem 12. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then (3.1) has a radially symmetric solution in $\mathbb{R}^N$ if and only if one of the following conditions is satisfied:*
(i) $f(u_1) = 0$;
(ii) $f > 0$ and $\int_{u_0}^{\infty} \frac{dt}{\sqrt{F(t)}} = \infty$;
(iii) $f < 0$ and $\int_{-\infty}^{u_0} \frac{dt}{\sqrt{F(t)}} = \infty$.

Hence we remark here an opposite condition to (3.14). For a proof, we indicate the original paper of Keller([Ke57]).

3.4 Relaxation of Monotonicity Condition. Multiplicity and Non-existence Results

In this long section we present the recent development of the problem in absence of the monotonicity of f. In fact, after the papers of Keller and Osserman, there were very few papers on problem (3.1)-(3.2) for a very long period. It was only in the 90's when the important work of Lazer, McKenna, W. Reichel, W. Walter opened a new direction in studying the problem, that of non-necessary monotonic f. As we will show, the results here are much different: one can has uniqueness, non-uniqueness or even non-existence.

The first result of non-uniqueness was obtained by McKenna, Reichel and Walter ([MKRW97]) for the function $f(u) = |u|^p$ on a ball in $\mathbb{R}^N$. In this paper, it is showed that for $1 < p < N^*$, where N^* is the critical Sobolev exponent(i.e. $N^* = \frac{N+2}{N-2}$ for $N \geq 3$ and $N^* = \infty$ for $N = 1, 2$), there exists exactly two blow-up solutions, a positive one and another one that changes sign. For $p \geq N^*$ there exists only one solution, which is positive (and is obtained in a very similar way as in theorem 8).

W. Reichel (see [Re97]) has extended the multiplicity result to convex and bounded smooth domains, for the same function and for some small perturbation of it. The technique used is variational and uses the Mountain-Pass theorem(see chapter 2).

We will present next the more general result in this direction, obtained by A. Aftalion and W. Reichel in [AfR97]. This extends the previous results to a wide class of nonlinearities, using a topological degree technique.

Consider a function $f : \mathbb{R} \to \mathbb{R}$ with the following properties:

(A) f is locally Lipschitz continuous and $f(0) = 0$;

(B) there exists $s_0 > 0$ such that f is positive and nondecreasing on $[s_0, \infty)$ and it satisfies condition (3.14);

(C) there exists $p \in (1, N^*)$ such that $0 < \lim\limits_{s \to -\infty} \frac{f(s)}{|s|^p} < \infty$.

By example, these hypotheses allows to consider the nonlinearities mentioned above and many more, as $f(s) = s^{p_1}$ for $s > 0$ and $f(s) = (-s)^{p_2}$ for $s < 0$, where $1 < p_1$ and $1 < p_2 < N^*$. In fact, conditions (3.14) and (C) are growth conditions near ∞ and $-\infty$ which are required in order to obtain solutions(we have seen already that (3.14) is needed in general). If the second growth condition is relaxed, then the sign-changing solution may not exist, as it is the case in the next result:

Theorem 13. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, symmetric with respect to all hyperplanes $x_i = 0$ (for example a ball) and f as before. If $p_2 > N^*$, then there is no sign-changing solution of (3.1)-(3.2)*

We will prove this at the end of this section, in a more general form.

The idea for proving a multiplicity result in the conditions (A), (B), (C) is to obtain two solution for the problem (3.1) with the boundary condition

$$u = c \ \ on \ \ \partial\Omega \tag{3.26}$$

where $c > s_0$ and then to pass to the limit as $c \to \infty$. In order to use the topological degree method, we will also consider the perturbed problem:

$$\Delta u = f(u) + k \ \ in \ \ \Omega \tag{3.27}$$

together with (3.26), where $k \geq 0$.

Lemma 12. *Let f as before. If u solves (3.26)-(3.27) for some $k \geq 0$ then $u < c$ in Ω.*

Proof. If there exists $x_0 \in \Omega$ such that $u(x_0) = \sup\limits_{x\in\Omega} u(x) > c$, then $f(u(x_0)) > 0$ by condition (B), but $\Delta u(x_0) \leq 0$, which is a contradiction. $\square$

Lemma 13. *(Apriori bound from above) If f satisfies (A) and (B), then, for every compact $K \subset \Omega$, there exists $C_K > 0$ such that $u \leq C(K)$ on K, for every solution u of (3.1)-(3.26).*

Proof. Consider v_α the solution of the Cauchy problem:

$$v_\alpha'' = \frac{1}{N} v_\alpha \tag{3.28}$$

$$v_\alpha(0) = \alpha, \quad v_\alpha'(0) = 0 \tag{3.29}$$

where $\alpha > s_0$. Consider the maximal interval $(0, R_\alpha)$ on which the solution v_α exists. Then $\lim\limits_{x\to R_\alpha} v_\alpha(x) = \infty$. We multiply in both sides by v_α' and integrate. Then

$$v_\alpha' = \frac{2}{N}(F(v_\alpha) - F(\alpha))$$

with the common notation. Hence by integrating on (α, ∞) and changing the variables, we get

$$R_\alpha = \int_\alpha^\infty \frac{\sqrt{N}}{\sqrt{2(F(s) - F(\alpha))}} ds \tag{3.30}$$

One can write

$$\begin{aligned} R_\alpha &= \int_0^\infty \frac{\sqrt{N}}{\sqrt{2(F(s+\alpha) - F(\alpha))}} ds \\ &= \int_0^\alpha \frac{\sqrt{N}}{\sqrt{2(F(s+\alpha) - F(\alpha))}} ds + \int_\alpha^\infty \frac{\sqrt{N}}{\sqrt{2(F(s+\alpha) - F(\alpha))}} ds \\ &= I_1 + I_2 \end{aligned}$$

But for $s > s_0$, F is increasing and convex, hence $F(s+\alpha) - F(\alpha) \geq \alpha f(s)$. It follows that

$$I_1 \leq \int_0^\alpha \frac{\sqrt{N} ds}{\sqrt{2sf(\alpha)}} = C\sqrt{\frac{\alpha}{f(\alpha)}}$$

and (3.14) implies $\lim\limits_{\alpha\to\infty} \frac{\alpha}{f(\alpha)} = 0$. Hence $\lim\limits_{\alpha\to\infty} I_1 = 0$. Similarly, from the convexity of F it follows that $F(s+2\alpha) \geq F(2\alpha) \geq 2F(\alpha)$, and $I_2 \leq C \int_0^\infty \frac{ds}{\sqrt{F(s+2\alpha)}} \to 0$ as $\alpha \to \infty$. Hence

$$\lim_{\alpha\to\infty} R_\alpha = 0 \tag{3.31}$$

Let now u be a solution of (3.1)-(3.26) and $x_0 \in \Omega$. Choose $\alpha > 0$ big such that $R_\alpha < \frac{1}{2}d(x_0)$. Then if $v_\alpha = v_\alpha(r)$, we have:

$$\Delta v_\alpha = v_\alpha'' + \frac{N-1}{r} v_\alpha' \leq N v_\alpha'' = f(v_\alpha)$$

for $0 < r := |x - x_0| < R_\alpha$. Hence v_α is a supersolution for (3.1) in $B(x_0, R_\alpha)$. By maximum principle we obtain that $v_\alpha \geq u$ in the same ball. For a general compact $K \subset \Omega$ we cover it with a finite number of such balls $B(x_0, R_\alpha)$ and we take C_K to be the maximum of the estimates on each ball. □

Remark. *As we shall see, the condition (3.31) it is particularly important in connection to what we usually call today the strong Keller-Osserman condition, for short (KO_s). That's why [AfR97] is considered the origin of this stronger growth condition. We will see many aspects concerning (KO_s) in the next section.*

In what follows we will need to prove some technical non-existence results for (3.26)-(3.27) for large k. These results will be essential when applying the topological degree technique.

Lemma 14. *Let α and $k > 0$ be given. Let $p > 1$. If the following Dirichlet problem:*

$$\Delta v + \alpha v^p + k = 0 \quad in \quad B(0, R) \tag{3.32}$$

$$v = 0 \quad on \quad \partial B(0, R) \tag{3.33}$$

has a positive solution, then

$$k \leq \lambda_1^{\frac{p}{p-1}} \alpha^{-\frac{1}{p-1}} R^{-2\frac{p}{p-1}} \tag{3.34}$$

where λ_1 is the first eigenvalue of the Laplacian in $B(0,1)$.

Proof. Let $v > 0$ be a solution of (3.32)-(3.33). We multiply in both sides of (3.32) by ϕ, the first eigenfunction of the Laplacian in $B(0, R)$, and we integrate. We obtain:

$$\begin{aligned} \int\limits_{B(0,R)} \alpha v^p \phi dx &= \frac{\lambda_1}{R^2} \int\limits_{B(0,R)} v \phi dx - k \\ &\leq \frac{\lambda_1}{R^2} \int\limits_{B(0,R)} v \phi dx \\ &\leq \frac{\lambda_1}{R^2} (\int\limits_{B(0,R)} \alpha v^p \phi dx)^{\frac{1}{p}} \alpha^{-\frac{1}{p}} \end{aligned}$$

hence

$$\int\limits_{B(0,R)} \alpha v^p \phi dx \leq \lambda_1^{\frac{p}{p-1}} \alpha^{-\frac{1}{p-1}} R^{-2\frac{p}{p-1}} \tag{3.35}$$

and the same inequality for $\frac{\lambda_1}{R^2} \int\limits_{B(0,R)} v \phi dx$. Using these we find (3.34). □

We need now a comparison principle which is very useful for differential inequalities:

Lemma 15. *(Comparison principle) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous and nondecreasing function and suppose that in an interval $[a, b]$ with $0 \leq a < b$ there exists two functions g, h of class C^2 such that:*

$$g^{''} + \frac{N-1}{r} g^{'} \leq f(g), \quad h^{''} + \frac{N-1}{r} h^{'} \geq f(h)$$

and $g(a) \leq h(a)$, $g^{'}(a) \leq h^{'}(a)$ (where by convention we assume $g^{'}(a) = h^{'}(a) = 0$ if $a = 0$). Then $g \leq h$ on $[a, b]$.

Proof. Consider for any $\varepsilon > 0$ the solution h_ε of the problem:

$$h_\varepsilon^{''} + \frac{N-1}{r}h_\varepsilon^{'} - f(h_\varepsilon) = h^{''} + \frac{N-1}{r}h^{'} - f(h)$$

$$h_\varepsilon(a) = h(a) + \varepsilon, \quad h_\varepsilon^{'}(a) = h^{'}(a)$$

From Dini's theorem we get that h_ε converges to h uniformly as $\varepsilon \to 0$. We compare g to h_ε. Let $[a, c] \subset [a, b]$ be the maximal subinterval with $g \leq h_\varepsilon$, for some fixed ε. Then $c > a$ and by multiplication to r^{N-1} we find:

$$\begin{aligned}(r^{N-1}(h_\varepsilon^{'} - g^{'}))^{'} &= r^{N-1}(h_\varepsilon^{''} + \frac{N-1}{r}h_\varepsilon^{'} - g^{''} - \frac{N-1}{r}g^{'}) \\ &\geq r^{N-1}(h^{''} + \frac{N-1}{r}h^{'} - f(h) + f(h_\varepsilon) - f(g)) \\ &\geq 0\end{aligned}$$

It follows that $h_\varepsilon^{'} \geq g^{'}$ on $[a, c]$, hence $h_\varepsilon - g$ is increasing on $[a, c]$. But $h_\varepsilon(a) > g(a)$, hence $c = b$. □

Proposition 3. *Let f satisfy (A), (B), (C). Then there exists a constant $K^* > 0$ which depends only on Ω such that (3.26)-(3.27) posed in Ω has no solution for $k \geq K^*$.*

Proof. We divide the proof into several steps.
Step 1: We start the proof by making a reduction of the problem. We first take $\Omega = B(0, R)$. Consider a modification of f on $(0, s_0)$ to a function $\overline{f}$ which is Lipschitz continuous and nondecreasing on $[0, s_0]$, with $\overline{f}(0) = 0$. This linking is possible since $f(s_0) > 0$. Define $g(s) := \overline{f}(s) + \frac{k}{2}$ for $s > 0$ and $g(s) := a|s|^p + \frac{k}{2}$ for $s \leq 0$, where $a > 0$ is a constant chosen such that $g(s) \leq f(s) + k$.
We prove non-existence for the problem:

$$\Delta u = g(u) \quad in \quad B(0, R) \tag{3.36}$$

and (3.26). Suppose we have already proved this. Then any solution v of (3.26)-(3.27) becomes a subsolution for (3.36)-(3.26). By lemma 12 $v \leq c$ in $B(0, R)$. On the other hand, since $\overline{f}$ is nonnegative, the constant function c is a supersolution for (3.36)-(3.26). Hence, existence for (3.26)-(3.27) and theorem 1 will imply existence for (3.36)-(3.27). It follows that we can not have a solution for (3.26)-(3.27).
We study next the problem (3.36)-(3.26). By the Gidas-Ni-Nirenberg theorem(see [GNN79]) its solution must be radially symmetric and radially increasing, hence it remains bounded by the boundary value c.
Step 2: Non-existence of nonnegative solutions.
Consider ϕ the solution of the following Cauchy problem:

$$\phi^{''} = \frac{1}{N}(\overline{f}(\phi) + \frac{k}{2}) \tag{3.37}$$

$$\phi(0) = 0, \quad \phi^{'}(0) = 0 \tag{3.38}$$

which exists in a maximal interval $(0, R_k)$. Following the same lines of the proof of lemma 13, we obtain that

$$R_k = \int_0^\infty \frac{\sqrt{N}ds}{\sqrt{2\overline{f} + ks}}$$

which is obviously decreasing in k and $\lim_{k\to\infty} R_k = 0$ by the monotone convergence theorem. On the other hand, ϕ'' is nondecreasing and $\phi''(0) = \frac{k}{2N} > 0$, hence ϕ' is also nondecreasing on $(0, R_k)$. Since $\phi'(0) = 0$, ϕ is also nondecreasing. Using these facts, one can easily prove that $\frac{1}{r}\phi' \le \phi''$. It follows that

$$\Delta\phi = \phi'' + \frac{N-1}{r}\phi' \le N\phi'' = \overline{f}(\phi) + \frac{k}{2} \tag{3.39}$$

On the other hand, if there exists a nonnegative solution u of (3.36)-(3.26), then

$$\Delta u = u'' + \frac{N-1}{r}u' = \overline{f}(u) + \frac{k}{2} \tag{3.40}$$

and $u(0) \ge 0$, $u'(0) = 0$. We now use the comparison principle given in lemma 15 together with (3.39)-(3.40) and we obtain that $\phi \le u$. Hence u blows-up before ϕ. But ϕ blows-up at R_k which goes to 0 as $k \to \infty$, contradiction for k large enough.

Step 3: Non-existence of sign-changing solutions.

We have proved in the lines of step 2 that any solution is increasing in r. Hence, if there exists such a solution u, then $u(0) < 0$ and $u(\rho) = 0$ for some $\rho \in (0, R)$. From lemma 13 we deduce that for large k, for example for $k \ge \alpha^{-\frac{1}{p-1}}\lambda_1^{\frac{p}{p-1}}(\frac{R}{4})^{-\frac{2p}{p-1}}$ we have $\rho \le \frac{R}{4}$. As in step 2, the solution ϕ of (3.37)-(3.38) satisfies for $r \ge \rho$ the same inequality as in step 2, hence $\phi(\cdot - \rho) \le u$ for any $r \ge \rho$. We arrive to the same contradiction.

Step 4: Extension to general domains.

Until now we have proved the proposition in the case of a ball. Let Ω be a general domain, such that $0 \in \Omega$ and let $R > 0$ such that $\overline{B(0,R)} \subset \Omega$. Let $k \ge k_R$, where k_R is given by the preceding steps, and assume that (3.26)-(3.27) has a solution u. Then $u \le c$ in $B(0,R)$ by lemma 12, hence u is a subsolution of (3.26)-(3.27) in $B(0,R)$. Since the constant function $u \equiv c$ is a supersolution, it follows that the problem on the ball has a solution, in contradiction with the result of the previous two steps. $\square$

The next step is to obtain an apriori bound from below. For this goal we use two of the most spectacular techniques in nonlinear partial differential equations: the moving plane method of Gidas, Ni and Nirenberg(see [GNN79]) and a scaling argument based on the ideas of Gidas and Spruck([$GS81_1$] and [$GS81_2$]).

The moving plane method is a technique of "going away" from the boundary in an uniform way:

Theorem 14. *Let f be as above and let u be a positive solution of the equation (3.1) with zero boundary condition. Then there exists a $t_0 > 0$ depending only on the geometry of Ω such that on $[0, t_0]$, the function $t \to u(x - t\nu(x))$ is increasing, where $\nu(x)$ is the outer normal vector in $x \in \partial\Omega$. In particular, this result implies that the critical points of u stay away from the boundary at a distance depending only on Ω.*

Proof. (sketched) Let $x = (x_1, x')$ and for $\lambda > 0$, $x_\lambda := (2\lambda - x_1, x')$, where $x' = (x_2, ..., x_N)$.

Fix $x_0 \in \partial\Omega$ and take the axis x_1 for simplicity to be parallel to the normal direction at the boundary in x_0. Consider T_{λ_0} the tangent plane to $\partial\Omega$ at x_0, where λ_0 is the first coordinate of x_0. The idea is to take all the parallel hyperplanes T_λ to T_{λ_0} and to reflect the part of Ω which lies between T_{λ_0} and T_λ(which will be denoted by Σ_λ) with respect to T_λ. In this way, we still stay inside Ω until a limit position T_l where the symmetric of Σ_l goes out of the boundary of Ω(see the figure).

We remark that the previous transform $x \to x_\lambda$ is exactly the symmetry with respect to T_λ. Define $u_\lambda(x) := u(x_\lambda)$ and $w_\lambda(x) := u(x_\lambda) - u(x)$. We remark that if we prove that $w_\lambda > 0$ in Σ_λ for some $\lambda > 0$, this implies the desired increasingness in the normal direction.
We have

$$\Delta w_\lambda = \Delta(u_\lambda - u) = f(u_\lambda) - f(u)$$

or

$$\Delta w_\lambda = c_\lambda(x) w_\lambda \tag{3.41}$$

where $c_\lambda(x) := \frac{f(u_\lambda)-f(u)}{u_\lambda - u} \in L^\infty(\Omega)$. To apply the maximum principle in (3.41) we need that $c_\lambda \leq \lambda_1(\Sigma_\lambda)$ where λ_1 means the first eigenvalue of the operator $-\Delta$ on the indicated domain. Since $\lambda_1(\Sigma_\lambda) \to \infty$ as $vol(\Sigma_\lambda) \to 0$, the maximum principle holds for a small domain.
Consider the largest interval (λ^*, λ_0) on which $w_\lambda > 0$ in Σ_λ and the maximum principle is applicable. It follows by continuity that $w_{\lambda_*} \geq 0$ and by maximum principle that either $w_{\lambda^*} > 0$ or $w_{\lambda^*} \equiv 0$. If $\lambda_* > L$, then $w_{\lambda^*} > 0$ in Σ_{λ^*} and there exists $\delta > 0$ and a compact set $K \subset \Sigma_{\lambda^*}$ such that $w_{\lambda^*} > \delta > 0$ on K and $vol(\Sigma_{\lambda^*} \setminus K)$ is small enough in order to apply the maximum principle. If in this case we move λ between $\lambda^* - \varepsilon$ and λ^* for a small ε, it follows that $w_\lambda > 0$ also for these λ, which contradicts the maximality of the interval (λ^*, λ_0). $\square$

The scaling argument has been developed by Gidas and Spruck in the early '80s in order to prove a very nice extension of the Liouville theorem in $\mathbb{R}^N_+$:

Theorem 15. *(Gidas-Spruck) Suppose that $u \geq 0$ is a classical solution of the problem:*

$$-\Delta u = u^p \quad in \quad \mathbb{R}^N_+ \tag{3.42}$$

$$u = 0 \quad on \quad \partial\mathbb{R}^N_+ \tag{3.43}$$

where $1 < p \leq \frac{N+2}{N-2}$. Then $u \equiv 0$.

The proof of this Liouville-type result uses also the moving plane method in order to prove that the solution u depends only on the first coordinate x_1, then ODE techniques. Based on this theorem and using again the same scaling argument, Gidas and Spruck have obtained a L^∞-boundedness result for the solutions of certain nonlinear equations. In the next proof we will need another theorem of Liouville-type that we do not prove:

Theorem 16. *(Chen-Li) Suppose $u \geq 0$ is a classical solution of*

$$-\Delta u = u^p \quad in \quad \mathbb{R}^N \tag{3.44}$$

where $1 < p \leq \frac{N+2}{N-2}$. Then $u \equiv 0$.

The proof is easy and its idea is to show first that the solution must be radially symmetric. For this we restrict to a ball which becomes larger and larger and apply the Gidas-Ni-Nirenberg theorem. For a complete proof the reader should consult [ChLi91].
We are now in position to prove the basic apriori bound from below.

Proposition 4. *Let f satisfy (A), (B), (C). Then there exists a constant $L > 0$ such that any solution u of (3.26)-(3.27) with $k \in (0, k^*)$, where k^* is given by proposition 3, satisfies $u \geq -L$ in Ω.*

Proof. Suppose by contradiction that there exists sequences $c_j \to \infty$ as $j \to \infty$, $(x_{c_j})_j \subset \Omega$, $(k_{c_j})_j \subset [0,k^*]$ and functions u_{c_j} with the following properties:

$$\Delta u_{c_j} = f(u_{c_j}) + k_{c_j} \quad in \quad \Omega \tag{3.45}$$

$$u_{c_j} = c_j \quad on \quad \partial\Omega \tag{3.46}$$

and

$$m_{c_j} = -\inf_{\Omega} u_{c_j} = -u_{c_j}(x_{c_j}) \to \infty \tag{3.47}$$

as $j \to \infty$. We will use for simplicity in the next lines only indices j.
From the moving plane method we deduce that the critical points of u_j are bounded away from the boundary uniformly in j, hence $d(x_j, \partial\Omega) \geq \delta > 0$. By subtracting convergent subsequences if necessary, we may suppose that $k_j \to \overline{k} \in [0,k^*]$ and $x_j \to \overline{x} \in \Omega$ with $d(\overline{x}, \partial\Omega) \geq \delta$.
We define the scaling

$$w_j(y) := \frac{1}{m_j} u_j(m_j^{-\frac{p-1}{2}} y + x_j) \tag{3.48}$$

Hence $w_j(0) = -1$ and $w_j \geq -1$. We compute:

$$\begin{aligned}
\Delta w_j(y) &= \frac{1}{m_j}(\Delta u_j)(m_j^{-\frac{p-1}{2}} y + x_j) m_j^{1-p} \\
&= \frac{1}{m_j^p}(f(u_j(m_j^{-\frac{p-1}{2}} y + x_j))) \\
&= \frac{1}{m_j^p}(f(m_j w_j(y)) + k_j)
\end{aligned} \tag{3.49}$$

Consider $R > 0$ small enough such that $R \leq \frac{\delta}{2}$ and let ψ be the solution of the Cauchy problem :

$$\psi^{''} = \frac{1}{N} f(\psi) \tag{3.50}$$

$$\psi^{'}(0) = 0, \quad \psi(R) = \infty \tag{3.51}$$

We choose R small such that $\psi(0) \geq s_0$. Define a similar scaling

$$\phi_j(r) := \frac{1}{m_j} \psi(m_j^{-\frac{p-1}{2}} r) \tag{3.52}$$

where $r \in B(0, m_j^{\frac{p-1}{2}} R)$. We compute

$$\Delta \phi_j := \frac{1}{m_j^p} \Delta \psi \tag{3.53}$$

From (3.50) and (3.51) one can easily prove that ψ, $\psi^{'}$ and $\psi^{''}$ are both nondecreasing, hence $\Delta\psi = \psi^{''} + \frac{N-1}{r}\psi^{'} \leq N\psi^{''}$. It follows that

$$\Delta \phi_j \leq \frac{1}{m_j^p} N \psi^{''} = \frac{1}{m_j^p} f(m_j \phi_j) \tag{3.54}$$

We compare (3.49) and (3.54). We remark that $\phi_j(0) = \frac{1}{m_j}\psi(0) \geq \frac{s_0}{m_j}$. Then

$$\Delta(w_j - \phi_j) \geq \frac{1}{m_j^p}(f(m_j w_j) - f(m_j \phi_j)) + \frac{k_j}{m_j^p}$$

For j large, we have that $m_j w_j \geq s_0$, hence f is nondecreasing. By a standard comparison argument that we have used in the proof of theorem 8, it follows that $w_j \leq \phi_j$ in $B(0, m_j^{\frac{p-1}{2}} R)$.
Let $K \subset \mathbb{R}^N$ be a fixed compact set. Then $\lim_{j\to\infty} \phi_j = 0$ on K and moreover

$$m_j \phi_j(r) = \psi(m_j^{-\frac{p-1}{2}} r)$$

is bounded uniformly in j on K, where we consider the extension of ψ by 0 on the whole space. Then, for $w_j \geq 0$ one has

$$|\Delta w_j| \leq \frac{1}{m_j^p} |f(m_j w_j) + k_j| \leq \frac{1}{m_j^p} C(K) \tag{3.55}$$

On the other hand, we use the assumption (C) on the nonlinearity f to derive that there exists some positive constants C_1, C_2 and C_3 such that $-C_1 \leq f(s) + k_j \leq C_2 |s|^p + C_3$, for all $s < 0$. We use this for $s = m_j w_j$ if $w_j < 0$ and by multiplying in both sides by m_j^{-p}, we have:

$$-C_1 \frac{1}{m_j^p} \leq \frac{1}{m_j^p} (f(m_j w_j) + k_j) = \Delta w_j \leq C_2 |w_j|^p + C_3 \frac{1}{m_j^p} \tag{3.56}$$

for $w_j < 0$.
Hence, for j large, Δw_j is bounded uniformly in $L^\infty(K)$. By the Calderon-Zygmund estimates on K(see for example [GT02], chapter 9) it follows that $w_j \in W^{2,p}(K)$ for all $p > 1$. Using the Sobolev embedding theorem, we obtain that $w_j \in C^{1,\alpha}(K)$, for any $\alpha \in (0,1)$. Hence we can subtract a subsequence(denoted also by w_j for simplicity) such that w_j converges locally uniformly to some function w. Moreover, since $w_j \leq \phi_j$ and $\phi_j \to 0$ as $j \to \infty$, it follows that $w \leq 0$. Hence

$$\Delta w = \lim_{j\to\infty} \Delta_{w_j} = \lim_{j\to\infty} \frac{f(m_j w_j)}{(m_j w_j)^p} w_j^p = C w^p$$

and $w \leq 0$. Then $\overline{w} := -w$ solves $\Delta u + C u^p = 0$ in $\mathbb{R}^N$ and $\overline{w} \leq 0$. But this implies $w \equiv 0$ by the Chen-Li theorem. This is a contradiction, since $w_j(0) = -1$ for all j, hence $w(0) = -1$. $\square$

We can now prove the main result of this section, the existence of two solutions for the problem (3.1)-(3.2). We will first derive the same result for (3.1)-(3.26) and then pass to the limit.
For convenience we recall the main properties of the Leray-Schauder topological degree for compact perturbations of the identity. Let in general X be a real Banach space. Consider the family $\mathcal{T} := \{(I - T, Y, 0) : Y \subseteq X$ is a bounded, open set, $T : \overline{Y} \to X$ compact operator and $(I - T)(x) \neq 0, \ \forall x \in \partial Y\}$. Such a triple is called an admissible triple.

Theorem 17. *In the conditions above, there exists an application $d : \mathcal{T} \to \mathbb{Z}$ which has the following properties:*
(P1) If $0 \in Y$, then $d(I, Y, 0) = 1$;
(P2) (additivity) If $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 are two disjoint open sets, $T : \overline{Y} \to X$ a compact operator such that $(I - T, Y, 0) \in \mathcal{T}$, then $(I - T, Y_1, 0) \in \mathcal{T}$, $(I - T, Y_2, 0) \in \mathcal{T}$ and

$$d(I - T, Y, 0) = d(I - T, Y_1, 0) + d(I - T, Y_2, 0) \tag{3.57}$$

(P3) (the existence theorem) If $d(I - T, Y, 0) \neq 0$ then there exists $x \in Y$ such that $Tx = x$;
(P4) (the homotopy invariance) If $H : [0,1] \times \overline{Y} \to X$ is a compact operator and $H(t, x) \neq x$, for all $t \in [0,1]$ and $x \in \partial Y$, then $d(I - H_t, Y, 0)$ is constant with respect to t. In particular one has $d(I - H_0, Y, 0) = d(I - H_1, Y, 0)$.

The construction of this topological degree is based on the Brouwer degree for finite dimensional spaces, applied for the Schauder approximation of X. For us this construction is not important; we need only the properties listed before. For the interested reader we suggest the monograph of Zeidler([Ze84]) or a small but very nice book of Brown([Br93]), where one can find the precise construction and further properties. Also the paper [Rab74] is interesting for giving some other applications in our context.
Now we state the desired result:

Theorem 18. *Let $\Omega \subset \mathbb{R}^N$ be a smooth and bounded domain(by domain we understand also convexity). If f satisfies (A), (B) and (C) then there exists at least two solutions of (3.1)-(3.2):one is nonnegative and another is sign-changing.*

Before starting the proof, we will introduce some technical lemmas which will be useful in proving that certain triples are admissible for the topological degree. First of all, since $u \equiv 0$ is a subsolution of (3.1)-(3.26) and for $c \geq s_0$, the function $u \equiv c$ is a supersolution for (3.1)-(3.26), it follows that there exists a nonnegative solution of (3.1)-(3.26) denoted by u_1. Let $v_1 := c - u_1$.

Lemma 16. *There exists a constant $M_1 > 0$ such that for any v solution of the problem*

$$-\Delta v = f(c-v) + k, \quad in \quad \Omega \tag{3.58}$$

$$v = 0 \quad on \quad \partial\Omega \tag{3.59}$$

we have $\|v\|_{C^1(\overline{\Omega})} \leq M_1$, $\forall k \in [0, k^*]$.

Proof. By lemma 12, any solution v of (3.58)-(3.59) is positive. We remark that if we put $u = c - v$, then (3.58)-(3.59) becomes (3.26)-(3.27) for u. By proposition 4, we have a uniform lower bound for u, i.e. $u \geq -L$, hence $v \leq c + L$. It follows that $\|v\|_\infty \leq c + L$, for all $k \in [0, k^*]$. Hence $(f(c-v)+k) \in \Lambda^\infty(\Omega)$ uniformly in k and from the Calderon-Zygmund estimates we derive that $v \in W^{2,p}(\Omega)$, $\forall p > 1$. Finally, from the Sobolev embedding theorem, it follows that $v \in C^1(\overline{\Omega})$ uniformly in k. $\square$

Lemma 17. *Fix $\mu \geq 0$. Then there exists a constant $M_2 > 0$, independent of $t \in [0,1]$, such that for any solution v of the problem*

$$-\Delta v + \mu v = t(\mu v + f(c-v)) + (1-t)(\mu v_1 + f(c - v_1)) \quad in \quad \Omega \tag{3.60}$$

$$v = 0 \quad on \quad \partial\Omega \tag{3.61}$$

with $0 \leq v \leq c$ and v_1 the nonnegative solution of the problem (3.1)-(3.26), we have $\|v\|_{C^1(\overline{\Omega})} \leq M_2$.

Proof. We remark that the right hand side is bounded, since $0 \leq v \leq c$. Then the proof follows the same lines of that of the previous lemma. $\square$

Now we start the proof of the theorem, which we divide into several steps for simplicity. The first(easy) step was already done, that of finding a nonnegative solution of the approximating problem (3.1)-(3.26).

Proof. (Theorem 18) Step 1: we prove that (3.1)-(3.26) has at least two solutions and one is sign-changing. For this we use the topological degree. Consider $v := c - u$, hence

$$-\Delta v = f(c - v) \tag{3.62}$$

$$v = 0 \quad on \quad \partial\Omega \tag{3.63}$$

From lemma 12 we find that any solution of (3.62)-(3.63) is positive. As before, we remark that v_1 is such a solution which is bounded by c. We next prove that there exists a solution v_2 crossing c. For this goal we define the compact operators $(U_t)_{t\in[0,1]}$ as follows: for $v \in C_0^1(\Omega)$, define $w = U_t v$ to be the solution of

$$-\Delta w + \mu w = f(c - v) + \mu v + tk^* \quad in \quad \Omega \tag{3.64}$$

$$w = 0 \quad on \quad \partial\Omega \tag{3.65}$$

where $C_0^1(\Omega) := \{v \in C^1(\Omega) : v = 0 \;\; on \;\; \partial\Omega\}$ and $\mu := \sup\limits_{[0,c]} |f^{'}|$. It is well-known that U_t are compact operators. Moreover, if $0 \le v_1 \le v_2 \le c$, since the function $s \to f(c - s) + \mu s$ is increasing in $[0, c]$, then $U_t v_1 \le U_t v_2$ by the classical maximum principle.
We want to find a fixed point of U_0 crossing c. For a constant $M > max(M_1, M_2)$, we define the sets

$$Y := \{v \in C_0^1(\Omega) : v > 0 \quad in \quad \Omega, \quad \|v\|_{C^1(\overline{\Omega})} < M \quad and \quad \frac{\partial v}{\partial n} < 0 \quad on \quad \partial\Omega\} \tag{3.66}$$

$$Y_1 := \{v \in Y : v < c \quad in \quad \Omega\} \tag{3.67}$$

which are open in $C_0^1(\Omega)$. Set $Y_2 := Y \setminus Y_1$.
We prove next that $d(I - U_t, Y, 0) = 0$, $\forall t \in [0, 1]$. If not, by (P3) there exists $v \in \overline{Y}$ and $t \in [0, 1]$ such that $U_t v = v$. Then v solves (3.58)-(3.59) for $k = tk^*$, hence $\|v\|_{C^1(\overline{\Omega})} < M$. Since $v > 0$, it follows that $\frac{\partial v}{\partial n} > 0$ on $\partial\Omega$, hence $v \in Y$. Then $(I - U_t, Y, 0) \in \mathcal{T}$ and U_1 has no fixed points, by the nonexistence result presented in lemma 14. Using (P3) and (P4) we obtain that $d(I - U_0, Y, 0) = d(I - U_1, Y, 0) = 0$.
We show that $d(I - U_0, Y_1, 0) = 1$. We know that there exists a solution $v_1 \in Y_1$ of (3.62)-(3.63). Hence $d(I - v_1, Y_1, 0) = 1$. Define the compact homotopy

$$H_t := tU_0 + (1 - t)v_1, \quad t \in [0, 1]$$

If $v \in \overline{Y_1}$ such that $v = H_t v$, then v is a solution of (3.60)-(3.61) and $\|v\|_{C^1(\overline{\Omega})} < M$. Then by maximum principle $v \in Y_1$ and it follows that $(I - H_t, Y_1, 0) \in \mathcal{T}$. By the homotopy invariance (P4) we obtain that $d(I - U_0, Y_1, 0) = d(I - H_1, Y_1, 0) = d(I - v_1, Y_1, 0) = 1$.
Then by (P2) we have:

$$d(I - U_0, Y_2, 0) = d(I - U_0, Y, 0) - d(I - U_0, Y_1, 0) = -1$$

and by (P3) there exists a solution v_2 of (3.62)-(3.63) such that v_2 is not in $\overline{Y_1}$. Hence v_2 crosses c and $u_2 := c - v_2$ is a sign-changing solution of (3.1)-(3.26).
Step 2: passing to the limit. Let $u_{1,c}$ and $u_{2,c}$ be two solutions of (3.1)-(3.26) for $c \ge s_0$. Then $u_{1,c}$ is the maximal solution and by standard comparison $u_{1,c_1} \le u_{1,c_2}$ for $c_1 \le c_2$. We prove that there exists $A > 0$ such that

$$u_{2,c} \ge u_{1,c} - A, \quad \forall x \in \Omega, \quad \forall c > s_0 \tag{3.68}$$

To prove this, we know that $u_{2,c} \geq -L$ in Ω, by proposition 4. Since f is nondecreasing on $[s_0, \infty)$, one can find $s^* > s_0$ such that $f(s) \leq f(s^*)$, for all $s \in [-L, s^*]$. Set $A := L + s^*$. Then $f(s) \leq f(s + A)$, for all $s \geq -L$, hence

$$f(u_{2,c}) \leq f(u_{2,c} + A)$$

We want to compare $u_{1,c}$ and $u_{2,c} + A$. But

$$\Delta(u_{2,c} + A) = \Delta u_{2,c} = f(u_{2,c}) \leq f(u_{2,c} + A)$$

and $\Delta u_{1,c} = f(u_{1,c})$. Hence $u_{1,c} \leq u_{2,c} + A$ since f is nondecreasing on $[s_0, \infty)$.
It follows that $u_{i,c}$ is bounded below in Ω and above locally in Ω, for $i = 1, 2$. Hence $u_{i,c} \in L^{\infty}_{loc}(\Omega)$ and by the Calderon-Zygmund estimates and Sobolev inequalities, $u \in C^{1,\alpha}_{loc}(\Omega)$. Then there exists subsequences, denoted for short also as $u_{1,c}$ and $u_{2,c}$, converging to u_1, respectively u_2, and by the usual bootstrap technique we obtain that u_1 and u_2 satisfy (3.1). Moreover, $u_1 \geq 0$ and u_2 is sign-changing. Finally, $u_{1,c} \to \infty$ on $\partial\Omega$ as $c \to \infty$ and $u_{2,c}$ will do the same, since we have (3.68). □

There are also cases where there are no blow-up solutions, even if the conditions are very close to ours. The following bifurcation result shows the surprisingly very thin line which exists between multiple existence and nonexistence.

Theorem 19. *Let $\Omega \subset \mathbb{R}^N$ be a bounded and smooth domain and f satisfies (A), (B), (C), except from $f(0) = 0$, replaced by the condition $m_f := min\{f(s) : s \in \mathbb{R}\} > 0$. Consider the problem*

$$\Delta u = \lambda f(u), \quad in \quad \Omega \tag{3.69}$$

$$u(x) \to \infty \quad as \quad x \to \partial\Omega \tag{3.70}$$

Then there exists a critical value λ^* depending only on f and Ω such that for $\lambda \in (0, \lambda^*)$ the problem (3.69)-(3.70) has at least two solutions and for $\lambda > \lambda^*$ (3.69)-(3.70) has no solution.

The idea of the proof is a very elegant one: instead of varying the nonlinearity, as in the statement, we make a smart change in order to obtain the same problem and to vary the domain. More precisely, let us consider the domains $\Omega_t := t\Omega$ for any $t > 0$ (we suppose for simplicity that $0 \in \Omega$) and the functions

$$u_t(x) := u(\frac{x}{t}), \quad t \in \Omega_t$$

If u solves (3.69)-(3.70) for some λ, then u_t solves

$$\Delta u_t = f(u_t) \quad in \quad \Omega_t \tag{3.71}$$

$$u_t = \infty \quad on \quad \partial\Omega_t \tag{3.72}$$

where $t = \sqrt{\lambda}$. Concerning the domains, Ω_t is convex, bounded, smooth and $\Omega_{t_1} \subset \Omega_{t_2}$ for $t_1 < t_2$. Moreover, $\cup_t \Omega_t = \mathbb{R}^N$. The main step of the proof is given in the following general result concerning (3.1)-(3.2):

Lemma 18. *Let Ω and f as in the theorem. Then there exists $0 < d_1 \le d_2$ depending only on f such that:*
(a) If $diam(\Omega) \le d_1$, then there exists two blow-up solutions of (3.1)-(3.2) and one is nonnegative;
(b) If $r(\Omega) > d_2$, there are no blow-up solutions, where we define $r(\Omega)$ as the radius of the largest ball contained in Ω.

Proof. (a) The only time when we use in the proof of theorem 18 the fact that $f(0) = 0$ is when we find the function identically 0 to be a subsolution of (3.1)-(3.26). If we find in another way a subsolution $\underline{u}$ for this problem, the whole proof of theorem 18 works the same, with the only change that the first solution will be greater than $\underline{u}$ and the other will cross $\underline{u}$.
To find a subsolution, we start with the solution U of the linear problem

$$\Delta U = -1 \ \ in \ \ \Omega$$

$$U = 0 \ \ on \ \ \partial\Omega$$

By usual estimates on the solution, we obtain that $U(x) \le C(diam(\Omega))^2$.
Define also

$$D := \sup_{c>0} \frac{c}{\sup\limits_{s\in[0,c]} f(s)}$$

which is a number in $(0, \infty)$ and it is attained in some point $\overline{c} > 0$, since $\lim\limits_{s\to\infty} \frac{s}{f(s)} = 0$. Set

$$\underline{u}(x) := \overline{c}(1 - \frac{U(x)}{max(U)}) \ \ for \ \ x \in \overline{\Omega}$$

Then $\underline{u} \le \overline{c}$ and

$$\Delta\underline{u} = -\overline{c}\frac{\Delta U}{max(U)} \ge \frac{\overline{c}}{C(diam(\Omega))^2} \ge f(\underline{u})$$

hence $\underline{u}$ is a subsolution for (3.1)-(3.26) for $\overline{c}$ and for $diam(\Omega) \le d_1 := \sqrt{\frac{C_1}{C}}$.
(b) We remark that $f - m_f$ satisfies (A), (B), (C), hence there exists $b > 0$ such that

$$f(s) \ge \overline{f}(s) + \frac{1}{2}m_f, \ \ \ for \ \ s > 0$$

and

$$f(s) \ge b|s|^p + \frac{1}{2}m_f, \ \ \ for \ \ s < 0$$

where $\overline{f}$ is the increasing modification of f in $[0, s_0]$ with $\overline{f}(0) = 0$, given first in the proof of proposition 3.
If we change the way of regarding proposition 3 by fixing the value of k, we deduce that (3.26)-(3.27) has no solution for large R(i.e. posed in a large ball). We apply this observation for $k = m_f$. It follows that there exists R^* such that (3.1)-(3.26) has no solution in any ball of radius $R \le R^*$ for $c \ge s_0$. If (3.1)-(3.2) has a solution u in a domain Ω with $r(\Omega) > R^*$, then u becomes a subsolution on a ball of radius R^* contained in Ω for (3.1)-(3.26) with $c \ge \sup\limits_{B_{R^*}} u$. This is a contradiction, since the function which is identically c proves to be always a supersolution. ∎

Proof. (Theorem 19) For t small, the diameter of Ω_t is also small, hence $diam(\Omega_t) \le d_1$. By part (a) of the last lemma it follows that (3.71)-(3.72) has at least two solutions. Set

$$t^* := \sup\{t : (3.71) - (3.72) hasasolution\} \tag{3.73}$$

By the definition, for $t > t^*$ there are no solutions. Also by part (b) of the lemma, $t^* < \infty$. We show that for $t < t^*$, (3.71)-(3.72) posed in Ω_t has two solutions. But any solution u_{t_1} of (3.71)-(3.72) in Ω_{t_1} is a subsolution of (3.71) posed in Ω_{t_2} with $t_2 < t_1$, with the boundary value $c = \sup_{\Omega_{t_2}} u_{t_1}$. But as we have remarked, the only difficulty is to find a subsolution, that we just did; the proof of theorem 18 applies then without essential changes. We also find the critical value $\lambda^* = t^{*2}$. □

Remark. *One can see easily from the proof of the last result that* $\lambda^* \to 0$ *as* $r(\Omega) \to \infty$ *and* $\lambda^* \to \infty$ *as* $diam(\Omega) \to 0$.

The same results can be generalized to the non-autonomous case:

$$\Delta u = g(x, u) \quad in \quad \Omega \tag{3.74}$$

and (3.2), where we only require $u \in C^1(\Omega)$ for a weak solution. The hypotheses on g are more technical, but the changes in the proof are not essential and are indicated in great detail in the paper [AfR97].

3.5 Symmetry Results

The idea of obtaining symmetry results for solutions of partial differential equations is very deep and very useful in practice, since we deal in most of the cases with domains possessing some symmetries. For elliptic equations, the first important and the most well-known results are those of Gidas-Ni-Nirenberg ([GNN79]), where they prove that any positive solution u of the problem

$$-\Delta u = f(u) \quad in \quad B(0, R) \tag{3.75}$$

$$u = 0 \quad on \quad \partial B(0, R) \tag{3.76}$$

where f is of class C^1, is radially symmetric and moreover $\frac{\partial u}{\partial r} < 0$. Another result of them is that if u is a classical(i.e. C^2) solution of (3.75) in the annulus $R_1 < |x| < R_2$, without any condition on the boundary, then $\frac{\partial u}{\partial r} < 0$ on the upper half of the annulus: $\frac{R_1+R_2}{2} \leq |x| < R_2$. We mention here that the radial symmetry of u does not remain valid in the case of the annulus.
It was a conjecture proposed by Haim Brezis saying that similar results are valid for solutions with blow-up on the boundary. This was proved very recently by Laurent Veron and Alessio Poretta(see [PV06]), who extended the key results in the paper of [GNN79]. In this section we will present in detail these symmetry results.
For a function $u \in C^1(B)$, where B is an arbitrary ball centered in origin, we define the radial derivative of u by

$$\frac{\partial u}{\partial r}(x) := \left\langle \nabla u(x), \frac{x}{|x|} \right\rangle \tag{3.77}$$

and the tangential gradient of u by

$$\nabla_\tau u(x) := \nabla u(x) - \frac{\langle \nabla u(x), x \rangle}{|x|^2} x \tag{3.78}$$

The main symmetry result for blow-up solution is

Theorem 20. *Assume that f is a Lipschitz continuous function and let u be a solution of*

$$-\Delta u = f(u) \quad in \quad B(0,R) \tag{3.79}$$

$$u(x) \to \infty \quad as \quad x \to \partial B(0,R) \tag{3.80}$$

where $R > 0$. If there holds

$$\lim_{|x| \to R} \frac{\partial u}{\partial r}(x) = \infty \tag{3.81}$$

and

$$|\nabla_\tau u(x)| = o(\frac{\partial u}{\partial r}(x)) \quad as \quad |x| \to R \tag{3.82}$$

then u is radially symmetric and $\frac{\partial u}{\partial r}(x) > 0$ on $B(0,R) \setminus \{0\}$.

Proof. The idea of the proof is to use again, as in [GNN79], the moving plane method, which has been already discussed in the previous section. However, here it is a different setting and we will present all the details of the technique for convenience.
We start with some general notations. For $0 < \lambda < R$, set $T_\lambda := \{x : x_1 = \lambda\}$, where we will often write $x = (x_1, x^{'})$, where $x^{'} = (x_2, ..., x_N)$. We also denote $\Sigma_\lambda := \{x \in B(0,R) : \lambda < x_1 < R\}$ the region between the hyperplane T_λ and the boundary of the ball. For $x = (x_1, x^{'}) \in \Sigma_\lambda$, we denote its symmetric with respect to T_λ by $x_\lambda := (2\lambda - x_1, x^{'})$. Let $u_\lambda(x) := u(x_\lambda)$ and $\Sigma^{'}_\lambda := \{x \in B(0,R) : 2\lambda - R < x_1 < \lambda\}$ which represents the reflection of Σ_λ with respect to T_λ.
We start the moving plane method from the point $P_0 = (R, 0)$ on the boundary. From the conditions (3.81) and (3.82), it follows that there exists a small ball of radius δ_0 such that

$$\frac{\partial u}{\partial x_1}(x) > 0, \quad \forall x \in B(0,R) \cap B(P_0, \delta_0) \tag{3.83}$$

Then u is increasing along the lines $x^{'} = constant$ in this intersection of balls. Hence we deduce that $u(x_\lambda) < u(x)$ for all λ such that $\Sigma_\lambda \subset B(P_0, \delta_0)$ and $\Sigma^{'}_\lambda \subset B(P_0, \delta_0)$ and for all $x \in \Sigma_\lambda$. But this happens for $\lambda_0 < \lambda < R$, where $\{x : x_1 = \lambda_0\}$ is the hyperplane contains the intersection surfaces of the two spheres $\partial B(0,R)$ and $\partial B(P_0, \delta_0)$. By elementary geometry, we obtain that $\lambda_0 = R - \frac{\delta_0^2}{2R}$.
Also, in this case we have

$$\frac{\partial u}{\partial x_1}(x) > 0 \tag{3.84}$$

for all $x \in \Sigma_\lambda$.
Let

$$\lambda^* := \inf\{\lambda > 0 : u(x_\lambda) < u(x) \quad and \quad (3.84) \quad holds, \forall x \in \Sigma_\lambda\} \tag{3.85}$$

Suppose that $\lambda^* > 0$. By construction $u(x_{\lambda^*}) \leq u(x)$ in Σ_{λ^*}. Define $K_{\lambda^*} := T_{\lambda^*} \cap \partial B(0,R)$, which is a compact set. We make now the crucial remark that what we did for P_0 works for any other boundary point, i.e for any $P \in \partial B(0,R)$ with $x_1(P) > 0$ there exists a small radius $\delta(P)$ such that (3.83) is valid in $B(P, \delta(P)) \cap B(0,R)$. By compactness of K_{λ^*} and this remark, a standard argument shows that there exists an ε-neighborhood V_ε of K_{λ^*} such that (3.83) holds in $V_\varepsilon \cap B(0,R)$. Set $D_\varepsilon := B(0, R - \frac{\varepsilon}{2}) \cap \Sigma_{\lambda^*}$. Define $w := u - u_{\lambda^*}$ and the function

$$a(x) := \frac{f(u) - f(u_{\lambda^*})}{u - u_{\lambda^*}}$$

Then by direct calculation we observe that $\Delta w - aw = 0$ in D_ε and $w \leq 0$ and w is not identically 0. By the classical maximum principle it follows that $w > 0$ and $\frac{\partial u}{\partial x_1} > 0$ on $T_{\lambda^*} \cap \partial D_\varepsilon$. Since ε is

arbitrarily small, we obtain that $u > u_{\lambda^*}$ in Σ_{λ^*}. On the other hand, by the strict inequality and the continuity of ∇u, there exists $\eta > 0$ such that (3.83) holds for every $x \in \{x : \lambda^* - \eta < x_1 < \lambda^* + \eta\} \cap B(0,R)$.
By minimality of λ^*, it follows that there exists a sequence $(\lambda_n)_n$ which increases to λ^* and a sequence $(x_n)_n$ converging to a point $x^* \in \overline{\Sigma_{\lambda^*}}$ such that $u(x_n) \leq u((x_n)_{\lambda_n})$. We obtain a contradiction by investigating all the possible positions of x^*.
Since $u > u_{\lambda^*}$ in Σ_{λ^*}, x^* does not belong to Σ_{λ^*}, hence it must be on $\partial\Sigma_{\lambda^*}$. From (3.83) for $\lambda^* - \eta < x_1 < \lambda^* + \eta$, we deduce that x^* is not on T_{λ^*}; otherwise, by using the Lagrange theorem, we have

$$0 \geq u(x_n) - u((x_n)_{\lambda_n}) = (x_n - (x_n)_{\lambda_n})\frac{\partial u}{\partial x_1}(x_n^{'}) > 0$$

for large n, since $\lim_{n\to\infty} x_n = \lim_{n\to\infty}(x_n)_{\lambda_n} = x^*$. But this is obviously a contradiction. Finally, if $x^* \in \partial\Sigma_{\lambda^*} \setminus T_{\lambda^*}$, then $x^* \in \partial B(0,R)$ and in this case $u(x_n) - u((x_n)_{\lambda_n}) \to \infty$.
This contradiction shows that $\lambda^* = 0$ and by changing x_1 into $-x_1$ it follows that the function u is symmetric with respect to the hyperplane $x_1 = 0$. By doing the same for every x_i for $1 \leq i \leq N$, we obtain the radial symmetry of u. □

As we see, we need for our function to satisfy some technical conditions in order to derive the symmetry of the solutions. In [PV06] it is given a sufficient condition in order (3.81)-(3.82) to hold. For this we recall the Keller-Osserman condition (3.14).

Theorem 21. *Assume that f is locally Lipschitz continuous, convex on $[a,\infty)$ for some $a > 0$ and satisfies (3.14). Then any solution of (3.79)-(3.80) is radially symmetric.*

Proof. We will not give a complete proof of this theorem; instead of this, we will present the main ideas of it. The proof starts by decomposing f into a sum of two functions f_1 and f_2, where f_1 is convex, increasing and satisfies (3.14) and $f_2 = 0$ on $[M,\infty)$ for some $M > 0$. Then there exists a number $K_0 > 0$ such that

$$|\Delta u - f_1(u)| = |f_2(u)| \leq K_0$$

By considering an auxiliary function $\phi(x) := \frac{R^2-|x|^2}{2N}$ and using an uniqueness result of [MV97], we find that $v - K_0\phi \leq u \leq v + K_0\phi$, where v is the unique blow-up solution of the equation

$$-\Delta v + f_1(v) = 0 \quad in \quad B(0,R) \tag{3.86}$$

The next step is to pass to spherical coordinates in $\mathbb{R}^N$ in order to use the rotation symmetry of the equation. We introduce the coordinates (r,σ) and some geodesics of the sphere given by the exponential map $\gamma_j(t) = exp(tA_j)(\sigma)$ where $(A_j)_{j=1,N-1}$ are orthogonal antisymmetric matrices. Then these geodesics are orthogonal. Fix $h > 0$. The function $u^h(x) = u(r, exp(hA_j)\sigma)$ is also a solution of (3.86) and it satisfies the same inequality $v - K_0\phi \leq u^h \leq v + K_0\phi$. Then

$$\lim_{|x|\to R}(u(x) - u^h(x)) = 0 \tag{3.87}$$

Let r_0 be such that $u(x) \geq M$ for $|x| \geq r_0$. Then $\Delta u^h = f_1(u^h)$ in the annulus $r_0 < |x| < R$ and for $|x| = r_0$ there exists a number $L > 0$ such that $|(u - u^h)(x)| \leq L|h|$. Let ψ be the unique harmonic function such that $\psi = 1$ on $\partial B(0,r_0)$ and $\psi = 0$ on $\partial B(0,R)$ and the auxiliary function

$$v^h := u^h + |h|L\psi \tag{3.88}$$

Then it is easy to check that $\lim_{|x|\to R} v^h(x) - u(x) = 0$ and, by maximum principle, that $v^h \geq u$.
The last step is to consider the Lie derivative along the vector fields $\sigma \to A_j\sigma$, defined as

$$L_{A_j}u(r,\sigma) := \frac{du(r, exp(tA_j\sigma))}{dt}|_{t=0}$$

and to remark that

$$|L_{A_j}u(r,\sigma)| \leq L\psi(x) \leq C(R-r)$$

hence (3.82) holds. To prove (3.81) the ideas are similar , but we change the auxiliary function v^h into $w^h := \frac{u^h + u^{-h} - 2u}{h^2}$. We leave the detail as an exercise for the reader. They can be also taken from [PV06]. □

Finally, let us remark that the convexity hypothesis is required only to insure uniqueness of blow-up solutions for the equation

$$-\Delta v + f_1(v) = 0 \ \ in \ \ B(0,R)$$

3.6 Keller-Osserman Type Conditions

In this section we will make a further investigation of the importance of the Keller-Osserman type conditions and their connection with the blow-up solutions for (3.1)-(3.2). We will present one of the deepest result in the theory, which gives a characterisation of existence in terms of integral conditions.
Let us consider a function $f : [0,\infty) \to [0,\infty)$ which is of class C^1 and $f(0) = 0$. As usual, we denote $F(t) := \int\limits_0^t f(s)ds$. Let's remark from the beginning that no monotonicity of f is assumed. Define

$$\phi(\alpha) := \frac{1}{\sqrt{2}} \int\limits_\alpha^\infty \frac{dt}{\sqrt{F(t) - F(\alpha)}} \tag{3.89}$$

We introduce the following two conditions, suggested by the paper [AfR97](see section 3.4):

Definition 3. *We say that f satisfies the Keller-Osserman condition ((KO) for short) if there exists some α such that $\phi(\alpha) < \infty$. We say that f satisfies the strong Keller-Osserman condition ((KOs) for short) if $\liminf_{\alpha\to\infty} \phi(\alpha) = 0$.*

We remark that (KO) is a little bit different from (3.14), in fact implies (3.14). Also obviously (KOs) implies (KO). If f is nondecreasing, then F is convex, hence if we assume (3.14), then we may write:

$$\phi(\alpha) = \int\limits_{\alpha+1}^\infty \frac{dt}{\sqrt{F(t) - F(\alpha)}} + \int\limits_\alpha^{\alpha+1} \frac{dt}{\sqrt{F(t) - F(\alpha)}}$$

But the first integral is finite from (3.14) and for the second we have:

$$\int\limits_\alpha^{\alpha+1} \frac{dt}{\sqrt{F(t) - F(\alpha)}} \leq \int\limits_\alpha^{\alpha+1} \frac{dt}{\sqrt{(t-\alpha)f(\alpha)}} < \infty$$

We have just proved that for a monotone function all the Keller-Osserman condition that we introduce are equivalent. Hence, this study is interesting only in absence of monotonicity.
A good question at this moment could be that if (KO) and (KOs) are or not equivalent in general and our definitions are without object. We are saved by the following elementary example that we develop in the next technical lemmas.

Lemma 19. *Suppose that there exists $\eta > 2$ and $A > 0$ such that $\frac{F(t)}{t^\eta}$ is increasing on (A, ∞), where F is a primitive of f. Then f satisfies (KOs).*

Proof. From elementary analysis we know that

$$\frac{1}{\sqrt{1-x}} = \sum_n a_n x^n$$

where $a_n = \frac{(2n)!}{(n!)^2 4^n} \leq \frac{C}{\sqrt{n}}$, by Wallis' formula. Hence we do this for $\phi(\alpha)$ (and we skip the constant $\frac{1}{\sqrt{2}}$)

$$\begin{aligned}
\phi(\alpha) &= \int_\alpha^\infty (F(t) - F(\alpha))^{-\frac{1}{2}} dt \\
&= \int_\alpha^\infty (F(t))^{-\frac{1}{2}} (1 - \frac{F(\alpha)}{F(t)})^{-\frac{1}{2}} dt \\
&= \sum_n a_n F(\alpha)^n \int_\alpha^\infty \frac{dt}{F(t)^{n+\frac{1}{2}}} \\
&= \sum_{n=0}^N a_n F(\alpha)^n \int_\alpha^\infty \frac{dt}{F(t)^{n+\frac{1}{2}}} + \sum_{n=N}^\infty a_n F(\alpha)^n \int_\alpha^\infty \frac{dt}{F(t)^{n+\frac{1}{2}}} \\
&= S_1 + S_2
\end{aligned}$$

But $S_1 \leq (\sum_{n=0}^N a_n) \int_\alpha^\infty \frac{dt}{\sqrt{F(t)}}$, which goes to 0 as $\alpha \to \infty$ by the condition that $\frac{F(t)}{t^\eta}$ is increasing for some $\eta > 2$. We also have

$$\frac{1}{F(t)} < (\frac{\alpha}{t})^\eta \frac{1}{F(\alpha)}$$

hence

$$\begin{aligned}
S_2 &\leq \sum_{n=N+1}^\infty a_n F(\alpha)^n \alpha^{-\eta(n+\frac{1}{2})} F(\alpha)^{-n-\frac{1}{2}} \int_\alpha^\infty t^{\eta(n+\frac{1}{2})} \\
&= \sum_{n=N+1}^\infty \frac{\alpha}{\sqrt{F(\alpha)}} \frac{a_n}{1 + \eta(n + \frac{1}{2})} \\
&\leq \frac{C\alpha}{\eta\sqrt{F(\alpha)}} \sum_{n=N+1}^\infty \frac{1}{n\sqrt{n}}
\end{aligned}$$

which goes to 0 as $\alpha \to \infty$ since $\lim_{\alpha\to\infty} \frac{\alpha}{\sqrt{F(\alpha)}} = 0$. Hence f satisfies (KOs). □

Lemma 20. *The function $f(t) = t^2(1 + cost)$ satisfies (KOs). Moreover, for this function we have*

$$\liminf_{\alpha\to\infty} \phi(\alpha) = 0, \quad \limsup_{\alpha\to\infty} \phi(\alpha) = \infty \tag{3.90}$$

Proof. By computing the primitive of f, we find

$$F(t) = \frac{1}{3}t^3 + t^2 sint + 2tcost - 2sint$$

hence the previous lemma applies for any $\eta \in (2,3)$ and for A sufficiently large. Hence f satisfies (KOs). For the second limit, we remark that if we take $\alpha_k := (2k+1)\pi$, then, by a limited Taylor development and taking into account the exact value of f, we have:

$$F(t) = F(\alpha_k) + (t-\alpha_k)f(\alpha_k) + \frac{1}{2}(t-\alpha_k)^2 f'(\alpha_k) + o((t-\alpha_k)^2)$$

hence $F(t) - F(\alpha_k) \sim \frac{1}{2}\alpha_k^2(t-\alpha_k)^2$. But the function in the right-hand side is not integrable, hence $\lim_{k\to\infty} \phi(\alpha_k) = \infty$ and we are done. $\square$

These two lemmas show that there exists oscillating functions satisfying (KOs), hence the condition is interesting.
We pass now to the main part of this section, that of characterizing these conditions by results of existence of blow-up solutions. For the beginning, we need some technical preparations, i.e. a sort of sub- and supersolution principle for blow-up solutions. Recall first theorem 1 and especially the fact that there exists a minimal solution for any pair $(\underline{u}, \overline{u})$ of sub- and supersolution. We state here a more refined property of the minimal solution, whose proof is similar and can be consulted in [Rad] or in [DDGR06].

Proposition 5. *Let $\Omega \in \mathbb{R}^N$ be a smooth, bounded domain and $f \in C^1(\mathbb{R})$, $g \in C(\partial\Omega)$. Assume there exists a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of the problem*

$$\Delta u = f(u) \quad in \quad \Omega \tag{3.91}$$

$$u = g \quad on \quad \partial\Omega \tag{3.92}$$

such that $\underline{u} \leq \overline{u}$. Then there exists a unique solution $u \in C(\overline{\Omega})$ of (3.91)-(3.92) such that $\underline{u} \leq u$ and for any $\omega \subset \Omega$ and any function $\overline{v} \in C(\overline{\omega})$ which satisfies

$$\Delta \overline{v} \leq f(\overline{v}) \quad in \quad \omega \tag{3.93}$$

$$\overline{v} \geq \underline{u} \quad in \quad \omega, \quad \overline{v} \geq u \quad on \quad \partial\omega \tag{3.94}$$

then $u \leq \overline{v}$ in ω. We call the solution u the minimal solution relative to $\underline{u}$.

The following remark will be crucial in the next theorems. If in the notations and conditions above $\Omega = B(0,R)$ and the subsolution $\underline{u}$ is radial, then the minimal solution associated is also radial. For this it suffices to apply the minimality principle with any rotation of u. We obtain several minimal solutions relative to the same subsolution, hence all must be equal. It follows that $u(x) = u(Ax)$ for all rotation matrices A, hence u is radial.
We need a similar principle for blow-up solutions:

Proposition 6. *(Minimality principle for blow-up solutions) Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $f \in C^1(\mathbb{R})$. Assume that there exists a function $\underline{u} \in C(\overline{\Omega})$ such that $\Delta\underline{u} \geq f(\underline{u})$ in Ω and a function $v \in C(\Omega)$ such that*

$$\Delta v \leq f(v) \quad in \quad \Omega, \quad \lim_{x\to\partial\Omega} v(x) = \infty$$

and suppose that $v \geq \underline{u}$. *Then there exists a unique solution* $u \in C^1(\Omega)$ *of (3.1)-(3.2) such that* $\underline{u} \leq u$ *and* $u \leq \overline{v}$ *in* ω *for any* $\omega \subset \Omega$ *and for any function* $\overline{v} \in C(\omega)$ *satisfying*

$$\Delta \overline{v} \leq f(\overline{v}) \quad in \quad \omega \tag{3.95}$$

$$\overline{v} \leq \underline{u} \quad in \quad \omega, \quad \lim_{x \to \partial\omega} \overline{v} = \infty \tag{3.96}$$

We call u *the minimal blow-up solution relative to* $\underline{u}$.

Proof. The uniqueness is clear from the minimality assumption. Let $n > \|\underline{u}\|_{L^\infty(\Omega)}$. From the minimality principle for solutions(proposition 5) there exists u_n the minimal solution relative to the subsolution $\underline{u}$ for the same equation, but with boundary condition $u = n$ on $\partial\Omega$. By the classical maximum principle it follows that the sequence $(u_n)_n$ is nondecreasing and $u_n \leq n$.
Consider now a smooth open set $\omega \subset \Omega$ such that $v \geq n$ on $\partial\omega$. By the minimality principle, $v \geq u_n$ in ω and this is valid for any ω, hence $(u_n)_n$ is bounded uniformly on compact sets in Ω. It follows, using the classical elliptic estimates, that there exists $u := \lim_{n\to\infty} u_n$ which is a solution of (3.1)-(3.2). Take again $\omega \subset \Omega$ and take $\overline{v} \in C(\omega)$ satisfying (3.95)-(3.96). Take $\tilde{\omega} \subset \omega$ such that $\overline{v} \geq n$ on $\partial\tilde{\omega}$. Again by proposition 5, we have that $u_n \leq \overline{v}$ on $\tilde{\omega}$. Hence $u_n \leq \overline{v}$ in ω and we are done by letting $n \to \infty$. □

Before stating the main results, we still need another technical lemma:

Lemma 21. *Let* $v \in C^2(0, R)$ *be a nondecreasing function which satisfies:*

$$v^{''} + \frac{N-1}{r} v^{'} = f(v) \quad in \quad (0, R) \tag{3.97}$$

Then, for any $0 < r_1 < r_2 < R$ *we have:*

$$\frac{1}{\sqrt{2}} \int_{v(r_1)}^{v_{r_2}} \frac{ds}{\sqrt{F(s) - F(v(r_1))}} \geq \frac{r_1}{N-2}\left(1 - \left(\frac{r_1}{r_2}\right)^{N-2}\right) \tag{3.98}$$

for $N \geq 3$, *or*

$$\frac{1}{\sqrt{2}} \int_{v(r_1)}^{v_{r_2}} \frac{ds}{\sqrt{F(s) - F(v(r_1))}} \geq r_1 log \frac{r_2}{r_1} \tag{3.99}$$

for $N = 2$.

Proof. By multiplying in both sides by r^{N-1} and integrating, we obtain

$$\frac{1}{2}((r^{N-1}v^{'})^2)^{'} = r^{2N-2} f(v) v^{'}$$

or, after integration

$$\frac{r^{2N-2}v^{'}(r)^2}{2} - \frac{r_1^{2N-2}v^{'}(r_1)^2}{2} = \int_{r_1}^{r} t^{2N-2} v^{'}(t) f(v(t)) dt$$

hence

$$r^{2N-2}v^{'}(r)^2 \geq 2r_1^{2N-2} \int_{r_1}^{r} v^{'}(t) f(v(t)) dt = 2r_1^{2N-2}(F(v(r)) - F(v(r_1)))$$

We extract the square root and after dividing by the right hand side we obtain:

$$\frac{1}{\sqrt{2}}\frac{v^{'}(r)}{\sqrt{F(v(r))-F(v(r_1))}} \geq (\frac{r_1}{r})^{N-1}$$

After integration on $[r_1, r_2]$ and a change of variables, one has exactly the desired inequalities. □

Now we pass to, in my opinion, the most important results from all this chapter. These were proved in the very recent paper [DDGV06].

Theorem 22. *The function f satisfies (KO) if and only if (3.1)-(3.2) has a nonnegative solution on some ball.*

Theorem 23. *The function f satisfies (KOs) if and only if (3.1)-(3.2) has a nonnegative solution on every smooth bounded domain.*

Proof. (Theorem 22) Suppose that f satisfes (KO). Then there exists a number $\alpha > 0$ such that $\phi(\alpha) < \infty$. We have two different cases.
Case 1 Suppose first that $\phi(\alpha) < \frac{1}{|N-2|}$, where $N \geq 3$. Consider the problem:

$$\Delta u = f(u) \quad in \quad B(0,1) \tag{3.100}$$

$$u = \alpha \quad on \quad \partial B(0,1) \tag{3.101}$$

Then $\underline{u} = 0$ is a subsolution, $\overline{u} = \alpha$ is a supersolution and we consider u the minimal solution with respect to $\underline{u}$. Since $\underline{u}$ is radial, by the previous discussion u is radial. Hence we may consider the problem

$$v^{''} + \frac{N-1}{r}v^{'} = f(v) \quad in \quad (0,R) \tag{3.102}$$

$$v^{'}(0) = 0, \quad v(0) = u(0) \tag{3.103}$$

where we consider R such that $(0,R)$ is the maximal interval of existence. We just need to show that $R < \infty$. If not, we can apply lemma 21 for $r_1 = 1$ and for any $r_2 > 1$. We have

$$\frac{1}{\sqrt{2}}\int_{r_1}^{r_2}\frac{ds}{\sqrt{F(s)-F(v(r_1))}} \geq \frac{1}{N-2}(1-(\frac{1}{r_2})^{N-2})$$

Since $v(1) = \alpha$, we obtain

$$\frac{1}{\sqrt{2}}\int_{\alpha}^{v(r_2)} \geq \frac{1}{N-2}(1-(\frac{1}{r_2})^{N-2})$$

which implies that $\phi(\alpha) \geq \frac{1}{N-2}$, which is a contradiction. Hence $R < \infty$ and it is easy to see that v becomes a radial solution of (3.1)-(3.2) in the ball $B(0,R)$.
Case 2 If $\phi(\alpha) \geq \frac{1}{N-2}$, we choose a large constant M such that $\frac{1}{M}\phi(\alpha) < \frac{1}{N-2}$. We apply the result of the first case, we obtain a solution u on $B(0,R)$ for the nonlinearity $M^2 f$. Hence the function $\tilde{u}(x) := u(\frac{x}{M})$ is a blow-up solution of our equation on $B(0,MR)$.
Conversely, suppose that (3.1) has a positive blow-up solution on some ball $B(0,R)$. We have to prove that there exists α such that $\phi(\alpha) < \infty$. But the zero function is still a subsolution for

(3.1)-(3.2) on $B(0,R)$, hence we may assume that u is the minimal solution relative to $\underline{u}=0$. By a remark from above, u is radially symmetric. Hence the function $v(r)=u(x)$, where $r=|x|$, solves

$$v^{''}+\frac{N-1}{r}v^{'}=f(v) \tag{3.104}$$

which writes equivalently

$$(r^{N-1}v^{'})^{'}=r^{N-1}f(v) \tag{3.105}$$

having $(0,R)$ as the maximal interval of definition. By multiplying (3.105) by $r^{N-1}v^{'}$ and integrating between 0 and r, we have

$$\frac{1}{2}r^{2N-2}v^{'}(r)^2=\int_0^r t^{2N-2}f(v(t))v^{'}(t)dt\leq r^{2N-2}(F(v(r))-F(v(0)))$$

By integrating again on $(0,R)$ we obtain:

$$\int_0^R \frac{v^{'}(r)}{\sqrt{2(F(v(r))-F(v(0)))}}dr\leq R$$

hence

$$\int_\alpha^{v(R)} \frac{ds}{\sqrt{2(F(s)-F(\alpha))}}\leq R$$

Since $v(R)=\infty$, it follows that $\phi(\alpha)\leq R<\infty$. □

Proof. (Theorem 23) Suppose first that f satisfies (KOs), i.e. $\liminf_{\alpha\to\infty}\phi(\alpha)=0$. We divide the proof of this part into two steps.

Step 1: We show first that (3.1)-(3.2) has a solution on every ball with radius sufficiently small. Using theorem 22 and the fact that (KOs) implies (KO), there exists a blow-up solution on a ball $B(0,R)$. Let

$$R_0:=\inf\{R>0:(3.1)-(3.2)\ \ has\ \ a\ \ solution\ \ on\ \ B(0,R)\}$$

and suppose that $R_0>0$. From (KOs) we deduce the existence of a sequence $\beta_n\to\infty$ such that $\phi(\beta_n)\to 0$. We consider the problem:

$$\Delta u_n=f(u_n)\ \ in\ \ B(0,\frac{R_0}{2}) \tag{3.106}$$

$$u_n=\beta_n\ \ on\ \ \partial B(0,\frac{R_0}{2}) \tag{3.107}$$

We remark that $\underline{u_n}\equiv 0$ is a subsolution and $\overline{u_n}\equiv\beta_n$ is a supersolution of (3.106)-(3.107). Hence there exists a minimal solution u_n relative to $\underline{u_n}\equiv 0$, which is radial. Then the function $v_n(r):=u_n(x)$ satisfies (3.104) on an interval $(0,R_n)$ with $R_n>R_0$ with the initial conditions $v_n(0)=u_n(0)$ and $v_n^{'}(0)=0$. We apply now lemma 21 for $r_1:=\frac{R_0}{2}$ and $r_2:=R_0$. We find

$$\frac{1}{\sqrt{2}}\int_{v_n(\frac{R_0}{2})}^{v_n(R_0)}\frac{ds}{\sqrt{F(s)-F(v_n(\frac{R_0}{2}))}}\geq\frac{R_0}{2(N-2)}(1-(\frac{1}{2})^{N-2})$$

But $R_n > \frac{R_0}{2}$, hence $v_n(\frac{R_0}{2}) = u_n(\frac{R_0}{2}) = \beta_n$. It follows that

$$\frac{1}{2}\int_{\beta_n}^{v_n(R_0)} \frac{ds}{\sqrt{F(s)-F(\beta_n)}} \geq C > 0$$

which implies that $\phi(\beta_n) \geq C > 0$, contradicting the choice of β_n. Hence $R_0 = 0$.

Step 2: We prove that (3.1)-(3.2) has a solution on every smooth bounded domain Ω. For $n \in \mathbb{N}$, we consider the problem

$$\Delta u = f(u) \quad in \quad \Omega \tag{3.108}$$

$$u = n \quad on \quad \partial\Omega \tag{3.109}$$

which has obviously $\underline{u} \equiv 0$ as a subsolution and $\overline{u} \equiv n$ as a supersolution. Let u_n be the minimal solution of (3.108)-(3.109). Fix $x \in \Omega$ and consider a small ball $B(x,r) \subset \Omega$ such that (3.1)-(3.2) has a solution u_r on this ball(this is possible from step 1). By the minimality principle for blow-up solutions, $u_n \leq u_r$ in $B(x,r)$, hence $(u_n)_n$ is uniformly bounded in $B(x,\frac{r}{2})$. From the maximum principle we deduce that the sequence $(u_n)_n$ is increasing. Hence there exists $u(x) := \lim_{n\to\infty} u_n(x)$, which is a blow-up solution.

For the converse, suppose that there exists a solution of (3.1)-(3.2) on every smooth bounded domain Ω. For every $n \in \mathbb{N}$, we denote by u_n the minimal blow-up solution of (3.1) in the ball $B(0,\frac{1}{n})$. Then u_n is radial and set $\beta_n := u_n(0)$. We show that $\beta_n \to \infty$ and $\phi(\beta_n) \to 0$. From the same calculations done in the previous steps, we obtain

$$0 \leq \int_{\beta_n}^{v_n(\frac{1}{n})} \frac{ds}{\sqrt{2(F(s)-F(\beta_n))}} \leq \frac{1}{n}$$

hence $0 \leq \phi(\beta_n) \leq \frac{1}{n}$, i.e. $\lim_{n\to\infty} \phi(\beta_n) = 0$. Suppose that (on a subsequence) $\beta_n \to \beta_0 < \infty$. Then $\phi(\beta_n) \to \phi(\beta_0)$, hence $\phi(\beta_0) = 0$, which is obviously impossible since the function under the integral sign is positive. □

3.7 Uniqueness and Asymptotic Behaviour

In this section we study the question of uniqueness of blow-up solutions of (3.1), together with the classical question in PDE of asymptotic behaviour of solutions. In our case, by studying the "asymptotic behaviour" we understand to try to give an answer at the question "How fast does converge the blow-up solutions to ∞ ?". The asymptotic behaviour results are often very important also in paractical applications, because they give a more precise estimate of the blow-up rate.

The uniqueness problem has been treated since 1957, when Keller studied in [Ke57] the uniqueness of blow-up solutions for the first time. In modern times, the two famous papers of C. Bandle and M. Marcus, [BM92] and [BM95] establishes uniqueness results and obtain estimates for the asymptotic behaviour of solutions using some special conditions on the function f:

(i) $f \in C^1(\mathbb{R})$, $f(s) > 0$ and $f'(s) > 0$ for some $s > s_0$;

(ii) $g(s) := \int_s^\infty \frac{d\tau}{\sqrt{2F(\tau)}} < \infty$, for all $s > s_0$, where $F(\tau) = \int_{s_0}^\tau f(s)ds$;

(iii) $\liminf_{s\to\infty} \frac{g(ts)}{g(s)} > 1$, for all $t \in (0,1)$.

Condition (iii) is obviously achieved for example when $\lim_{s\to\infty}\frac{tf(s)}{f(ts)} > 1$, for all $t \in (0,1)$. Condition (ii) is of Keller-Osserman type, and condition (iii) can be seen as a superlinearity condition at ∞. A typical function satisfying all these is $f(s) = Cs^p$ for $p > 1$ and $C > 0$.
More recent, the conditions imposed on the nonlinearity were simplified, and it has been seen that uniqueness has a strong relation, in the most general cases, with the convexity of f.
Laurent Veron and Moshe Marcus have proved in two papers ([MV97] and [MV03]) some very general results of existence, uniqueness and asymptotic behaviour. The generality of these results comes from the fact that they consider much more general domains, even non-smooth or domains with corners. Here the topology plays a key-role and we will not enter in this in detail. We will discuss more on this in the last section.
We prove first a very interesting and simple uniqueness result, appeared in [MV03], which deals only with an increasing nonlinearity. But also from here we will remark the importance of convexity.

Theorem 24. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain(not necessarily smooth) and $f \in C(\mathbb{R})$, such that $f(0) = 0$, and f is nondecreasing and convex. Suppose that there exists a maximal solution $\overline{u}$ of (3.1) in Ω and that there exists two constants $C > 0$ and $\delta > 0$ such that*

$$0 \le \overline{u}(x) \le Cu(x), \quad \forall x \in \Omega, \quad d(x, \partial\Omega) \le \delta \tag{3.110}$$

and for any blow-up solution u of (3.1). Then there exists at most one blow-up solution.

Remark. *This theorem becomes a real uniqueness result for example if f satisfies (KO), and in this case the maximal solution is a blow-up one and it exists. In this case the previous theorem says that there exists exactly one blow-up solution.*

Proof. Let

$$\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) \le \delta\} \tag{3.111}$$

and assume that there exists a blow-up solution u such that $\overline{u} \ne u$. Set $w := u - \frac{1}{2C}(\overline{u} - u)$. Hence

$$\begin{aligned} -\Delta w + f(w) &= -(1 + \frac{1}{2C})\Delta u + \frac{1}{2C}\Delta\overline{u} + f(w) \\ &= f((\frac{1}{2C} + 1)u - \frac{1}{2C}\overline{u}) - (1 + \frac{1}{2C})f(u) + \frac{1}{2C}\overline{u} \end{aligned}$$

But by convexity of f we obtain

$$f(u) \le \frac{2C}{1+2C} f((\frac{1}{2C} + 1)u - \frac{1}{2C}\overline{u}) + \frac{1}{1+2C} f(\overline{u})$$

hence $-\Delta w + f(w) \ge 0$. We have obtained a supersolution for (3.1) in Ω and moreover $w \ge \frac{1+C}{2C}u$ in Ω_δ.
Consider $\tilde{\Omega}$ a smooth bounded domain such that $\overline{\Omega} \subset \tilde{\Omega}$. By the maximum principle, $u \ge v$, where v is the solution of

$$\Delta v = f(v) \ \ in \ \ \tilde{\Omega} \tag{3.112}$$

with zero boundary condition in $\tilde{\Omega}$. Set

$$w_\lambda := \lambda u + (1 - \lambda)v \tag{3.113}$$

for any $\lambda \in [0, 1]$. Then

$$-\Delta w_\lambda + f(w_\lambda) = f(\lambda u + (1 - \lambda)v) - \lambda f(u) - (1 - \lambda)f(v) \le 0$$

hence w_λ is a subsolution of (3.1) in Ω. For $\lambda < \frac{1+C}{2C}$, since $w > \lambda u$ in Ω_δ, it follows that the set

$$A := \{x \in \Omega : w_\lambda(x) \geq w(x)\}$$

is compact in Ω. By the maximum principle in this set, since f is nondecreasing, it follows that $A = \varnothing$, hence $w_\lambda < w$ in Ω. Then there exists a blow-up solution u_1 such that $w_\lambda \leq u_1 \leq w$ in Ω. It follows that $u_1 \leq u - \frac{1}{2C}(\overline{u} - u)$, or equivalently

$$\overline{u} - u_1 \geq (1 + \frac{1}{2C})(\overline{u} - u) \tag{3.114}$$

We replace u by the blow-up solution u_1. By the same arguments, but starting from u_1, we obtain a new blow-up solution u_2 such that

$$\overline{u} - u_2 \geq (1 + \frac{1}{2C})(\overline{u} - u_1) \tag{3.115}$$

By iterating this process, we construct a sequence $(u_n)_n$ of blow-up solutions with $u_0 = u$ and such that (3.115) holds with u_2 and u_1 replaced by u_n and u_{n-1}. Hence

$$\overline{u} - u_n \geq (1 + \frac{1}{2C})^n(\overline{u} - u) \quad in \quad \Omega \tag{3.116}$$

and u_n is uniformly bounded below by v. We have a contradiction by passing to the limit as $n \to \infty$. $\square$

We remark here the crucial importance of convexity in obtaining good inequalities. In the same setting, there exists another theorem of Veron and Marcus which gives sufficient conditions for uniqueness in more general domains. We just state it and the interested reader can find a proof in [MV03] and further applications:

Theorem 25. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain such that $\partial\Omega$ is locally the graph of a continuous function. Suppose that $f \in C^1(\mathbb{R})$, such that $f(0) = 0$ and f is nondecreasing. Then there exists at most one solution to (3.1)-(3.2) if one of the following conditions is satisfied:*
(i) f is convex and there exists $L \geq 0$ such that $f(a+b) \geq f(a) + f(b) - L$, for all a, $b \geq 0$;
(ii) there exists $L \geq 0$ such that

$$f(ra + sb) \geq rf(a) + sf(b) - (r + s - 1)L, \quad \forall a, b \geq 0, \quad \forall r, s \geq 1 \tag{3.117}$$

Next we prove some asymptotic behaviour estimates strongly connected to the results in the previous section. They are very important because they are simple and very general; we will suppose only very few facts on the nonlinearity f.

Proposition 7. *Let u be a nonnegative radial and monotone blow-up solution of the problem (3.1)-(3.2) in the unit ball $B(0,1)$, with f satisfying (KO). Then*

$$\int_{u(r)}^{\infty} \frac{dt}{\sqrt{F(t)}} \sim \sqrt{2}(1 - r) \tag{3.118}$$

as $r \to 1$.

Proof. By writing the equation in radial form, multiplying in both sides by $r^{N-1}u'$ and integrating by parts, we obtain:

$$\frac{u'(r)^2}{2} = F(u(r)) - G(r) \tag{3.119}$$

where

$$G(r) := \frac{2N-2}{r}\int_0^r (\frac{s}{r})^{2N-1} F(u(s))ds$$

We prove next that $G(r) = o(F(u(r)))$ as $r \to 1$. For this we compute:

$$\begin{aligned}
\frac{G(r)}{F(u(r))} &= \frac{2N-2}{r}\int_0^r (\frac{s}{r})^{2N-1}\frac{F(u(s))}{F(u(r))}ds \\
&\leq \frac{2N-2}{r}\int_0^{1-\varepsilon} (\frac{s}{r})^{2N-1}\frac{F(u(s))}{F(u(r))}ds + \frac{2N-2}{r}\int_{1-\varepsilon}^{1} (\frac{s}{r})^{2N-1}\frac{F(u(s))}{F(u(r))}ds \\
&\leq C\varepsilon + \frac{2N-2}{r}\int_0^{1-\varepsilon} (\frac{s}{r})^{2N-1}\frac{F(u(s))}{F(u(r))}ds \\
&\leq C\varepsilon + C\frac{F(u(1-\varepsilon))}{F(u(r))} \to 0
\end{aligned}$$

as $r \to 1$. Hence $G(r) = o(F(u(r)))$. From (3.119) we find that

$$\int_r^1 \frac{u'(r)}{\sqrt{F(u(r))}}dr = \int_r^1 \sqrt{2 - \frac{G(r)}{F(u(r))}}dr$$

which implies by a change of variable

$$\int_{u(r)}^{\infty} \frac{dt}{\sqrt{F(t)}} = \int_r^1 \sqrt{2 - \frac{G(r)}{F(u(r))}}dr \sim \int_r^1 \sqrt{2}dr = \sqrt{2}(1-r)$$

□

This is a result on a ball. But we will use it in the proof of the next result, which is a blow-up rate on general smooth domains.

Theorem 26. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and let f satisfy (KOs). Let u be a blow-up solution of (3.1) in Ω. Then*

$$\lim_{x\to x_0} \frac{\int_{u(x)}^{\infty} \frac{dt}{\sqrt{2F(t)}}}{d(x,x_0)} = 1 \tag{3.120}$$

for all $x_0 \in \partial\Omega$.

Proof. Let $x_0 \in \partial\Omega$. Since $\partial\Omega$ is smooth, there exists an interior ball $B(x_1, r) \subset \Omega$ such that $\overline{B(x_1, r)} \cap \partial\Omega = x_0$. Fix $\eta < 1$ but very close to 1 and denote by $\underline{u} := u|_{B(x_1,\eta r)}$. Let $\tilde{B}$ be another ball centered in x_1 such that $B(x_1, \eta r) \subset \tilde{B} \subset B(x_1, r)$. From theorem 23, there exists a solution v of (3.1)-(3.2) posed in $\tilde{B}$. By the minimality principle, $v \geq \underline{u}$ on $\tilde{B}$.
Let $M > 0$ be such that $M\tilde{B} = B(x_1, r)$, hence $M < 1$ and $M > \eta$. Define

$$v_M(x) = v(\frac{x - x_1}{M} + x_1) \tag{3.121}$$

It follows that

$$\Delta v_M = \frac{1}{M^2} f(v_M) \quad in \quad B(x_1, r) \tag{3.122}$$

$$v_M = \infty \quad on \quad \partial B(x_1, r) \tag{3.123}$$

Then $u(x_1) \le v(x_1) = v_M(x_1)$, hence

$$M \int_{u(x_1)}^{\infty} \frac{dt}{\sqrt{F(t)}} \ge M \int_{v_M(x_1)}^{\infty} \frac{dt}{\sqrt{F(t)}} \sim \sqrt{2} r$$

where we have used the result of proposition 7. Then

$$\liminf_{x \to x_0} \frac{1}{d(x, x_0)} \int_{u(x)}^{\infty} \frac{dt}{\sqrt{F(t)}} \ge \sqrt{2}$$

We need to obtain a converse inequality. To this goal consider an exterior ball $B(x_2, R^{'}) \subset \mathbb{R}^N \setminus \Omega$ such that $\overline{B(x_2, R^{'})} \cap \partial\Omega = x_0$. Consider also a very large ball $B(x_2, R^{''})$ such that $\Omega \subset B(x_2, R^{''})$. Let $A := B(x_2, R^{''}) \setminus B(x_2, R^{'})$. From theorem 23 we deduce that there exists a minimal solution v of (3.1)-(3.2) in A. By the minimality principle, $u \ge v$ in Ω and, since v solves a problem of type (3.1)-(3.2) on an annulus, it is easy to see(by adapting straightforwardly the proof of proposition 7) that v has the same asymptotic behaviour as in the case of a ball. Hence

$$\limsup_{x \to x_0} \frac{1}{d(x, x_0)} \int_{u(x)}^{\infty} \frac{dt}{\sqrt{F(t)}} \le \limsup_{x \to x_0} \int_{v(x)}^{\infty} \frac{dt}{\sqrt{F(t)}} \le \sqrt{2}$$

which ends the proof. □

In order to obtain uniqueness results, it is necessary to prove first some results of "similar behaviour on the boundary".

Proposition 8. *Assume that f is a $C^1(\mathbb{R})$-function which is convex on $[a, \infty)$ for some $a > 0$. Then two radially symmetric boundary blow-up solutions u and v of (3.1) in $B(0, R)$ satisfy $u(r) \sim v(r)$ on $\partial B(0, R)$.*

Proof. Let $R = 1$ and fix a blow-up solution u. From the proof of proposition 7 we obtain:

$$(1 - r) - \int_{u(r)}^{\infty} \frac{dt}{\sqrt{2F(t)}} \le C \int_{r}^{1} \frac{u^{'}(s)}{\sqrt{F(u(s))}} ds$$

We introduce the function $w(r)$ such that $1 - r = \int_{w(r)}^{\infty} \frac{dt}{\sqrt{2F(t)}}$. From the previous inequality we find

$$\int_{w(r)}^{u(r)} \frac{dt}{\sqrt{2F(t)}} \le C \int_{u(r)}^{\infty} \frac{t}{F(t)} dt \tag{3.124}$$

hence, by increasingness of F near ∞, we have

$$(u(r) - w(r)) \frac{1}{\sqrt{2F(u(r))}} \le C \int_{u(r)}^{\infty} \frac{t}{F(t)} dt$$

and

$$(1 - \frac{w(r)}{u(r)}) \leq C\frac{\sqrt{2F(u(r))}}{u(r)} \int_{u(r)}^{\infty} \frac{t}{F(t)}dt \tag{3.125}$$

Since f is convex at a neighborhood of ∞, there exists $\alpha > 0$ such that for u and v large, $\frac{F(v)}{v^2} \geq \alpha^2 \frac{F(u)}{u^2}$. From this, for r sufficiently close to 1 and from (3.125), we obtain

$$(1 - \frac{w(r)}{u(r)}) \leq \frac{C}{\alpha} \int_{u(r)}^{\infty} \frac{dt}{\sqrt{2F(t)}} \to 0 \tag{3.126}$$

as $r \to 1$. Hence every blow-up solution u has a fixed blow-up rate $w(r)$. It follows that $u(r) \sim v(r)$ as $r \to 1$. □

We remark that the same result can be obtained by replacing the convexity condition near ∞ on f by the condition that $\frac{f(u)}{u}$ is increasing at a neighborhood of ∞. The proof is similar, since both conditions insure that the number α such that $\frac{F(v)}{v^2} \geq \alpha^2 \frac{F(u)}{u^2}$ exists(in fact, this α comes from the increasingness of the function $t \to \frac{F(t+a)-F(a)-tf(a)}{t^2}$, which is true in both cases). We will prove the subsequent uniqueness results in both cases. We start with the second one, which is much simpler.

Theorem 27. *Suppose that $f(0) = 0$ and the function $(0, \infty) \ni u \to \frac{f(u)}{u}$ is increasing. Then there exists at most one blow-up solution of (3.1) in a smooth bounded domain Ω.*

Proof. Suppose that u_1 and u_2 are two solutions of (3.1)-(3.2). From proposition 8 we have $u_1 \sim u_2$ on $\partial\Omega$. Set

$$\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$$

For any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$(1 - \varepsilon)u_1(x) \leq u_2(x) \leq (1 + \varepsilon)u_1(x), \quad \forall x \in \Omega_\delta \tag{3.127}$$

For $v = (1 + \varepsilon)u_1$, we remark that $\Delta v = (1 + \varepsilon)f(u_1) \leq f((1 + \varepsilon)u_1)$ in $\Omega \setminus \Omega_\delta$, hence $\Delta v \leq f(v)$. By (3.127), $v \geq u_2$ on $\partial\Omega_\delta$. From the maximum principle we obtain that $v \geq u_2$ in $\Omega \setminus \Omega_\delta$, hence in the whole Ω. Hence

$$(1 - \varepsilon)u_1 \leq u_2 \leq (1 + \varepsilon)u_1 \quad in \quad \Omega$$

for any $\varepsilon > 0$. This implies $u_1 \equiv u_2$. □

A similar result is true if we suppose that f is convex near ∞. The proof follows the same ideas, but it is more complicated technically. We ask the reader to consult the paper [DDGR06] for it.

3.8 Final Comments and Open Questions

The investigation of blow-up boundary solution is a very active area of research in the last years. There were proved many results, and there are still many open questions, even some with in appearance elementary character.

In this chapter we have tried to present the main developents of the problem (3.1)-(3.2). We have to say that this is not the only interesting problem concerning the blow-up solutions; there are many recent works studying more general equations than (3.1), or posed in more general, nonsmooth domains.
For example, in the last four years V.Radulescu and F. Carstea proved in a series of papers results about blow-up solutions for a class of logistic equations, i.e.

$$\Delta u + au = b(x)f(u) \quad in \quad \Omega \tag{3.128}$$

where $a \in \mathbb{R}$, $b \in C^{0,\alpha}(\overline{\Omega})$ and nonnegative and $f \in C^1[0,\infty)$ satisfying $f(0) = 0$, $\frac{u}{u}$ increasing and (KO). The authors mentioned above had obtained a deep result that we state here. If we denote by $\Omega_0 := \{x \in \Omega : b(x) = 0\}$ and $\lambda_1(\Omega_0)$ the first eigenvalue of $-\Delta$ in $H_0^1(\Omega_0)$, then

Theorem 28. *The equation (3.128) has at least a blow-up positive solution if and only if* $a < \lambda_1(\Omega_0)$.

The proof of this results uses a theorem of Alama and Tarantello, which states a similar result for the positive solutions of the usual Dirichlet problem for the logistic equation, and few comparison lemmas. The proof appears in [CR03] or [CR04], and in the other papers of V.Radulescu and F. Carstea mentioned in the references the reader can find more extensions.
The other directions of study is for the equations with gradient terms, i.e. for example

$$\Delta u + |\nabla u| = p(x)f(u) \quad in \quad \Omega \tag{3.129}$$

where $p \in C^{0,\alpha}(\overline{\Omega})$ is a nonnegative function and f is nondecreasing, $f(0) = 0$ and $f > 0$ on $(0,\infty)$. We assume also that f is sublinear at ∞, i.e. $\sup\limits_{s\geq 1} \frac{f(s)}{s} < \infty$. In this case the results are more striking:

Theorem 29. *In a smooth bounded domain, there is no blow-up solutions of (3.129).*

This result is proved in [GR04], where it is given also a criterion for existence of entire positive blow-up solution(i.e for $\Omega = \mathbb{R}^N$), fact which was not possible, as we have seen, for the equation (3.1). The presence of the potential p is in this case dominating on the effect of the gradient terms. More results on these problems can be found in [Gh06] and the references therein.
There is another direction of study: to keep the equation (3.1) and to change the domain. In the two papers [MV97] and [MV03] there are many results about existence and uniqueness of blow-up solutions of (3.1) in Lipschitz domains or, more general, in domains whose boundaries are locally continuous graphs. Here the topology plays a key role. There is also a result of asymptotic behaviour in this setting.
We end this last chapter by stating few open problems:
Open problem 1: Find a function f satisfying (KO) but such that on a certain domain there do not exists blow-up solutions. This is connected with the question that if (KO) and (KOs) are equivalent or not. We were able to produce a characterization theorem for both conditions, but it is not clear if these theorems are sharp. In connection with this, it will be an interesting result to produce some new examples of functions which satisfy (KOs) or to give a better characterization result than lemma 19.
Open problem 2: Does there exists a function $f \in C^1[0,\infty)$, with $f(0) = 0$ and $f \geq 0$ which satisfies (KO), but such that $\liminf\limits_{\alpha\to\infty} \phi(\alpha) > 0$? In spite of its elementary character, this is still an

open question, up to our knowledge. It is not easy to produce such an example.

Open problem 3: Establish the uniqueness results as in section 3.7 theorem 27 if we suppose only that f is increasing. In fact, both results that we have proved for $\frac{f(u)}{u}$ increasing are no longer valid: if u_1 and u_2 are two blow-up solutions for (3.1) and f is increasing, as we know, it is still open question if $u_1 \sim u_2$ near $\partial\Omega$ or not. After this, it is also a question if $u_1 \sim u_2$ implies equality of them and therefore uniqueness.

Open problem 4: There are many results about the existence of a maximal solution of a certain equation, for example for (3.1). For example for the special nonlinearity $f(u) = |u|^{p-1}u$, $p > 1$, there always exists a maximal solution. In the paper [La] there is given a characterization of the situations when this solution is a blow-up one. There are many types of nonlinearities for which such a result is not known, for example $f(u) = e^{au}$ in nonsmooth domains. For a more complete presentation of this problem, the reader should consult [MV03].

References

1.[AfR97]-A.Aftalion, W.Reichel-Existence of two boundary blow-up solutions for semilinear elliptic equations, J. Diff. Equations 141(1997), 400-421;

2.[BM92] -C.Bandle, M.Marcus-Large solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behaviour, J. Anal.Math. 58(1992), 9-24;

3.[BM95] -C.Bandle, M.Marcus-Asymptotic behaviour of solutions and their derivative for semilinear elliptic problems with blow-up on the boundary, Ann. Inst. H. Poincare Anal. Non Lineaire, 12(1995), 155-171;

4.[Br93] -R.Brown-A topological introduction to nonlinear analysis, Birkhauser, Boston, 2004;

5.[ChLi91]-W.Chen, Congming Li- Classification of solutions of some nonlinear elliptic equations, Duke Math. J. 63(1991), no. 3, 615-622;

6.[CR03]-F.Carstea, V.Radulescu-Solutions with boundary blow-up for a class of nonlinear elliptic problems, Houston J. of Math. 3(2003), 821-829;

7.[CR04]-F.Carstea, V.Radulescu-Extremal singular solutions for degenerate logistic-type equations in anisotropic media, C.R.Acad. Sci. Paris, ser.I, t. 339(2004), 119-124;

8.[DDGR06]-S.Dumont, L.Dupaigne, O.Goubet, V.Radulescu-Boundary blow-up solutions to $\Delta u = f(u)$, in press, 2006;

9.[Gh06]-M.Ghergu-Problemes avec singularites sur la frontiere pour des equations elliptiques, these, Univ. de Savoie, Chamberry, 2006;

10.[GNN79]-B.Gidas, W.M.Ni, L.Nirenberg-Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68(1979), 209-243;

11.[GR04]-M.Ghergu, V.Radulescu-Nonradial blow-up solutions of sublinear elliptic equations with gradient term, Comm. on Pure Appl. Analysis, 3(2004), 465-474;

12.[$GS81_1$]-B.Gidas, J.Spruck-Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math. 34(1981), no. 4, 525-598;

13.[$GS81_2$]-B.Gidas, J.Spruck-A priori bounds for positive solutions of nonlinear elliptic equations, Comm. Partial Differential Equations 6(1981), no. 8, 883-901;

14.[GT02]-D.Gilbarg, N.Trudinger-Elliptic partial differential equations of second order, Springer, Berlin, 2002;

15.[Ke56]-J.B.Keller-Electrohydrodynamics I:The equilibrium of a charged gas in a container, J. Rational Mech. Anal. 5(1956), 715-724;

16.[Ke57]-J.B.Keller-On solutions to $\Delta u = f(u)$, Comm. Pure and Applied Math. 10(1957), 503-510;

17.[LMK94]-A.Lazer, P.J.McKenna-Asymptotic behavior of solutions of boundary blowup problems, Differential Integral Equations 7(1994), no. 3-4, 1001-1019;

18.[LN74]-C.Loewner, L.Nirenberg-Partial differential equations invariant under conformal or projective transformations, in Contributions in Analysis, Acad. Press, New York, 1974, 245-272;

19.[MKRW97]-P.J.McKenna, W.Reichel, W.Walter-Symmetry and multiplicity for nonlinear elliptic differential equations with boundary blow-up, Nonlinear Anal. 28(1997), no. 7, 1213-1225;

20.[ML18]-Max von Laue-1918(in German);

21.[MV97]-M.Marcus, L.Veron-Uniqueness and asymptotic behaviour of solutions with boundary blow-up for a class of nonlinear elliptic equations, Ann. Inst. H.Poincare, Ann. Non Lineaire, 14(1997), no.2, 237-274;

22.[MV03]-M.Marcus, L.Veron-Existence and uniqueness results for large solutions of general nonlinear elliptic equations, J. Evol. Equations, 3(2004), 637-652;

23.[Os57]-R.Osserman-On the inequality $\Delta u \geq f(u)$, Pacific J. of Math. 7(1957), 1641-1647;

24.[RaC06]-V.Radulescu, F.Carstea-Nonlinear problems with boundary blow-up:a Karamata regular variation theory approach, Asymptotic Anal. 46(2006), 275-298;

25.[Rab74]-P.H. Rabinowitz-Pairs of positive solutions of nonlinear elliptic partial differential equations, Indiana Univ. Math. J. 23(1973/74), 173-186;

26.[Rad]-V.Radulescu-Treatment methods for nonlinear elliptic problems, Lecture Notes, Univ. of Craiova;

27.[Re97]-W. Reichel-Symmetry of solutionsof semilinear elliptic equations with boundary blow-up, preprint, 1997;

28.[PV06]-A.Poretta, L.Veron-Symmetrie des grandes solutions d'equations elliptiques semi lineaires, C.R.Acad. Sci. Paris, Ser I 342(2006), 483-487;

29.[Ze84]-E. Zeidler-Nonlinear functional analysis and its applications. I. Fixed-point theorems. Springer-Verlag, New York, 1986.