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Chordal Graphs, Alexander Duality and Cohen–Macaulayness

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Introduction

The aim of this thesis is to present some recent results in the field of algebraic combinatorics. Combinatorial commutative algebra is a relatively new area of research, starting in the 70's with the works of M. Hochster and R. Stanley. One of the main features of this new domain is the strong interaction between several apparently divergent fields: commutative algebra, graph theory, convex geometry (polytopes), algebraic topology to name just a few. It was our wish to highlight some of these interactions in this thesis.

The initial topic we intended to cover was a result of Herzog, Hibi and Zheng. In [10] they presented a new proof of a classical theorem of Dirac (see [3]) on chordal graphs. In [10] they rephrase Dirac's original work in terms of the new geometrical notion of quasi-forest which they introduce. However, the tools used come from commutative algebra via the Alexander duality for simplicial complexes.

Alexander duality in itself proves to be a powerful device to switch between algebraic and/or combinatorial properties of graphs or simplicial complexes.

A property that comes often to our sight is the Cohen-Macaulay property. Using techniques developed in the previously mentioned article, in [11] Herzog, Hibi and Zheng classify all Cohen-Macaulay chordal graphs. Although at a first sight there is little commutative algebra present there, and mostly graph theory techniques are used, at some crucial steps some results from commutative algebra are implicitely needed.

It seemed just fair to present also some results about Alexander duality and contexts in which notions as Cohen-Macaulayness, chordal graphs, linear resolutions, shellability interact. Results from the works of Fröberg([5]), Eagon and Reiner([7]), Terai([14]) and Hochster([12]) are outlined.

The thesis consists of four sections. The first section covers some Commutative Algebra background needed later. We talk about the standard and the fine gradings on the polynomial ring $S = k[x_1, \ldots, x_n]$ over a commutative field k. The minimal free resolution of a finitely generated graded module over S is introduced, and in analogy to the local case the graded and fine Betti numbers are defined. In the last subsection we talk about pefect S-modules, a class which contains the Cohen-Macaulay S-modules. It is shown that, if I is a homogeneous ideal in S, then I is a Cohen-Macaulay ideal if and only if it is perfect. The Hilbert-Burch Theorem gives the general structure for perfect ideals of codimension(i.e. height) 2 in a Noetherian ring in terms of Fitting ideals. Further readings were some proofs and other information about these topics can be found are Bruns and Herzog [2] and Eisebud [8].

The second section deals with chordal graphs in combinatorics and commutative algebra. In subsection 2.1 the chordal graphs are introduced. We present here the work of Dirac from his article [3], where he gives necessary and sufficient conditions for a graph to be chordal, and he even gives a constructibility criterion. Starting with a simplicial complex Δ, Stanley's idea was to define a square-free monomial ideal I_Δ, called the Stanley-Reisner ideal, in order to study the geometric or combinatorial properties via commutative algebra's methods. The nice thing is that starting with a square free monomial ideal in S we can recover the simplicial complex Δ from which it showed up. Further in this section other similar constructions are presented: the facet ideal, the edge ideal of a graph, the complementary simplicial complex and some basic properties are listed and/or proved.

In section 2.3 we go deeper into the study of the Alexander duality. Starting with a simplicial complex we can build another simplicial complex $\Delta^\vee$ by considering the complements of the non-faces in Δ. For Δ, the monomials that generate I_Δ are x_F, with F a minimal non-face, while the associated primes are generated by monomials x_G with G the complement in $[n]$ of a facet $F \in \mathcal{F}(\Delta)$. For $\Delta^\vee$, the Stanley-Reisner ideal $I_{\Delta^\vee}$ is generated by monomials x_G, with $G = [n] \setminus F$, for some facet $F \in \mathcal{F}(\Delta)$, while an associated prime is generated by a monomial x_F for some F, a minimal non-face. Thus Alexander duality maps generators of I_Δ into associated primes of $I_{\Delta^\vee}$ and viceversa.

The information on (fine) Betti numbers is gathered by the Betti-polynomial. Hochster in [12] gives a formula for the Betti polynomials of the Stanley-Reisner ring of a simplicial complex. This is used in [7] by Eagon and Reiner to express this polynomial in terms of homology of links of faces in the Alexander dual. A nice consequence they derive, using a criterion of Reisner for Cohen-Macaulayness, is that I_Δ has linear resolution if and only if $\Delta^\vee$ is Cohen-Macaulay over the field k. Fröberg in [5] and in [6] shows more: I_Δ is generated by quadratic monomials and it has linear resolution if and only if Δ is the flag complex $\Delta(G)$ associated to some chordal graph G. Using these results, Eagon and Reiner in [5] provide a new characterization of chordal graphs: a graph G is chordal if and only if the Alexander dual of the associated flag complex is vertex-decomposable, equivalently it is Cohen-Macaulay over some/any field k. Another result only mentioned without a proof belongs to Terai [14] who gives a formula for computing the projective dimension of $I_{\Delta^\vee}$.

The third section introduces the quasi-forests and gives an algebraic proof for Dirac's theorem. At the beginning, the definition of Taylor's resolution for a monomial ideal I is mentioned for further reference. Using the Taylor complex and the cyclic syzygies they define, Bruns and Herzog in [1] describe the structure of the perfect monomial ideals in S of codimension 2. These ideals are described by a relation tree assigned uniquely to a minimal set of generators of $syz_2(S/I)$. An example of the construction of such a tree is presented. As a corollary to the general theory presented, in [1] the authors also show that the second Betti number of S/I is independent of the field k.

In [17] Zheng introduces the quasi-trees and quasi-forests, extending some definitions of Faridi [4]. In [10] Herzog, Hibi and Zheng give an if and only if condition for a simplicial complex to be a quasi-forest in terms of some minors of a matrix they introduce. They remark the similarity with a matrix of Taylor relations and thus conclude in [10] by proving that a simplicial complex is a quasi-forest if and only if the projective dimension of the complementary simplicial complex is 1.

In the last subsection we give the proof from [10] to Dirac's Theorem. It is true that it is not easier then the original one in [3]. On the contrary. Herzog, Hibi, Zheng show that a graph is chordal if and only if it is the 1-skeleton of a quasi-forest.

In the last section we present some results on classes of Cohen-Macaulay graphs. Villarreal [16], Herzog and Hibi [9] give a complete description of Cohen-Macaulay trees, cycles, and bipartite graphs. In the last subsection we prove in detail a recent result of Herzog, Hibi and Zheng [11] concerning the classification of Cohen-Macaulay chordal graphs. With a long and technical proof, using Dirac's theorem in the form from [10], they show that a graph G is Cohen-Macaulay over some field if and only if it is over any field if and only if it is unmixed. Furthermore, this is equivalent to the fact that the vertex set is a disjoint union of the (vertices of) faces in the flag complex associated the graph which have a free vertex.

I am indebted to some people who attracted me to the area of combinatorial commutative algebra. My interest in the topics presented in this thesis started at the School in Commutative Algebra-"Monomial Algebras", Eforie 2003 where professor Jürgen Herzog and Xinxian Zheng described their work from [9], [10] and [17]. This topic is also a consequence of the graduate course of "Combinatorics in Commutative algebra" held at Şcoala Normală Superioară Bucureşti by professors Dorin Popescu and Cristodor Ionescu.

1. Commutative Algebra background

1.1. Gradings on $k[x_1, ..., x_n]$.

Let $(G, +)$ be a commutative group.

Definition 1.1.1. A *graded G-ring R* is a ring R together with a decomposition $R = \bigoplus_{g\in G} R_g$ (as a $\mathbb{Z}$-module) such that $R_g R_h \subset R_{g+h}$ for all $g, h \in G$.

Definition 1.1.2. Let R be a graded G-ring. A *graded G-module M* is an R-module M together with a decomposition $M = \bigoplus_{g\in G} M_g$ (as a $\mathbb{Z}$-module) such that $R_g M_h \subset M_{g+h}$ for any $g, h \in G$. One calls M_g the g^{th} *homogeneous* (or *graded*) *component of M.*

Suppose M is a G-graded R-module. The elements $x \in M_g$ are called *homogeneous (of degree g).* According to this definition the zero element is homogeneous of arbitrary degree. The degree of a homogeneous element $x \in M$ will be denoted be $\deg x$. Any element $x \in M$ has a unique presentation $x = \sum_{g\in G} x_g$ as a sum of its homogeneous components.

Remark 1.1.3. Note that if R_0 is a ring with $1 \in R_0$, then all summands M_g are R_0-modules and that $M = \bigoplus_{g\in G} M_g$ is a direct sum decomposition of M as an R_0-module.

Definition 1.1.4. Let R be a G-graded ring. An R-module homomorphism $\varphi\colon M \to N$ of G-graded R-modules is called *homogeneous* (or *of degree 0*) if $\varphi(M_g) \subset N_g$ for any $g \in G$.

Example 1.1.5. A G-graded module $M = \bigoplus_{g\in G} M_g$ can be *shifted* "d positions" obtaining another G-graded module $M[d]$ defined by $M[d]_g = M_{g+d}$. If $x \in R$ is a homogeneous element of degree d, then the multiplication by x, $\phi_x\colon M[-d] \to M$ is a homogeneous homomorphism.

Definition 1.1.6. Let M be a graded R-module and N a submodule of M. Then N is called *a graded submodule* if it is a graded module such that the inclusion map is a homogeneous homomorphism.

Remark 1.1.7. This is equivalent to the condition $N_g = N \cap M_g$ for all $g \in G$. In other words, N is a graded submodule of M if and only if N is generated by all the homogeneous elements of M which belong to N. In particular, if $x \in N$, then all homogeneous components of x belong to N. Furthermore, M/N is G-graded in an obvious way.

Definition 1.1.8. A graded submodule of a graded ring R is called *graded ideal.*

Definition 1.1.9. Let R be a G-graded ring. A graded ideal $\mathfrak{m}$ of R is called **maximal* if every graded ideal that properly contains $\mathfrak{m}$ equals R. The ring R is called **local* if it has a unique *maximal ideal.

Remark 1.1.10. In the *local graded ring $R/\mathfrak{m}$ all nonzero homogeneous elements are invertible.

Remark 1.1.11. With respect to its finitely generated graded modules M, a *local graded ring $(R, \mathfrak{m})$ behaves like a local ring.

Let $m_1, \ldots, m_n$ be a minimal homogeneous system of generators of M and let $F_0 = \sum_{i=1}^{n} R(-\deg m_i)$, the i^{th} summand being generated by an element e_i satisfying $\deg e_i = \deg m_i$. The R-module F_0 is free of rank n and the assignment $e_i \mapsto m_i$ induces a surjective homomorphism φ_0 of graded modules. $\ker \varphi_0$ is a graded submodule of F_0. Suppose that $\ker \varphi_0 \not\subset \mathfrak{m}F_0$. Then one sees easily that there exists a homogeneous element $u \in \ker \varphi_0$, $u \notin \mathfrak{m}F_0$ and one of the coefficients a_i in the decomposition $u = \sum a_i e_i$ is not in $\mathfrak{m}$, call it a_j. But each a_i is homogeneous, and so a_j is a unit. It follows that the given system of generators is not minimal, which is a contradiction. It can be shown that all homogeneous systems of generators for M have the same number of elements. Iterating the construction of F_0 and φ_0, one obtains an augmented free resolution of M.

Definition 1.1.12. The resolution of M obtained in the above remark is called a *minimal G-graded free resolution of M.*

Proposition 1.1.13. *A minimal G-graded resolution is unique up to graded isomorphism.*

Let us specialize these notions to give a concrete example. In the sequel, suppose k is a commutative field and let $S = k[x_1, \ldots, x_n]$ be the ring of polynomials with coefficients in k. For S, one has the decomposition

$$S = \bigoplus_{d \in \mathbb{N}^n} S_d \text{ with } S_d = \{f \in k[x_1, \ldots, x_n] | \deg f = d\}$$

where by $\deg f$ we mean the standard notion of degree for the polynomial f.

Definition 1.1.14. This is the so called *standard grading* of $S = k[x_1, \ldots, x_n]$.

From now on throughout this thesis by *graded* we mean graded in the standard grading unless specified. The graded ideals of S are the ideals generated by homogeneous polynomials in the usual sense.

We can give S another grading by taking $G = \mathbb{Z}^n$ and using the decomposition

$$S = \bigoplus_{\alpha \in \mathbb{N}^n} S_\alpha = \bigoplus_{\alpha \in \mathbb{N}^n} kx^\alpha \text{ where } x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \text{ for } \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n.$$

Definition 1.1.15. This is the so called $\mathbb{N}^n$-grading on S.

This is also called *the fine grading on S* because with the notations above we have

$$S_d = \bigoplus_{|\alpha|=d} kx^{\alpha} \text{ where } |\alpha| = \alpha_1 + \cdots + \alpha_n, \text{ for } \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n.$$

Proposition 1.1.16. *The homogeneous ideals in S with the fine grading are generated by monomials in the indeterminates $x_1, \ldots, x_n$.*

This is why they are called *monomial ideals*. It can be proved (see [13]) that for any monomial ideal $I \subset S$ there is a unique minimal system of generators which we shall denote by $G(I)$. The ring S with both gradings presented here is a *local ring whose *maximal ideal is the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$.

Definition 1.1.17. A $\mathbb{Z}^n$-graded resolution is also called a *multigraded resolution*.

1.2. **Betti numbers.**

For a finitely generated module M over a local ring $(R, \mathfrak{m}, k)$ the Betti numbers $\beta_i(M)$ give the ranks of the i^{th} resolvent in a minimal free resolution of M. One can show (see [1] Corollary 1.3.2) that $\beta_i(M) = \dim_k \mathrm{Tor}_i^R(M, k)$.

We shall keep our notation for S to denote $k[x_1, \ldots, x_n]$. If M is some finitely generated graded S-module we proved in the previous section that we can build a minimal free resolution

$$\cdots \to S^{\beta_i} \to \cdots \to S^{\beta_0} \to M \to 0.$$

The numbers $\beta_i(M)$ are also called Betti numbers, and, as in the local case, one can prove that $\beta_i(M) = \dim_k \mathrm{Tor}_i^R(M, k)$.

This free resolution can be written in the form

$$\cdots \to \bigoplus_j S(-j)^{\beta_{ij}} \to \cdots \to \bigoplus_j S(-j)^{\beta_{0j}} \to M \to 0$$

after we collect the terms with the same "shift" and make the maps in the resolution homogeneous. It can be proved (see [1] Proposition 1.5.16) that the numbers β_{ij} above are uniquely determined. They are called *the graded Betti numbers of M*.

A similar result can be formulated if we work with the fine grading. One obtaines the *fine Betti numbers $\beta_{i\alpha}$*.

Definition 1.2.1. The *(Castelnuovo-Mumford) regularity* of a finite graded S-module M is the number

$$\operatorname{reg} M = \max\{i + j \mid {}^*H^i_{\mathfrak{m}}(M)_j \neq 0\},$$

where $\mathfrak{m}$ is the maximal ideal in S generated by the indeterminates.

Definition 1.2.2. Let q be an integer. Then M is called q-regular if $q \geq \operatorname{reg} M$, in other words ${}^*H^i_{\mathfrak{m}}(M)_{j-i} = 0$ for all i and all $j \geq q$.

The notion of regularity measures the "complexity" of the minimal graded free resolution of M, as it was shown by Eisenbud and Goto. The following theorem gives a description of regularity in terms of the graded Betti numbers. Denoting by $M_{\geq q}$ the truncated graded R-module $\bigoplus_{j \geq q} M_j$, one has

Theorem 1.2.3. *(Eisenbud-Goto) The following conditions are equivalent:*
a) M is q-regular;
b) ${}^*Tor_i^S(M, k)_{j+i} = 0$ for all i and all $j > q$;
c) $M_{\geq q}$ admits a linear S-resolution, i.e. a graded resolution of the form

$$0 \to S(-q-\ell)^{c_\ell} \to \cdots \to S(-q-1)^{c_1} \to S(-q)^{c_0} \to M_{\geq q} \to 0.$$

Proof. See Theorem 4.3.1 in [2]. □

1.3. **Perfect modules. The Hilbert-Burch theorem.**

The notion of *grade* was given by Rees the following meaning:

Definition 1.3.1. Let R be a Noetherian ring and $M \neq 0$ a finitely generated R-module. Then the *grade of* M is given by

$$\text{grade}(M) = \min\{i |\ \text{Ext}^i_R(M, R) \neq 0\}.$$

For an ideal $I \subset R$, by convention, $\text{grade}(I) = \text{grade}(R/I)$.

Remark 1.3.2. It follows from Theorem 1.2.10(e) in [2] that

$$\text{grade}(M) = \text{grade}(\text{Ann}M, R),$$

where in the right member of the equality by grade we mean the biggest length of a regular sequence on R with elements in $\text{Ann}M$. One sees thus that for ideals, the two notions of grade coincide.

Remark 1.3.3. Since one can compute $\text{Ext}^i_R(M, R)$ from a projective resolution of M, one has $\text{grade}M \leq \text{projdim}\, M$.

Definition 1.3.4. Let R be a Noetherian ring. A non-zero finitely generated R-module M is called *perfect* if $\text{projdim}\, M = \text{grade}\, M$. An ideal $I \subset R$ is called *perfect* if R/I is a perfect module.

Remark 1.3.5. One can show that an ideal generated by a regular sequence in a Noetherian ring R is perfect.

The following theorem gives examples of perfect modules.

Theorem 1.3.6. *Let R be a Cohen-Macaulay ring, and M a finitely generated R-module of finite projective dimension.*
a) If M is perfect, then it is a Cohen-Macaulay module.
b) The converse holds when R is local.

Proof. See Theorem 2.1.5 in [2]. □

The implication from b) in the previous theorem also holds when referred to homogeneous ideals $I \in S$, as it is shown in the next

Theorem 1.3.7. *(*[12] *Theorem 3.5) With the above notations, if I is a homogeneous proper ideal, then S/I is Cohen-Macaulay if and only if* $\text{projdim}\, S/I = \text{grade}\, I$*(= height$I$).*

Proof. One implication results from Remark 1.3.2 and Theorem 1.3.6 above. For the other note that if S/I is Cohen-Macaulay, then any localization at a maximal ideal (including $\mathfrak{m} = (x_1, \ldots, x_n)$) is Cohen-Macaulay (see Theorem 2.1.3 in [2]). Thus $(S/I)_\mathfrak{m} \cong S_\mathfrak{m}/I_\mathfrak{m}$. Now use Theorem 1.3.6a) for this local ring and obtain that $I_\mathfrak{m}$ is perfect in $S_\mathfrak{m}$. Then $\text{projdim} S_\mathfrak{m}/I_\mathfrak{m} = \text{grade}(I_\mathfrak{m}, S_\mathfrak{m})$. Now, by Theorem 1.5.15 in [2] we have $\text{projdim}\, S/I = \text{projdim}(S/I)_\mathfrak{m}$ and $\text{grade}(I, S) = \text{grade}(I_\mathfrak{m}, S_\mathfrak{m})$. Putting these together, it yields that $\text{projdim}\, S/I = \text{grade}(I, S)$ and S/I is therefore a perfect ring, i.e. I is a perfect ideal in S. □

The following theorem gives a characterization of perfect ideals of grade 2 in terms of the ideal of the n-minors of a $n \times (n+1)$ matrix φ (ideal called *the* n^{th} *Fitting ideal* $I_n(\varphi)$ of φ).

Theorem 1.3.8. *(Hilbert-Burch) Let R be a Noetherian ring and I an ideal with a free resolution*

$$F_\bullet \colon 0 \longrightarrow R^n \xrightarrow{\varphi} R^{n+1} \longrightarrow I \longrightarrow 0.$$

Then there exists an R-regular element a such that $I = aI_n(\varphi)$. If I is projective, then $I = (a)$, and if $\operatorname{projdim} I = 1$*, then $I_n(\varphi)$ is perfect of grade 2.*

Conversely, if $\varphi \colon R^n \to R^{n+1}$ is a R-linear map with $\operatorname{grade} I_n(\varphi) \geq 2$*, then $I = I_n(\varphi)$ has the free resolution $F_\bullet$.*

Proof. This theorem is proved in [2] as Theorem 1.4.7. ☐

2. Chordal graphs in combinatorics and commutative algebra

2.1. Dirac's work on chordal graphs.

The graphs we are working with are considered to be finite and without loops or multiple edges. However, the results from this subsection are valid also for infinite graphs.

In this section we intend to present some of the results of G. A. Dirac on chordal graphs as they appeared in [3]. Further on, in Section 3.4 we shall rephrase these results in terms of the so called quasi-forests, a notion to be introduced later.

Definition 2.1.1. If the graph C is a circuit and x and y are two distinct vertices of C which are not joined by any edge belonging to C, then an edge which joins x and y is called a *chord* of C. A graph in which every circuit with more than three vertices has at least a chord, is called a *chordal graph*.

Remark 2.1.2. For chordal graphs some other names are (have been) used in the literature: *rigid circuit graphs* ([3]), *triangulated graphs* ([16]).

Definition 2.1.3. A graph in which every two distinct vertices are connected is called a *clique*. A graph with only one vertex is considered to be a clique, but the empty graph is not.

Example 2.1.4. Trees, forests and cliques are examples of chordal graphs.

Recall that by removing a set of vertices $\mathfrak{F}$ from the graph G one obtains a new graph $G - \mathfrak{F}$ whose vertices are the vertices of G which are not in $\mathfrak{F}$ and whose edges are the one in G that connect two vertices not in $\mathfrak{F}$.

Remark 2.1.5. If G is a chordal graph and H is the subgraph obtained after removing some vertices, then H is chordal, too.

Definition 2.1.6. If G is a connected graph and $\mathfrak{F}$ is a set of vertices contained in G, then $\mathfrak{F}$ is called a *cut-set* of G if the graph $G - \mathfrak{F}$ is disconnected. A cut-set is called a *minimal cut-set* of G if no proper subset of $\mathfrak{F}$ is a cut-set of G, and it is called a *relatively minimal cut-set* of G if G contains two vertices which are separated by $\mathfrak{F}$ but by no proper subset of $\mathfrak{F}$.

Remark 2.1.7. It is obvious that any minimal cut-set is also relatively minimal. However, the converse is not true.

Remark 2.1.8. We can extend the definition above for arbitrary graphs asking $\mathfrak{F}$ to be included in some connected component, and when removed disconnects it.

Theorem 2.1.9. *(Dirac) A graph is a chordal graph if and only if every pair of vertices that belong to the same relatively minimal cut-set are joined by at least one edge.*

In other words, G is chordal if and only if any relatively minimal cut-set $\mathfrak{F}$ in G is a clique.

Proof. Let G be a connected chordal graph, $\mathfrak{F}$ a relatively minimal cut-set and i and i' are two vertices that belong to $\mathfrak{F}$ and are not joined by an edge. Let a_1 and a_2 be two vertices which are separated by $\mathfrak{F}$, but by no proper subset oh $\mathfrak{F}$, and let G_1 and G_2 denote the connected components of $G - \mathfrak{F}$ to which a_1 and a_2 respectively belong. The vertex i is joined by an edge to at least one vertex of G_1 and to al least one vertex of G_2, and so is i', because no proper subset of $\mathfrak{F}$ separates a_1 and a_2. For $j = 1, 2$, let Y_j be a path with the least number of vertices among the paths which have i and i' as their end-vertices, and whose intermediate vertices belong all to G_j. The paths Y_1 and Y_2 exist because G_1 and G_2 are connected. $Y_1 \cup Y_2$ is then a circuit with at least four vertices and it has no chord. This contradicts the assumption that G is a chordal graph.

Conversely, suppose that G is a connected graph in which every pair of vertices which belong to the same relatively minimal cut-set are joined by at least one edge, but G is not chordal. Let C be a circuit contained in G which has more that three vertices, but no chord. Let v_1 and v_2 denote two vertices of C which are not joined by any edge belonging to C, and let the two paths connecting v_1 and v_2 which together make up C be denoted by Y_1 and Y_2. v_1 and v_2 are not joined to each other by any edge in G because C has no chord. Consequently G contains at least one cut-set separating v_1 and v_2 (all vertices which are adjacent to v_1 form such a cut-set, for example). Therefore G contains a cut-set $\mathfrak{F}$ such that $\mathfrak{F}$, and no proper subset of $\mathfrak{F}$, separates v_1 and v_2. $\mathfrak{F}$ is a relatively minimal cut-set, therefore, by hypothesis, every pair of vertices belonging to $\mathfrak{F}$ are joined by at least one edge. Y_1 and Y_2 each contain at least one vertex belonging to $\mathfrak{F}$ because $\mathfrak{F}$ separates v_1 and v_2, and $(Y \cap \mathfrak{F}) \cap (Y' \cap \mathfrak{F}) = \emptyset$ because $Y \cup Y' = C$. Every edge joining a vertex of $Y \cap \mathfrak{F}$ to a vertex of $Y' \cap \mathfrak{F}$ is however a chord of G, and this contradicts the hypothesis that G has a chord. This concludes our proof. □

Corollary 2.1.10. *In a chordal graph every pair of vertices belonging to the same minimal cut-set are joined by at least one edge.*

Proof. The conclusion follows from Dirac's Theorem above and the fact that any minimal cut-set is also relatively minimal. □

Theorem 2.1.11. *If G and F are chordal graphs and $G \cap F$ is a clique or empty, then $G \cup F$ is a clique, too.*

Proof. If $G \cap F$ contains no circuit with more than three vertices, then it is a chordal graph. If $G \cup F$ contains such circuits, then let C be one of them. If $C \subset G$ or $C \subset F$ then C has a chord. If $C \not\subset F$ and $C \not\subset G$, then C contains at least one vertex from each of $G \setminus (G \cap F)$ and $F \setminus (G \cap F)$, v and w respectively say. v and w are separated by the vertices of $G \cap F$. It follows, exactly as in the second part of the proof of Theorem 2.1.9, that C has a chord. Thus every circuit of $G \cup F$ with more than three vertices has a chord. $\square$

Remark 2.1.12. Theorem 2.1.9 shows that any chordal graph which is not a clique can be built up from two smaller mutually disjoint chordal graphs by identifying a clique in one with a similar clique in the other. It follows that any chordal graph which is not a clique can be obtained by applications of this process starting from a set of cliques. Theorem 2.1.11 shows that conversely, whenever the process is applied to two mutually disjoint chordal graphs, the result is still a chordal graph. It's worth mentioning that the union of two chordal graphs whose intersection is neither empty nor a clique may of course still be a chordal graph.

2.2. **Ideals associated to simplicial complexes and the Alexander dual.**

In the following, let S denote the polynomial ring in n variables $k[x_1, \ldots, x_n]$ over a field k. We also denote by $[n]$ the set $\{1, 2, \ldots, n\}$ and by $P([n], i)$ the set of all i-element subsets of $[n]$. For $F \subset [n]$, F^c denotes $[n] \setminus F$. If $F = \{i_1, \ldots, i_p\} \subset [n]$, x_F denotes the square-free monomial $x_{i_1} \cdots x_{i_p}$.

If X is any finite set, we denote by $|X|$ its cardinality.

Definition 2.2.1. A *simplicial complex* Δ over a set of vertices $[n]$ is a collection of subsets of $[n]$ with the property that $\{i\} \in \Delta$ for all $i = 1, \ldots, n$ and if $F \in \Delta$, then all subsets of F are also in Δ (including the empty set). An element of Δ is called a *face* of Δ, and the *dimension* of a face F of Δ is defined as $|F| - 1$.

The faces of dimension 0 and 1 are called *vertices* and *edges*, respectively. By convention, $\dim(\emptyset) = -1$.

The maximal faces of Δ with respect to inclusion are called *facets*. The dimension of a simplicial complex is the maximal dimension of its facets. A simplicial complex is called *pure* if all its facets have the same dimension.

A simplicial complex can be given either by enumerating all its faces, or just by giving the facets $\mathcal{F}(\Delta) = \{F_1, \ldots, F_t\}$. In the latter case we write

$$\Delta = \langle F_1, \ldots, F_t \rangle.$$

A simplicial complex with only one facet is called a *simplex*.

A *subcomplex* of the simplicial complex Δ is a simplicial complex whose facets are faces of Δ.

Definition 2.2.2. The *i-skeleton* of a simplicial complex Δ is the simplicial complex $skel_\Delta(i)$ whose facets are the i-dimensional faces of Δ.

Therefore, $skel_\Delta(i)$ is pure. By taking the i-skeleton, one loses information about the facets of dimension smaller than i.

To a simplicial complex Δ we can attach in a natural way two square-free monomial ideals in S.

Definition 2.2.3. The *Stanley-Reisner ideal* $I_\Delta \subset S$ is the ideal generated by all monomials x_F, with $F \notin \Delta$.

One sees easily that the minimal generators of I_Δ are x_F with F a minimal non-face of Δ. The ring S/I_Δ is called the *Stanley-Reisner* ring of Δ and it is denoted by $k[\Delta]$.

Remark 2.2.4. It can be proved (see [13] and [16]) that a minimal (or associated) prime of I_Δ is generated by a monomial of the form $x_{[n]\setminus F}$ where $F \in \mathcal{F}(\Delta)$.

Definition 2.2.5. The *facet ideal* of the simplicial complex Δ is the ideal $I(\Delta)$ generated by all monomials x_F, where $F \in \mathcal{F}(\Delta)$. Suppose $\Delta = \langle F_1, \ldots F_t \rangle$. Then

$$I(\Delta) = (x_{F_1}, \ldots, x_{F_t}).$$

If Δ is a graph G, $I(G)$ is called the *edge ideal* of G.

Remark 2.2.6. A minimal prime ideal over $I(G)$ is generated by a monomial x_F where F is a minimal vertex cover of G. (see [13] and [16]).

Definition 2.2.7. A graph G is called *unmixed* if all its minimal vertex covers have the same number of elements.

Remark 2.2.8. If $I \subset (x_1, \dots x_n)^2 \subset S$ is a monomial ideal generated by square-free monomials, there is exactly one simplicial complex Δ on $[n]$ such that $I = I_\Delta$. Indeed, if $G(I) = \{x_{F_1}, \dots, x_{F_t}\}$ with $F_1, \dots, F_t \subset [n]$, then

$$\Delta = \{F | F_i \not\subset F \text{ for any } i = 1, \dots, t\}.$$

When we compute the Stanley-Reisner ideal of Δ, we see that the minimal non-faces of Δ are exactly $F_1, \dots, F_t$.

In the following we present some constructions of simplicial complexes derived from a given one.

Definition 2.2.9. Suppose Δ is a pure $(d-1)$-dimensional simplicial complex. We define

$$\bar{\Delta} = \langle F | F \notin \Delta,\ F \in P([n], d) \rangle.$$

Lemma 2.2.10. *Let Δ be a $(d-1)$-dimensional pure simplicial complex, and let Γ be the simplicial complex such that $I(\Delta) = I_\Gamma$. Then*

$$\bar{\Delta} = skel_\Gamma(d-1).$$

Proof. If F is a facet of $\bar{\Delta}$, then $F \notin \Delta$. Therefore $x_F \in I(\Delta)$, hence $x_F \notin I_\Gamma$. This means that $F \in skel_\Gamma(d-1)$. The converse is straightforward. □

Definition 2.2.11. The *Alexander dual* of a simplicial complex Δ is

$$\Delta^\vee = \{[n] \setminus F | F \notin \Delta\}.$$

Remark 2.2.12. One sees easily that $(\Delta^\vee)^\vee = \Delta$. The facets of $\Delta^\vee$ are the complementary of the minimal non-faces of Δ.

Definition 2.2.13. Define

$$\Delta^c = \langle [n] \setminus F | F \in \mathcal{F}(\Delta) \rangle.$$

The facets of Δ^c are the complements of the facets in Δ. Indeed, two sets $F_1, F_2 \in [n]$ are comparable with respect to inclusion if and only if F_1^c, F_2^c are comparable.

Recall that for a monomial ideal $I \subset S$, $G(I)$ denotes the minimal set of monomial generators. Next, we express the Stanley-Reisner ideal of the Alexander dual of a simplicial complex Δ in terms of a facet ideal.

Lemma 2.2.14. *Let Δ be a simplicial complex. Then*

$$I_{\Delta^\vee} = I(\Delta^c).$$

Proof. $\Delta^\vee = \langle F^c | F$ minimal nonface of $\Delta \rangle$. Then $x_G \in G(I_{\Delta^\vee})$ if and only if G is a minimal subset of $[n]$ such that $G = (G^c)^c \notin \Delta^\vee$, equivalently G is a minimal subset of $[n]$ such that $G^c \in \Delta$, equivalently $G^c \in \mathcal{F}(\Delta)$, hence G is a facet of Δ^c. □

Recall that a set of vertices F of the graph G on the vertex set $[n]$ is called a clique if any two different vertices of F are linked by an edge in G.

Definition 2.2.15. Starting with a graph G we can build a simplicial complex

$$\Delta(G) = \{F \subset [n] | F \text{ is a clique in G } \}$$

called *the flag complex* associated to G.

The facets of $\Delta(G)$ are the maximal cliques of G. A complex of the form $\Delta(G)$ has the following interesting property.

Proposition 2.2.16. *All minimal non faces of $\Delta(G)$ consist of 2 elements. Therefore the Stanley-Reisner ideal $I_{\Delta(G)}$ is generated by quadratic monomials.*

Proof. Suppose there is $F = \{i_1, \dots, i_p\}$, $p > 2$ a minimal non-face in $\Delta(G)$. Therefore, $\{i_1, \dots, i_p\}$ is not a clique (i.e. there exist $1 \leq u < v \leq p$ such that $\{u, v\}$ is not an edge in G), but all proper subsets of F are cliques. Since $|F| > 2$, $\{u, v\}$ is a clique, hence $\{u, v\}$ is an edge in G, contradiction. $\square$

There is a converse result.

Proposition 2.2.17. *Let Δ be a simplicial complex whose minimal non-faces consist of 2 elements. Then there is a graph G such that $\Delta = \Delta(G)$.*

Proof. Take G to be the 1-skeleton of G. $\square$

Definition 2.2.18. A simplicial complex Δ is called *flag* if any minimal non-face consists of 2 elements.

According to the above theorem, this is equivalent to Δ being of the form $\Delta(G)$ for some graph G. Using the Alexander dual, we see that Δ is flag if and only if $\Delta^\vee$ is a graph.

2.3. **Algebraic properties via Alexander duality.**

There are in the literature some very nice results relating combinatorial or algebraic properties of a simplicial complex Δ to those of the Alexander dual $\Delta^\vee$. We present here some of them that will be needed in the following sections.

In the following, Δ denotes a simplicial complex on $[n]$, and k a field. If

$$0 \to S^{\beta_h} \to \ldots \to S^{\beta_1} \to S \to k[\Delta] \to 0$$

is a minimal free resolution of $k[\Delta]$, then the resolvents may also be given the $\mathbb{N}^n$-grading, so as to make the maps in the resolution homogeneous, and $\mathrm{Tor}_i^S(k[\Delta], k)$ inherits this grading. For a given grade $\alpha \in \mathbb{N}^n$, let $\mathrm{Tor}_i^S(k[\Delta], k)_\alpha$ denote the α-graded component of $\mathrm{Tor}_i^S(k[\Delta], k)$. One can put together these parts and define the *Betti polynomial*

$$T_i(k[\Delta], t) = \sum_\alpha \dim_k \mathrm{Tor}_i^S(k[\Delta], k)_\alpha t^\alpha$$

where $t^\alpha = \prod_i t_i^{\alpha_i}$. We present here without proof the formula found by M.Hochster for these Betti polynomials.

Theorem 2.3.1. *(Hochster* [12]*)*

$$T_i(k[\Delta], t) = \sum_{W \subset [n]} \dim_k \tilde{H}_{|W|-i-1}(\Delta_W; k) t^W,$$

where Δ_W denotes the simplicial complex on the vertex set W defined by

$$\Delta_W = \{W' \subset W | W' \in \Delta\}.$$

Here $\tilde{H}(\cdot; k)$ denotes the homology with coefficients in the field k, and $t^W = \prod_{i \in W} t_i$.

Proof. See Hochster's original article [12] or in Bruns and Herzog [2] Theorem 5.5.1 □

In [7] Eagon and Reiner notice that one may view the reduced homologies in Hochsters's formula as the reduced cohomologies of links of faces of $\Delta^\vee$, the Alexander dual of Δ.

Notice that, if one thinks of Δ as an ordered ideal in the Boolean algebra $2^{[n]}$, then $\Delta^\vee$ is obtained by taking the order filter $2^{[n]} \setminus \Delta$ and then applying the order-reversing map $F \mapsto [n] \setminus F$ to each of these sets, yielding another order ideal $\Delta^\vee$.

Definition 2.3.2. If F is a face of the simplicial complex Δ on $[n]$, then the *link* of the face F is the simplicial complex on the vertex set $[n] \setminus F$ defined by

$$\mathrm{link}_\Delta F = \{G \in \Delta | G \cup F \in \Delta, G \cap F = \emptyset\}.$$

Definition 2.3.3. Given the vertex $v \in \Delta$, the *deletion of vertex v* is the simplicial complex given by

$$\mathrm{del}_\Delta v = \{G \in \Delta | v \notin G\}.$$

Proposition 2.3.4.

$$T_i(k[\Delta], t) = \sum_{F \in \Delta^\vee} \dim_k \tilde{H}_{i-2}(\operatorname{link}_{\Delta^\vee} F; k) t^{[n]\setminus F}$$

Proof. Given $W \subset [n]$ appearing as a term in Hochster's sum from Theorem 2.3.1, let $F = [n] \setminus W$. Note that if W is a face of Δ, then Δ_W will be a simplex and hence have no reduced homology, therefore we may assume W is not a face of Δ. By the definition of $\Delta^\vee$, then F is a face of $\Delta^\vee$, so F appears in the sum on the right-hand side term in this proposition.

Therefore, it suffices to show that:

$$\dim_k \tilde{H}_{i-2}(\operatorname{link}_{\Delta^\vee} F; k) = \dim_k \tilde{H}_{|W|-i-1}(\Delta_W; k).$$

To see this, note that the complementation map

$$\{W' \subset [n] | W' \subset W\} \to \{F' \subset [n] | F \subset F'\}$$

given by $W' \mapsto [n] \setminus F'$ identifies the Boolean algebra 2^W with the interval $[F, [n]]$ in the Boolean algebra $2^{[n]}$, and has the property that W' is a face of Δ if and only if $F' = [n] \setminus V'$ is not a face of $\Delta^\vee$. Thus, the map gives an isomorphism between the complexes $\operatorname{link}_{\Delta^\vee}$ and $(\Delta_W)^\vee$, if we consider both to have the same vertex set W. Applying the following lemma and the isomorphism between the reduced homology and the reduced cohomology over a field k, the conclusion follows. □

Lemma 2.3.5. *For any simplicial complex Δ on vertex set $[n]$, we have*

$$\tilde{H}_{i-2}(\Delta^\vee; k) = \tilde{H}^{n-i-1}(\Delta; k).$$

Proof. see [7] □

Definition 2.3.6. The simplicial complex Δ is called *Cohen Macaulay* (or *Gorenstein*) *over the field* k if the Stanley-Reisner ring $k[\Delta]$ is Cohen-Macaulay (respectively Gorenstein).

If Δ is Cohen-Macaulay (or Gorenstein) over any field k, then it is called *Cohen-Macaulay* (respectively *Gorenstein*).

We mention here without proof a criterion that Reisner obtained for Δ to be Cohen-Macaulay over k (see Corollary 5.3.9 in [2]).

Theorem 2.3.7. *The following conditions are equivalent:*
a) Δ is Cohen Macaulay over k;
b) $\tilde{H}_i(\operatorname{link}_\Delta F; k) = 0$ for all $F \in \Delta$ and all $i < \dim \operatorname{link}_\Delta F$.

Definition 2.3.8. An ideal $I \subset S$ has *linear resolution* if there is a minimal free resolution for S/I in which all non-zero entries in the matrices $\varphi_i\colon S^{\beta_i} \to S^{\beta_{i-1}}$ for all $i \geq 2$ are of degree 1 in the standard grading on S.

Eagon and Reiner used Proposition 2.3.4 to prove the following nice result:

Theorem 2.3.9. *I_Δ has linear resolution if and only if $\Delta^\vee$ is Cohen-Macaulay over k.*

Proof. It is easy to see that I_Δ has linear resolution if and only if its minimal generators all have the same degree t and for each i we have $\mathrm{Tor}_i^S(k[\Delta], k)$ is homogeneous of degree $t+i$ in the standard grading. The first of these conditions is equivalent to $\Delta^\vee$ being pure. Using Proposition 2.3.4 the second condition is equivalent to $\mathrm{link}_{\Delta^\vee} F$ having no homology over k except on its top dimension for all faces F of $\Delta^\vee$. Thus these two conditions, by Reisner's criterion 2.3.7 are equivalent to $\Delta^\vee$ being Cohen-Macaulay over k. $\square$

Definition 2.3.10. The simplicial complex Δ is called *shellable* if it is pure and we can order its facets $F_1, \ldots, F_t$ in such a way that for each $i \geq 2$ the intersection $F_i \cap (\cup_{j<i} \bar{F_j})$ between F_i and the subcomplex generated by the previous facets is a subcomplex of codimension 1 inside F_i.

Theorem 2.3.11. *([2] Theorem 5.1.13) If Δ is shellable, it is Cohen-Macaulay over any field k.*

In [5] Fröberg gives the following characterization of Stanley-Reisner ideals generated by quadratic monomials and with a linear resolution:

Theorem 2.3.12. *([5], Theorem 1) A Stanley-Reisner ideal I_Δ generated by quadratics has linear resolution if and only if $\Delta = \Delta(G)$ for some chordal graph G, where recall that $\Delta(G)$ means the flag complex associated to G.*

For chordal graphs besides the results presented in Theorem 2.1.9 it is also known the following characterization:

Theorem 2.3.13. *A graph G is chordal if and only if there is an elimination ordering $v_1, \ldots, v_n$ on the vertices, i.e. for all i there are edges between all pairs of v_i's neighbors in $G \setminus \{v_1, \ldots, v_{i-1}\}$. In this case v_i is called a simplicial vertex of $G \setminus \{v_1, \ldots, v_{i-1}\}$.*

Definition 2.3.14. A simplicial complex Δ is said to be *vertex decomposable* if it satisfies the following recursive definition: either $\Delta = \{\emptyset\}$, or there exists some vertex $v \in \Delta$ for which both subcomplexes $\mathrm{del}_\Delta v$ and $\mathrm{link}_\Delta v$ are vertex decomposable.

Theorem 2.3.15. *A vertex-decomposable simplicial complex is shellable.*

Theorem 2.3.16. *The following are equivalent:*
a) $\Delta(G)^\vee$ is vertex decomposable;
b) $\Delta(G)^\vee$ is Cohen-Macaulay over any field k;
c) $\Delta(G)^\vee$ is Cohen-Macaulay over some field k;
d) G is chordal.

Proof. The implications a)$\Rightarrow$b)$\Rightarrow$c) are trivial.

c)$\Rightarrow$d) If G is not chordal, then there exists some subset W of vertices which form a cycle in G having no chord. It can be proven (Fröberg [5]) that $\Delta(G)$ is homemorphic to a circle. Using the identity proved along with proposition 2.3.4, one gets that

$$0 = \tilde{H}_{|W|-4}(\mathrm{link}_{\Delta(G)^\vee} F; k) = \tilde{H}_1(\Delta(G)_W; k) \neq 0,$$

so that $\Delta(G)^\vee$ is not Cohen-Macaulay over k, contradiction.

d)$\Rightarrow$a) If G id chordal, let $v_1, \ldots, v_n$ be an elimination ordering for its vertices. A vertex decomposition for $\Delta(G)^\vee$ starting with v_1 will then follow from lemma 2.3.17 below. We have to show that the lemma implies that both subcomplexes $\operatorname{link}_{\Delta(G)^\vee} v_1$ and $\operatorname{del}_{\Delta(G)^\vee} v_1$ are vertex decomposable. By induction and part 1 of the lemma we have that $\operatorname{link}_{\Delta(G)^\vee} v_1 = (\Delta(G \setminus v_1))^\vee$ is vertex decomposable, since $G \setminus v_1$ is chordal because G is chordal. By part 2 of the lemma, since v_1 is simplicial, $\operatorname{del}_{\Delta(G)^\vee} v_1$ is the complex generated by a collection of codimension one faces of a simplex, and all such complexes are easily seen to be vertex-desomposable. $\square$

Lemma 2.3.17. *1) For any vertex v in a graph G, we have $\operatorname{link}_{\Delta(G)^\vee} v = (\Delta(G \setminus v))^\vee$ as complexes on the vertex set $[n] \setminus \{v\}$.*

2) For any simplicial vertex v in a graph G, the deletion $\operatorname{del}_{\Delta(G)^\vee} v$ is the simplicial complex on the vertex set $[n] \setminus \{v\}$ having as facets the faces $[n] \setminus \{v, v'\}$ as v' runs over all non-neighbors of v in G.

Before going further we recall another equivalent definition of shellability, as it appears in [2], Definition 5.1.11.

Proposition 2.3.18. *The simplicial complex Δ is shellable if and only if it is pure and there exists an ordering of the facets $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$ such that for all $0 < j < i$ and $x \in F_i \setminus F_j$ there is some $k < i$ such that $F_i \setminus F_k = \{x\}$.*

Definition 2.3.19. An ideal $I \subset S$ is said to have *linear quotients* if $I = (f_1, \ldots, f_m)$ and for all $i > 0$ the colon ideals $(f_1, \ldots, f_{i-1})\colon f_i$ are generated by linear forms. For a monomial ideal I we require also that the f_i's belong to the unique minimal set of monomial generators $G(I)$ of I.

Proposition 2.3.20. *The simplicial complex Δ is shellable if and only if $I_{\Delta^\vee}$ has linear quotients.*

Proof. $I = (f_1, \ldots, f_m)$ has linear quotients if for all $i > 1$ and any $j < i$, there exists $k < i$ such that $f_k / \operatorname{lcm}(f_i, f_k)$ is a monomial of degree 1, say x_ℓ and $x_\ell | f_{x_j}$. But by Lemma 2.2.14 $I_{\Delta^\vee} = (x_{F_1^c}, \ldots, x_{F_m^c})$, and the conclusion follows. $\square$

Theorem 2.3.21. *(Terai,* [14]*)* $\operatorname{projdim} k[\Delta] = \operatorname{reg}(I_{\Delta^\vee})$

3. Quasi-forests and Dirac's theorem

3.1. **Taylor's resolution of a monomial ideal.**

In her Ph.D thesis [15] Diana K. Taylor described a way to produce a finite free resolution of J, when J is an ideal in the ring R generated by monomials in $a_1, \ldots, a_n$ (an R-sequence) such that either J contains a power of each a_i, for $1 \leq i \leq n-1$, or every permutation of $a_1, \ldots, a_n$ is an R-sequence.

We limit ourselves to the case when R is the polynomial ring $S = k[x_1, \ldots, x_n]$. As $a_1, \ldots, a_n$ will shall take the indeterminates $x_1, \ldots, x_n$ which remain an S-sequence after any permutation. An ideal J generated by monomials in the indeterminates $x_1, \ldots, x_n$ is therefore a monomial ideal of S.

Let $m_1, \ldots, m_t$ be monomials in the x_i-s. We define the *Taylor complex* $\mathbb{T}_\bullet$ as follows. Let F_s be the free module on basis elements e_I, where I is a subset of length s, $I \subset \{1, 2, \ldots, t\}$. Set

$$m_I = \text{least common multiple}\{m_i | i \in I\}.$$

For each pair of subsets I, H such that I has s elements and H has $s-1$ elements, let $I = \{i_1, \ldots, i_s\}$ and suppose $i_1 < i_2 < \ldots < i_s$. Define $c_{I,H} = 0$ if $H \not\subset I$ and $c_{I,H} = (-1)^k m_I / m_H$ if $I = H \cup \{i_k\}$ for some k. Finally define $d_s \colon F_s \to F_{s-1}$ by sending

$$e_I \mapsto \sum_H c_{I,H} e_H.$$

Let $\mathbb{T}(m_1, \ldots, m_t)_\bullet \colon 0 \to F_t \xrightarrow{d_t} \ldots \xrightarrow{d_1} F_0 \longrightarrow 0.$

$$c_{I,H} = \begin{cases} 0, & \text{if} \quad H \not\subset I, \\ \\ (-1)^k m_I / m_H, & \text{if} \quad I = H \cup \{i_k\}. \end{cases}$$

Theorem 3.1.1. *The complex $\mathbb{T}(m_1, \ldots, m_t)_\bullet$ is a free resolution of the ideal $J = (m_1, \ldots, m_t)$.*

Proof. This theorem is proved in Taylor's thesis [15]. For some hints on how to prove the particular case presented here check Exercise 17.11 in [8]. $\square$

3.2. **Perfect monomial ideals of codimension 2.**

Let I be generated by the monomials $m_1, \dots, m_t$ in S and consider the start

$$\Lambda^2 S^t \xrightarrow{\varphi_2} S^t \xrightarrow{\varphi_1} S \longrightarrow S/I \longrightarrow 0$$

of the Taylor resolution. We denote by $e_1, \dots, e_t$ the canonical basis of S^t. Then $\varphi_1(e_i) = m_i$ for all $i = 1, \dots, t$ and

$$\varphi_2(e_i \wedge e_j) = \frac{\operatorname{lcm}(m_i, m_j)}{m_i} e_i - \frac{\operatorname{lcm}(m_i, m_j)}{m_j} e_j, \text{ for any } 1 \le i < j \le t.$$

The Taylor resolution is multigraded with $\deg e_i = \deg m_i$ and $\deg e_i \wedge e_j = \deg \operatorname{lcm}(m_i, m_j)$.

We choose some $\ell \in \{1, \dots, t\}$ and some $i_1, \dots, i_\ell \in \{1, \dots, t\}$. Set

$$s(i_1, \dots, i_\ell) = \sum_{k=1}^{\ell} \frac{\operatorname{lcm}(m_{i_1}, \dots, m_{i_\ell})}{\operatorname{lcm}(m_{i_k}, m_{i_{k+1}})} \quad (\text{where } i_{\ell+1} = i_1).$$

Then $\varphi_2(s(i_1, \dots, i_\ell)) = 0$. We call the elements $s(i_1, \dots, i_\ell)$ *cyclic syzygies.*

In the following, let $R = \mathbb{Z}[x_1, \dots, x_n]$.

Proposition 3.2.1. *Let U be a subset of $\{(i, j | 1 \le i < j \le t)\}$ and set $F = \sum_{(i,j) \in U} Re_i \wedge e_j$. Then $\ker(\varphi_2|_F)$ is generated by cyclic syzygies.*

Proof. We prove the proposition by induction on the number of elements of U. If $U = \emptyset$, then there is nothing to prove. Let us assume that $|U| > 0$, and that the property holds for all U with fewer elements. $\ker(\varphi_2|_F)$ is a multigraded submodule, therefore it is generated by homogeneous elements.

Let $x \in F \cap \ker(\varphi_2)$ be a homogeneous (in the fine grading) element. $x = \sum_{(i,j) \in U'} a_{ij} e_i \wedge e_j$ with $a_{ij} \neq 0$. We may assume that $U = U'$, otherwise we may apply directly the induction hypothesis. Consider the graph Γ with edges $(i, j) \in U$. If Γ contains no cyles, then the elements $\varphi_2(e_i \wedge e_j)$, $(i, j) \in U$ are linearly independent over R. Otherwise, there is a cycle in Γ, say $(i_1, i_2), (i_2, i_3), \dots, (i_{\ell-1}, i_\ell), (i_\ell, i_1)$. Write $x = \sum_{k=1}^{\ell} \alpha_k v_k e_{i_k} \wedge e_{i_{k+1}} + r$, where $\alpha_k \in \mathbb{Z}$ and v_k is a monomial for all k and where r is a linear combination in the remaining basis elements $e_i \wedge e_j$. Since x is homogeneous, we must have

$$v_1 \operatorname{lcm}(m_{i_1}, m_{i_2}) = \cdots = v_\ell \operatorname{lcm}(m_{i_\ell}, m_{i_1}).$$

Let v denote the common value above. Then there exist a monomial $\tilde{v}$ such that $v = v_k \operatorname{lcm}(m_{i_k}, m_{i_{k+1}}) = \tilde{v} \operatorname{lcm}(m_{i_1}, \dots, m_{i_\ell})$ for any k. Hence

$$v_k = \frac{\tilde{v} \operatorname{lcm}(m_{i_1}, \dots, m_{i_\ell})}{\operatorname{lcm}(m_{i_k}, \dots, m_{i_{k+1}})}$$

for all k, and it follows that the new cycle $x' = x - \alpha_1 \tilde{v} s(i_1, \dots, i_\ell) \in \ker \varphi_2$ is a linear combination of fewer basis elements than x. Therefore, by our induction hypothesis the proof is complete. $\square$

Corollary 3.2.2. *There exists a multigraded exact sequence*

$$F \xrightarrow{\psi_2} R^t \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0$$

with a free R-module F such that $\ker \psi_2 \subset (x)F$, *where* $(x) = (x_1, \dots, x_n)$.

Proof. Let $U \subset \{(i,j)|1 \le i < j \le t\}$ such that the elements $\{\varphi_2(e_i \wedge e_j)|(i,j) \in U\}$ generate $\ker \varphi_1$ minimally, i.e. none of them can be left out. Set $F = \sum_{(i,j)\in U} Re_i \wedge e_j$ and $\psi_2 = \varphi_2|_F$. Suppose that $\ker \psi_2 \not\subset (x)F$. Then, by Proposition 3.2.1 there exists a cyclic syzygy $s(i_1, \ldots, i_\ell) \in F \setminus (x)F$. Such a cyclic syzygy contains a coefficient ± 1 , a contradiction to the minimality of U. $\square$

Remark 3.2.3. The Corollary 3.2.2 says that the sequence

$$F \xrightarrow{\psi_2} R^t \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0$$

can be continued to become a minimal multigraded free resolution of R/I.

Corollary 3.2.4. *Let k be a field and J a monomial ideal in $S = k[x_1, \ldots, x_n]$. Then, the multigraded Hilbert series of* $\mathrm{Tor}_2^S(S/J, k)$ *is independent of k.*

Proof. We let I be the ideal in $\mathbb{Z}[x_1, \ldots, x_n]$ generated by the same monomials as J, and tensor the complex of Corollary 3.2.2 with k (over $\mathbb{Z}$). This yields the start of a minimal multigraded resolution of S/J. $\square$

Corollary 3.2.5. *The second Betti number of S/J is independent of the field k.*

The results proved up to now enable us to proceed towards a complete classification of perfect monomial ideals $I \subset S$ of grade 2. For such an ideal a minimal multigraded free resolution of S/I has the form:

$$0 \longrightarrow S^{t-1} \xrightarrow{\varphi_2} S^t \xrightarrow{\varphi_1} S \longrightarrow S/I \longrightarrow 0$$

where the relations of the generators of I given by φ_2 are Taylor relations and S^{t-1} has a basis of the form $e_i \wedge e_j, (i,j) \in U$. Let Γ be the graph associated to U as described in the proof of Proposition 3.2.1. Then Γ is a graph with t vertices, $t-1$ edges and no cycles, therefore it is a tree. In the matrix of φ_2 one has $2(t-1)$ nonzero entries, and on each line we have exactly 2 nonzero entries. We choose $2(t-1)$ variables and place them into an $(t-1) \times t$ matrix ψ_2 such that the nonzero entries of ψ_2 and φ_2 are in the same positions. It is easy to see that the $t-1$ minors of ψ_2 are monomials (up to sign). Furthermore, I arises from the Fitting ideal $I_{t-2}(\psi_2)$ by the substitution that sends each entry of ψ_2 into the corresponding entry of φ_2. We can draw the following conclusion from these observations:

Theorem 3.2.6. *The generic types of perfect monomial ideals $I = (m_1, \ldots, m_t)$ of grade 2 are in bijective correspondence with the trees with t vertices.*

Remark 3.2.7. Note that, in general, the tree Γ is not uniquely determined by I, but it is uniquely determined by the Taylor relations that generate $syz_1(I) = syz_2(R/I)$.

Example 3.2.8. If I is a perfect monomial ideal of grade 2, then it admits a multigraded resolution of the form

$$0 \longrightarrow S^{t-1} \xrightarrow{\varphi} S^t \longrightarrow I \longrightarrow 0.$$

We present here an example in order to understand exactly how these trees appear. Let $I = (x_4x_5x_6, x_1x_5x_6, x_1x_2x_6, x_1x_2x_5)$ be a monomial ideal in $S =$

$k[x_1, x_2, x_3, x_4, x_5]$. One can prove that ht($I$)=2 and that I is perfect. We are looking for a presentation of I of the kind

$$0 \longrightarrow S^3 \xrightarrow{\varphi} S^4 \longrightarrow I \longrightarrow 0.$$

Our ideal I can be presented by the matrix

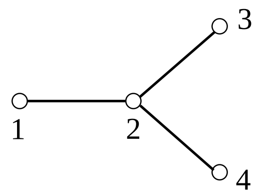
$$\begin{pmatrix} x_1 & -x_4 & 0 & 0 \\ 0 & x_2 & -x_5 & 0 \\ 0 & x_2 & 0 & x_6 \end{pmatrix} \text{ or by } \begin{pmatrix} x_1 & -x_4 & 0 & 0 \\ 0 & x_2 & -x_5 & 0 \\ 0 & 0 & x_5 & -x_6 \end{pmatrix}$$

$$\text{or } \begin{pmatrix} x_1 & -x_4 & 0 & 0 \\ 0 & x_2 & 0 & -x_6 \\ 0 & 0 & x_5 & -x_6 \end{pmatrix}.$$

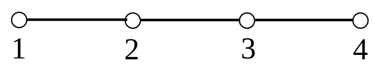
The second matrix is obtained from the first by substracting the second line from the third line in the first matrix; and the third matrix is obtained from the second one by adding the third line to the second line. It is obvious that since the first matrix is a representation for I, then so are the other two: the lines in these matrixes represent coordinates of the vectors of a basis of $syz_1(I)$.

How can we actually build the trees? For a given choice R of $t-1$ Taylor relations which generate $syz_1(I)$ we define a tree Ω with t vertices with $\{i,j\} \in E(\Omega)$ if $m_{ji}e_i - m_{ij}e_j \in R$ for $i < j$, where we denote $m_{pq} = m_p/\gcd(m_p, m_q)$ for any $1 \le p, q \le t$. Ω is called *the relation tree of R*.

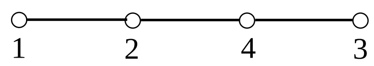
For our example, the relation tree for the first matrix is



while for the other matrices we have:

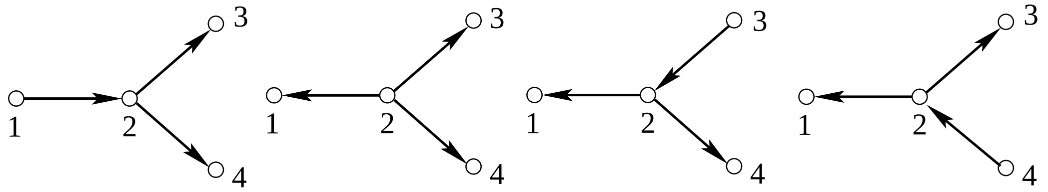


or



Next, we want to describe how the generators m_i of I can be computed from the m_{ij} and the relation tree. To this end we introduce for each $i = 1, \dots, t$ an orientation to make Ω a directed(oriented) graph which we denote by Ω_i. Let's fix some vertex i. For any other vertex j of the tree Ω there is a unique directed walk from i to j.

For the first of our relation trees above we get the following directed graphs:



By the Hilbert-Burch Theorem 1.3.8 we have

$$m_i = (-1)^i \det(A_i),$$

where the matrix A_i is obtained from the relation matrix A by deleting its i^{th} column. Computing $\det(A_i)$, one sees that

$$m_i = \prod_{(k,j)} m_{kj},$$

where the product is taken over all oriented edges (k, j) of Ω_i.

3.3. **Quasi-forests and relation trees of ideals of projective dimension 1.**

In [4] S. Faridi extends the notions of *tree* and *forest* from graphs to simplicial complexes. In [17] X. Zheng gives a further extention of these notions introducing the so-called *quasi-trees* and *quasi-forests*.

In the following, let Δ be a simplicial complex.

Definition 3.3.1. A facet $F \in \mathcal{F}(\Delta)$ is called a *leaf*, if either F is the only facet of Δ, or there exists a facet $G \in \mathcal{F}(\Delta)$ such that $G \neq F$ and $H \cap F \subset G \cap F$ for each $H \in \mathcal{F}(\Delta)$ with $H \neq F$. A facet G with this property is called a *branch* of F.

Definition 3.3.2. A vertex i of Δ is called a *free vertex* if it belongs to exactly one facet.

Remark 3.3.3. The condition for $G \neq F$ to be a branch of F is equivalent to the fact that the inclusion-ordered set of "traces" of F on the other facets $\{H \cap F | H \in \mathcal{F}(\Delta), H \neq F\}$ has a unique maximal element.

Remark 3.3.4. For a given leaf F of Δ the branch G is not necessarily unique. Denote $F = (1, 2, 3, 4)$, $G = (1, 2, 3, 5)$, $H = (1, 2, 6)$, $J = (1, 2, 7)$ and $G_1 = (1, 2, 3, 8)$. For instance, in the simplicial complex given by the set of facets $\Delta = \langle F, G, H, J\rangle$, F is a leaf and G is the only branch that "supports" it. But in $\Delta_1 = \langle F, G, H, J, G_1\rangle$ the leaf F is supported by both branches G and G_1. It is also possible for a simplicial complex to contain no leaves at all. Take $\Delta_2 = \langle (1, 2, 3), (1, 3, 4), (1, 2, 4)\rangle$ for instance.

Remark 3.3.5. An easy to check necessary criterion for a facet of Δ to be a leaf is the presence of a free vertex. In the example before Δ_2 had no free vertices, thus no leaves. This condition is not sufficient:

$$\Delta = \langle (1, 2, 3), (3, 4, 5), (5, 6, 7)\rangle$$

is a pure simplicial complex, the facet $(3, 4, 5)$ has a free vertex 4, but it is not a leaf.

Remark 3.3.6. For a graph G, F is a facet if F is an isolated vertex or an edge. Therefore, the leaves are the isolated vertices and the edges $\{i, j\}$ such that one of the ends is a free vertex.

Definition 3.3.7. A simplicial complex Δ is *connected* if for any two facets F and G there exists a sequence of facets $F = F_0, \dots, F_n = G$, such that $F_i \cap F_{i+1} \neq \emptyset$ for all $i = 0, \dots, n - 1$.

Definition 3.3.8. A connected simplicial complex Δ is called a *tree* if every nonempty subcomplex of Δ (including Δ) has a leaf.

Equivalently, Δ is a tree if every nonempty connected subcomplex of Δ has a leaf.

Definition 3.3.9. ([4]) A simplicial complex Δ with the property that every connected component of Δ is a tree is called a *forest*. In other words, a forest is a simplicial complex with the property that every nonempty subcomplex has a leaf.

Definition 3.3.10. ([17]) A conected simplicial complex Δ is called a *quasi-tree*, if there exists an order $F_1, \ldots, F_t$ of the facets, such that F_i is a leaf of $\langle F_1, \ldots, F_i \rangle$ for all $i = 1, \ldots, t$. A simplicial complex Δ with the property that every connected component is a quasi-tree is called a *quasi-forest.*

Remark 3.3.11. ([17]) Note that a quasi-tree may have different leaf orders. It is clear that a tree (as in Definition 3.3.9) is a quasi-tree. However, the converse is not true. The complex $\Delta = \langle (1,2,3), (2,3,4), (3,4,5), (2,4,6) \rangle$ is a quasi-tree, but it is not a tree, because the subcomplex $\langle (1,2,3), (3,4,5), (2,4,6) \rangle$ has no leaf. Still, for 1-dimensional simplicial complexes (i.e. graphs), the notions of tree and quasi-tree coincide.

Let Δ be a simplicial complex on $[n]$ with $\mathcal{F}(\Delta) = \{F_1, \ldots, F_t\}$ its facets. We introduce the $\binom{t}{2} \times t$ matrix

$$M_\Delta = (a_k^{(i,j)})_{1 \le i < j \le t, 1 \le k \le t}$$

whose entries $a_k^{(i,j)}$ are $a_i^{(i,j)} = x_{F_i \setminus F_j}$, $a_j^{(i,j)} = x_{F_j \setminus F_i}$, and $a_k^{(i,j)} = 0$ if $k \neq i, k \neq j$ and for all $1 \le i < j \le t, 1 \le k \le t$. Recall that for some $F = \{i_1, \ldots, i_s\} \subset [n]$, x_F denotes the monomial $x_{i_1} \cdots x_{i_s}$. And another piece of notation: if u and v are monomials in S, we write $|u| = v$ if $u = \pm v$.

Lemma 3.3.12. *A simplicial complex $\mathcal{F}(\Delta) = \{F_1, \ldots, F_t\}$ on $[n]$ is a quasi-forest if and only if the matrix M_Δ contains a $(t-1) \times t$ submatrix $M_\Delta^\sharp$ with the property that for each $1 \le j \le t$, if $M_\Delta^\sharp(j)$ is the $(t-1) \times (t-1)$ submatrix of $M_\Delta^\sharp$ obtained by removing the j^{th} column from $M_\Delta^\sharp$, then $|\det(M_\Delta^\sharp(j))| = x_{[n]}/x_{F_j}$.*

Proof. "$\Rightarrow$" Let Δ be a quasi-forest on $[n]$ and fix a leaf ordering. Let $t > 1$ and let $F_k, k \neq t$ be a branch of F_t and $\Delta' = \Delta \setminus F_t$. Since Δ' is a quasi-forest (by definition) on $[n] \setminus (F_t \setminus \cup_{i<t} F_i)$ generated by $t-1$ facets, by our induction hypothesis it follows that M_Δ contains a $(t-2) \times t$ submatrix with the property that, for each $1 \le j < t$, if $M'(j,t)$ is the $(t-2) \times (t-2)$ submatrix obtained by removing the j^{th} and the t^{th} columns from M', then $|\det(M'(j,t))| = x_{[n] \setminus (F_t \setminus F_k)}/x_{F_j}$. Let $M_\Delta^\sharp$ denote the $(t-1) \times t$ submatrix of M_Δ obtained by adding the (k,t) row to M'. On the added row only the entries $a_t^{(k,t)} = x_{F_t \setminus F_k}$ and $a_k^{(k,t)} = x_{F_k \setminus F_t}$ are nonzero. If $1 \le j < t$ one has $|\det(M_\Delta^\sharp(j))| = x_{F_t \setminus F_k} \cdot |\det M'(j,t)| = x_{F_t \setminus F_k} \cdot x_{[n] \setminus (F_t \setminus F_k)}/x_{F_j} = x_{[n]}/x_{F_j}$. If $j = t$, $|\det(M_\Delta^\sharp(t))| = |x_{F_k \setminus F_t} \cdot \det(M'(k,t))| = x_{F_k \setminus F_t} \cdot x_{[n] \setminus (F_t \setminus F_k)}/x_{F_k} = x_{[n]}/x_{F_t}$.

" $\Leftarrow$ " Suppose that the matrix M_Δ contains a $(t-1) \times t$ submatrix with the property that, for each $1 \le j \le t$, if $M_\Delta^\sharp(j)$ is the $(t-1) \times (t-1)$ submatrix of $M_\Delta^\sharp$ obtained by removing the j^{th} column from $M_\Delta^\sharp$, then $|\det(M_\Delta^\sharp(j))| = x_{[n]}/x_{F_j}$. Let Ω denote the graph on $[t]$ whose edges are those $\{i,j\}$ with $1 \le i < j \le t$ such that the $(i,j)^{th}$ row of M_Δ belongs to $M_\Delta^\sharp$. Then Ω contains no cycles.

If $C = \{E_1, E_2, \ldots, E_p\}$ were a cycle in Ω with $\{E_1, E_2\} = \{i_1, j_1\} \in E(C)$, the set of edges of the cycle, then in the matrix $M_\Delta^\sharp(i_1)$ the $(i,j)^{th}$ rows with $(i,j) \in E(C)$ are linearly dependent (over $S = k[x_1, \ldots, x_n]$), equivalently

$\det M_\Delta^\sharp(i_1) = 0$, a contradiction. Indeed, if we multiply each line (i,j) of $M_\Delta^\sharp$ with the monomial $x_{F_i\cap F_j}$ we obtain a matrix $M_\Delta^\flat$ which on the $(i,j)^{th}$ line has the entries x_{F_i} and x_{F_j} on the i^{th}, respectively the j^{th} column. And obviously $\det M_\Delta^\sharp(i_1) = 0$ if and only if $\det M_\Delta^\flat(i_1) = 0$. And now if we add to the line in $M_\Delta^\flat$ that corresponds to the edge $\{E_1, E_2\}$ each line that corresponds to an edge $\{E_q, E_{q+1}\}$ multiplied by $(-1)^q \cdot x_{F_{E_q}\cap F_{E_{q+1}}}$ we obtain only zeros on the $\{i_1, j_1\}$ line in $M_\Delta^\flat(i_1)$, thus $\det M_\Delta^\sharp(i_1) = 0$.

Ω is a graph with t vertices, $t-1$ edges and without cycles. Therefore Ω is connected, and it is a tree. Then, by the following Lemma 3.3.13 it must have a free vertex v which has only v' as its neighbor. This implies that on the v^{th} column in $M_\Delta^\sharp$ there is only one nonzero entry. Without loss of generality we can assume that it is the t^{th} column that contains exactly one nonzero entry and that the $(k,t)^{th}$ row appear in $M_\Delta^\sharp$. Then, for each $1 \leq j < t$, the monomial $x_{F_t\setminus F_k}$ (lying on the t^{th} column on the $(k,t)^{th}$ row) divides $|\det(M_\Delta^\sharp(j))| = x_{[n]}/x_{F_j}$. It follows that F_t is a leaf of Δ and F_k is a branch of F_t. Let $\Delta' = \Delta \setminus F_t$ and $M_{\Delta'}^\sharp$ the $(t-2)\times(t-1)$ submatrix of $M_{\Delta'}^\sharp$, which is obtained by removing the $(k,t)^{th}$ row and the t^{th} column from $M_\Delta^\sharp$. Since Δ' is a simplicial complex on $[n]\setminus(F_t\setminus F_k)$ and since $x_{F_t\setminus F_k}\cdot(x_{[n]\setminus(F_t\setminus F_k)}/x_{F_j}) = x_{[n]}/x_{F_j}$ for $1 \leq j \leq t-1$, working with induction on t it follows that Δ' is a quasi-forest. $\square$

Lemma 3.3.13. *Let G be a graph. If G is a tree, it has a free vertex.*

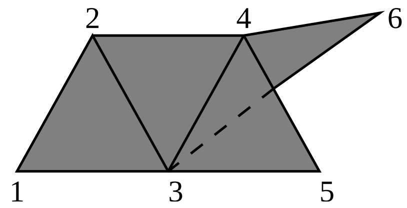
Proof. If the graph G has t vertices, being a tree it must have $t-1$ edges. We use the identity
$$\sum_{v\in G} \deg\, v = 2|E(G)|.$$
G does not have any isolated vertices because it is connected. If all vertices had degree at least 2, then $2(t-1) \geq 2t$, contradiction. Therefore G must have a vertex of degree 1, i.e. a free vertex. $\square$

Corollary 3.3.14. *A simplicial complex Δ is a quasi-forest if and only if*
$$\operatorname{projdim} I(\Delta^c) = 1.$$

Proof. Let $\mathcal{F}(\Delta) = \{F_1, \dots F_t\}$ be the facets of Δ. By Lemma 3.3.12, the simplicial complex Δ is a quasi-forest if and only if M_Δ contains a $(t-1)\times t$ submatrix $M_\Delta^\sharp$ whose ideal of maximal minors is $I(\Delta^c)$. Hence, if Δ is a quasi-forest, the Hilbert-Burch Theorem 1.3.8 implies that $\operatorname{projdim} I(\Delta^c) = 1$. Conversely, suppose $\operatorname{projdim} I(\Delta^c) = 1$, and let A be a $(t-1)\times t$ relation matrix of this ideal consisting of Taylor relations. By the Hilbert-Burch Theorem again, $I(\Delta^c)$ is the ideal of maximal minors of A. Since $M_\Delta = M_{\Delta^c}$, it follows that A is a submatrix of M_Δ. Hence Δ is a quasi-forest. $\square$

Let's come back to the perfect monomial ideal
$$I = (x_4x_5x_6, x_1x_5x_6, x_1x_2x_6, x_1x_2x_5)$$
from Example 3.2.8. I may be viewed as a facet ideal $I = I(\Delta^c)$, where
$$\Delta = \langle(1,2,3), (2,3,4), (3,4,5), (3,4,6)\rangle.$$

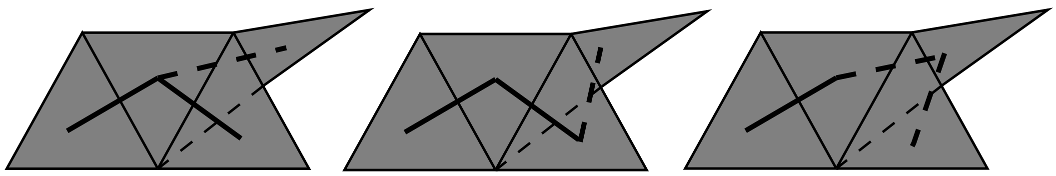
Δ has the following geometric realization:



This is a quasi-forest (actually a quasi-tree), as it should be by Corollary 3.3.14.

From the proof of Lemma 3.3.12 we notice that all possible relation trees Ω of $I(\Delta^c)$ can be recovered from the quasi-forest $\Delta = \langle F_1, \ldots F_t \rangle$ as follows: start with some leaf F_i of Δ, and let F_j be a branch of F_i. Then $\{i, j\}$ will be an edge of Ω. According to Corollary 3.4.4, $\langle \mathcal{F}(\Delta) \setminus \{F_i\} \rangle$ is again a quasi-forest. Then remove the leaf F_i, and continue in the same way with the remaining quasi-forest in order to find the other edges of Ω. Of course, at each step of the procedure there may be different choices. This gives us the different possible relation trees.

Geometrically a relation tree is obtained from a given quasi-forest by connecting the barycenters of the leaves and branches according to the above rules. In our example we get:



3.4. **An algebraic proof of Dirac's theorem.**

In this section we present a characterization of chordal graphs which is somehow a restatement of the original result of Dirac presented in Theorem 2.1.9. We shall use the notion of quasi-forest introduced before. This new characterization given in Theorem 3.4.3 belongs to Herzog, Hibi, and Zheng [10].

Recall that for a graph G on $[n]$ and with $E(G)$ its edge set, a *clique* is a subset F of $[n]$ such that $\{i,j\} \in E(G)$ for any $i,j \in F$ with $i \neq j$. We write $\Delta(G)$ for the simplicial complex on $[n]$ whose faces are the cliques of G. It is clear that G is the 1-skeleton of $\Delta(G)$, and that if Γ is a simplicial complex with $G = skel_\Gamma(1)$, then Γ is a subcomplex of $\Delta(G)$. Hence, in a certain sense, $\Delta(G)$ is the largest simplicial complex whose 1-skeleton is G. A simplicial complex is called *flag* if all its minimal nonfaces are of dimension 1, i.e. consist of two elements.

Lemma 3.4.1. *Let G be a graph and Δ the simplicial complex defined by $I_\Delta = I(\bar{G})$. Then*
a) $\Delta = \Delta(G)$;
b) $G = skel_\Delta(1)$;
c) Δ is a quasi-forest if and only if G is a chordal graph.

Proof. a) Since the 1-skeleton of $\Delta(G)$ is G, it follows that $I(\bar{G}) \subset I_{\Delta(G)}$. Conversely, let F be a minimal nonface of $\Delta(G)$. If $|F| > 2$, then each subset $G \subset F$ with $|G| = 2$ is an edge of G and therefore F is a clique in G and hence $F \in \Delta(G)$, a contradiction. Thus, for every minimal nonface F of $\Delta(G)$ one has $|F| = 2$. This shows that $I_{\Delta(G)} = I(\bar{G})$. Therefore $\Delta = \Delta(G)$.

b) is a consequence of a) and the remarks before the proof.

c) Theorem 2.3.12 of Fröberg guarantees that the complementary graph G of $\bar{G}$ is a chordal graph if and only if $I(\bar{G}) = I_\Delta$ has a 2-linear resolution. By Theorem 2.3.21 $\operatorname{reg}(I_\Delta) = \operatorname{projdim} k[\Delta^\vee] = \operatorname{projdim} I_{\Delta^\vee} + 1$, and so, using Theorem 1.2.3, the ideal $I(\bar{G})$ has a linear resolution if and only if $\operatorname{projdim} I_{\Delta^\vee} = 1$. Since by Lemma 2.2.14 $I_{\Delta^\vee} = I(\Delta^c)$, using Corollary 3.3.14 the conclusion follows. $\square$

The following result will prove useful in the proof of Dirac's theorem.

Lemma 3.4.2. *A quasi-forest is a flag complex.*

Proof. Let Δ be a quasi-forest on $[n]$ and fix a leaf ordering of the facets $F_1, \ldots, F_t$ of Δ. We prove the lemma by induction on t. The case $t = 1$ is obvious. Let $t > 1$. Since $\Delta' = \langle F_1, \ldots, , F_{t-1}\rangle$ is a quasi-forest, by the induction hypothesis it follows that Δ' is flag. Let F_k with $k < t$ be a branch of F_t. Then $F_i \cap F_t \subset F_k \cap F_t$ for any $i < t$ and thus Δ' consists of all faces G of Δ with $G \cap (F_t \setminus F_k) = \emptyset$. Suppose H is a minimal nonface of Δ having at least 3 elements of $[n]$. We can prove that H is a minimal nonface of Δ', i.e. $H \cap (F_t \setminus F_k) = \emptyset$. Since H is a nonface, there is some $p \in H$ with $p \notin F_t$. If $q \in F_t$ belongs also to H, then $\{p,q\} \in \Delta$. Thus there is some F_j with $j \neq t$ such that $\{p,q\} \in F_j$. Hence $\{q\} \subset F_j \cap F_t \subset F_k \cap F_t$. Thus $q \in F_k$ and $H \cap (F_t \setminus F_k) = \emptyset$, as desired. But Δ' is flag and we get a contradiction. Therefore Δ is also flag. $\square$

Theorem 3.4.3. *(Dirac) A finite graph G on $[n]$ is a chordal graph if and only if G is the 1-skeleton of a quasi-forest on $[n]$.*

Proof. The statements b) and c) in Lemma 3.4.1 imply that a chordal graph is the 1-skeleton of a quasi-forest. Conversely, suppose that G is the 1-skeleton of a quasi-forest Γ. Since by Lemma 3.4.2 Γ is flag, the ideal I_Γ is generated by all monomials x_F with $|F| = 2$ and $F \notin \Gamma$. This shows that $I_\Gamma = I(\bar{G})$, and so $\Gamma = \Delta(G)$ by Lemma 3.4.1a). Hence G is chordal by Lemma 3.4.1c). $\square$

Corollary 3.4.4. *Let Δ be a quasi-forest and F a leaf of Δ. Then $\langle \mathcal{F}(\Delta) \setminus \{F\} \rangle$ is again a quasi-forest.*

Proof. Let $\Delta' = \langle \mathcal{F}(\Delta) \setminus \{F\} \rangle$. Let G be the 1-skeleton of Δ and G' the 1-skeleton of Δ'. Then G' is the subgraph of G obtained after removing all free vertices of F and all edges containing these vertices. Since Δ is a quasi-forest, by Dirac's Theorem 3.4.3 G is chordal; it follows that G' is chordal, too. And, again by 3.4.3, Δ' results to be a quasi-forest. $\square$

We present in the next theorem a result similar to Dirac's theorem 3.4.3, which gives a characterization of the ℓ-skeleton of a quasi-forest:

Theorem 3.4.5. *Let Δ be a pure ℓ-dimensional simplicial complex on the vertex set $[n]$ and Γ its 1-skeleton. Then the following conditions are equivalent:*

a) Δ is the ℓ-skeleton of a quasi-forest

b) (i) Γ is a chordal graph and

(ii) Δ is the ℓ-skeleton of $\Delta(\Gamma)$.

Proof. The implication b) $\Rightarrow$ a) follows from Lemma 3.4.1c). For the implication a) $\Rightarrow$ b), suppose that Δ is the ℓ-skeleton of a quasi-forest Σ. Then Γ is also the 1-skeleton of Σ. As in the proof of Theorem 3.4.3 we conclude that $\Sigma = \Delta(\Gamma)$. This implies b)(ii). Finally, by Dirac's theorem Γ is chordal. $\square$

4. The Cohen-Macaulay property for chordal graphs

For easy reference we recall here some of the notions that will be used in this section.

The graphs we'll be working with are supposed to be finite, without loops, multiple edges or isolated vertices.

Let k be a field. A graph G is called *Cohen-Macaulay over k* if the edge ideal $I(G) = (x_i x_j | \{i, j\} \in E(G))$ of G is a Cohen-Macaulay ideal in $S = k[x_1, ..., x_n]$, in other words $S/I(G)$ is a Cohen-Macaulay ring.

Suppose G is Cohen-Macaulay over k. Then we say that G is of *type r over k* if r is the Cohen-Macaulay type of $S/I(G)$, that is, r is the minimal number of generators of the canonical module of $S/I(G)$. The Cohen-Macaulay type of a Cohen-Macaulay module R can also be computed as the socle dimension of the residue class ring of R modulo a maximal regular sequence.

A ring R is Gorenstein if the Cohen-Macaulay type of R is 1. We say that the graph G is *Gorenstein* over k if $S/I(G)$ is *Gorenstein over k*. Finally, we say that the graph G is *Gorenstein* if G has the corresponding property over any field.

4.1. **The general classification problem.**

Given a field k the general problem is to classify all the graphs which are Cohen-Macaulay over k.

In this generality the problem is as hard as to classify all Cohen-Macaulay simplicial complexes, because given a simplicial complex Δ one can naturally construct a finite graph G such that G is Cohen-Macaulay if and only if Δ is Cohen-Macaulay. In fact, if P is the face poset of Δ (i.e. the poset consisting of all faces of Δ, ordered by inclusion), then Δ is Cohen-Macaulay if and only if the order complex $\Delta(P)$ is Cohen-Macaulay. Since the order complex $\Delta(P)$ is flag, it follows that there is a finite graph G such that $I(G)$ coincides with the Stanley-Reisner ideal of $\Delta(P)$.

Therefore, one cannot expect a general classification theorem. On the other hand, some positive results have been obtained recently for several classes of graphs: trees, cycles, bipartite graphs and chordal graphs. We briefly mention here without proof the results concerning the first three classes mentioned above. The classification of chordal graphs will be the main topic of the following section.

Theorem 4.1.1. [16]*(Theorem 6.3.4) Let T be a tree with vertex set V and edge E. Then T is Cohen-Macaulay if and only if $|V| \leq 2$ or $2 < |V| = 2r$ and there are vertices $x_1, ..., x_r, y_1, ..., y_r$ so that deg$(x_i) = 1$, deg$(y_i) \geq 2$ and $\{x_i, y_i\} \in E$ for $i = 1, ..., r$.*

Theorem 4.1.2. [16]*(Corollary 6.3.5) If G is a tree, then G is Cohen-Macaulay if and only if G is unmixed.*

Corollary 4.1.3. [16] *(Corollary 6.3.6) The only Cohen-Macaulay cycles are a triangle and a pentagon.*

One knows that bipartite graphs are characterized by the lack of odd length cycles. Since trees are acyclic, any tree is such a graph. We go further in our survey and check the more general case of bipartite graphs.

Theorem 4.1.4. [16] *(Theorem 6.4.4) Let G be a Cohen-Macaulay bipartite graph. If G is not a discrete graph, then there is a vertex $v \in V(G)$ such that deg$(v) = 1$.*

Corollary 4.1.5. [16] *(Corollary 6.4.5) If G is a Cohen-Macaulay bipartite graph, then $G\backslash\{v\}$ is Cohen-Macaulay for some vertex v.*

Definition 4.1.6. The *complementary simplicial complex* Δ_G of the graph G is

$$\Delta_G = \{A \subset V | A \textit{ is an independent set in } G\},$$

where V is the vertex set of G.

Notice that Δ_G is exactly the Stanley-Reisner simplicial complex of $I(G)$.

Theorem 4.1.7. [16] *(Theorem 6.4.7) If G is a Cohen-Macaulay bipartite graph, then the Stanley-Reisner simplicial complex Δ_G of $I(G)$ is shellable.*

In [9] Herzog and Hibi give a complete classification of all bipartite Cohen-Macaulay and Gorenstein graphs. In the following we shall present their results.

Let G be a finite bipartite graph on the vertex set $W \cup W'$ with $W = \{i_1, ..., i_s\}$ and $W' = \{j_1, ..., j_t\}$ where $s \leq t$ (i.e. any edge of G is of the form $\{i, j\}$ with $i \in W$ and $j \in W'$). For each subset U of W we write $N(U)$ for the set of those vertices $j \in W'$ for which there is a vertex $i \in U$ such that $\{i, j\}$ is an edge in G.

The "Marriage Problem" (check for a proof in [16], Theorem 6.1.8) says that if $|U| \leq |N(U)|$ for all subsets U of W, then there is a subset $W'' = \{j_{l_1}, ..., j_{l_s}\} \subset W'$ with $|W''| = s$ such that $\{i_k, j_{l_k}\}$ is an edge of G for $k = 1, 2, ..., s$.

Let G be a finite bipartite graph on the vertex set $W \cup W'$ and suppose that G is unmixed. Since each W and W' is a minimal vertex cover, one has $|W| = |W'|$. Let $W = \{x_1, ..., x_n\}$ and $W' = \{y_1, ..., y_n\}$. Since $(W \setminus U) \cup N(U)$ is a vertex cover of G for all subsets U of W and since G is unmixed, it follows that $|U| \leq |N(U)|$ for all subsets U of W. Thus, the "Marriage Problem" enables us to assume that G satisfies the following condition:

$$(\sharp)\ \{x_i, y_i\} \textit{ is an edge of } G \textit{ for all } 1 \leq i \leq n.$$

Furthermore, suppose that G is a Cohen-Macaulay graph. Then, it can be proved (see [9] Lemma 3.3) that, after a suitable relabeling of the vertices $y_1, ..., y_n$, the edge set of G satisfies the following two conditions:

$$(\sharp)\ \{x_i, y_i\}\ is\ an\ edge\ of\ G\ for\ all\ 1 \leq i \leq n.$$

$$(\sharp\sharp)\ If\ \{x_i, y_j\}\ is\ an\ edge\ of\ G,\ then\ i \leq j.$$

Adding a third condition one obtains the necessary and sufficient conditions for a bipartite graph to be Cohen-Macaulay .

Theorem 4.1.8. [9]*(Theorem 3.4) Let G be a finite bipartite graph on the vertex set $W \cup W'$, where $W = \{x_1, ..., x_n\}$, $W' = \{y_1, ..., y_n\}$ and suppose that the edge set of G satisfies the conditions ($\sharp$) and ($\sharp\sharp$) above. Then G is a Cohen-Macaulay graph if and only if the following condition ($\sharp\sharp\sharp$) holds:*

$$(\sharp\sharp\sharp) If\ \{x_i, y_j\}\ are\ edges\ of\ G\ with\ i < j < k,\ then\ \{x_i, y_k\}\ is\ an\ edge\ of\ G.$$

Corollary 4.1.9. [9]*(Corollary 3.5) Let G be a finite graph and Δ_G the simplicial complex whose Stanley-Reisner ideal coincides with $I(G)$. Then G is Cohen-Macaulay if and only if Δ_G is pure and strongly connected.*

It is given also a complete characterization of bipartite Gorenstein graphs.

Theorem 4.1.10. [9]*(Corollary 3.6) A Cohen-Macaulay bipartite graph G is Gorenstein if and only if G is the disjoint union of edges.*

4.2. **The characterization of all Cohen-Macaulay chordal graphs.**

Before passing to the main theorem we need to mention the following technical results.

Lemma 4.2.1. *Let R be a Noetherian ring, $S = R[x_1, ..., x_n]$ the polynomial ring over R, k an integer with $0 \le k < n$, and J the ideal*

$$J = (I_1x_1, ..., I_kx_k, \{x_ix_j\}_{1\le i<j\le n}) \subset S,$$

where $I_1, ..., I_k$ are ideals in R. Then the element $x = \sum_{i=1}^n x_i$ is a non-zerodivisor on S/J.

Proof. For a subset $T \subset [n]$ we let L_T be the ideal generated by all monomials x_ix_j with $i, j \in T$ and $i < j$. Let $I_T = \sum_{j\in T} I_j$ and $X_T = (\{x_j\}_{j\in T})$. One sees easily that

$$L_T = \bigcap_{\ell\in T} X_{T\setminus\{\ell\}}.$$

Hence we get $J = (I_1x_1, ..., I_kx_k, L_{[n]}) = \bigcap_{T\subset[k]}(I_T, X_{[k]\setminus T}, L_{[n]}) =$

$$= \bigcap_{T\subset[k]} (I_T, X_{[k]\setminus T}, L_{[n]\setminus([k]\setminus T)}) = \bigcap_{T\subset[k],\ \ell\in[n]\setminus([k]\setminus T)} (I_T, X_{[k]\setminus T}, X_{([n]\setminus([k]\setminus T))\setminus\{\ell\}}) =$$

$$= \bigcap_{T\subset[k],\ \ell\in[n]\setminus([k]\setminus T)} (I_T, X_{[n]\setminus\{\ell\}})$$

For an ideal J in a Noetherian ring R, if $J = Q_1 \bigcap \cdots \bigcap Q_s$ is the primary decomposition and $\sqrt{Q_i} = P_i$, for all $i = 1, \ldots, s$, then x is a non-zerodivisor of J if and only if $x \notin P_1 \bigcup \cdots \bigcup P_s$. If $J = J_1 \bigcap \cdots \bigcap J_t$ and for each $1 \le i \le t$ as above $\sqrt{J_i} = \bigcap_{j=1}^{s_i} P_{ij}$, then $\sqrt{J} = \bigcap_{i,j} P_{ij}$ and therefore $P_1 \bigcup \cdots \bigcup P_s \subset \bigcup_{i,j} P_{ij}$. Hence, in order to prove that x is a non-zerodivisor modulo J it suffices to show that x is a non-zerodivisor modulo each of the ideals $(I_T, X_{[n]\setminus\{\ell\}})$.

To see this, we first pass to the residue class ring modulo I_T, and hence if we replace R by R/I_T, it remains to be shown that x is a non-zerodivisor on $R[x_1, \ldots, x_n]/(x_1, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_n)$. But this is obvious: if $xf = x_1f_1 + \cdots + x_{\ell-1}f_{\ell-1} + x_{\ell+1}f_{\ell+1} + \cdots x_nf_n$, then $x_\ell f(0, \ldots, x_\ell, \ldots, 0) = 0$, and $f(0, \ldots, x_\ell, \ldots, 0) = 0$, hence $f \in (x_1, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_n)$. □

Lemma 4.2.2. *Let $\varphi : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_{n-1}]$ be the k-algebra homomorphism given by $\varphi(x_i) = x_i$ for any $i = 1, \ldots, n-1$ and $\varphi(x_n) = -x_1 - x_2 - \cdots - x_{n-1}$. Let x denote $x_1 + \cdots + x_n$. Then*

a) the ideal $I = (\{x_ix_j | 1 \le i < j \le n\})$ is mapped by φ into the ideal

$$J = (x_1, \ldots, x_{n-1})^2 = (x_1^2, \ldots, x_{n-1}^2, \{x_ix_j | 1 \le i < j \le n-1\});$$

b) $\varphi(x) = 0$ and therefore φ induces a homomorphism

$$\psi : k[x_1, \ldots, x_n]/(I, x) \to k[x_1, \ldots, x_{n-1}]/J;$$

c) ψ is onto and it is a k-algebra isomorphism.

Proof. Easy verification. □

Recall that for a simplicial complex Δ a vertex i is called a *free vertex* if it belongs to precisely one facet of Δ . For a given graph G the associated *flag complex* $\Delta(G)$ is the simplicial complex whose faces are the cliques of G.

Theorem 4.2.3. *Let k be any field and let G be a chordal graph on the vertex set $[n]$. Let $F_1, \ldots, F_m$ be the facets of $\Delta(G)$ which admit a free vertex. Then the following conditions are equivalent:*

a) G is Cohen-Macaulay,

b) G is Cohen-Macaulay over k,

c) G is unmixed ,

d) $[n]$ is the disjoint union of $F_1, \ldots, F_m$.

Proof. The implication a) $\Rightarrow$ b) is trivial.

b) $\Rightarrow$ c): Since any Cohen-Macaulay ring is height unmixed, it follows that G is unmixed.

c) $\Rightarrow$ d): Let $E(G)$ be the edge set of the chordal graph G. Let $F_1, \ldots, F_m$ be the facets of $\Delta(G)$ with free vertices. Fix a free vertex v_i of F_i and set $W = \{v_1, \ldots, v_m\}$. Since all v_i are free, it yields that these vertices are distinct. Suppose that $B = [n] \setminus (\cup_{i=1}^m F_i) \neq \emptyset$ and write $G|_B$ for the induced subgraph of G on B. In particular, $G|_B$ is unmixed. Notice that $\{v_i, b\} \notin E(G)$ for all $1 \leq i \leq m$ and for all $b \in B$.

Take X (included in B) a minimal vertex cover of $G|_B$. Then $((\cup_{i=1}^m F_i) \setminus W) \cup X$ is a vertex cover of G and it is minimal due to the preceding remark and due to the fact that from the set of vertices $(\cup_{i=1}^m F_i) \setminus W$ we can not remove any more vertices and still obtain a vertex cover, because otherwise one could not cover all the edges which contain the removed free vertices. Since the induced graph $G|_B$ is again chordal, by induction on the number of vertices, it follows that if $H_1, \ldots, H_s$ are the facets of $\Delta(G|_B)$ with free vertices, then B is the disjoint union $B = \cup_{j=1}^s H_j$. Let v'_j be a free vertex of H_j and set $W' = \{v'_1, \ldots, v'_s\}$. Since $((\cup_{i=1}^m F_i) \setminus W) \cup (B \setminus W')$ is a minimal vertex cover of G and since G is unmixed, every minimal vertex cover of G consists of $n - (m + s)$ vertices.

We claim that $F_i \cap F_j = \emptyset$ if $i \neq j$. In fact, if, say, $F_i \cap F_j \neq \emptyset$ and $w \in [n]$ satisfies $w \in F_i$ for all $1 \leq i \leq \ell$ (where $\ell \geq 2$) and $w \notin F_i$ for all $\ell < i \leq m$ (hence $w \neq v_i$ for any $1 \leq i \leq m$), then $Z = (\cup_{i=1}^m F_i) \setminus \{w, v_{\ell+1}, \ldots, v_m\}$ is a vertex cover of the induced subgraph $G' = G|_{[n]\setminus B}$, and this is a minimal one by the same argument as above. Let Y be a minimal vertex cover of G with $Z \subset Y$. Since $Y \cap B$ is a vertex cover of $G|_B$, one has $|Y \cap B| \geq |B| - s$. Moreover, $|Y \cap ([n] \setminus B)| \geq n - |B| - (m - \ell + 1) > n - |B| - m$. Hence $|Y| = |Y \cap B| + |Y \cap ([n] \setminus B)| > n - (m + s)$, a contradiction.

Consequently, a subset Y of $[n]$ is a minimal vertex cover of G if and only if $|Y \cap F_i| = |F_i| - 1$ for all $1 \leq i \leq m$ and $|Y \cap H_j| = |H_j| - 1$ for all $1 \leq j \leq s$.

Since $\Delta(G|_B)$ is a quasi–forest, it has a leaf which must have a free vertex and it is thus one of the facets $H_1, \ldots, H_s$. Suppose H_1 is leaf of $\Delta(G|_B)$. Notice that if δ and δ' (where $\delta \neq \delta'$) are two free vertices of H_1 with $\{\delta, a\} \in E(G)$ and $\{\delta', a'\} \in E(G)$, where a and a' belong to $[n] \setminus B$ such that $a \neq a'$ and

$\{a, a'\} \in E(G)$, then either $\{\delta, a'\} \in E(G)$ or $\{\delta', a\} \in E(G)$ because G is chordal and $\{\delta, \delta'\} \in E(G)$.

We intend to build a subset $A \subset [n] \setminus B$ such that:

i) $\{a, b\} \notin E(G)$ for all $a, b \in A$ with $a \neq b$,

ii) for each free vertex δ of H_1 one has $\{\delta, a\} \in E(G)$ for some $a \in A$, and

iii) for each $a \in A$, one has $\{\delta, a\} \in E(G)$ for some free vertex δ of H_1.

One sees easily that a subset $A \subset [n] \setminus B$ satisfying ii) and iii) above exists: by the above argument, for each free vertex $\delta \in H_1$ there is some $u_\delta \in [n] \setminus B$ with $\{\delta, u_\delta\} \in E(G)$ and take A to be the set containing all vertices u obtained in this manner. Now, if $\{a, a'\} \in E(G)$, $\{\delta, a\} \in E(G)$ and $\{\delta, a'\} \notin E(G)$ for some a, $a' \in A$ with $a \neq a'$ and for a free vertex δ of H_1, then every free vertex δ' of H_1 with $\{\delta', a'\} \in E(G)$ must satisfy $\{\delta', a\} \in E(G)$. Repeating this technique one obtains a subset $A \subset [n] \setminus B$ satisfying i), ii) and iii), as required.

If $s > 1$, then H_1 has a branch. Let $w_0 \notin H_1$ be a vertex belonging to a branch of the leaf H_1 of $G|_B$. Thus $\{\xi, w_0\} \in E(G)$ for all nonfree vertices ξ of H_1 : since ξ is not free, it belongs to some other facet of $\Delta(G|_B)$. But w_0 belongs also to all branches of H_1; therefore indeed $\{\xi, w_0\} \in E(G)$. We claim that either $\{a, w_0\} \notin E(G)$ for all $a \in A$, or one has some $a \in A$ with $\{a, \xi\} \in E(G)$ for every nonfree vertex ξ of H_1. To see why this is true, suppose $\{a, w_0\} \in E(G)$ and $\{\delta, a\} \in E(G)$ for some $a \in A$ and for some free vertex δ of H_1. Then one has a cycle (a, δ, ξ, w_0) of length four for every nonfree vertex ξ of H_1. Since $\{\delta, w_0\} \notin E(G)$ (δ is free), one has $\{a, \xi\} \in E(G)$.

Let X be a minimal vertex over of G such that $X \subset [n] \setminus (A \cup \{w_0\})$ (respectively $X \subset [n] \setminus A$) if $\{a, w_0\} \notin E(G)$ for all $a \in A$ (respectively if one has some $a \in A$ with $\{a, \xi\} \in E(G)$ for every nonfree ξ of H_1). In the first case $\{a, w_0\} \notin E(G)$ for any $a \in A$; if $\gamma \in H_1$ is free there is some $a \in A \setminus X$ with $\{\gamma, a\} \in E(G)$, else if $\gamma \in H_1$ is not free there is $\{w_0, \gamma\} \in E(G)$. In the second case there is $\{a, w_0\} \in E(G)$, and if $\gamma \in H_1$ is free there is some $a \in A$, hence $a \notin X$ with $\{\gamma, a\} \in E(G)$. If $\gamma \in H_1$ is not free, then $\{\gamma, a\} \in E(G)$ for any $a \in A$. To conclude, for each vertex $\gamma \in H_1$ there is $w \notin X$ with $\{\gamma, w\} \in E(G)$. Hence $H_1 \subset X$, in contrast to our considerations before. This contradiction guarantees that $B = \emptyset$. Hence $[n]$ is the disjoint union $[n] = \cup_{i=1}^m F_i$, as required.

Finally, suppose that $s = 1$. Then H_1 is the only facet of $\Delta(G|_B)$. Then $X = \cup_{i=1}^m (F_i \setminus v_i)$ is a minimal vertex cover of G with $H_1 \subset X$, a contradiction.

d) $\Rightarrow$ c) Let $F_1, \ldots, F_m$ denote the facets of $\Delta(G)$ with free vertices and for each $1 \leq i \leq m$ write F_i also for the set of vertices of F_i. Given a minimal vertex cover $X \subset [n]$ of G, one has $|X \cap F_i| \geq |F_i| - 1$ for all i since F_i is a clique of G. If, however, for some i one has $|X \cap F_i| = |F_i|$, i.e. $F_i \subset X$, then $X \setminus \{v_i\}$ is a vertex cover of G for any free vertex v_i of F_i. This contradicts the fact that X is a minimal vertex cover of G. Thus $|X \cap F_i| = |F_i| - 1$ for all i. Since $[n]$ is the disjoint union $[n] = \cup_{i=1}^m F_i$, it follows that $|X| = n - m$ and G is unmixed, as desired.

c) and d) $\Rightarrow$ a) We know that G is unmixed. Moreover, if $v_i \in F_i$ is a free vertex, then $[n] \setminus \{v_1, \ldots, v_m\}$ is a minimal vertex cover of G. In particular, it follows that $\dim S/I(G) = m$.

For $i = 1, \ldots, m$, we set $y_i = \sum_{j \in F_i} x_j$. We shall show that $y_1, \ldots, y_m$ is a regular sequence on $S/I(G)$. This yields that G is a Cohen-Macaulay graph.

Let $F_i = \{i_1, \ldots, i_k\}$ and assume that $i_{\ell+1}, \ldots, i_k$ are the free vertices of F_i. Let $G' \subset G$ be the induced subgraph of G on the vertex set $[n] \setminus \{i_1, \ldots, i_k\}$. Then $I(G) = (I(G'), J_1 x_{i_1}, \ldots, J_\ell x_{i_\ell}, J)$ where $J_j = (\{x_r | \{r, i_j\} \in E(G)\})$ for $j = 1, \ldots, \ell$ and $J = (\{x_{i_r} x_{i_s} | 1 \le r < s \le k\})$.

Since $[n]$ is the disjoint union of $F_1, \ldots, F_m$, it follows that all generators of the ideal $(I(G'), y_1, \ldots, y_{i-1})$ belong to $K[\{x_i\}_{i \in [n] \setminus F_i}]/(I(G'), y_1, \ldots, y_{i-1})$. Thus, if we set

$$R = K[\{x_i\}_{i \in [n] \setminus F_i}]/(I(G'), y_1, \ldots, y_{i-1}),$$

then

$$\begin{aligned} &(S/I(G))/(y_1, \ldots, y_{i-1})(S/I(G)) \\ &\cong R[x_{i_1}, \ldots, x_{i_k}]/(I_1 x_{i_1}, \ldots, I_\ell x_{i_\ell}, \{x_{i_r} x_{i_s} | 1 \le r < s \le k\}), \end{aligned}$$

where for each j, the ideal I_j is the image of J_j under the residue class map onto R. Thus Lemma 4.2.1 implies that y_i is regular on

$$(S/I(G))/(y_1, \ldots, y_{i-1})(S/I(G)).$$

□

Corollary 4.2.4. *Let G be a Cohen-Macaulay chordal graph, and let $F_1, \ldots, F_m$ be the facets of $\Delta(G)$ which have a free vertex. Let i_j be a free vertex of F_j for every $j = 1, \ldots, m$ and let G' be the induced subgraph of G on the vertex set $[n] \setminus \{i_1, \ldots, i_m\}$. Then*

a) the type of G is the number of maximal independent subsets of G',

b) G is Gorenstein if and only if G is a disjoint union of edges.

Proof. a) Let $F \subset [n]$ and $S = k[x_1, \ldots, x_n]$. If J is the ideal generated by all monomials $x_i x_j$ with $i, j \in F$ and $i < j$, and $x = \sum_{i \in F} x_i$, notice that by Lemma 4.2.2 for any $i \in F$ one has

$$(S/J)/x(S/J) \cong S_i/(\{x_j | j \in F, j \neq i\})^2,$$

where $S_i = k[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$.

Thus, if we factor out the Cohen-Macaulay ring $S/I(G)$ by a maximal regular sequence as in the proof of Theorem 4.2.3, we obtain an Artinian ring of the form

$$A = T/(P_1^2, \ldots, P_m^2, I(G'')).$$

Here we denote by $P_j = (\{x_s | \ s \in F_j, s \neq i_j\})$, G'' is the subgraph of G consisting of all edges which do not belong to any F_j, and T is the polynomial ring over k in the set of variables $X = \{x_s | \ s \in [n], s \neq i_j \text{ for all } j = 1, \ldots, m\}$. In doing the above factorisation we used from d) in Theorem 4.2.3 that $[n]$ is the disjoint union of the facets F_i. It is obvious that A is obtained from the polynomial ring T by factoring out the squares of all variables of T and all $x_i x_j$ with $\{i, j\} \in E(G')$. Therefore A has a k-basis of squarefree monomials corresponding to the independent subsets of G', and the socle of A is generated as a k-vector space by the monomials corresponding to the maximal independent subsets of G'.

b) If G is a disjoint union of edges, then $I(G)$ is a complete intersection, and hence Gorenstein. Conversely, suppose that G is Gorenstein. Then A is Gorenstein. Since A is an Artinian ring with monomial relations, A is Gorenstein if and only if A is a complete intersection. This is the case only if $E(G') = \emptyset$, in which case G is a disjoint union of edges. $\square$

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