Residue functionals on the algebra of adiabatic pseudo-differential operators

by

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Abstract

We consider the deformation $\Psi_a(X)$ ([10], [5]) of the algebra of pseudo-differential operators on the total space of a fibration of closed manifolds $X \to M$, defined by using the blow-up technique of Melrose. We compute the Hochschild homology of this algebra and of certain ideals and quotients. We prove a general criterion, from which we deduce that the ideals involved are H-unital. In particular, we study the long exact sequence induced by the inclusion of the ideal of "smoothing" adiabatic operators. The homology of this ideal is related to the cohomology of the base. We define (filtered) higher analogues of the Residue trace of Wodzicki and Guillemin, which are strongly related to the homology of the smoothing ideal. They are the explicit leading terms of certain Hochschild cochains on the algebra of adiabatic symbols. We use them to compute the boundary map in the homology long exact sequence.

Thesis Supervisor: Richard B. Melrose
Title: Professor

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Contents

Chapter 1

Introduction

Let $\Psi(Y)$ be the algebra of pseudo-differential operators of integral order on a closed manifold Y^n. Related to this algebra, consider the following three functionals:

- the trace Tr, defined on smoothing operators;
- the residue trace Tr_R [16], defined on the whole algebra; Tr_R is a genuine trace, i.e. it vanishes on commutators;
- the index map $Index$, defined on elliptic symbols between two bundles over Y.

These functionals have interesting interpretations in the framework of Hochschild (co)-homology. Namely, the trace Tr is a generator of $HH^{\bullet}(\Psi^{-\infty}(Y))$, which is concentrated in dimension 0. The residue trace Tr_R is a generator of $HH^0(\Psi^{\mathbb{Z}}(Y))$. Let us now interpret the index map. To the knowledge of the author, this interpretation is due to Melrose and Nistor [13]. Consider an elliptic operator $\tilde{A}$ between two trivial $\mathbb{C}^N$ bundles on Y. Let $\tilde{B}$ be a pseudo-inverse, i.e.

$$Id_2 - \tilde{A}\tilde{B} = P_{Coker \tilde{A}}, \quad Id_1 - \tilde{B}\tilde{A} = P_{\ker \tilde{A}},$$

where P are projections, and Id are the identity operators in the two bundles. Let A, B be the images of $\tilde{A}, \tilde{B}$ in the symbol algebra $\Psi(Y)/\Psi^{-\infty}(Y)$. Then $tr(A \otimes B) = \sum a_{ij} \otimes b_{ji}$ defines a Hochschild cycle on $\Psi(Y)/\Psi^{-\infty}(Y)$. Here tr is the generalized trace functional [8].

More generally, let $\mathcal{E}, \mathcal{F}$ be two vector bundles over Y. Let $\tilde{A} : \mathcal{E} \to \mathcal{F}$ be elliptic, and $\tilde{B}$ a pseudo-inverse. We can assume that $\mathcal{E}, \mathcal{F}$ are sub-bundles in two trivial $\mathbb{C}^N$ bundles, and choose $P_{\mathcal{E}}, P_{\mathcal{F}}$ projections. Then $P_{\mathcal{F}}\tilde{A}P_{\mathcal{E}}$, respectively $P_{\mathcal{E}}\tilde{B}P_{\mathcal{F}}$ are extensions of $\tilde{A}, \tilde{B}$ to $\mathbb{C}^N$. Define $tr(A \otimes B) := tr(P_2\tilde{A}P_1 \otimes P_1\tilde{B}P_2)$.

Lemma 1.0.1 *If $\dim Y \geq 2$, then the class of $tr(A \otimes B)$ in $HH_1(\Psi(Y)/\Psi^{-\infty}(Y))$ does not depend on the choices made.*

Proof: Choose a different trivialization of the first $\mathbb{C}^N$ bundle, and let $C = (c_{ij}) : Y \to M_N(\mathbb{C})$ be the matrix of change of coordinates. We want to show that $tr(A \otimes B)$ is homologous to $tr(CA \otimes BC^{-1})$. We have

$$b(tr(CA \otimes B \otimes C^{-1})) \;=\; tr(C \otimes C^{-1} - CA \otimes BC^{-1} + A \otimes B).$$

Observe that $tr(C \otimes C^{-1})$ belongs to the Hochschild complex of the subalgebra $\mathcal{D}(Y)$ of *differential* operators. This subalgebra has zero homology in dimensions smaller than $\dim Y$ (see for instance [2]), whence the lemma. $\square$

Note that if Y is a circle and $C(e^{it}) = e^{it}$, then the cycle $C \otimes C^{-1}$ does *not* vanish in $HH_1(\Psi(Y))$.

In [17], Wodzicki proved that the short exact sequence

$$0 \to \Psi^{-\infty}(Y) \to \Psi(Y) \to \Psi(Y)/\Psi^{-\infty}(Y) \to 0$$

induces a long exact sequence in homology, hence a map $\delta : HH_1(\Psi(Y)/\Psi^{-\infty}(Y)) \to HH_0(\Psi^{-\infty}(Y))$. Let Q be a positive elliptic scalar operator. Seeley [15] proved the existence of a holomorphic group $\{Q^z\}_{z\in\mathbb{C}}$ of pseudo-differential operators of order z. The derivative at $z = 0$ of conjugation by Q^z defines an exterior derivation D_Q on $\Psi(Y)$. This induces a map of order -1 on $HH_\bullet(\Psi(Y))$, denoted e_{D_Q} [8]. Using the commutation formula

$$Q^z[A, B] = [Q^z A, B] + zQ^z \frac{Q^{-z}BQ^z - B}{z} A, \tag{1.1}$$

one can easily show that

$$Index(A) = Tr(\delta(tr(A \otimes B))) = Tr_R(e_{D_Q}(tr(A \otimes B)))$$

by evaluating $Tr(Q^z[A, B])$ at $z = 0$. This provides the interpretation of $Index$ and motivates the definition of a 1-cocycle $Ind = Tr \circ \delta$.

Our results concern the case where the algebra $\Psi(X)$ is replaced by the *adiabatic limit*. Let $X \to M$ be a fibration of closed manifolds. Roughly speaking, the adiabatic limit algebra of this fibration is the algebra $\Psi_a(X)$ of pseudo-differential operators with symbols of the form $a(x, y, t\xi, \eta, t)$ in local coordinates. Here t is a parameter in $[0, \infty)$, y, η are coordinates on the cotagent bundle of the fibers, and x, ξ on the base. The definition of $\Psi_a(X)$ is made precise and coordinate-free in Chapter 2. We observe that $C^\infty[0, \infty)_t[t^{-1}]$ is central in $\Psi_a(X)$, hence the homology (over C) of $\Psi_a(X)$ will be “doubled” by the presence of $dt \in HH_1$.

We introduce two outer derivations on the adiabatic algebra. For one of them, we construct a holomorphic group $\{Q^z\}_{z\in\mathbb{C}}$ of adiabatic operators. We do not prove the existence of the complex powers of *any* positive adiabatic operator of order 1. The other derivation is, in some sense, the derivative with respect to t. These derivations play an important rôle in our computations. The family Q^z is also used in the definition of the residues in Chapter 7.

The adiabatic algebra has two filtrations: F_i is the space of operators of order at most i, and T_j is the space of operators vanishing at least to order j at $t = 0$. Denote by $\Psi_a^{-\infty}(X)$ the ideal of operators in $F_{-\infty}$, i.e. of order $-\infty$. Then the graded algebra of $\Psi_a^{-\infty}(X)$ with respect to the filtration T_j is isomorphic to the algebra of Laurent series whose coefficients are families of suspended pseudo-differential operators [12]. This means that the Fourier transform of an element in $T_j/T_{j+1}\Psi_a^{-\infty}(X)$ is a family of operator on the fibers of $X \to M$.

In Chapter 3, we compute the homology of the symbol algebra, $\Psi_a(X)/\Psi_a^{-\infty}(X)$, using the spectral sequence argument from [2]. The adiabatic symbols live naturally on a modified cotangent bundle, ${}^aT^*X \to X\times[0,\infty)$. There is a natural map ϕ_a from $T^*X\times[0,\infty)$ to ${}^aT^*X$ which is an isomorphism for $t>0$, but fails to be an isomorphism at $t=0$. To compute d_1, we replace the symplectic duality operator $*$ of Brylinski [1] with its adiabatic version, $*_a$, obtained from $\phi_a^{-1*} * \phi_a^*$ by means of a connection ∇ in $X\to M$.

The main difference from the case of the algebra $\Psi(X)$ is that now the "smoothing" ideal, $\Psi_a^{-\infty}(X)$, has homology in dimensions from 0 to $n+1$. Except in dimension 0, the homology is concentrated near $t=0$. This is computed in Chapters 4 and 5 using the spectral sequence associated to the filtration T_j. The adiabatic duality $*_a$ is again involved in the computation. The isomorphism with a certain cohomology space is realized by the map $*_aK$, where K is an analogue of the Hochschild-Kostant-Rosenberg map HKR:

$$a_0\otimes\ldots\otimes a_k \mapsto Tr_V(\hat{a}_0\nabla^t\hat{a}_1\wedge\ldots\wedge\nabla^t\hat{a}_k). \tag{1.2}$$

Here $\hat{a}$ denotes the Fourier transform in the fibers of ${}^aT^*M_{|t=0}$ of the smoothing adiabatic operator a, and $\nabla^t=\nabla+dt\wedge\frac{d}{dt}$.

In Chapter 6, we prove that a large class of algebras, which are deformations of H-unital algebras, are themselves H-unital. The residual ideals of the two filtrations of $\Psi_a(X)$ both fit into this class. This implies the existence of a long exact sequence of Hochschild homology groups induced from the short exact sequence

$$0\to\Psi_a^{-\infty}(Y)\to\Psi_a(Y)\to\Psi_a(Y)/\Psi_a^{-\infty}(Y)\to 0.$$

Comparing with the previous discussion, the boundary maps in this long exact sequence are therefore higher dimensional analogues of the 1-cocycle Ind. It would be interesting to exhibit higher Hochschild chains arising from geometrical objects, by analogy with Lemma 1.0.1, and then describe their image under the boundary maps.

In Chapter 7 we compute the boundary maps δ in our identifications of Hochschild homology. For this, we introduce some higher analogues of the residue trace. Ignoring the complications arising from the "doubling" phenomenon, this functionals are defined in dimensions up to $\dim M$, with values in $\Lambda^\bullet(M)$. Namely, let $a\in T_iC_k(\Psi_a(X))$ and let $a_0\otimes\ldots\otimes a_k$ be the image of a in $T_iC_k(\Psi_a(X))/T_{i+1}C_k(\Psi_a(X))$. Define $R_i(a)$ by

$$a\mapsto Res_{z=0}\int_{{}^aT^*M_{|t=0}/M} *_aTr_V(\widehat{Q^z a_0}\nabla^t\hat{a}_1\wedge\ldots\wedge\nabla^t\hat{a}_k).$$

This map can be rewritten

$$Res_{z=0}\int_{{}^aT^*M_{|t=0}/M} *_aK. \tag{1.3}$$

We replace the commutation (1.1) by a suitable identity in higher dimensional Hochschild homology. We prove that our residue functionals are the first components (in the T_j filtration) of some Hochschild cochains. This means that while we restrict the order of vanishing, we allow arbitrary operator orders in our residue.

Theorem 1.0.2 *The map $R_i : T_iC_k(\Psi_a(X)) \to \Lambda^{n-k}(M)$ has the following properties:*

1. *R_i vanishes on chains of sufficiently small total operator order, hence it descends to $T_iC_k(\Psi_a(X)/\Psi_a^{-\infty}(X))$.*

2. *If $a \in T_iC_k(\Psi_a(X)/\Psi_a^{-\infty}(X))$ is Hochschild exact, then $R_i(a)$ is deRham exact.*

3. *If $a \in T_iC_k(\Psi_a(X)/\Psi_a^{-\infty}(X))$ is Hochschild closed, then $R_i(a)$ is deRham closed.*

Therefore R_i descends to Hochschild homology, with values in the deRham cohomology of the base M. For comparison, the only such cochain known in $\Psi(Y)$ is the residue trace, in dimension 0. The proof of this result is based on the relationship between formulas 1.2 and 1.3. Note that if $M = \{pt\}$, then $\Psi_a(X) \cong C^\infty([0,\infty), \Psi(X))$, and our residues are only defined in dimensions 0 and 1. The former is just the residue trace, and the latter the "double" of the residue in dimension 0.

In Appendix A, we describe an explicit chain representing the image of the "undoubled" part of $HH_n(\Psi_a^{-\infty}(X))$ in $HH_n(\Psi_a(X))$, in the case when the Euler class of M is non-zero.

In Appendix B, we give a condition for the homology of a sheaf of algebras $\mathcal{A}$ to be computable by Čech cohomology. We would like to stress that the Hochschild complex of $\mathcal{A}$ is *not* a sheaf on the same space as $\mathcal{A}$. In the case of the algebra $C^\infty(Y)$, one way around this problem is to choose a different complex for Hochschild homology, and check that it is a complex of sheaves and quasi-isomorphic to the Hochschild complex. This is done in [3]. It is not clear how to make the corresponding change in the definition of the Hochschild complex for more general algebras.

Finally, in Appendix C, we note that the periodic cyclic homology of the algebra of adiabatic symbols does not depend on the star product; in other words, it is the same for *any* deformation of the commutative product on homogeneous symbols.

Chapter 2

Preliminaries

2.1 The adiabatic algebra

Let $X^{m+n} \xrightarrow{\pi} M^n$ be a fibration of compact manifolds. Denote by TX/M the vertical sub-bundle of TX. Let $t \in [0, \infty)$ be a parameter. Consider the following class of 1-parameter families of vector fields on X:

Definition 2.1.1 *A smooth family $v : [0, \infty) \to \Gamma(X, TX)$ of vector fields is called* adiabatic *if $v(0) \in \Gamma(X, TX/M)$.*

Mazzeo and Melrose [9] remarked that the adiabatic vector fields are the sections of a vector bundle over $X \times [0, \infty)$, called ${}^{a}TX$. The existence of ${}^{a}TX$ follows by exhibiting local trivializations, i.e. by proving that the adiabatic vector fields form a locally free sheaf of $C^\infty(X)$-modules: if $\{V_i, H_j\}$ form a local basis of TX over $U \times D \subset X$, where $D \subset M$, then $\{V_i, tH_j\}$ form a basis for the adiabatic vector fields over $U \times D \times [0, \infty)$. In general, the vector bundles $TX \times [0, \infty)$ and ${}^{a}TX$ are isomorphic as bundles over $X \times [0, \infty)$, but *not* canonically isomorphic.

Definition 2.1.2 *Let*

$$\phi^a : {}^{a}TX \to TX \times [0, \infty)$$

be the tautological map of vector bundles which transforms a section of ${}^{a}TX$ into itself, viewed as a time-dependent section of TX. Let $\phi_a : T^*X \times [0, \infty) \to {}^{a}T^*X$ be the dual map.

Following [10], [5], we construct an algebra of pseudo-differential operators in which the adiabatic vector fields are the differential operators of order exactly 1. This algebra is defined as a space of conormal distributions on a "radial blow-up" of $X \times X \times [0, \infty)$. More precisely, let $X \times_M X$ be the fiber product in $X \times X$. For $Y \subset \partial Z$, let us denote by $S^+N(Y)$ the positive half-sphere of the normal bundle NY of Y inside Z.

Definition 2.1.3 ([10]) *Let*

$$X_a^2 = [X^2 \times [0, \infty); X \times_M X \times \{0\}].$$

As a set, this is defined as $(X^2 \times [0,\infty) \setminus X \times_M X \times \{0\}) \cup S^+N(X \times_M X \times \{0\})$. *The* front face, *denoted* ff, *of the blow-up, is* $S^+N(X \times_M X \times \{0\})$.

This set has a natural C^∞ structure, defined by gluing the two parts along the normal geodesic flow of some Riemannian metric. As a manifold with corners, it has one boundary component of codimension 1 if $n = 0$, three if $n = 1$ and two otherwise.

There exists a natural blow-down map $\beta : X^2_a \to X^2 \times [0,\infty)$. Let Δ be the diagonal in X^2. We define Δ_a, the *lifted diagonal*, to be the closure in X^2_a of $\beta^{-1}(\Delta \times (0,\infty))$.

Definition 2.1.4 *Let* Ω_a *be the pull-back on* X^2_a *via* $\pi_R \circ \beta$ *of the density bundle* $\Omega = \Omega^{m+n}({}^a\!T^*X)$.

Note that the lifted diagonal is transversal to the front face. Let $2X^2_a$ be the double of X^2_a across the front face. By definition, a distribution on X^2_a, conormal to the lifted diagonal, is *smooth up to the front face* if it is the restriction of a distribution on $2X^2_a$, conormal to $2\Delta_a$.

Proposition 2.1.5 *The set of adiabatic differential operators (defined as compositions of adiabatic vector fields) is exactly the space of the* Ω_a*-valued distributions on* X^2_a*, supported on the lifted diagonal and smooth up to the front face.*

Proof: The action of a distribution $A \in \mathcal{D}'(X^2_a)$ on $f \in C^\infty(X \times [0,\infty))$ is defined via the left and right projections π_L, π_R and the blow-down map β:

$$f \mapsto (\pi_L \circ \beta)_*(A(\pi_R \circ \beta)^* f).$$

The support condition implies that this action is local, hence A defines a differential operator. One can check directly in local coordinates that the differential operators that lift to X^2_a are exactly the adiabatic differential operators. Conversely, from Schwartz's kernel theorem, differential operators define distributions on X^2 supported on the diagonal. The distributions coming from adiabatic operators lift via β to X^2_a. These operations are inverse to each other. $\Box$

Definition 2.1.6 *Let* $T_0\Psi_a(X)$ *be the space of those* Ω_a*-valued distributions on* X^2_a *which are classical conormal to the lifted diagonal (i.e. of integral order and 1-step poly-homogeneous), vanish rapidly to the boundary faces of* X^2_a *other than the front face, and are smooth to the front face. They form naturally a module over* $C^\infty([0,\infty))$. *The adiabatic algebra,* $\Psi_a(X)$, *is by definition* $T_0\Psi_a(X)[t^{-1}]$, *the space of adiabatic operators of* t*-order 0, with* t^{-1} *adjoined.*

Note that the function t is not a zero-divisor, hence by adjoining t^{-1}, the space $T_0\Psi_a(X)$ injects in $\Psi_a(X)$.

Definition 2.1.7 *Let* $\dot{\Psi}_a(X)$ *denote the subspace of distributions in* $\Psi_a(X)$ *which vanish rapidly at* all *boundary faces of* X^2_a. *Let*

$$\Psi_{(0,\infty)}(X) := C^\infty((0,\infty), \Psi(X))$$

be the algebra of one-parameter smooth families of pseudo-differential operators on X. The topology on $\Psi(X)$ is a direct limit of Frechét topologies, described in Section 2.3.

Remark 2.1.8 *The blow-down map β induces an isomorphism of $\dot{\Psi}_a(X)$ with $\dot{C}^\infty([0,\infty), \Psi(X))$, the ideal of rapidly vanishing families in $\Psi_{(0,\infty)}(X)$. Thus $\dot{\Psi}_a(X)$ has a natural algebra structure.*

Theorem 2.1.9 (Melrose [11]) *$\Psi_a(X)$ has an algebra structure such that $\dot{\Psi}_a(X)$ with the above product becomes an ideal of $\Psi_a(X)$.*

Proof: The composition rule is defined by constructing a triple blow-up space X_a^3, such that there exist three maps to X_a^2 which are b-submersions. Associativity follows from the action of $\Psi_a(X)$ on $C^\infty(X \times [0,\infty))$. $\square$

2.2 Ideals, quotients and filtrations

Proposition 2.2.1 *The map ϕ_a from Definition 2.1.2 induces a canonical splitting*

$$ {}^aTX_{|t=0} \cong TX/M \oplus \pi^*TM. $$

Proof: First, there is a natural inclusion map $TX/M \times [0,\infty) \to {}^aTX$, which views a time-dependent family of vertical vector fields as an adiabatic family. Hence TX/M is naturally a subspace of ${}^aTX_{|t=0}$. Obviously, ϕ^a kills this subspace. We claim that the null-space of ϕ_a is isomorphic to π^*TM. Indeed, consider the following maps of vector bundles over $X \times [0,\infty)$:

$$ TX/M \times [0,\infty) \to TX \times [0,\infty) \xrightarrow{t\cdot} {}^aTX \xrightarrow{\phi_a} TX \times [0,\infty), $$

where the first map is inclusion and the second is multiplication by t. Restricting at $t=0$, we get an exact sequence. Hence $Null(\phi_a) \cong im(t\cdot) \cong TX/(TX/M) \cong \pi^*TM$. The last isomorphism is induced by π_*. $\square$

Corollary 2.2.2 *${}^aT^*X_{|t=0}$ is canonically isomorphic to $\pi^*T^*M \oplus (TX/M)^*$.*

$\square$

Proposition 2.2.3 *The normal bundle to the lifted diagonal in X_a^2 is canonically isomorphic to ${}^aTX$.*

Proof: Note that Δ_a is diffeomorphic to $X \times [0,\infty)$. Recall that for any $t > 0$, ${}^aTX_{|t}$ is isomorphic to TX via ϕ^a, and that the blow-down map β is a local diffeomorphism around any $t > 0$. Define a map from ${}^aTX$ to $TX^2_{a|\Delta_a}$ in the interior of X_a^2 by $v \mapsto \beta_*^{-1}(v, 0)$. This map extends to be non-degenerate and transversal to Δ_a at $t = 0$. $\square$

Proposition 2.2.4 *The interior of the front face of X_a^2 is naturally diffeomorphic to $X \times_M X \times_M \pi^* TM$. As such, it has a vector bundle structure.*

Proof: From the definition, ff fibers over $X \times_M X$. Define the "0-section" in this bundle as the class of the tangent vector ∂_t. Now take an adiabatic vector field v which lives in π^*TM at $t = 0$ (see Proposition 2.2.1) and lift it via $\pi_R \circ \beta$ to X_a^2. The restriction of the lift $\tilde{v}$ at the front face is tangent to ff and depends only on the restriction of v at $t = 0$. Moreover, the projection of $\tilde{v}$ onto the 0-section $X \times_M X$ vanishes, since it coincides with the lift of v viewed as a family of vector fields vanishing at $t = 0$. On the other hand, the map ${}^a TX \to TX_a^2$, given by lifting vector fields, is pointwise injective. It follows that for any $(x, y) \in X \times_M X$, we have a linear isomorphism $(\pi^*TM)_x \to T(ff_{(x,y)})$. This map preserves Lie brackets since lifting vector fields does. Choose a basis $v_1, \dots, v_n$ of ${}^a T_x X$. We get a *commuting* set of vector fields on $(ff_{(x,y)})$, which span $T(ff_{(x,y)})$ at every interior point. The forms α_i in the dual basis are closed, and since $ff_{(x,y)}$ is contractible, they are also exact. Let f_i be the primitives that vanish at the 0-section. Define a map $f : (ff_{(x,y)})^\circ \to (\pi^*TM)_x$ by $p \mapsto \sum f_i(p) v_i$; f must be injective, since df_i are nowhere zero. It is easy to see that this map does not depend on the choice of the basis v_i. Moreover, $\partial/\partial f_i = \tilde{v}_i$. Hence, if $f(p) = v$, then $p = c_{\tilde{v}}(1)$, where $c_{\tilde{v}}$ is the integral curve of the vector field $\tilde{v}$ starting at the 0-section. Since the vector fields $\tilde{v}_i$ are tangent to ff, which is a compact manifold with boundary, they must also be tangent to the boundary of ff. It follows that their integral curves live on ff and that an integral curve which touches the boundary is completely contained in it. This shows that the map f is also surjective. □

Definition 2.2.5 *The algebra $\Psi_a(X)$ is filtered by the* order*, defined as the order of singularity at the lifted diagonal. We denote this (increasing) filtration by $F_i, i \in Z$. The* smoothing ideal *$F_{-\infty}\Psi_a(X)$ is denoted by $\Psi_a^{-\infty}(X)$. The quotient $S_a(X) = \Psi_a(X)/\Psi_a^{-\infty}(X)$ is called the* adiabatic symbol algebra*:*

$$0 \to \Psi_a^{-\infty}(X) \to \Psi_a(X) \to S_a(X) \to 0. \tag{2.1}$$

Definition 2.2.6 *In this paper, we denote by $S^{\mathbb{Z}}(V)$ the poly-homogeneous symbols of integer order, by $\mathcal{S}(V)$ the Schwartz (rapidly vanishing) symbols, and by $S(V) := S^{\mathbb{Z}}(V)/\mathcal{S}(V)$ the formal symbols on a vector bundle V.*

The filtration F_i is complete on S_a. Let $G_i^F S_a(X) = F_i S_a(X)/F_{i-1} S_a(X)$ be the filtration quotients. Not surprisingly, the associated graded algebra $G^F S_a(X)$ can be identified with the set of symbols on the appropriate cotangent bundle:

Proposition 2.2.7 *There is a natural identification*

$$G_i^F S_a(X) \cong C^\infty_{(i)}({}^a T^* X \setminus \{0\})[t^{-1}] = S^i({}^a T^* X)[t^{-1}].$$

Hence $G^F S_a(X) \cong S^{\mathbb{Z}}({}^a T^* X)/S^{-\infty}({}^a T^* X)[t^{-1}] = S({}^a T^* X)[t^{-1}]$.

Proof: The subscript i denotes the homogeneity degree in the cotangent fibers. Remark 2.1.8 implies that Ψ_a coincides with $\Psi_{(0,\infty)}(X)$ when localized around $t \neq 0$. The symbol map on $\Psi_{(0,\infty)}(X)$ gives the desired identification, which does not depend on the localization. This identification extends down to $t = 0$, as one can see in local coordinates. Alternately, by Proposition 2.2.3, this follows from the definition of conormal distributions. The identification involves Fourier transform in the fibers of ${}^aTX$, with respect to the density Ω_a pulled back from X_a^2. □

There is another (decreasing) filtration on $\Psi_a(X)$, denoted $T_j, j \in \mathbb{C}$.

Definition 2.2.8 *We say that $A \in T_j\Psi_a(X)$ if A vanishes at order j at 0.*

This filtration descends to $\Psi^{-\infty}(X)$, $S_a(X)$ and $I_a(X)$.

Definition 2.2.9 *Let $A^\partial(X) = \Psi_a(X)/\dot{\Psi}_a(X)$ be the* formal adiabatic algebra. *Let $\mathcal{V}^\psi$ be the* vertical algebra *associated with $\Psi_a(X)$,*

$$\mathcal{V}^\psi = T_0\Psi_a(X)/T_1\Psi_a(X).$$

Definition 2.2.10 *Let $I_a(X) = \Psi_a^{-\infty}(X)/(\dot{\Psi}_a(X) \cap \Psi_a^{-\infty}(X))$ be the* formal smoothing ideal. *Let $\mathcal{V} = T_0(I_a(X))/T_1(I_a(X))$ be the* vertical smoothing ideal *associated to $I_a(X)$. Let $S_a^\partial(X) = S_a(X)/\dot{C}^\infty([0,\infty))S_a(X)$ be the* formal symbol algebra.

Proposition 2.2.11 *Fourier transform identifies $\mathcal{V}$ with $\mathcal{S}(X \times_M X \times_M {}^aT^*X, \Omega_R)$.*

Proof: An element in $\mathcal{V}$ is the restriction to the front face of a non-singular smoothing adiabatic operator. On ff, Ω_a (see Definition 2.1.4) decomposes as $\Omega_R \otimes \Omega({}^aTX)$. We identify such a restriction with its Fourier transform in the vector fibers. □

Proposition 2.2.12 *If $X \to M$ is the identity fibration, i.e. $X = M$, then there exist also canonical isomorphisms $\mathcal{V}^\psi(M) \cong S^{\mathbb{Z}}({}^aT^*X, \Omega_R)$, $\mathcal{V}^\psi(M)/\mathcal{V}(M) \cong S({}^aT^*X, \Omega_R)$.*

Proof: The proof of Proposition 2.2.11 applies. Note that this is *not* true if $X \neq M$. □

2.3 Hochschild Homology

Let A be an unital algebra (over $\mathbb{C}$). Let $C_i(A) = A^{\otimes i+1}$ be the space of *Hochschild chains*. For $j = 0, \ldots, i$, define $b_j : C_i(A) \to C_{i-1}(A)$ by

$$a_0 \otimes \ldots \otimes a_j \otimes a_{j+1} \otimes \ldots \otimes a_i \mapsto a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_i.$$

Define the *Hochschild boundary map* by $b = \sum_{j=0}^{i}(-1)^j b_j$. By a direct computation, $b^2 = 0$. The homology of the complex $(C_\bullet, b)$ is the *Hochschild homology* of the algebra A (relative to A).

If A is not unital, let $\bar{A}$ be the augmented algebra $A \oplus \mathbb{C}$. We define the chain spaces of A by $C_k(A) = \ker(\bar{A}^{\otimes k+1} \to \mathbb{C}^{\otimes k+1})$. This definition, applied to an unital algebra, gives different chain spaces from the ones above. Nevertheless, there are canonical chain maps which induce isomorphisms on homology.

If A is a topological algebra with a Frechét topology, we replace in the definition of the chain spaces the tensor product with the projective tensor product.

We define the topology on $T_0\Psi_a^{-\infty}(X)$ by the following semi-norms: choose boundary defining functions ρ_i for the hyper-surfaces of X_a^2 other than ff. Define $\|A\|_{I,j} = \|\rho^I A\|_{C^j}$. The ideal $\Psi_a^{-\infty}(X)$ becomes a direct limit of Frechèt spaces.

Choose an embedding $TX = N(\Delta) \to X^2$, such that TX/M maps to $X \times_M X$. This induces a map ${}^a TX \to X^2 \times [0,\infty)$, which lifts to X_a^2. The restriction at $t=0$ of this map is the map of vector bundles induced by $TX/M \to X \times_M X$ (see Propositions 2.2.4 and 2.2.1). Choose a cut-off function on X^2, supported inside the image of TX. Let ψ denote its pull-back to X_a^2. Then every adiabatic operator A splits in $A = \psi A + (1-\psi)A$. By Fourier transform, ψA becomes an element of $S^{\mathbb{Z}}({}^a T^*X)$. This space is a direct limit of Frechèt spaces. For instance, on $T_0F_0\Psi_a(X)$, the topology is the C^∞ topology on the radial compactification of ${}^a T^*X$. As for $(1-\psi)A$, it belongs to $\Psi_a^{-\infty}(X)$. This construction allows us to define a direct limit topology on $\Psi_a(X)$. This topology is independent of the choices made. It has the property that the isomorphism from Remark 2.1.8 becomes a topological isomorphism.

The homology of all the algebras we consider is in the topological sense. Namely, the chain spaces are defined as the direct limit of the projective tensor products of terms in bounded filtrations.

The filtrations F_i and T_j are compatible with multiplication in $\Psi_a(X)$. They induce filtrations on the Hochschild chain spaces in the following way: A pure tensor $a_0 \otimes \ldots \otimes a_k$ is said to belong to F_i if $a_0 \in F_{i_0}, \ldots, a_k \in F_{i_k}$ and $i_0 + \ldots + i_k = i$. Then F_i is defined as the closure of the linear span of pure tensors in F_i. The other filtration is defined similarly. The boundary map b is compatible with these two filtrations. Hence we get filtrations on the Hochschild complexes of $\Psi_a(X)$, $\Psi_a^{-\infty}(X)$, $S_a(X)$ etc. We will denote the associated spectral sequences by ${}^F E$, respectively ${}^T E$, or simply by E when no confusion can arise.

Definition 2.3.1 *Let $\mathcal{L}$ denote the field of Laurent formal series in the variable t, $\mathcal{L} = C[[t]][t^{-1}]$.*

Clearly $\Psi_a(X)/\dot\Psi_a(X)$ is an $\mathcal{L}$-vector space. However, our definition of Hochschild homology involves only tensors over $\mathbb{C}$.

Definition 2.3.2 (Hochschild-Kostant-Rosenberg,[7]) *Let A be a commutative algebra. Define HKR : $C_k(A) \to \Omega^k_{A/C}$ by $a_0 \otimes \ldots \otimes a_k \mapsto (1/k!) a_0 da_1 \wedge \ldots \wedge da_k$.*

It is easy to see that *HKR* vanishes on boundaries, hence descends to Hochschild homology.

Proposition 2.3.3 *Let Y be a compact manifold, possibly with boundary. The map HKR induces isomorphisms $HH_k(C^\infty(Y)) \to \Lambda^k(Y)$ and $HH_k(\dot C^\infty(Y)) \to \dot\Lambda^k(Y)$, where the dot means rapidly vanishing objects to the boundary of Y.*

This result is known as the *Hochschild-Kostant-Rosenberg* theorem, even though it was proved in [4] for the algebra $C^\infty(Y)$ when Y is a closed manifold. The other cases follow from this one, by considering for instance the double of a manifold with boundary, and the H-unital ideals of rapidly vanishing functions.

2.4 The action of derivations on Hochschild Homology

This section is standard, except for Proposition 2.4.2, part 2. See for instance [8].

Let d be a derivation on an algebra A. We can define the inner product, respectively the Lie derivative with respect to d on $HH(A)$ by the following chain maps:

Definition 2.4.1

$$\begin{aligned} a_0 \otimes a_1 \otimes \ldots \otimes a_n &\overset{e_d}{\longmapsto} (-1)^{n+1} d(a_n) a_0 \otimes \ldots \otimes a_{n-1}; \\ a_0 \otimes a_1 \otimes \ldots \otimes a_n &\overset{L_d}{\longmapsto} \sum_i a_0 \otimes \ldots \otimes d(a_i) \otimes \ldots \otimes a_n. \end{aligned}$$

If d is an inner derivation, then e_d and L_d both vanish on $HH(A)$ [8]. We will need the following relations:

Proposition 2.4.2 *1.* $[L_{d_1}, e_{d_2}] = e_{[d_1,d_2]}$.

2. $e_{d_1} e_{d_2} = -e_{d_2} e_{d_1}$.

Proof: 1 was discovered in [14] and is true at chain level. For 2, we use the following lemma [11]:

Lemma 2.4.3 *Every cycle $a' \in C_k(A)$ is homologous to a cycle $a \in C_k(A)$, such that $b_i(a) = 0, \forall i = 0, \ldots, k$, where $b = \sum_{i=0}^k (-1)^i b_i$ is the Hochschild boundary map.*

Proof: We recall that if A is not unital, then $C_k(A)$ is defined to be $\ker(\bar{A}^{\otimes k+1} \to C^{\otimes k+1})$. For the sake of simplicity, make the notation $a' = a_0 \otimes \ldots \otimes a_k$. We assume that $b(a') = 0$. Then

$$a_0 a_1 \otimes 1 \otimes a_2 \otimes \ldots \otimes a_k = b(a_0 a_1 \otimes 1 \otimes 1 \otimes a_2 \otimes \ldots \otimes a_k)$$

is exact. Subtracting this element from a' does not change the homology class. Rename this cycle a'. The new a' has the property $b_0(a') = 0$. Repeat the process on positions $1, 2, \ldots, k-1$. □

Let $[a]$ be a class in $HH(A)$, represented by a cycle a as in Lemma 2.4.3. To alleviate notation, write $a = a_0 \otimes \ldots \otimes a_n$. Then

$$\begin{aligned} (e_{d_1} e_{d_2} + e_{d_2} e_{d_1}) a &= -(d_2 a_{n-1} d_1 a_n + d_1 a_{n-1} d_2 a_n) a_0 \otimes \ldots \otimes a_{n-2} \\ &= -[d_1 d_2 (a_{n-1} a_n) - (d_1 d_2 a_{n-1}) a_n - a_{n-1} (d_1 d_2 a_n)] a_0 \\ &\quad \otimes \ldots \otimes a_{n-2}. \end{aligned}$$

The first two terms vanish by the hypothesis on a. The third one is exact:

$$a_{n-1}(d_1 d_2 a_n)a_0) \otimes \ldots \otimes a_{n-2} = (-1)^{n-1} b((d_1 d_2 a_n)a_0 \otimes \ldots \otimes a_{n-1}).$$

□

2.5 The derivative with respect to t

Recall (Corollary 2.2.2) that ${}^{a}T^*X_{|t=0}$ is canonically isomorphic to $(TX/M)^* \oplus \pi^*T^*M$. A connection in $X \to M$, i.e. a horizontal inclusion $\pi^*TM \xrightarrow{I} TX$, induces an extension of this splitting of ${}^{a}T^*X$ over the interval $[0, \infty)$. To see this, observe that the map

$$V \oplus \pi^*TM \times [0, \infty) \longrightarrow TX \times [0, \infty), \quad (\eta, \xi, t) \mapsto (\eta, tI(\xi), t)$$

satisfies the defining condition for ${}^{a}TX$, hence it defines an isomorphism between $TX/M \oplus \pi^*TM \times [0, \infty)$ and ${}^{a}TX$, as claimed. Now, choosing the inclusion I is the same as choosing a dual projection map $T^*X \xrightarrow{A} \pi^*T^*M$. In local coordinates (x, y, ξ, η, t), respectively $(x, y, \tilde{\xi}, \tilde{\eta}, t)$, adapted to the bundle structures for ${}^{a}T^*X$ and $T^*X \times [0, \infty)$, ϕ_a has the expression:

$$(x, y, \xi, \eta, t) \stackrel{\phi_a}{\longmapsto} (x, y, tA(x, y, \xi, \eta), \eta, t). \tag{2.2}$$

Definition 2.5.1 *Let $\mathcal{M}$ be the radial vector field on π^*T^*M. If we fix the projection A, $\mathcal{M}$ becomes a vector field on ${}^{a}T^*X$.*

Proposition 2.5.2 *The map*

$$D = t\frac{d}{dt} + \mathcal{M} \tag{2.3}$$

is an (outer) derivation on $S_a(X)$ and $A^\partial(X)$ (after Fourier transform).

Proof: Recall the algebra map $\beta : \Psi_a(X)_{|t>0} \to \Psi_{(0,\infty)}(X)$ from Remark 2.1.8. This last algebra has the obvious derivation $t\frac{d}{dt}$.

Lemma 2.5.3

$$\phi_{a*}(\frac{d}{dt}) = \frac{d}{dt} + t^{-1}\mathcal{M}.$$

Proof: In the local coordinate systems (2.2),

$$\begin{aligned} \phi_{a*}(\frac{d}{dt}) &= \sum \frac{\partial \tilde{\xi}_i}{\partial t}\frac{d}{d\tilde{\xi}_i} + \frac{\partial \tilde{\eta}_j}{\partial t}\frac{d}{d\tilde{\eta}_j} + \frac{d}{dt} \\ &= t^{-1}\sum \tilde{\xi}_i \frac{d}{d\tilde{\xi}_i} + \frac{d}{dt}. \end{aligned}$$

□

The action of $t\frac{d}{dt}$ on $S_{(0,\infty)}(X)$ is t times partial differential with respect to the parameter t. From the lemma, the induced derivation on $S_a(X)_{|_{t>0}}$ is $t\frac{d}{dt}+\mathcal{M}$. This extends down to $t=0$ and, by continuity, stays a derivation. The fact that it is outer is obvious, since $D(t)=1$, whereas t is central in $\Psi_a(X)$. The same argument works for $A^\partial(X)$. □

Observe that D preserves all ideals like I_a, etc. As in Definition 2.4.1, we get an action e_D on the Hochschild homology of these algebras. Moreover, D preserves the filtrations F_j and T_i, hence it induces maps of spectral sequences.

2.6 The conjugation by $\log Q$

We can extend the definition of $\Psi_a(X)$ by allowing in Definition 2.1.6 1-step conormal distributions of any complex order. We will call elements in this algebra *adiabatic operators of complex order*. There are obvious extensions of the symbol algebra, the vertical algebra and the vertical ideal (see Definitions 2.2.9, 2.2.10).

Let $Q \in T_0\Psi_a^1(X)$ be a positive elliptic adiabatic operator of order 1.

Theorem 2.6.1 *There exists a holomorphic family $D(z)$ of adiabatic operators, such that $D(0)=Id$, $D(z)D(\tau)=D(z+\tau)$, and $Q-D(1)\in\dot{\Psi}_a^{-\infty}(X)$.*

Proof: Denote by Q_σ the image of Q in $S_a(X)$.

Proposition 2.6.2 *There exists a unique holomorphic family $(Q_\sigma^z)_{z\in\mathbb{C}}$ of symbols of order z, such that $Q_\sigma^0=Id$, $Q_\sigma^1=Q$ and $Q_\sigma^{z+\tau}=Q_\sigma^zQ_\sigma^\tau$.*

Proof: Follow the construction of the complex powers from [6], Theorem 5.5. Let $r=\sigma(Q)$. Start with a family $A_0(z)$, such that $\sigma(A_0(z))=r^z$. In particular, $A_0(z)$ is invertible. We can assume that $A_0(0)=Id$. Then $F_0(z,\tau):=A_0(z)A_0(\tau)A_0(z+\tau)^{-1}-1$ belongs to $S_a^{-1}(X)$, and the principal symbol $f(z,\tau):=\sigma(F(z,\tau))$ is a 2-cocycle in the holomorphic group cohomology $H^2(\mathbb{C},C^\infty_{(-1)}({}^aT^*X\setminus\{0\}))$. This cohomology space is 0, hence there exists $h:\mathbb{C}\to C^\infty_{(-1)}({}^aT^*X\setminus\{0\})$ holomorphic, such that $F(z,\tau)=h(z)+h(\tau)-h(z+\tau)$ and $h(0)=0$. Moreover, h is unique up to a linear factor. We can fix this linear factor by asking that $\sigma(Q^{-1}A_0(1)-Id)=h(1)$. With this condition, h is unique. Let $H_0(z)$ be a holomorphic family of symbols in $S^{-1}({}^aT^*X)$ of principal symbol $h(z)$, such that $H_0(0)=0$. Let $A_1(z)=A_0(z)(Id-H_0(z))$. This family has the properties that $A_1(0)=Id$, $Q^{-1}A_1(1)-Id\in S_a^{-2}(X)$, and $F_1(z,\tau):=A_1(z)A_1(\tau)A_1(z+\tau)^{-1}-1$ belongs to $S_a^{-2}(X)$. Repeat the process to improve the family $A_1(z)$. The step i of this process only changes the terms of homogeneity at most i. Hence, the limit $Q_\sigma^z=\lim_{i\to\infty}A_i(z)$ is well-defined, and has the desired properties. □

In conclusion, Guillemin's proof [6] applies to the algebra of adiabatic symbols without any change. From the proof, it follows that the family Q^z is unique.

Consider now the boundary algebra $A^\partial(X)=\Psi_a(X)/\dot{\Psi}_a(X)$. Let Q_∂ denote the image of Q in this algebra. We will construct the complex powers of Q_∂ in the boundary algebra of complex order, following a suggestion of R.B. Melrose.

Lemma 2.6.3 *Let Q_0 be the image of Q in $\mathcal{V}^\psi = T_0A^\partial(X)/T_1A^\partial(X)$. There exists a (unique) holomorphic family Q^z in this algebra, with the properties $Q^0 = Id$, $Q^1 = Q$, $Q^zQ^\tau = Q^{z+\tau}$.*

Proof: After Fourier transform in the horizontal cotangent directions, $\mathcal{V}^\psi$ becomes a subalgebra of the algebra of families of pseudo-differential operators on the fibers of $X \times_M {}^\alpha T^*M \to {}^\alpha T^*M$. Apply Theorem 5.2 from [6] to construct (over each point in ${}^\alpha T^*M$) the complex powers Q_0^z of this family. Consider also the image of the symbol constructed in Proposition 2.6.2. From the uniqueness part, it follows that the two families of symbols coincide. Hence, the family Q_0^z consists of complex-order elements in $\mathcal{V}^\psi$. All the other properties follow from loc.cit., Theorem 5.2. □

This Lemma allows us to imitate the proof of Proposition 2.6.2 to construct the complex powers of Q_∂ in $A^\partial(X)$. We replace the principal symbol map with the "principal t-symbol". If $A \in T_iA^\partial(X)$, this map associates to A its image in $T_iA^\partial(X)T_{i+1}A^\partial(X) \cong t^i\mathcal{V}^\psi$. The rest of the proof is identical with Proposition 2.6.2, so we skip the details.

So far, we have constructed:

- The unique holomorphic family of adiabatic symbols Q_σ^z
- The unique holomorphic family of boundary operators Q_∂^z
- For $t > 0$, the unique family of pseudo-differential operators Q^z on X. This was constructed by Seeley [15]. It belongs to the algebra of complex order pseudo-differential operators by [6], Theorem 5.2.

We would like to glue these pieces together, or rather to show that they are compatible. This means to show that Q^z has a Taylor expansion at $t = 0$ which is exactly Q_∂^z. We can achieve this after perturbing Q.

By uniqueness, the Taylor expansion at $t = 0$ of Q_σ^z coincides with the image of Q_∂^z in $S_a^\partial(X)$. Let $B(z)$ be a holomorphic family of adiabatic operators which extends both Q_σ^z and Q_∂^z in the obvious sense. For $t > 0$, we have

$$B(z)B(\tau) = B(z+\tau) + H(z,\tau), \tag{2.4}$$

where $H(z,\tau)$ belongs to $\dot\Psi^{-\infty}(X)$. Hence for each $t > 0$ fixed, $H(z,\tau)$ is a smoothing operator. For z close to 0, $B(z)$ is invertible. Let $P = \partial_z B(z)_{|z=0}$. Differentiating (2.4) with respect to τ at $\tau = 0$, we get

$$B(z)P = \partial_z B(z) + S(z),$$

where $S(z) = \partial_\tau H(z,\tau)_{|\tau=0}$ is smoothing. Let $C(z)$ be the solution of the following ODE:

$$\partial_z C(z) = C(z)S(z)B^{-1}(z), \quad C(0) = Id.$$

Note that $C(z)S(z)B^{-1}(z) \in \dot\Psi^{-\infty}(X)$. This implies that $C(z) - Id \in \dot\Psi^{-\infty}(X)$. Let $D(z) = C(z)B(z)$. Then $\partial_z(D(z)) = D(z)P$. Hence, for each fixed τ, $D(\tau)D(z)$ and

$D(z+\tau)$ are both solutions of the ODE

$$\partial_z Y(z) = Y(z)P, \quad Y(0) = D(\tau).$$

This means that they coincide, i.e. $D(z)$ is the family of complex powers of $D(1)$. This family coincides with $B(z)$ modulo $\dot{\Psi}^{-\infty}(X)$. The semigroup property allows us to extend $D(z)$ to the whole complex plane. $\square$

Note that the symbol of $D(1)$ is the same as the symbol of Q. We can assume that they are both equal to r, where $r : {}^{a}T^*X \to \mathbb{R}$ is the norm function of some metric on ${}^{a}T^*X$. In the sequel, we will replace Q by $D(1)$. Hence, by Q^z we mean the family $D(z)$.

Definition 2.6.4 *Let D_Q be the following (outer) derivation on $S_a(X)$:*

$$a \overset{D_Q}{\longmapsto} \frac{d}{dz}\left(Q^{-z}aQ^z\right)_{|z=0} = \lim_{z\to 0} D_z(a)_{|z=0}, \tag{2.5}$$

where $D_z(a) = \frac{Q^{-z}aQ^z - a}{z}$.

Proposition 2.6.5 $[D_Q, D]$ *is an inner derivation, where D is defined in (2.3).*

Proof: We need to prove that $\frac{d}{dz}(DQ^z)_{|z=0}$ belongs to $S_a(X)$. Write

$$Q^z = r^z(\sum_{j=0}^{\infty} a_{-j}(z)),$$

where $a_{-j} : \mathbb{C} \to C^{\infty}_{(-j)}({}^{a}T^*X\setminus\{0\})$ are holomorphic. Since $Q^0 = Id$, we have $a_0(0) = 1$, and a_{-j} is a multiple of z for all $j \geq 1$. The terms in which $\frac{d}{dz}$ does not differentiate r^z are of the form $r^z\mathcal{H}(\mathbb{C}, S_a(X))$, hence their evaluation at $z = 0$ belongs to $S_a(X)$. For $j \geq 1$, the terms where $\frac{d}{dz}$ does not differentiate a_{-j} will vanish at $z = 0$. We are left with $D(\frac{d}{dz}(r^z)a_0(z))$. Since $a_0(0) = 1$, the term $\frac{d}{dz}(r^z)D(a_0(z))$ vanishes at $z = 0$. Finally, $\frac{d}{dz}(zD(r)r^{z-1})_{|z=0} = D(r) \in S_a(X)$. $\square$

Lemma 2.6.6 *D_Q decreases order by at least 1. If A is an operator of order i, then*

$$\sigma_{i-1}(D_Q A) = \frac{1}{r^2}\{r^2, \sigma_i(A)\}_t,$$

where $\{,\}_t$ is the adiabatic Poisson bracket.

Proof: The symbol of Q^z is r^z. The lemma follows from the composition formula for adiabatic operators. $\square$

Chapter 3

The homology of the algebra of adiabatic symbols

In this section we compute $HH_\bullet(S_a(X))$ by using the spectral sequence ${}^F E$.

3.1 The E_2 term

Proposition 3.1.1 $E_1^{i,k}(S_a) \simeq \Lambda_{(i)}^{i+k}({}^aT^*X \setminus \{0\})[t^{-1}]$.

Proof: The subscript (i) stands for homogeneous forms of homogeneity i. From Proposition 2.2.7, $GS_a(X) \cong S({}^aT^*X)[t^{-1}]$ is the commutative algebra of formal symbols on ${}^aT^*X$, with t^{-1} adjoined. This algebra has a filtration by order which is preserved by multiplication, hence, in a similar way, we form a spectral sequence for its Hochschild complex. The symbol isomorphism from Proposition 2.2.7 induces an isomorphism between $E_0(S_a)$ and $E_0(GS_a)$. This isomorphism commutes with the first differential d_0, which is obtained in both sides by commutative contractions. Hence

$$E_1^{i,k}(S_a) \overset{\sigma}{\simeq} E_1^{i,k}(S({}^aT^*X)[t^{-1}]).$$

By Proposition 2.3.3,

$$HH_k(S({}^aT^*X)[t^{-1}]) = \Lambda_S^k({}^aT^*X)[t^{-1}].$$

Since the algebra $S({}^aT^*X)[t^{-1}]$ is *graded*, its spectral sequence degenerates at E_1 and is convergent. This means that $E_1^{i,k}(S({}^aT^*X)[t^{-1}]))$ is the part of homogeneity i of $HH_{i+k}(S({}^aT^*X)[t^{-1}]))$, which (by the above result) is $\Lambda_{(i)}^{i+k}({}^aT^*X \setminus \{0\})[t^{-1}]$. $\square$

The algebra $C^\infty[0,\infty)[t^{-1}]$ of Laurent functions in t is central in $\Psi_a(X)$, hence the multiplication in the adiabatic algebra is local in t. As in Remark 2.1.8, the blow-down map β induces an inclusion of algebras $\dot{\Psi}_a(X) \to \Psi_{[0,\infty)}(X)$. This induces a map between the symbol algebras and hence between their Hochschild complexes:

$$(C_\bullet(\dot{S}_a(X)), b) \to (C_\bullet(S_{[0,\infty)}(X)), b). \tag{3.1}$$

Here $S_{[0,\infty)}(X))$ has the product induced from $\Psi_{[0,\infty)}(X)$. This map preserves the filtration by the order, therefore it descends to maps of spectral sequences. Note that all the terms of these spectral sequences are $HH_*(C^\infty[0,\infty)[t^{-1}])$-modules, hence local in t. As in Proposition 3.1.1, we see that

$$\begin{aligned} E_1^{i,k}(\dot{S}_a(X)) &\cong \dot{\Lambda}_{(i)}^{i+k}({}^aT^*X \setminus \{0\})[t^{-1}]; \\ E_1^{i,k}(S_{[0,\infty)}(X))) &\cong \Lambda_{(i)}^{i+k}([0,\infty) \times T^*X \setminus \{0\})[t^{-1}]. \end{aligned}$$

Recall (Definition 2.1.2) the canonical map $T^*X \times [0,\infty) \xrightarrow{\phi_a} {}^aT^*X$. For $t > 0$ this is an isomorphism. Then the above map of E_0 and E_1 terms is just ϕ_a^*, the pull-back by ϕ_a. Denote the differential in the spectral sequence of S_a by a superscript a. We get

$$d_1^a = \phi^{*-1} d_1 \phi^*. \tag{3.2}$$

The right-hand side is defined only for $t > 0$. This equality is valid *a priori* in $E_1^{i,k}(\dot{S}_a(X))$; however, for $t > 0$, there is a canonical identification of $S_a(X)$ with $\dot{S}_a(X)$. We will extend (3.2) by continuity down to $t = 0$. Choose a connection in $X \to M$ as in Section 2.5.

Definition 3.1.2 *Let*

$$d_v = d - dt \wedge \mathcal{L}_{\frac{d}{dt}}$$

be the deRham differential in the vertical directions in a product decomposition of ${}^aT^*X$.

Definition 3.1.3 *Let $*$ be the symplectic duality operator on $\Lambda(T^*X)$ introduced by Brylinski [1]. Let $*_\phi$ be the conjugation of $*$ by ϕ_a:*

$$*_\phi = \phi_a^{*-1} * \phi_a^*$$

Proposition 3.1.4 *For $t > 0$, the differential d_1^a can be written in terms of the product structure on ${}^aT^*X$ as*

$$d_1^a = *_\phi (d_v - t^{-1} dt \wedge \mathcal{L}_{\mathcal{M}}) *_\phi .$$

Proof: The differential d_1 for the algebra $S(X) = \Psi^{\mathbb{Z}}(X)/\Psi^{-\infty}(X)$ was computed in [1] and [2] to be equal to $*d*$, where d is the deRham differential on $T^*X \setminus 0$. Hence, in the spectral sequence of the families algebra $S_{[0,\infty)}(X) = \Psi_{(0,\infty)}(X)/\Psi^{-\infty}_{(0,\infty)}(X)$, $d_1 = *(d - dt \wedge \frac{\partial}{\partial t})*$. From formula (3.2) and Lemma 2.5.3, it follows that

$$\begin{aligned} d_1^a &= *_\phi (\phi_a^{*-1} (d - dt \wedge \mathcal{L}_{\frac{d}{dt}}) \phi_a^*) *_\phi \\ &= *_\phi (d - dt \wedge \mathcal{L}_{\phi_{a*}(\frac{d}{dt})}) *_\phi \\ &= *_\phi (d - dt \wedge (\frac{\partial}{\partial t} + t^{-1} \mathcal{L}_{\mathcal{M}})) *_\phi \end{aligned}$$

$\square$

Remark 3.1.5 *The operator $d_v - t^{-1}dt \wedge \mathcal{L}_{\mathcal{M}}$ is canonically defined (regardless of the product decomposition), even though its two components are not. Also, this operator extends continuously on $S_a(X)$ down to $t = 0$.*

Let ${}^v\Lambda$ denote forms that do not contain dt.

Proposition 3.1.6 *The involution $*_\phi$ extends to $\Lambda^\bullet({}^a T^*X)[t^{-1}]$. Its restriction to ${}^v\Lambda^{i+j}_{(i)}$ acts as follows:*

$$\begin{aligned} {}^v\Lambda^{i+j}_{(i)}({}^a T^*X \setminus \{0\})[t^{-1}] \quad \xrightarrow{*_\phi} \quad & {}^v\Lambda^{2m+2n-i-j}_{(m+n-j)}({}^a T^*X \setminus \{0\})[t^{-1}] \\ & \oplus dt \wedge {}^v\Lambda^{2m+2n-i-j-1}_{(m+n-j)}({}^a T^*X \setminus \{0\})[t^{-1}] \\ & \oplus dt \wedge {}^v\Lambda^{2m+2n-i-j+1}_{(m+n-j+1)}({}^a T^*X \setminus \{0\})[t^{-1}]. \end{aligned} \tag{3.3}$$

Proof: First observe that $*_\phi$ commutes with $C^\infty({}^a T^*X)[t^{-1}]$ and with $dt\wedge$. Let $\mu = dx^I \wedge dy^J \wedge d\tilde{\xi}^K \wedge d\tilde{\eta}^L \in {}^v\Lambda^{i+j}({}^a T^*X)[t^{-1}]$ in the local coordinates system (2.2). Then $\phi^*\mu = \nu + t^{-1}dt \wedge \imath_{\Sigma A_i \frac{\partial}{\partial \xi_i}} \nu$ for

$$\nu = t^{|K|} dx^I \wedge dy^J \wedge dA(x, y, \xi, \eta)^K \wedge d\eta^L.$$

We have used the fact that $\frac{\partial A_i}{\partial \xi_j} = \delta^i_j$, since A is a projection. Then

$$*\phi^*\mu = *\nu - t^{-1}dt \wedge \Sigma A_i dx_i \wedge *\nu.$$

In the same way, $\phi^{-1*} * \nu = \beta - t^{-1}dt \wedge \imath_{\mathcal{M}}\beta$ for some β. Hence

$$*_\phi\mu = \phi^{-1*} * \phi^*\mu = \beta - t^{-1}dt \wedge \imath_{\mathcal{M}}\beta - t^{-2}dt \wedge \Sigma\xi_i dx_i \wedge \beta. \tag{3.4}$$

This shows that $*_\phi$ does indeed extend up to $t = 0$. Since $*_\phi^2 = 1$ for $t > 0$, the same must be true for all t. The map ϕ_a is homogeneous of degree 1, hence it preserves homogeneity by pull-back. As noted in [1], the symplectic duality operator $*$ maps $\Lambda^{i+j}_{(i)}(T^*X)$ to $\Lambda^{2n+2m-i-j}_{(m+n-j)}(T^*X)$, and since it commutes with dt, it maps $dt \wedge \Lambda^{i+j-1}_{(i)}(T^*X)$ to $dt \wedge \Lambda^{2n+2m-i-j+1}_{(m+n-j+1)}(T^*X)$. Hence, the three components of $*_\phi$ from formula (3.4) belong to the spaces in the right-hand side of (3.3). $\square$

Corollary 3.1.7 *$d^a_1 = *_\phi(d_v - t^{-1}dt \wedge \mathcal{L}_{\mathcal{M}})*_\phi$ in the chosen product structure.*

Proof: The two sides agree for $t > 0$ and are well-defined for $t \geq 0$. The result follows by continuity. $\square$

Definition 3.1.8 *Fix a product structure on ${}^a T^*X$. Define the* adiabatic duality operator *$*_a$ on $\Lambda^\bullet({}^a T^*X)$ by*

$$*_a := (1 + t^{-1}dt \wedge \imath_{\mathcal{M}})*_\phi. \tag{3.5}$$

From the definition, $*_a$ is an isomorphism with inverse $*_\phi(1 - t^{-1}dt \wedge \imath_{\mathcal{M}})$. Also, it is clear that $*_a$ consists of the first and third terms of $*_\phi$ from formula (3.4).

Proposition 3.1.9 *The conjugate of d_1^a by $*_a$ is d_v, the vertical deRham differential in the chosen product structure.*

Proof:

$$\begin{aligned}(*_a)^{-1}d_v(*_a) &= *_\phi(1-t^{-1}dt\wedge\imath_{\mathcal{M}})d_v(1+t^{-1}dt\wedge\imath_{\mathcal{M}})*_\phi\\ &= *_\phi(d_v - t^{-1}dt\wedge d_v\imath_{\mathcal{M}} - t^{-1}dt\wedge\imath_{\mathcal{M}}d_v)*_\phi\\ &= *_\phi(d_v - t^{-1}dt\wedge\mathcal{L}_{\mathcal{M}})*_\phi\end{aligned}$$

which equals d_1^a, by Corollary 3.1.7. □

Hence by conjugation with $*_a$, $d_1^a : E_1^{i,j}(S_a) \to E_1^{i-1,j}(S_a)$ becomes

$$\begin{array}{c} {}^v\Lambda^{2m+2n-i-j}_{(m+n-j)}({}^a T^*X\setminus\{0\})[t^{-1}] \;\oplus\; {}^v\Lambda^{2m+2n-i-j+1}_{(m+n-j+1)}({}^a T^*X\setminus\{0\})[t^{-1}]dt \\ \downarrow d_v \\ {}^v\Lambda^{2m+2n-i-j+1}_{(m+n-j)}({}^a T^*X\setminus\{0\})[t^{-1}] \;\oplus\; {}^v\Lambda^{2m+2n-i-j+2}_{(m+n-j+1)}({}^a T^*X\setminus\{0\})[t^{-1}]dt \end{array} \tag{3.6}$$

Let ${}^aS^*X \to [0,\infty)$ be the sphere bundle of ${}^aT^*X$ viewed as a bundle over $[0,\infty)$. For any bundle $B \to [0,\infty)$, denote by $C^\infty([0,\infty), H_v^*(B))$ the space of sections in the (trivial) fiber cohomology bundle.

Theorem 3.1.10

$$E_2^{i,j}(S_a) \cong \begin{cases} C^\infty([0,\infty), H_v^{m+n-i}({}^aS^*X\times S^1))[t^{-1}] & \text{if } j = m+n \\ C^\infty([0,\infty), H_v^{m+n-i}({}^aS^*X\times S^1))[t^{-1}]dt & \text{if } j = m+n+1 \\ 0 & \text{otherwise.}\end{cases}$$

Proof: We have seen that $E_2^{i,j}(S_a)$ is isomorphic to the homology of the complex (3.6). Since d_v preserves homogeneity and commutes with $dt\wedge$, this complex splits.

Lemma 3.1.11 *The homology of the complex*

$$(\Lambda^\bullet_{(\bullet)}({}^aT^*X\setminus\{0\})[t^{-1}], d_v)$$

is concentrated in homogeneity $(\bullet) = (0)$.

Proof: Let $\mathcal{R}$ be the radial vector field on $({}^aT^*X)$. Then, for $\nu \in \Lambda^\bullet_{(k)}({}^aT^*X\setminus\{0\}[t^{-1}])$, we have $\mathcal{L}_{\mathcal{R}}\nu = k\nu$. Assume ν is d_v-closed. Observe that

$$[\mathcal{R}, \frac{d}{dt}] = 0. \tag{3.7}$$

This follows from the fact that, in the product structure that we have chosen, translations in the t direction act linearly on the fibers of ${}^aT^*X$. Then

$$k\nu = \mathcal{L}_{\mathcal{R}}\nu = d\imath_{\mathcal{R}}\nu + \imath_{\mathcal{R}}d\nu = d_v\imath_{\mathcal{R}}\nu + \imath_{\mathcal{R}}d_v\nu = d_v(\imath_{\mathcal{R}}\nu).$$

Hence if $k \neq 0$, then ν is exact. □

Lemma 3.1.12

$$h_k(\Lambda^\bullet_{(0)}({}^aT^*X \setminus \{0\})[t^{-1}], d_v) \cong C^\infty([0,\infty), H^k_v({}^aS^*X \times S^1))[t^{-1}].$$

Proof: Recall that, in order to define d_v, we have fixed an isomorphism

$${}^aT^*X \cong (\pi^*T^*M \oplus (TX/M)^*) \times [0,\infty).$$

Choose a metric on the fibers of $\pi^*T^*M \oplus (TX/M)^*$. Let S^* be the sphere bundle inside this vector bundle, and let r be the radial function. This induces a splitting

$$(\Lambda^\bullet_{(0)}(\pi^*T^*M \oplus (TX/M)^*), d) \cong (\Lambda^\bullet(S^*), d) \oplus r^{-1}dr \wedge (\Lambda^{\bullet-1}(S^*), d).$$

and hence our complex splits as

$$\begin{aligned} {}^v\Lambda^\bullet_{(0)}({}^aT^*X \setminus \{0\})[t^{-1}], d_v)) &\cong ({}^v\Lambda^\bullet(S^* \times [0,\infty))[t^{-1}], d_v) \\ &\quad \oplus r^{-1}dr \wedge ({}^v\Lambda^{\bullet-1}(S^* \times [0,\infty))[t^{-1}], d_v) \end{aligned}$$

These two isomorphic terms explain the factor of S^1 in the statement of the lemma. There exist natural evaluation and inclusion maps

$$\begin{array}{rcl} h_*({}^v\Lambda^\bullet(S^* \times [0,\infty))[t^{-1}], d) & \xrightarrow{ev} & C^\infty([0,\infty), H^\bullet(S^*))[t^{-1}], \\ a & \mapsto & (t \mapsto [a_{|t}]) \\ C^\infty([0,\infty), H^\bullet(S^*))[t^{-1}] & \xrightarrow{in} & h_*({}^v\Lambda^\bullet(S^* \times [0,\infty))[t^{-1}], d), \\ f \otimes [a] & \mapsto & [f \otimes a]. \end{array}$$

Clearly, $ev \circ in = Id$. So, in order for both of them to be isomorphisms, we need to check only that ev is injective. In other words, if for every t, $[a_{|t}]$ is exact, then we need to find a Laurent family of primitives for a. We can do this by using Hodge theory. Choose a metric on the total space of S^*. There is an isomorphism of spaces of L_2 forms:

$$B\Lambda^\bullet(S^*) \xrightarrow{h} (Z\Lambda^{\bullet-1}(S^*))^\perp$$

which, by elliptic regularity, interchanges smooth forms. Then $h(a)$ is the desired family of primitives for a. □

The theorem follows immediately from Lemmas 3.1.11 and 3.1.12. □

Remark 3.1.13 *Since any two choices of metrics or projections are homotopic, they induce the same maps in cohomology, so the isomorphism in Theorem 3.1.10 does not depend on the choices made.*

3.2 Action of derivations and degeneracy

The derivation D introduced in (2.3) acts by e_D on $C_\bullet(S_a(X))$. Since it commutes with b and preserves the filtration F_i, the action descends to the spectral sequence.

Proposition 3.2.1 *The effect of e_D on $*_a E_1(S_a(X))$, and hence on $E_\infty(S_a)$, is contraction by the vector field $t\frac{d}{dt}$.*

Proof: The following commutations hold trivially on $E_1(\Psi_{[0,\infty)})$:

$$HKR\, e_{t\frac{d}{dt}} = \imath_{t\frac{d}{dt}} HKR, \quad * \,\imath_{t\frac{d}{dt}} = \imath_{t\frac{d}{dt}} * .$$

Since HKR is invariant by pull-backs, we can pull it back via the map (3.1). It follows that

$$HKR\, e_D = \imath_{\phi_*(t\frac{d}{dt})} HKR, \quad *_\phi \imath_{\phi_* t\frac{d}{dt}} = \imath_{\phi_* t\frac{d}{dt}} *_\phi,$$

which implies $*_\phi HKR\, e_D = \imath_{\phi_*(t\frac{d}{dt})} *_\phi HKR$, and so

$$\begin{aligned}(1 + t^{-1}dt \wedge \imath_{\mathcal{M}}) *_\phi HKR\, e_D &= (1 + t^{-1}dt \wedge \imath_{\mathcal{M}})(\imath_{t\frac{d}{dt}} + \imath_{\mathcal{M}}) *_\phi HKR \\ &= \imath_{t\frac{d}{dt}}(1 + t^{-1}dt \wedge \imath_{\mathcal{M}}) *_\phi HKR. \end{aligned} \tag{3.8}$$

This identity extends by continuity down to $t = 0$. $\square$

Proposition 3.2.2 *The effect of L_D on $[a] \in E_\infty(S_a)$ is the Lie derivative $\mathcal{L}_{t\frac{d}{dt}}[a]$ applied to the cohomology class $[a]$ representing a at $E_2 = E_\infty$ in the presentation of Theorem 3.1.10. In particular, if a is a class of homogeneity k in t, then $L_D(a) = ka$.*

Proof: From the definitions, $HKR \circ L_D = \mathcal{L}_{\phi_*(t\frac{d}{dt})} HKR$, i.e L_D acts as the Lie derivative $\mathcal{L}_{\phi_*(t\frac{d}{dt})}$ on E_1, i.e. on forms. This Lie derivative commutes with $*_\phi$ and with $(1 + t^{-1}dt \wedge \imath_{\mathcal{M}})$. Finally, $\mathcal{L}_{\mathcal{M}} = 0$ on cohomology. $\square$

Corollary 3.2.3 *The map e_D is injective on $E_2^{\bullet,m+n+1}(S_a(X))$ and vanishes on the rest of $E_2(S_a(X))$. This implies that the spectral sequence ${}^F E(S_a(X))$ degenerates at E_2.*

$\square$

Definition 3.2.4 *Let $t^{-1}dt\wedge$ be the following operation on $C_\bullet(\Psi_a(X))$:*

$$a_0 \otimes \ldots \otimes a_n \overset{t^{-1}dt\wedge}{\longmapsto} \sum_{i=1}^{n} (-1)^i t^{-1} a_0 \otimes \ldots \otimes t \otimes \ldots \otimes a_n,$$

where t is inserted on position i. Let $\alpha = e_D t^{-1}dt\wedge$, $\beta = t^{-1}dt \wedge e_D$.

Proposition 3.2.5 *1. $(t^{-1}dt\wedge)^2 = 0$.*

2. $e_D t^{-1}dt \wedge + t^{-1}dt \wedge e_D = 1$, $(t^{-1}dt\wedge)^2 = 0$.

3. $C_\bullet(S_a(X))$ and $HH(S_a)$ split as $im(e_D t^{-1}dt\wedge) \oplus im(t^{-1}dt \wedge e_D)$.

Proof: The first two statements follow by direct computation. From part 1, $\alpha\beta = 0$. Then part 2 implies that $\alpha = \alpha(\alpha+\beta) = \alpha^2$ and $\beta = (\alpha+\beta)\beta = \beta^2$, so α and β are idempotents. Since $1 = \alpha+\beta = (\alpha+\beta)^2 = \alpha+\beta+\beta\alpha$, we get also $\beta\alpha = 0$. This proves part 3. □

Proposition 3.2.6 *Using the identification from Theorem 3.1.10 of $HH(S_a(X))$ with cohomology spaces, the action of e_{D_Q} is twice the cup product with $r^{-1}dr$, the generator of $H^1(S^1)$.*

Proof: First, notice that, by Propositions 2.6.5 and 2.4.2, e_{D_Q} commutes with $t^{-1}dt\wedge$ and with e_D. This implies that e_{D_Q} will preserve the spaces of 0 and 1-forms in t. Let $a \in E_2^{i,m+n}$ be a representative for a homology class with no dt. Then, by Theorem 3.1.10, $e_{D_Q}(a)$ is determined by its part in homogeneity $i-1$.

From Lemma 2.6.6, it follows that $HKR(\sigma(e_{D_Q}(a))) = \frac{1}{r^2}dr^2 \wedge HKR(a)$, which implies the desired formula after applying the conjugation $(1+t^{-1}dt\wedge \imath_{\mathcal{M}})*_\phi$. The case where the class of a is a multiple of dt follows by multiplication with $t^{-1}dt$. □

3.3 Convergence of the spectral sequence

Theorem 3.3.1 *The spectral sequence for S_a is convergent, in the sense that*

$$HH_k(S_a) \cong E_2^{k-m-n,m+n} \oplus E_2^{k-m-n-1,m+n+1}.$$

Moreover, this isomorphism preserves the filtration by powers of t.

The proof of this theorem is surprisingly complicated. One can construct abstractly a *formal* extension of any element in E_∞ to a closed "chain". The issue is finding asymptotically sumable extensions, hence bounding from below the t-order. We claim that we can achieve this with the bound equal to the t-degree of the starting element in E_∞.

The proof uses the notation, some statements and ideas from [12].

The graded algebra of $\Psi_a(X)$ with respect to the filtration T_j by t-degree is a fibered version of the suspended algebra, that we call the *vertical algebra.* Recall Definition 2.2.9.

Definition 3.3.2 *Let $S\mathcal{V}^\psi$ be the symbol algebra of $\mathcal{V}^\psi$.*

We will identify elements in $\mathcal{V}^\psi$ with the Fourier transform of their Schwartz kernel. Then we see that the algebra of polynomial functions on ${}^aT^*M$ is central in $\mathcal{V}^\psi$.

Definition 3.3.3 *(see [12], Section 5) Let $B_{(sus)} \subset S\mathcal{V}^\psi$ be the ideal consisting of the symbols that vanish rapidly at the vertical sub-bundle of ${}^aT^*X$. Let $F_{(sus)}$ be the quotient of $S\mathcal{V}^\psi$ by $B_{(sus)}$.*

These algebras are filtered by the operator order, hence we get filtrations on the three Hochschild chains spaces which are preserved by the boundary map. We obtain a spectral sequence for each Hochschild complex. We can adapt Propositions 6 and 7 and Lemma 7 from [12] directly to our case, so we will assume without proof the following result:

Proposition 3.3.4 *The spectral sequences $E(F_{(sus)})$ and $E(B_{(sus)})$ degenerate at E_2.*

Since the polynomial functions on ${}^{a}T^*M$ commute with these algebras, we can define the following two types of natural operations on the spectral sequences: contractions by vector fields with polynomial coefficients, and exterior multiplication by forms with polynomial coefficients. These operations define splittings of the homologies and of the spectral sequences, which are preserved by the differentials. We can also consider the action of the Lie derivative in the direction of $\mathcal{M}$, the radial vector field in ${}^{a}T^*M$. This replaces the dilation argument from [12]. The different spectral sequences split as eigenspaces of this action, and using this fact, we can prove the following:

Proposition 3.3.5 *The spectral sequence $E(S\mathcal{V}^{\psi})$ degenerates at E_2.*

Convergence of these sequences is automatic since the topology of the chain spaces is defined as an inverse limit.

Another important fact that we use is that there is no analogue of the index map for the suspended algebra (Lemma 8 of [12]). Namely, the short exact sequence

$$0 \to \mathcal{V} \to \mathcal{V}^{\psi} \to S\mathcal{V}^{\psi} \to 0$$

induces a long exact sequence in homology (by H-unitality, see Chapter 6) which actually splits into short exact sequences. See Proposition 7.2.2.

Consider now the the graded algebra of $\Psi_a(X)$ with respect to the filtration T_j. It is canonically isomorphic to $\mathcal{V}^{\psi} \otimes \mathcal{L}$. In the statements above, we can replace the algebras by their tensor product with $\mathcal{L}$. The convergence part still holds. To see this, work with chains of bounded t-order in each factor. Here and in the sequel, the *order* of a chain $a_0 \otimes \ldots \otimes a_k$ with respect to a decreasing filtration T_i is at least i if all the components of the chain belong to T_i.

We can collect these statements into the following:

Proposition 3.3.6 *Let $a \in G_i C_\bullet(\Psi_a(X))$ be a chain of order j, such that $\sigma(a)$ survives at E_2 in the (vertical) spectral sequence by operator order. Then there exists $A \in T_i C_\bullet(\Psi_a(X))$, $\sigma(A) = a + T_{i+1}$, such that $b(A) \in T_{i+1}$ (i.e. A is a cycle in the vertical sense). Moreover, we can assume that $ord(A) = ord(a)$, i.e. all the factors in A have the same lower bound for their t-order as a.*

$\square$

Proposition 3.3.7 *Let $a \in T_i F_j C_k(\Psi_a(X))$ be a chain which survives at $E_2^{j,k-j}(S_a(X))$, and $k - j \neq m+n, m+n+1$. Then there exists $x \in T_{i-1}F_{j+1}C_{k+1}$ $(\Psi_a(X))$ such that $bx - a \in T_i F_{j-1} C_k(\Psi_a(X))$. Moreover, we can assume $ord(x) \geq \min(ord(a), 0)$.*

Proof: Let ν be the form corresponding to a in $E_2^{j,k-j}(S_a(X))$. Hence $d_v(\nu) = 0$. From the proof of Lemma 3.1.11, it follows that

$$\nu = d_v\left(\frac{\imath_{\mathcal{R}}\nu}{k-j-m-n}\right)$$

(the constant is $k-j-n-m-1$ when ν contains dt). The assumption on $k-j$ means exactly that this constant is non zero. We ignore this constant in the rest of this proof. The passage between E_1 and E_2 is done via the isomorphism $(1+t^{-1}dt\wedge\imath_{\mathcal{M}})*_\phi$ (Proposition 3.1.9). Then $\imath_{\mathcal{R}}\nu$ corresponds to

$$c = *_\phi(1-t^{-1}dt\wedge\imath_{\mathcal{M}})\imath_{\mathcal{R}}\nu = \mathcal{R}^{\#_\phi}\wedge *_\phi(1-t^{-1}dt\wedge\imath_{\mathcal{M}})\nu,$$

where $\mathcal{R}^{\#_\phi}$ is the 1-form dual to $\mathcal{R}$ under ω_ϕ. We see that this 1-form is in T_{-1}, since $\phi^*\mathcal{R} = \mathcal{R}$, $\mathcal{R}^{\#} = \alpha$, the canonical 1-form, and $\phi_*\alpha \in T_{-1}$.

Now find a chain x_0 in $G_{i-1}F_{j+1}$ represented at $E_1(S_a(X))$ by c, which satisfies the condition on the order. By Proposition 3.3.6, x_0 is the symbol of some x which will satisfy the desired conditions. □

Proof of Theorem 3.3.1: We must prove two statements:

1. $\bigcap_i F_iHH_k(S_a(X)) = 0$ and
2. $F_iHH_k(S_a(X)) \to E_2^{i,k-i}$ is surjective.

The first statement follows by repeatedly using Proposition 3.3.7, with two special cases in homogeneity $k-m-n, k-m-n-1$. In this way, for an element $a \in \bigcap_i F_iHH(S_a(X))$, we construct x such that $b(x) = a$. The actual t-filtration of x is not important, provided that its t-order is uniformly bounded. For the second statement, start with $a \in T_jF_iC_k$ so that $\sigma(a)$ represents a class in $E_2^{i,m+n}(S_a(X))$, hence assuming that a contains no dt. This means that we can change the sub-principal symbol of a in T_j so that $b(a) \in F_{i-2}$.

From Proposition 3.3.6, there exists $A \in T_jC_k(\Psi_a(X))$, $\sigma(A) = a$, such that $b(A) \in T_{j+1}$. Observe that we can assume that the sub-principal symbols of A and a agree up to a chain $\mu \in T_jF_{i-1}$ with the property $b(\mu) \in F_{i-2}$. This follows from the proof of Proposition 3.3.6. Now $b(A)$ represents a form which is the zero element in $E_2^{i-2,m+n+1}(S_a(X))$. We can assume that this form contains no dt (if not, project onto the no-dt part). As in Proposition 3.3.7, it follows that there exists $x \in T_jF_{i-1}$ such that $b(A) - b(x_1) \in T_jF_{i-3}$ and $ord(x) \geq \min\{ord(a), 0\}$. Repeated applications of Proposition 3.3.7 will yield an infinite sequence $x_p \in F_{i-p}$ of chains of uniformly bounded orders and of t-degree j. Due to the decreasing operator orders and bounded t-orders, such a sequence is sumable. It follows that $b(A - \sum x_p) = 0$. The dt case is similar. □

Corollary 3.3.8 *$HH(S_a^\partial(X))$ splits in eigenspaces of the L_D action with integer eigenvalues.*

Proof: First, note that the spectral sequence associated to $S_a^\partial(X)$ also degenerates at E^2 and is convergent, by the same proof. Since α and β commute with b and L_D, the splitting form Proposition 3.2.5 is inherited by Hochschild homology and is preserved by L_D. Via the edge inclusion, $E_2^{k-m-n-1,m+n+1}(S_a^\partial(X))$ is a subspace of $HH_k(S_a^\partial(X))$. By Proposition 3.2.1, β acts the identity and α acts as 0 on this subspace, which therefore coincides with $im(\beta)$. Then the inclusion $im(\alpha) \to HH_k(S_a^\partial(X))$, followed by the edge surjection $HH_k(S_a^\partial(X)) \to E_2^{k-m-n,m+n}(S_a^\partial(X))$, is an isomorphism which commutes with L_D. Using Proposition 3.2.2, we conclude that the action of L_D on $HH(S_a^\partial(X))$ is semi-simple with integer eigenvalues. $\square$

Chapter 4

The semi-classical limit algebra

Consider the case where $X = M$, i.e. the fibers of the fibration $X \to M$ are points. In this case, the adiabatic algebra is called the *semi-classical limit* algebra. Recall the definitions of $I_a(X)$, $A^\partial(M)$ and $S_a^\partial(M)$ (2.2.10, 2.2.9) and the notations for the spaces of symbols (2.2.6). Consider the short exact sequence

$$0 \to I_a(M) \overset{\imath}{\hookrightarrow} A^\partial(M) \overset{p}{\to} S_a^\partial(M) \to 0. \tag{4.1}$$

This sequence is compatible with the filtration T_j, hence it induces a short exact sequence of the associated graded algebras. By Proposition 2.2.12 and Corollary 2.2.11, this is:

$$0 \to \mathcal{S}({}^aT^*M_{|t=0}) \otimes \mathcal{L} \overset{\imath}{\hookrightarrow} S^{\mathbb{Z}}({}^aT^*M_{|t=0}) \otimes \mathcal{L} \overset{p}{\to} S({}^aT^*M_{|t=0} \setminus \{0\}) \otimes \mathcal{L} \to 0. \tag{4.2}$$

In this section, $E(I_a(M))$ will stand for the spectral sequence with respect to the filtration T_j (Definition 2.2.8) of the Hochschild complex of $I_a(M)$, etc.

Proposition 4.0.9

$$\begin{aligned}
E_1^{i,j}(I_a(M)) &\cong E_1^{i,j}(\mathcal{S}({}^aT^*M_{|t=0})\hat{\otimes}\mathcal{L}) \\
&\cong t^i\Lambda_{\mathcal{S}}^{i+j}({}^aT^*M_{|t=0}) \oplus \Lambda_{\mathcal{S}}^{i-j-1}({}^aT^*M_{|t=0})t^{i-1}dt; \\
E_1^{i,j}(A^\partial(M)) &\cong E_1^{i,j}(S^{\mathbb{Z}}({}^aT^*M_{|t=0})\hat{\otimes}\mathcal{L}) \\
&\cong t^i\Lambda_{S^{\mathbb{Z}}}^{i+j}({}^aT^*M_{|t=0}) \oplus \Lambda_{S^{\mathbb{Z}}}^{i-j-1}({}^aT^*M_{|t=0})t^{i-1}dt; \\
E_1^{i,j}(S_a^\partial(M)) &\cong E_1^{i,j}(S({}^aT^*M_{|t=0})\hat{\otimes}\mathcal{L}) \\
&\cong t^i\Lambda_{S}^{i+j}({}^aT^*M_{|t=0}) \oplus \Lambda_{S}^{i-j-1}({}^aT^*M_{|t=0})t^{i-1}dt.
\end{aligned}$$

Proof: This proposition is entirely similar to Proposition 3.1.1, so we shall only sketch the proof for $I_a(M)$. The first isomorphism follows from the fact that the induced product on $GI_a(M)$ is the usual commutative product on functions. The second isomorphism is realized by *HKR* (Definition 2.3.2). □

Let d_1 be the first differential in the above spectral sequences.

As a vector space, the algebra $A^\partial(M)$ is isomorphic to $S^{\mathbb{Z}}({}^aT^*M_{|t=0})\hat{\otimes}\mathcal{L}$. The product on $A^\partial(M)$ is a star product, say $\star_t$. It is the push forward under the canonical map

$$T^*M \times [0,\infty) \xrightarrow{\phi} {}^{a}T^*M.$$

of the product of symbols on T^*M in some quantization, that we choose to be the so-called *Riemann-Weyl* quantization. In local coordinates, if $\star$ is given by:

$$a \star b = ab + \frac{1}{2i}\{a,b\} + P_2(a,b) + \dots,$$

then

$$a \star_t b = ab + t\frac{1}{2i}\{a,b\} + t^2 P_2(a,b) + \dots.$$

Proposition 4.0.10 *On $E_1^{i,j}(I_a(M))$, $E_1^{i,j}(A^\partial(M))$ and $E_1^{i,j}(S_a^\partial(M))$, we have*

$$d_1 = (*_a)^{-1} d_v *_a . \tag{4.3}$$

Proof: We first prove the assertion for $S_a^\partial(M)$.

Lemma 4.0.11 *Formula (4.3) holds on $E_1^{i,j}(S_a^\partial(M))$.*

Proof: Let $a \in T_i F_s C_{i+j}(S_a^\partial(M))$ be a chain of pure homogeneity s in the fibers of ${}^{a}T^*M$, which survives at ${}^{T}E_1^{i,j}$. Notice that a also survives at ${}^{F}E_1^{s,i+j-s}(S_a^\partial(M))$. The identification of ${}^{T}E_1$ and ${}^{F}E_1$ with forms on ${}^{a}T^*M$ is realized by the same map (*HKR*). From the definition, $d_1[a] = HKR([b(a)]_{T_{i+1}/T_{i+2}})$ is represented by the part of $b(a)$ of degree $i+1$ in t. From the structure of the product on the boundary algebras, this is exactly the part of pure homogeneity $s-1$ of $b(a)$, i.e. it is equal to $d_1^a[a]$. Now use Proposition 3.1.9. $\square$

Observe that d_1 and $(*_a)^{-1}d_v *_a$ are operators with polynomial coefficients in the fibers of ${}^{a}T^*M_{|t=0}$. We claim that we can recover d_1 from its action on homogeneous forms, i.e. ${}^{T}E_1^{i,j}(S_a^\partial(M))$. Indeed, we can obtain the coefficients of a polynomial-coefficient differential operator from its action on polynomials. $\square$

In the particular case of the trivial fibration $M \to M$, the local coordinates formula (3.4) simplifies as follows:

Proposition 4.0.12 *Let $\mu = \mu_1 + t^{-1}dt \wedge \mu_2$, where $\mu_1, \mu_2 \in {}^{v}\Lambda^k({}^{a}T^*M) \otimes t^i$. Then*

$$*_a\mu \in {}^{v}\Lambda^{2n-k}({}^{a}T^*M) \otimes t^{i+k-n} \oplus {}^{v}\Lambda^{2n-k+1}({}^{a}T^*M) \wedge t^{i+k-n-2}dt.$$

Proof: Use the notation from Section 3.1. Formula (3.4) shows

$$(*_\phi + t^{-1}dt \wedge \imath_{\mathcal{M}} *_\phi)\mu = \beta - t^{-2}dt \wedge \alpha \wedge \beta,$$

for some β, where α is the canonical form $\sum \xi_i dx_i$. The fibration $M \to M$ has a unique connection. The corresponding projection map A is now $A(x,\xi) = \xi$. This implies

$$\beta = t^{k-n} * \mu.$$

$\square$

Proposition 4.0.13 *Conjugation by $*_a^M$ on ${}^T E_1$ induces the following identifications for the E_2 terms:*

$$\begin{aligned}
{}^T E_2^{i,k}(I_a(M)) &\cong t^{2i+k-n} H_{\mathcal{S}}^{2n-i-k}({}^a T^* M_{|t=0}) \\
&\quad \oplus t^{2i+k-n-2} dt \otimes H_{\mathcal{S}}^{2n-i-k+1}({}^a T^* M_{|t=0}); \qquad (4.4)\\
{}^T E_2^{i,k}(A^\partial(M)) &\cong t^{2i+k-n} H_S^{2n-i-k}({}^a T^* M_{|t=0}) \\
&\quad \oplus t^{2i+k-n-2} dt \otimes H_S^{2n-i-k+1}({}^a T^* M_{|t=0}); \qquad (4.5)\\
{}^T E_2^{i,k}(S_a^\partial(M)) &\cong t^{2i+k-n} H^{2n-i-k}({}^a S^* M_{|_{t=0}} \times S^1) \\
&\quad \oplus t^{2i+k-n-2} dt \otimes H^{2n-i-k+1}({}^a S^* M_{|_{t=0}} \times S^1); \qquad (4.6)
\end{aligned}$$

Proof: The subscripts $\mathcal{S}$, S stand for cohomology of forms with Schwartz, respectively with symbol coefficients. The result is a consequence of Propositions 4.0.10 and 4.0.12. □

We can now compute the effect of derivations on these E_2 terms.

Proposition 4.0.14 *The operator e_D acts on $E_2(I_a(M))$, $E_2(S_a^\partial(M))$ and $E_2(A^\partial(M))$ by contraction with the vector field $t\frac{d}{dt}$.*

Proof: The proof of Proposition 3.2.1 applies word by word. □

Proposition 4.0.15 *The operator L_D acts semisimply on $E_2^{i,k}(I_a(M))$, $E_2^{i,k}(S_a^\partial(M))$ and $E_2^{i,k}(A^\partial(M))$, with eigenvalues $2n-i-k$ on $\ker(e_D)$ and $2n-i-k-1$ on $im(e_D)$.*

Proof: Imitate the proof of Proposition 3.2.2 to obtain that L_D acts as the Lie derivative $\mathcal{L}_{\mathcal{M}+t\frac{d}{dt}}$. Hence, e_D has eigenvalues $2n-i-k$ and $2n-i-k-1$ on $E_2^{i,k}$, corresponding to the splitting from Proposition 4.0.13. By Proposition 4.0.14, this splitting is given by the image and nullspace of e_D. □

Corollary 4.0.16 *The spectral sequences from Proposition 4.0.13 degenerate at E_2.*

Proof: By naturality, the boundary maps in the spectral sequences commute with the maps e_D and L_D. In particular, they preserve the nullspace and the image of e_D. On $\ker(e_D) \cap {}^T E_2^{i,k}$, L_D acts as multiplication by $2n-i-k$. This shows that for $s \geq 2$, the map $d_s : \ker(e_D) \cap {}^T E_2^{i,k} \to \ker(e_D) \cap {}^T E_2^{i+s,k-s-1}$ must vanish. Similarly, d_s vanishes on the image of e_D. □

Chapter 5

The formal smoothing ideal

Recall the definition of $I_a(X)$ (2.2.10). In this chapter, $E(I_a(X))$ will represent the spectral sequence with respect to the filtration T_j (Definition 2.2.8) of the Hochschild complex of $I_a(X)$.

5.1 Relationship with the semi-classical limit

We will construct an algebra map $\phi : \Psi_a^{-\infty}(M) \to \Psi_a^{-\infty}(X)$, and show that it induces an isomorphism $E_1(I_a(M)) \cong E_1(I_a(X))$. This will imply that ϕ induces an isomorphism in Hochschild homology.

There exists a natural fibration $X_a^2 \xrightarrow{\pi} M_a^2$. The fibers of this fibration are $F \times F$, where F is the fiber of $X \to M$. Choose a smooth family dv of densities of volume 1 in each fiber of $X \to M$.

Definition 5.1.1 *Define $\phi : \Psi_a^{-\infty}(M) \to \Psi_a^{-\infty}(X)$ by*

$$A \mapsto (\pi^* A) \otimes dv_R,$$

where dv_R is the density dv in the second fiber $X \to M$.

Proposition 5.1.2 *ϕ is a map of algebras.*

Proof: Let $A, B \in \Psi_a^{-\infty}(M)$. Then

$$\begin{aligned} \phi(A)\phi(B) &= (\pi_m)_*(\pi_l^*((\pi^* A) \otimes dv_R) \cdot \pi_r^*((\pi^* B) \otimes dv_R)) \\ &= \pi^*((\pi_m)_*(\pi_l^* A \cdot \pi_r^* B)) \otimes dv_R \\ &= \phi(AB). \end{aligned}$$

□

5.2 A connection on the vertical algebra

Consider the following diagram of fibrations:

$$\begin{array}{ccc} X & \xleftarrow{p} & \pi^*({}^a T^*M) \\ \pi \downarrow & & \pi \downarrow \\ M & \xleftarrow{p} & {}^a T^*M \end{array}$$

A connection in $X \longrightarrow M$, i.e. a horizontal distribution H in X, induces a connection in $\pi^*({}^a T^*M) \longrightarrow {}^a T^*M$, by

$$H_x = p_*^{-1}(H_{p(x)}).$$

This is indeed a connection, since p is a submersion. Note that when we restrict ourselves over a point $x \in M$, the restricted fibration

$$p^{-1}\pi^{-1}(x) \longrightarrow p^{-1}(x)$$

is canonically isomorphic to the product $X_x \times {}^a T^*M_x$, and the induced connection is the canonical trivial connection on the product.

Let $\Gamma_{S^{\mathbb{Z}}}(\Lambda^1({}^a T^*M))$ denote the space of 1-forms on ${}^a T^*M$ which have symbolic coefficients when expressed in a local basis $(dx_i, t^{-1}d\xi_i)$ adapted to the cotangent bundle structure. One can easily check that this space is preserved under changes of coordinates in M. The connection defines a derivation

$$\mathcal{S}(\pi^*({}^a T^*M)) \xrightarrow{\nabla} \Gamma_{S^{\mathbb{Z}}}(\Lambda^1({}^a T^*M)) \otimes \mathcal{S}(\pi^*({}^a T^*M)).$$

Recall Definitions 2.2.9, 2.2.10, and the identification of $\mathcal{V}$, via Fourier transform, with $\mathcal{S}(X \times_M X \times_M {}^a T^*M, \Omega_R)$. This is the space of symbols on $X \times_M X \times_M {}^a T^*M$ with values in the density bundle of the second fiber term. Let $\pi_R, \pi_L : X \times_M X \times_M {}^a T^*M \to X \times_M {}^a T^*M$ be the projections on the first, respectively the second X term. Then the elements in $\mathcal{V}^\psi$ are the distributions on $X \times_M X \times_M {}^a T^*M$ with values in Ω_a, which are conormal to the X-diagonal inside the 0-section of this bundle. The action of $\mathcal{V}^\psi$ on $\mathcal{V}$ is $\pi_L^*(C^\infty(X))$-linear. After Fourier transform, $\mathcal{V}^\psi$ acts $S^{\mathbb{Z}}({}^a T^*M)$-linearly on $\mathcal{S}(X \times_M {}^a T^*M)$:

$$\mathcal{V}^\psi \otimes \mathcal{S}(X \times_M {}^a T^*M) \ni (A, f) \mapsto \pi_{R*}(A\pi_L^*(f)) \in \mathcal{S}(X \times_M {}^a T^*M)$$

This action is faithful. We extend by duality the derivation ∇ to $\mathcal{V}^\psi$:

$$(\nabla A)(f) = \nabla(Af) - A(\nabla(f)).$$

Proposition 5.2.1 *The operator ∇ is well-defined and preserves $\mathcal{V}$.*

Proof: Over a coordinate patch in M, the fibration is trivial. Choose a density dv in the second fiber term which is independent of the base point in ${}^a T^*M$. Then

$\nabla = d_{{}^{c}T^*M} + \alpha$, and $A = a \otimes dv$. Choose a vector field $V \in \Gamma_{S^{\mathbb{Z}}}(T({}^{c}T^*M))$. We have

$$\begin{aligned}(\nabla_V A)(f) &= (V+\alpha(V))\int_{X/M} af\,dv - \int_{X/M} a(V+\alpha(V))f\,dv \\ &= \int_{X/M}((V+\alpha(V)_L + \alpha(V)_R)a)f\,dv,\end{aligned}$$

where the sub-indices L, R mean that the derivative in the direction of $\alpha(V)$ is in the first, respectively the second X factor. The proposition follows by inspection of this formula. □

Proposition 5.2.2 ∇ *commutes with multiplication by* $S^{\mathbb{Z}}({}^{c}T^*M)$*, in the following sense: for every* $A \in \mathcal{V}^{\psi}$*,* $V \in \Gamma_{S^{\mathbb{Z}}}(T({}^{c}T^*M))$ *and* $g \in S^{\mathbb{Z}}({}^{c}T^*M)$*, we have*

$$\nabla_{gV} A = g\nabla_V(A).$$

Proof: Let $f \in \mathcal{S}(X \times_M {}^{c}T^*M)$. We use the fact that $\nabla_{gV} f = g\nabla_V f$ and we commute A with g since the action of $\mathcal{V}^{\psi}$ is $S^{\mathbb{Z}}({}^{c}T^*M)$-linear:

$$\begin{aligned}(\nabla_{gV}A)(f) &= \nabla_{gV}(Af) - A(\nabla_{gV}(f)) = g\nabla_V(Af) - A(g\nabla_V(f)) \\ &= g\nabla_V(Af) - gA(\nabla_V(f)) = (g\nabla_V(A))(f).\end{aligned}$$

□

Definition 5.2.3

$$\nabla^t = \nabla + \frac{\partial}{\partial t}dt.$$

Then Proposition 5.2.2 carries over to ∇^t in the following form:

Proposition 5.2.4 *Let* $A \in \mathcal{V}^{\psi} \otimes \mathcal{L}$*,* $V \in \Gamma_{S^{\mathbb{Z}}}(T({}^{c}T^*M)) \otimes \mathcal{L} \oplus S^{\mathbb{Z}}({}^{c}T^*M) \otimes \mathcal{L}\frac{\partial}{\partial t}$ *and* $g \in S^{\mathbb{Z}}({}^{c}T^*M) \otimes \mathcal{L}$*. Then*

$$\nabla^t{}_{gV}(A) = g\nabla^t{}_V(A).$$

□

Proposition 5.2.5 ∇^t *is a derivation on* $\mathcal{V} \otimes \mathcal{L}$*.*

Proof:

$$\begin{aligned}\nabla^t(AB)(f) &= \nabla^t(AB(f)) - AB(\nabla^t(f)) \\ &= \nabla^t(AB(f)) - A\nabla^t(B(f)) + A\nabla^t(B(f)) - AB(\nabla^t(f)) \\ &= (\nabla^t A)(B(f)) + A(\nabla^t B)(f)\end{aligned}$$

□

5.3 The analogue to *HKR*

Now we restrict our attention to $I_a(X)$. The product in $I_a(X)$ is a deformation of the vertical product. It follows that $E_0(I_a(X)) \cong E_0(\mathcal{V} \otimes \mathcal{L})$. Moreover, this isomorphism commutes with d_0.

Proposition 5.3.1 *By using the Künneth formula in Hochschild homology ([12]), we obtain*

$$\begin{aligned} E_1^{i,j}(I_a(X)) &\cong E_1^{i,j}(\mathcal{V} \otimes \mathcal{L}) \\ &= T_i(HH_{i+j}(\mathcal{V} \otimes \mathcal{L}))/T_{i+1}(HH_{i+j}(\mathcal{V} \otimes \mathcal{L})) \\ &= t^i \otimes HH_{i+j}(\mathcal{V}) \oplus t^{i-1}dt \otimes HH_{i+j-1}(\mathcal{V}). \end{aligned}$$

Proof: Similar to Proposition 4.0.9. □

On the vertical ideal $\mathcal{V}$ there exists a trace functional with values in $\mathcal{S}({}^{a}T^*M)$, defined by push-forward under the projection map π of the restriction to the diagonal of each fiber of kernels in $\mathcal{V}$:

Definition 5.3.2

$$A \overset{Tr_V}{\longmapsto} \int_{X \times_M X \times_M {}^{a}T^*M_{|t=0}/{}^{a}T^*M_{|t=0}} A_{|\Delta_a \cap ff}.$$

Using Tr_V, we can define an analogue of the *HKR* map:

$$C_k(\mathcal{V} \otimes \mathcal{L}) \xrightarrow{K} \Lambda_{\mathcal{S}}({}^{a}T^*M) \otimes \Lambda\mathcal{L}.$$

Definition 5.3.3 *Let* $A = A_0 \otimes A_1 \otimes \ldots \otimes A_k \in C_k(\mathcal{V}^\psi \otimes \mathcal{L})$*, such that* $A_0 \nabla^t A_1 \wedge \ldots \wedge \nabla^t A_k$ *is of order* $-m - \epsilon$*. Define*

$$K(A) = Tr_V(A_0 \nabla^t A_1 \wedge \ldots \wedge \nabla^t A_k).$$

Proposition 5.3.4 *The map* K *is well-defined.*

Proof: For $V_1, \ldots, V_k \in \Gamma_S(T({}^{a}T^*M)) \otimes \mathcal{L} \oplus S^{\mathbb{Z}}({}^{a}T^*M) \otimes \mathcal{L}\frac{\partial}{\partial t}$, define

$$K(A)(V_1, \ldots, V_k) = Tr_V\left(\sum_{\sigma \in S_k} A_0 \nabla^t{}_{V_1}(A_1) \ldots \nabla^t{}_{V_k}(A_k)\right)$$

This is skew-symmetric in $V_1, \ldots, V_k$. Proposition 5.2.4 and the fact that $\mathcal{V} \otimes \mathcal{L}$ and Tr_V commute with $S^{\mathbb{Z}}({}^{a}T^*M) \otimes \mathcal{L}$ show that the right-hand side is $S^{\mathbb{Z}}({}^{a}T^*M) \otimes \mathcal{L}$-linear, hence it defines a differential form. □

Proposition 5.3.5 *If* $A \in F_{-m-k-\epsilon}C_k(\mathcal{V}^\psi \otimes \mathcal{L})$*, then* $K \circ bA = 0$.

Proof:

$$\begin{aligned}
&K(b(A))\\
&= Tr_V[A_0A_1\nabla^tA_2\wedge\ldots\wedge\nabla^tA_k - A_0\nabla^t(A_1A_2)\wedge\ldots\wedge\nabla^tA_k+\ldots\\
&\quad +(-1)^{k-1}A_0\nabla^tA_1\wedge\ldots\wedge\nabla^t(A_{k-1}A_k)+(-1)^kA_kA_0\nabla^tA_1\wedge\ldots\wedge\nabla^tA_{k-1}]\\
&= Tr_V[A_0A_1\nabla^tA_2\wedge\ldots\wedge\nabla^tA_k - A_0(A_1\nabla^tA_2+\nabla^tA_1A_2)\wedge\ldots\wedge\nabla^tA_k\\
&\quad \ldots+(-1)^{k-1}A_0\nabla^tA_1\wedge\ldots\wedge(A_{k-1}\nabla^tA_k+\nabla^tA_{k-1}A_k)\\
&\quad +(-1)^kA_kA_0\nabla^tA_1\wedge\ldots\wedge\nabla^tA_{k-1}]\\
&= (-1)^kTr_V[A_kA_0\nabla^tA_1\wedge\ldots\wedge\nabla^tA_{k-1} - A_0\nabla^tA_1\wedge\ldots\wedge\nabla^t(A_{k-1})A_k]\\
&= 0,
\end{aligned}$$

since Tr_V vanishes on commutators of operators of total order less than $-m-\epsilon$. □

It follows that K descends to the Hochschild homology of the vertical algebra.

Proposition 5.3.6 *For $A, B \in I_a(M)$,*

$$\nabla^t(\phi(A))\phi(B) = \phi(d(A)B).$$

Proof: Let $f \in \mathcal{S}(X\times_M T^*M)$. Then $\phi(B)f = \pi^*(B\int f dv)$. It follows that

$$\begin{aligned}
\nabla^t(\phi(A))\phi(B)f &= \nabla^t(\phi(A)\phi(B)f) - \phi(A)\nabla^t(\phi(B)f)\\
&= \nabla^t(\phi(AB)f) - \phi(A)\nabla^t(\pi^*(B\int f dv))\\
&= \nabla^t(\pi^*(AB\int f dv) - \phi(A)(\pi^*d(B\int f dv)\\
&= \pi^*d(AB\int f dv) - \pi^*(Ad(B\int f dv))\\
&= \pi^*((dA)(B\int f dv))\\
&= \phi(d(A)B)f.
\end{aligned}$$

Since f was arbitrary, the proposition follows. □

Proposition 5.3.7 *$K\circ\phi = (k!)HKR$. Hence, K is surjective.*

Proof: Let $A = A_0\otimes A_1\otimes\ldots\otimes A_k \in C_k(\mathcal{V}(M)\otimes\mathcal{L})$. Then

$$\begin{aligned}
K(\phi(A)) &= Tr_V[\phi(A_0)\nabla^t\phi(A_1)\wedge\ldots\wedge\nabla^t\phi(A_k)]\\
&= Tr_V[\nabla^t\phi(A_1)\wedge\ldots\wedge(\nabla^t\phi(A_k))\phi(A_0)]\\
&= Tr_V[\nabla^t\phi(A_1)\wedge\ldots\wedge\nabla^t\phi(A_{k-1})\wedge\phi(d(A_k)A_0)]]\\
&= \ldots = Tr_V[\phi(d(A_1)\wedge\ldots\wedge d(A_k)A_0)]\\
&= d(A_1)\wedge\ldots\wedge d(A_k)A_0\\
&= k!HKR(A).
\end{aligned}$$

We used Proposition 5.3.6 and commutativity of the trace. □

5.4 $E_1(I_a(X))$

Since *HKR* is an isomorphism on Hochschild homology [4], by Proposition 5.3.7, K must be surjective. In order to prove that K is an isomorphism at E_1, we need to prove injectivity. Recall the decomposition $b = \sum_{i=0}^{k}(-1)^i b_i$ of the Hochschild differential on C_k.

Let g be a metric on M. Let ψ be a cut-off function, $\psi(0) = 0$, supported inside the injectivity radius of M. For two points x_1, x_2 in M at distance less than the injectivity radius, let $\tau_{x_2}^{x_1}$ denote the parallel transport along the shortest geodesic from x_1 to x_2 in $X \to M$, with respect to the connection ∇, and also in ${}^{a}T^*M \to M$ with respect to ∇^{LC}. Fix again a vertical density dv on the fibers of $X \to M$.

Let $P_i = (\xi_i, u_i, v_i)$ denote a typical point in $X \times_M X \times_M {}^{a}T^*M$ over a base point $x_i \in M$, $i = 0, \ldots, k$. Recall the identification

$$C_k(\mathcal{V} \otimes \mathcal{L}) \cong \mathcal{S}((X \times_M X \times_M {}^{a}T^*M)^{k+1}) \otimes \mathcal{L}(t_0, \ldots, t_k) \otimes dv_0 \otimes \ldots dv_k.$$

Definition 5.4.1 *Let a be a cycle in $C_k(\mathcal{V} \otimes \mathcal{L})$,*

$$a = A(t_0, \xi_0, u_0, v_0, \ldots, t_k, \xi_k, u_k, v_k) dv_0 \otimes \ldots \otimes dv_k.$$

For $l \in \{0, \ldots, k\}$, we say that a satisfies $\mathcal{P}_l$ if the following two conditions are satisfied:

1. *$b_j(a) = 0$ for $j = 0, \ldots, l-1$.*
2. *The function A is independent of $v_0, \ldots, v_{l-1}$.*

Proposition 5.4.2 *If a satisfies $\mathcal{P}_l$, then it is homologous to another cycle a', which satisfies $\mathcal{P}_{l+1}$.*

Proof: Define $h_l : C_k(\mathcal{V} \otimes \mathcal{L}) \to C_{k+1}(\mathcal{V} \otimes \mathcal{L})$ by

$$\begin{aligned} h_l(a)(t_0, P_0, \ldots, t_{k+1}, P_{k+1}) = \psi(d(x_l, x_{l+1})) e^{-\|\tau_{x_l}^{x_{l+1}} \xi_{l+1} - \xi_l\|^2} dv_l \\ \otimes a(t_0, P_0, , \ldots, t_l, \xi_l, u_l, \tau_{x_l}^{x_{l+1}} v_{l+1}, \ldots, t_{k+1}, P_{k+1}). \end{aligned}$$

Lemma 5.4.3 *$h_l(a) \in C_{k+1}(\mathcal{V} \otimes \mathcal{L})$.*

Proof: First, note that ψ is 0 at the points where τ is not defined. We need to check that $h_l(a)$ is Schwartz in ξ_{l+1}. This follows from the invariance of the Schwartz space under linear isomorphisms. Indeed, trivialize the fibration above the injectivity ball around x_l by parallel transport. Then $h_l(a)$ is Schwartz in ξ_l and in $\xi_{l+1} - \xi_l$. □

Let a' be defined by the following equation:

$$\begin{aligned} b(h_l(a)) &= (-1)^l a + (-1)^{l+1} \psi(d(x_l, x_{l+1})) e^{-\|\tau_{x_l}^{x_{l+1}} \xi_{l+1} - \xi_l\|^2} dv_l \\ &\quad \otimes \left(\int_{X_{x_{l+1}}} A(t_0, P_0, \ldots, t_l, \xi_l, u_l, \tau_{x_l}^{x_{l+1}} z, t_{l+1}, \xi_{l+1}, z, v_{l+1}, \ldots) dz \right. \end{aligned}$$

$$
\begin{aligned}
& - \int_{X_{x_{l+1}}} A(t_0, P_0, \ldots, t_l, \xi_l, u_l, z, t_l, \xi_l, z, \tau_{x_l}^{x_{l+1}} v_{l+1}, \ldots,) dz \Bigg) \\
& - h_l(b(a)) \\
= \; & (-1)^l (a - a')
\end{aligned}
$$

Since, by hypothesis, $b(a) = 0$, we have $h_l(b(a)) = 0$. In addition, since a satisfies $\mathcal{P}_l$, a' must also satisfy $\mathcal{P}_l$. Thus $b_l(a') = 0$ and that a' does not depend on v_l. Therefore, a' satisfies $\mathcal{P}_{l+1}$. □

Corollary 5.4.4 *Every cycle $a \in C_k(\mathcal{V} \otimes \mathcal{L})$ is homologous to a cycle which satisfies $\mathcal{P}_k$.*

Proof: By induction over l, we show that a satisfies $\mathcal{P}_l$ for every l. $\mathcal{P}_0$ is the empty condition. The induction step is proved in Proposition 5.4.2. □

Definition 5.4.5 *Let a be a cycle in $C_k(\mathcal{V} \otimes \mathcal{L})$, which satisfies $\mathcal{P}_k$. For $l \in \{1, \ldots, k+1\}$, we say that a satisfies $\mathcal{R}_l$ if the the function A is independent of $u_1, \ldots, v_{l-1}$.*

Proposition 5.4.6 *If a satisfies $\mathcal{R}_l$, then it is homologous to another cycle a' which satisfies $\mathcal{R}_{l+1}$.*

Proof: Define $q_l : C_k(\mathcal{V} \otimes \mathcal{L}) \to C_{k+1}(\mathcal{V} \otimes \mathcal{L})$ by

$$
\begin{aligned}
q_l(a)(t_0, P_0, \ldots, t_{k+1}, P_{k+1}) &= \psi(d(x_l, x_{l+1})) e^{-\|\tau_{x_l}^{x_{l+1}} \xi_{l+1} - \xi_l\|^2} dv_l \\
& \otimes a(t_0, P_0, , \ldots, t_l, \xi_l, u_l, \tau_{x_l}^{x_{l+1}} u_{l+1}, \ldots, t_{k+1}, P_{k+1}).
\end{aligned}
$$

As in Lemma 5.4.3, one can prove that $q_l(a)$ is well defined. Since a satisfies $\mathcal{P}_k$, it follows that

$$
\begin{aligned}
b(q_l(a)) &= (-1)^l a - (-1)^l \psi(d(x_l, x_{l+1})) e^{-\|\tau_{x_l}^{x_{l+1}} \xi_{l+1} - \xi_l\|^2} dv_l \\
& \otimes \left(\int_{X_{x_{l+1}}} A(t_0, P_0, \ldots, t_l, \xi_l, \tau_{x_l}^{x_{l+1}} u_{l+1}, t_{l+1}, \xi_{l+1}, z, \ldots) dz \right) \\
&= (-1)^l (a - a'),
\end{aligned}
$$

where the last equality is the definition of a'. Note that a' satisfies $\mathcal{P}_k$ and $\mathcal{R}_l$ since a does, and moreover it is independent of u_l. □

Corollary 5.4.7 *Every cycle $a \in C_k(\mathcal{V} \otimes \mathcal{L})$ is homologous to a cycle which satisfies $\mathcal{P}_k$ and $\mathcal{R}_{k+1}$.*

Proof: By induction, as in Corollary 5.4.4. □

Corollary 5.4.8 *If $a \in C_k(\mathcal{V} \otimes \mathcal{L})$ is a cycle, then $K(a)$ is independent of ∇.*

Proof: From Corollary 5.4.7, a is homologous to some a', which lies in the image of the pull-back map ϕ, i.e $a' = \phi(c)$. By Proposition 5.3.5, we have $K(a) = K(a')$. Finally, by Proposition 5.3.7, it follows that $K(a') = HKR(c)$, hence $K(a)$ is independent of ∇. $\square$

Proposition 5.4.9 *The map (using Corollary 5.3.1)*

$$K : E_1(I_a(X)) \cong HH_*(\mathcal{V} \otimes \mathcal{L}) \longrightarrow \Lambda_{\mathcal{S}}({}^a T^* M) \otimes \Lambda\mathcal{L}$$

is injective. Hence, together with Proposition 5.3.7, it follows that K is an isomorphism.

Proof: Let a be a cycle in $C_k(\mathcal{V} \otimes \mathcal{L})$ so that $[a] \in \ker(K)$. By Corollary 5.4.7, we can assume that a satisfies $\mathcal{P}_k$ and $\mathcal{R}_{k+1}$. This means that

$$a \in \mathrm{im}(\phi : C_k(\mathcal{V}(M) \otimes \mathcal{L}) \longrightarrow C_k(\mathcal{V}(X) \otimes \mathcal{L}).$$

Say $a = \phi(\bar{a})$. Then, by Proposition 5.3.7, we have

$$0 = K(a) = K(\phi(\bar{a})) = HKR(\bar{a}).$$

Since *HKR* is an isomorphism, it follows that $\bar{a}$ is exact in $C_\bullet(\mathcal{V}(M) \otimes \mathcal{L})$. As ϕ is an algebra map, it commutes with the Hochschild differential, and so a is also exact. $\square$

Theorem 5.4.10 *The map ϕ induces an isomorphism*

$$HH_*(I_a(M)) \xrightarrow{\phi} HH_*(I_a(X)).$$

Proof: Propositions 5.3.7 and 5.4.9 imply that K is an isomorphism, hence the map ϕ induces an isomorphism between $E_1(I_a(M))$ and $E_1(I_a(X))$. The result follows using a general property of spectral sequences. We only need to notice that the two spectral sequences are (tautologically) convergent. $\square$

Corollary 5.4.11 *The effect of e_D on $E_\infty(I_a(X))$ is contraction by the vector field $t\frac{d}{dt}$. The space $E_\infty(I_a(X))$ splits as the direct sum of the null-space and the image of e_D. The effect of L_D on $\ker(e_D) \cap {}^T E_\infty^{i,k}(I_a(X))$, respectively on $im(e_D) \cap {}^T E_\infty^{i,k}(I_a(X))$, is multiplication by $2n - i - k$, respectively $2n - i - k - 1$.*

Proof: Follows from Propositions 4.0.15, 4.0.14 and Theorem 5.4.10. We have implicitly used the fact that D commutes with the map ϕ from Definition 5.1.1. Alternately, we could prove this directly along the lines of Propositions 4.0.15 and 4.0.14, where *HKR* is replaced by the map K from Definition 5.3.3. $\square$

Chapter 6

H-unitality of the ideal $\Psi_a^{-\infty}(X)$

Let A be an algebra over $\mathbb{C}$. Recall the definition of the Hochschild complex $C_\bullet(A)$ and the boundary map $b : C_i(A) \to C_{i-1}(A)$, $b = \sum_{j=0}^{i}(-1)^j b_j$ (Section 2.3). The *bar resolution* of A is the complex $(C_\bullet(A), b')$, where $b' : C_i(A) \to C_{i-1}(A)$ is given by $b' = \sum_{j=0}^{i-1}(-1)^j b_j$. Recall that if A is a topological algebra, then $C_\bullet(A)$ is defined by using a topological tensor product.

Definition 6.0.12 (Wodzicki [18]) *The algebra A is called* H-unital *if the bar resolution of A is acyclic.*

Every unital algebra is H-unital. Wodzicki [18] proved that H-unital algebras are *excisive* for Hochschild homology. In other words, let I be an algebra. Then every extension

$$0 \to I \to A \to A/I \to 0$$

leads to a long exact sequence in Hochschild homology. In turn, this property implies that I is H-unital.

In this chapter we prove that $I_a(X)$ is H-unital by using a general criterion. We note that the ideal $\dot{\Psi}_a^{-\infty}(X)$ is also H-unital. We then deduce that $\Psi_a^{-\infty}(X)$ is H-unital. This implies the existence of a long exact sequence in Hochschild homology deduced from the short exact sequence of algebras

$$0 \to \Psi_a^{-\infty}(X) \hookrightarrow \Psi_a(X) \to S_a(X) \to 0. \tag{6.1}$$

6.1 A sufficient condition for H-unitality

Theorem 6.1.1 *Let A be an algebra with a decreasing filtration F_iA, $i \in \mathbb{Z}$, adapted to the product on A, so that*

1. *The graded algebra GA is H-unital;*

2. *$\bigcap F_iA = \{0\}$, $\bigcup F_iA = A$ and*

$$\varinjlim_{i\to-\infty} \varprojlim_{j\to\infty} F_iA/F_jA \cong A;$$

3. *For* $a = a_0 \otimes \ldots \otimes a_k \in C_k(GA)$, *let* $\deg(a) = \min\{p; \exists a_i \notin F_{p+1}A\}$. *Assume that for any* b'*-closed* a, *there exists* c *such that* $b'(c) = a$ *and* $\deg(c) \geq \min(deg(a), 0)$.

Then A *itself is* H*-unital.*

Proof: The filtration is inherited by the chain spaces $C_k(A)$. We can form a spectral sequence for the homology of $(C_\bullet(A), b')$, whose E_0 term is isomorphic to the E_0 term of the similar sequence for GA. This natural isomorphism commutes with d_0, hence it descends to E_1. But

$$E_1^{i,j}(GA) \cong G_i h_{i+j}(C_\bullet(GA), b') = 0,$$

from the first hypothesis, hence $E_1(A) = 0$. This implies $E_\infty(A) = 0$.

Lemma 6.1.2 *The second and the third conditions imply the convergence of the spectral sequence.*

Proof: Condition 2 implies a similar statement with A replaced by $C_k(GA)_{\deg \geq p}$. Note that this would not be true without the restriction on the degree. Together with condition 3, this yields the result. □

Therefore,

$$h(C_\bullet(A), b') = 0.$$

□

6.2 H-unitality of $I_a(X)$

Proposition 6.2.1 $I_a(X)$ *is* H*-unital.*

Proof: Observe that the filtration $T_k I_a(X)$ satisfies the second assumption of Theorem 6.1.1. Therefore, it is enough to prove that the first and the third assumptions hold. The associated graded algebra $GI_a(X)$ is just $\mathcal{V} \otimes \mathcal{L}$. We shall construct an explicit chain contraction $s_k : C_k \to C_{k+1}$. Use the notations from Section 5.4.

Let $a \in C_k(\mathcal{V} \otimes \mathcal{L})$. Define $s_k(a)$ by

$$\begin{aligned} &s_k(a)(t_0, P_0, \ldots, t_{k+1}, P_{k+1}) \\ &\quad = \psi(d(x_0, x_1)) e^{-\|\tau_{x_0}^{x_1}\xi_1 - \xi_0\|^2} dv_0 \otimes a(t_0, \tau_{x_1}^{x_0} u_0, v_1, t_2, P_2, \ldots, t_{k+1}, P_{k+1}). \end{aligned}$$

Like in Lemma 5.4.3, we see that $s_k(a)$ is well defined.

Lemma 6.2.2

$$b' s_k + s_{k-1} b' = Id.$$

Proof: Recall the decomposition of b' on C_k in alternate sum of contractions, $b' = \sum_{i=0}^{k-1}(-1)^i b_i$. Then

$$\begin{aligned} b_0 s_k(a)(t_0, P_0, \ldots, t_k, P_k) &= \int_{X_{v_0}/M} \left(s_k(a)_{|(t_0,P_0)=(t_1,P_1)} \right) \\ &= a(t_0, P_0, \ldots, t_k, P_k) \end{aligned}$$

and

$$b_i s_k = s_{k-1} b_{i-1}$$

for $i = 1, \ldots, k$. □

This implies that $h(C_\bullet(\mathcal{V} \otimes \mathcal{L}), b') = 0$, proving condition 1. At this stage, it suffices to remark that s_k does not decrease the degree, as defined in condition 3 of Theorem 6.1.1. □

6.3 H-unitality of $\dot{\Psi}_a^{-\infty}(X)$

This ideal is isomorphic to $\dot{\Psi}_{[0,\infty)}^{-\infty}(X)$ via the blow-down map $\beta : X_a^2 \to X^2$. Fix a density dx on X. Then

$$C_k(\dot{\Psi}_{[0,\infty)}^{-\infty}(X)) \cong \dot{C}^\infty([0,\infty)^{k+1} \times X^{2(k+1)}) \otimes dy_0 \otimes \ldots \otimes dy_k.$$

Define a chain contraction

$$s_k : C_k(\dot{\Psi}_{[0,\infty)}^{-\infty}(X)) \longrightarrow C_{k+1}(\dot{\Psi}_{[0,\infty)}^{-\infty}(X))$$

by

$$\begin{aligned} & s_k(a)(t_0, x_0, y_0, \ldots, t_{k+1}, x_{k+1}, y_{k+1}) \\ & \quad = e^{-(\frac{1}{t_0} - \frac{1}{t_1})^2} dy_0 \otimes a(t_0, x_0, y_1, \ldots, t_{k+1}, x_{k+1}, y_{k+1}). \end{aligned}$$

Like in Lemma 6.2.2, we can prove the following:

Proposition 6.3.1 *The operator s_k is a well-defined chain contraction, hence*

$$h(C_\bullet(\dot{\Psi}_a^{-\infty}(X)), b') = h(C_\bullet(\dot{C}^\infty([0,\infty))\Psi_{[0,\infty)}^{-\infty}(X)), b') = 0.$$

□

6.4 Algebraic conditions for H-unitality

A short exact sequence of algebras

$$0 \to I \hookrightarrow A \to B \to 0$$

(I is an ideal in A, $B \cong A/I$) induces a double complex with three columns

$$0 \to (C_\bullet(I), b') \overset{\imath}{\hookrightarrow} (C_\bullet(A), b') \overset{p}{\to} (C_\bullet(B), b') \to 0 \tag{6.2}$$

which is exact only on the bottom row. Nevertheless, there is a simple algebraic condition for the existence of a long exact sequence in homology:

Theorem 6.4.1 *A three-column double complex leads to a long exact sequence in the homologies of the columns if and only if the total homology of the double complex vanishes.*

Proof: Follows from the analysis of the spectral sequence of the filtration by columns. This spectral sequence degenerates at E_3. One can check by diagram chasing that $E_3 = 0$ if and only if there is a long exact sequence. In this case, the boundary map is the inverse of d_2. The spectral sequence is convergent since it is concentrated in the first quadrant. Hence the total homology is 0 if and only if $E_\infty = E_3 = 0$. $\square$

Theorem 6.4.2 *If I is H-unital, then*

$$h(C_\bullet(A), b') \cong h(C_\bullet(B), b').$$

Proof: Apply Theorem 6.4.1 to the complex (6.2). The assumption is that the homology of the first column vanishes.

The fact that the map p induces an isomorphism $h(C_\bullet(A), b') \cong h(C_\bullet(B), b')$ is equivalent to the existence of the long exact sequence. By Theorem 6.4.1, this is equivalent to the vanishing of the total homology.

Consider the filtration by rows of the complex (6.2). It induces a spectral sequence whose E_1 term lives only in the middle column because $\imath$ is an injection and p a surjection.

We can construct a filtration on $E_1^{1,k} = \ker(p_k)/\mathrm{Im}(\imath_k)$ by defining an element $a \in A^{\otimes k+1}$ to be in F_j if it can be written as a tensor product such that at least $k+1-j$ of its terms belong to I. This filtration is preserved by the differential d_1, which is just b'. We can form a spectral sequence which computes E_2 and we shall prove that the first term, $\bar{E}_1(E_1)$, of this spectral sequence, vanishes.

Note that the filtration F_j is increasing and on $E_1^{1,k}$ it stabilizes at F_k, since the elements in the kernel of p contain at least one factor from the ideal. Also, since we mod out by $\mathrm{Im}(\imath_k)$, the smallest non-zero term is F_0.

Lemma 6.4.3

$$\bar{E}_1(E_1) = 0.$$

Proof: In the proof of this lemma, use the notation $\bar{d}_0$ for the first differential in $\bar{E}(E_1)$. Since I is an ideal in A, it follows that any contraction involving at most one element in the ideal decreases the filtration order, hence can be neglected when we compute $\bar{d}_0$.

Note that $\bar{E}_0^{j,k-j}(E_1) \cong I^{\otimes k+1-j} \overset{s}{\otimes} B^{\otimes j}$, where $\overset{s}{\otimes}$ stands for the tensor product in any possible order of the $k+1$ spaces.

This complex endowed with the differential $\bar{d}_0$ splits according to the number of connected components of terms from I in the tensor product. Denote by G^s the part with s components. Then

$$\begin{aligned}(G^s_\bullet[-1], \bar{d}_0) \quad \cong \quad & (C^{\mathrm{Aug}}_\bullet(B)[-1], 0) \otimes (C_\bullet(I)[-1]), b') \\ & \otimes (C_\bullet(B)[-1], 0) \otimes \ldots \otimes (C_\bullet(B)[-1], 0) \\ & \otimes (C_\bullet(I)[-1], b') \otimes (C^{\mathrm{Aug}}_\bullet(B)[-1], 0)\end{aligned}$$

(s factors of $(C_\bullet(I), b')$), where *Aug* stands for augmentation by $\mathbb{C}$, and $C_\bullet[-1]$ is the suspension (right shift) of the chain complex $C_\bullet$.

By the Eilenberg-Zilber theorem, the homology of this tensor product is the tensor product of the homologies of the factors. Note that $s \geq 1$, since we saw above that $F_k(E_1^{1,k}) = E_1^{1,k}$. In addition

$$h(C_\bullet(I), b') = 0$$

by assumption. This means that $(G^s_\bullet[-1], \bar{d}_0)$ is also acyclic. □

Since the filtration F_j is finite in each degree, it follows that the spectral sequence $\bar{E}(E_1)$ converges to the homology of (E_1, d_1), i.e. to E_2. Hence, E_2 of the initial double complex is 0. Thus, $E_\infty = 0$ and therefore the double complex is acyclic (again, convergence is a consequence of the spectral sequence being concentrated in the first quadrant). □

Remark 1: The use of the Eilenberg-Zilber theorem is not indispensable. Indeed, we can filter G^s by the lexicographic order of strings of I's and construct a new spectral sequence to prove that (G^s, d_0) is acyclic. This approach is probably preferable when dealing with completed tensor products, as in the case of the adiabatic algebra.

Remark 2: The proof of Theorem 6.4.2 can be translated almost verbatim to provide a proof of the existence of the long exact sequence in Hochschild homology of a short exact sequence of algebras starting with an H-unital ideal. Again, the case of topological algebras can be treated well in this way.

6.5 H-unitality of $\Psi_a^{-\infty}(X)$

Proposition 6.5.1 *$\Psi_a^{-\infty}(X)$ is H-unital.*

Proof: Apply Theorem 6.4.2 to the short exact sequence

$$0 \to \dot{\Psi}_a^{-\infty}(X) \hookrightarrow \Psi_a^{-\infty}(X) \to I_a(X) \to 0. \tag{6.3}$$

Propositions 6.3.1 and 6.2.1 state that $\dot{\Psi}_a^{-\infty}(X)$ and $I_a(X)$ are both H-unital, so by Theorem 6.4.2, $\Psi_a^{-\infty}(X)$ must also be H-unital. □

Corollary 6.5.2 *By the result of Wodzicki [18], which can be adapted to topological algebras by the remarks above, the short exact sequence (6.1) induces a long exact* *sequence in the Hochschild homologies of the three terms. In particular, there exists a boundary map*

$$\delta : HH_k(S_a(X)) \longrightarrow HH_{k-1}(\Psi_a^{-\infty}(X)). \tag{6.4}$$

$\square$

Chapter 7

The residue functional

7.1 Definition of the residue

Recall Definitions 2.2.9, 5.3.3. Since $Q^z \in T_0$, multiplication by Q^z descends to $\mathcal{V}^\psi$ and extends to chains in $C_\bullet(\mathcal{V}^\psi \otimes \mathcal{L})$ by $Q^z(a_0 \otimes \ldots \otimes a_h) = (Q^z a_0) \otimes \ldots \otimes a_h$. Recall Definition 3.1.8. We denote by $*_a^M$ the operator $*_a$ on ${}^a T^*M$. Let ΩM be the orientation bundle of M.

Definition 7.1.1 *For $z \in C$, $Re(z)$ sufficiently negative, define*

$$\begin{aligned} F_z : C_h(\mathcal{V}^\psi \otimes \mathcal{L}) &\to \Lambda^{n-h}(M, \Omega M) \otimes \mathcal{L} \oplus \Lambda^{n-h+1}(M, \Omega M) \otimes \mathcal{L} dt \\ A &\mapsto \int_{{}^a T^*M_{|t=0}/M} *_a^M (K(Q^z A)). \end{aligned}$$

Proposition 7.1.2 *The dt-free part of F_z, $\imath_{\frac{d}{dt}} dt \wedge F_z$, is well-defined and holomorphic on $F_i C_h(\mathcal{V}^\psi \otimes \mathcal{L})$ for $Re(z) < h - m - n - i$. The dt part of F_z, $\imath_{\frac{d}{dt}} F_z$, is well-defined and holomorphic on $F_i C_h(\mathcal{V}^\psi \otimes \mathcal{L})$ for $Re(z) < h - m - n - i - 1$.*

Lemma 7.1.3 *The operator F_z is local in M and does not depend on ∇.*

Proof: The fact that F_z is local in M means that for $V \subset U$ open sets in M and V relatively compact in U, $F_z(A)_{|V}$ depends only on $A_{|(\pi^{-1}(U))^{h+1}}$. This is obvious from the definition. Since F_z is local, we can work over a coordinate patch $U \subset M$. Let (x_i, ξ_j), $(x_i, \tilde{\xi}_j)$ be local coordinates adapted to the cotangent structure in T^*M and ${}^a T^*M_{|t=0}$ over U. The structure map ϕ_a takes the form $(x_i, \xi_j) \mapsto (x_i, t\xi_j)$. Let Id be the local isomorphism $(x_i, \xi_j) \mapsto (x_i, \xi_j)$. Let $*^M$ be Brylinski's duality operator on $\Lambda(T^*M)$. We denote by $*^U$ the conjugate of $*^M$ by Id. On h-forms,

$$\imath_{\frac{d}{dt}} dt \wedge (*_\phi^U + t^{-1} dt \wedge \imath_{\mathcal{M}} *_\phi^U) = t^{h-n} *^U \imath_{\frac{d}{dt}} dt \wedge . \tag{7.1}$$

Using this and the fact that the integral along the fiber vanishes on forms that are not multiples of $d\xi_1 \wedge \ldots d\xi_n$, we get

$$\imath_{\frac{d}{dt}} dt \wedge F_z(A) \;=\; \int_{U \times \mathbb{R}^n / U} t^{h-n} *^U \imath_{\frac{d}{dt}} dt \wedge K(Q^z A) \tag{7.2}$$

$$= \quad t^{h-n}\int_{U\times\mathbb{R}^n/U} *^U Tr_V(\widehat{Q^z a_0}\nabla_\xi \hat{a}_1 \wedge \ldots \wedge \nabla_\xi \hat{a}_h),$$

where $\nabla_\xi = \sum \nabla_{\xi_i} d\xi_i$. Recall that we Fourier transform the kernels in the horizontal tangent directions. Now just observe that ∇_{ξ_i} is actually independent of ∇. This is due to the fact that the connection is lifted to $X \times_M X \times_M {}^aT^*M \to {}^aT^*M$ from $X \times_M X \to M$. This proves that the dt-free part of F_z is independent of ∇. The other case follows by replacing formula (7.1) with (7.4). $\square$

Proof of proposition 7.1.2: Using Lemma 7.1.3, we can assume that we work in local coordinates and that formula (7.2) holds. From the properties of symbols, the assumption on the order of A, and the assumption $Re(z) + i - h < -m - n$, it follows that

$$\widehat{Q^z a_0}\nabla_\xi \hat{a}_1 \wedge \ldots \wedge \nabla_\xi \hat{a}_h \in d\xi^h F_{z+i-h}\mathcal{V}^\psi \otimes \mathcal{L} \tag{7.3}$$

and hence is of trace class (recall that all operators of order less than $-m$ are of vertical trace class). Taking the trace increases homogeneity by m, hence

$$Tr_V(\widehat{Q^z a_0}\nabla_\xi \hat{a}_1 \wedge \ldots \wedge \nabla_\xi \hat{a}_h) \in d\xi^h S^{z+i-h+m}({}^aF) \otimes \mathcal{L}$$

and so

$$*^M Tr_V(\widehat{Q^z a_0}\nabla_\xi \hat{a}_1 \wedge \ldots \wedge \nabla_\xi \hat{a}_h) \in \Lambda^{2n-h}_{(z+i-h+m+n)}({}^aF) \otimes \mathcal{L}.$$

This shows that for $Re(z) + i - h + m + n < 0$, this form is of negative homogeneity and hence integrable along the fibers of ${}^aT^*M_{|t=0}/M$. Holomorphy is preserved by all the operations involved, as long as they make sense. This finishes the proof of the first assertion. The second one is similar and uses the identity

$$\imath_{\frac{d}{dt}}(*^U_\phi + t^{-1}dt \wedge \imath_{\mathcal{M}} *^U_\phi) = t^{h-n-1} *^U \imath_{\frac{d}{dt}} + t^{h-n-2}\alpha \wedge *^U \imath_{\frac{d}{dt}} dt \wedge \tag{7.4}$$

instead of 7.1. Here α is the pull-back via Id^{-1} of the canonical form on T^*M, and has homogeneity 1. $\square$

Remark 7.1.4 *From (7.2), we see that $\imath_{\frac{d}{dt}} dt \wedge F_z$ shifts the t-degree by $h - n$ on h-chains.*

Proposition 7.1.5 *Let A_z be a holomorphic family of chains in $F_iC_h(\mathcal{V}^\psi \otimes \mathcal{L})$. Then $F_z(A_z)$ is meromorphic, with at most simple poles at the real integers.*

Proof: Denote by $\mathcal{H}(\mathcal{B})$ the holomorphic functions in the band $\mathcal{B} = \{-1 < Re(z) < 1\}$. We shall show that $F_z(A_z)$ has at most a simple pole at 0 in the band $\mathcal{B}$. Choose a *total symbol* map for adiabatic operators, i.e. a map $q : \Psi_a(X) \to S({}^aT^*X)[t^{-1}]$, which extends the symbol map. Let A^j_z be the component of order $z+j$ of $q(\widehat{Q^z a_0}(z) \otimes \hat{a}_1(z) \ldots \otimes \hat{a}_h(z))$. By Proposition 7.1.2, $F_z((\mathcal{V}^\psi)^{h-m-n-1} \otimes \mathcal{L}) \in \mathcal{H}(\mathcal{B})$. Therefore only a finite number of A^j_z's are significant for the poles of $F_z(A_z)$ in $\mathcal{B}$. Fix a metric $|r| = |(\xi, \eta)|$ on ${}^aT^*X$. Choose B^j_z such that $q(B^j_z) = A^j_z$. Let ψ be a cut-off function, $\psi(r) \equiv 0$ for small r, $\psi(r) \equiv 1$ for $r \geq 1$.

Lemma 7.1.6

$$\int_{{}^aT^*M_{|t=0}/M} *_a^M Tr_V(K(B_z^j)) \equiv \int_{{}^aT^*X_{|t=0}/M} *_a^X \psi(\xi,\eta) HKR(A_z^j) \pmod{\mathcal{H}(\mathbb{C})}.$$

Proof: Both terms are local, so we can prove the statement in local coordinates. The left-hand side is independent of the connection. We claim that the part in $*_a^X HKR(A_z^j)$ which is a multiple of the fiber volume form dV of ${}^aT^*X_{|t=0}/M$, is also independent of ∇. Indeed,

$$*_a^X = (1 + t^{-1}dt \wedge \imath_{\mathcal{M}})\phi_a^{*-1} * \phi_a^*,$$

where ϕ_a is given by (2.2). Let μ denote a monomial form in local coordinates. First, if μ does not contain a multiple of dV, then neither does $(1 + t^{-1}dt \wedge \imath_{\mathcal{M}})\phi_a^{*-1}\mu$. Secondly, if μ contains any $dx, d\eta, dy$, then $*\mu$ does not contain multiples of dV [1]. Finally, if μ contains some $dx, d\eta, dy$, then so do all monomials in $\phi_a^*\mu$. Therefore, only those μ that contain only $d\xi$'s survive in $\int_{{}^aT^*X_{|t=0}/M} \psi(\xi,\eta) *_a^X \mu$. Moreover, the result is seen to be independent of the map A in (2.2), which can therefore be assumed of the form $A(x,y,\xi,\eta) = \xi$. Then

$$\begin{aligned} &Tr_V(K(B_z^j)) \\ &= \int_{{}^aT^*X_{|t=0}/{}^aT^*M_{|t=0}} \psi(\xi,\eta) HKR(A_z^j)\omega_{X/M}^m \pmod{\mathcal{H}(\mathbb{C}, \Lambda_{\mathcal{S}}({}^aT^*M)))}, \end{aligned}$$

whence the lemma. $\square$

Lemma 7.1.7 *Let $\mu_j(z)$ be an entire form of pure homogeneity j on ${}^aT^*X \setminus 0$. Then $\int_{{}^aT^*X_{|t=0}/M} \psi(r) r^z \mu_j(z)$ has only one simple pole at $z = -j$, and*

$$Res_{z=-j} \int_{{}^aT^*X_{|t=0}/M} \psi(r) r^z \mu_j(z) = -\int_{{}^aS^*X_{|t=0}/M} \imath_{\mathcal{R}} \mu_j(0).$$

Proof: Let $\mathcal{R}$ be the radial vector field on ${}^aT^*X$. In polar coordinates,

$$\int_{{}^aT^*X_{|t=0}/M} \psi(r) r^z \mu_j(z) = \int_0^\infty \psi(r) r^{z+j-1} dr \int_{{}^aS^*X_{|t=0}/M} \imath_{\mathcal{R}} \mu_j(z).$$

The first factor equals $-1/(z+j) \pmod{\mathcal{H}(\mathbb{C})}$. The second one is entire. $\square$

From Lemma 7.1.6, $F_z(A)$ has the same poles in $\mathcal{B}$ as a finite sum

$$\sum_j \int_{{}^aT^*X_{|t=0}/M} \psi(r) r^z \mu_j(z),$$

where $\mu_j(z)$ is entire in z and of homogeneity j in the fibers of ${}^aT^*X_{|t=0}$. By Lemma 7.1.7, the proposition follows. $\square$

Definition 7.1.8 *For $A \in C_h(\mathcal{V}^\psi \otimes \mathcal{L})$, define*

$$R(A) = Res_{z=0}(F_z(A)).$$

Definition 7.1.9 *Let $A = A_0 \otimes \ldots \otimes A_k \in C_k(\mathcal{V}^\psi \otimes \mathcal{L})$. Define*

$$e_{Q^z}(A) = (-1)^n z Q^z \left(\frac{Q^{-z} A^k Q^z - A_k}{z} \right) A_0 \otimes A_1 \otimes \ldots \otimes A_{k-1}.$$

If $A \in C_k(\Psi_a(X))$, define $e_{Q^z}(A)$ by the same expression, where now all products are in the adiabatic algebra.

Lemma 7.1.10 *$Q^z b = bQ^z + zQ^z e_{D_z}$ on $C_\bullet(\mathcal{V}^\psi \otimes \mathcal{L})$. This identity also holds on $C_\bullet(\Psi_a(X))$, with respect to adiabatic multiplication.*

Proof: Direct computation on chains $A = A_0 \otimes \ldots \otimes A_k$. □

Proposition 7.1.11 *$R \circ b = 0$ on $C_k(\mathcal{V}^\psi \otimes \mathcal{L})$.*

Proof: Using Lemma 7.1.10,

$$\begin{aligned} &R(b(A)) \\ &= Res_{z=0} \int_{{}^aT^*M_{|t=0}/M} *_a^M Tr_V(K(b(Q^z A))) + Res_{z=0}(zF_z(e_{Q^z}(A))). \end{aligned} \tag{7.5}$$

Note that $e_{Q^z}(A)$ is holomorphic, including at $z = 0$. By Proposition 7.1.5, the second term vanishes. By Proposition 5.3.5, the first integrand vanishes for small $Re(z)$, so it extends to be identically zero by analytic continuation. □

It follows that R induces a map

$$R : HH_k(\mathcal{V}^\psi \otimes \mathcal{L}) \longrightarrow \Lambda(M, \Omega) \otimes \Lambda\mathcal{L}. \tag{7.6}$$

We claim that on this space, $R \circ e_D = \imath_{t\frac{d}{dt}} R$. Indeed, $\mathcal{V}^\psi \otimes \mathcal{L}$ is a module over $\mathcal{L} \otimes Polyn({}^aT^*M)$, and hence $HH_k(\mathcal{V}^\psi \otimes \mathcal{L})$ is a module over $HH_k(\mathcal{L} \otimes Polyn({}^aT^*M))$, i.e. over the ring of differential forms with polynomial coefficients. Then, by exactly the same reasoning as in Chapter 3, we can prove that formula (3.8) holds with HKR replaced by K. Note that the claim is false at chain level.

This observation shows that the map (7.6) takes $im(\alpha), im(\beta)$ onto forms with no dt, respectively multiples of dt.

Proposition 7.1.12 *Let $A \in F_{h-n-m} HH_h(\mathcal{V}^\psi \otimes \mathcal{L}) \cap im(\alpha)$. Then*

$$R(A) = \imath_{\frac{d}{dt}} dt \wedge R(A) = - \int_{{}^aS^*X_{|t=0}/M} \imath_{\mathcal{R}} *_a^X HKR(\sigma(A)).$$

Let $A \in F_{h-n-m-1} C_h(\mathcal{V}^\psi \otimes \mathcal{L})$. Then $R(A)$ is a multiple of dt and

$$R(A) = - \int_{{}^aS^*X_{|t=0}/M} \imath_{\mathcal{R}} *_a^X HKR(\sigma(A)).$$

Proof: We know that in the first case, $R(A)$ does not contain dt. Apply Proposition 7.1.2 with $i = h - n - m - 1$, respectively $i = h - n - m - 2$. It follows that $\imath_{\frac{d}{dt}} dt \wedge F_z$, respectively F_z, is holomorphic for such chains around $z = 0$, which implies that $\imath_{\frac{d}{dt}} dt \wedge R(A)$, respectively $R(A)$ depend only on $\sigma(A)$. The result follows from Lemmas 7.1.6 and 7.1.7. □

Remark 7.1.13 *This Proposition implies that we can improve Remark 7.1.4 as follows: The map (7.6) shifts t-degree by $h - n$ when applied to $HH_h(\mathcal{V}^\psi \otimes \mathcal{L}) \cap im(\alpha)$, and by $h - n - 1$ when applied to $HH_h(\mathcal{V}^\psi \otimes \mathcal{L}) \cap im(\beta)$.*

From Proposition 7.1.2, it follows that R vanishes on $\mathcal{V} \otimes \mathcal{L}$, so we think about it as being defined on symbols, with a natural extension to vertical operators. Using the projection $T_j S_a(X) \to \mathcal{V}^\psi$, we can extend the definition of R to $C_\bullet(S_a(X))$.

Proposition 7.1.14 *Let $A \in F_{h-n-m}C_h(S_a(X))$ or $F_{h-n-m-1}C_h(S_a(X))$ represent a class $[A]$ in $HH_h(S_a(X))$. Then $R(A)$ is deRham closed in M. If A is Hochschild exact, then $R(A)$ is deRham exact.*

Proof: We have seen in Theorem 3.1.10 that the differential form

$$[A]_{E_2} = *_a^X HKR(\sigma(A))$$

is closed and homogeneous of homogeneity 0. Hence $\mathcal{L}_\mathcal{R}[A]_{E_2} = 0$. Using (3.7), this implies $(\imath_\mathcal{R} d_v + d_v \imath_\mathcal{R})[A]_{E_2} = 0$, hence $\imath_\mathcal{R}[A]_{E_2}$ is closed. Proposition 7.1.12 shows that

$$d_v R(a) = d_v \int_{{}^aS^*X_{|t=0}/M} \imath_\mathcal{R}[A]_{E_2} = \int_{{}^aS^*X_{|t=0}/M} d_v \imath_\mathcal{R}[A]_{E_2} = 0.$$

If $[A] = 0$, then $[A]_{E_2} = d_v\beta$ is exact, and, since d_v preserves homogeneity, we can assume that β is also homogeneous of homogeneity 0. Therefore $\mathcal{L}_\mathcal{R}\beta = 0$. This implies $\imath_\mathcal{R}[A]_{E_2} = \imath_\mathcal{R} d_v \beta = -d_v \imath_\mathcal{R}\beta$, and so

$$R(a) = \int_{{}^aS^*X_{|t=0}/M} \imath_\mathcal{R}[A]_{E_2} = -d_v \int_{{}^aS^*X_{|t=0}/M} \imath_\mathcal{R}\beta.$$

□

7.2 The boundary map

Recall (2.5) the definition $D_Q = \frac{d}{dt}(D_z)$. Let b_0 denote the Hochschild differential in $C_\bullet(\mathcal{V}^\psi \otimes \mathcal{L})$. We would like to give a formula for the boundary map δ of the short exact sequence (6.1) in terms of the presentations found in Chapters 3 and 4 for Hochschild homology as cohomology groups. As with all spectral sequences, we can in principle observe only the top part, say $\delta_0 : T_i/T_{i+1}(HH(S_a(X))) \to T_i/T_{i+1}(HH(I_a(X)))$, of δ. This might be zero on some element even though δ itself does not vanish on that element.

Remark 7.2.1 *The boundary map commutes with e_D, L_D, $t^{-1}dt\wedge$, α and β, since these operations are maps of complexes on $C_\bullet(\Psi_a(X), b)$.*

This means that δ preserves the dt and no-dt parts and the t-homogeneity in the presentation by cohomology spaces.

Proposition 7.2.2 *If $\dim M \neq 0$, then $\delta_0 = 0$.*

Proof: If $M = \{pt\}$, then the adiabatic algebra becomes isomorphic to the algebra $\Psi_{[0,\infty)}(X)$ of one-parameter families of pseudo-differential operators. In that case, $\delta = \delta_0$ since t is a "flat parameter". If $\dim M \neq 0$, we claim that the vertical boundary map vanishes. This is essentially Lemma 8 from [12]. For convenience, we reproduce the proof. Let $A \in C_k(\mathcal{V}^\psi(X)/\mathcal{V}(X) \otimes \mathcal{L})$ be a cycle. Let $\tilde{A} \in C_k(\mathcal{V}^\psi(X) \otimes \mathcal{L})$ be an extension to the full vertical algebra. Then, using Proposition 5.4.9,

$$\delta_0(A) = \int_{{}^aT^*M_{|t=0}/M} *_a^M K(b_0\tilde{A}).$$

From Corollary 5.4.8, this is independent of ∇. The vector fields tangent to the fibers of ${}^aT^*M_{|t=0} \to M$ are outer derivations on the algebra $\mathcal{V}^\psi(X) \otimes \mathcal{L}$. By naturality, the boundary map commutes with the action L_V of such a vector field: $[\delta_0, L_V] = 0$. But $L_V(A)$ has order 1 less than A. After repeated applications, the order of $L_{V_1} \dots L_{V_s}\tilde{A}$ will be less than $-m-k-1$, so, by Proposition 5.3.5, $K(b(L_{V_1} \dots L_{V_s}\tilde{A})) = 0$. Hence, $L_{V_1} \dots L_{V_s}\delta_0(A) = 0$. Since $V_1, \dots, V_s$ were arbitrary and $\delta_0(A)$ is a Schwartz form on ${}^aT^*M_{|t=0}$, the proposition follows. □

Therefore, the first significant part of δ is

$$\delta_1 : T_i/T_{i+1}(HH(S_a(X))) \to T_{i+1}/T_{i+2}(HH(I_a(X))),$$

which increases the t-filtration by 1.

Theorem 7.2.3 $\delta_1 = (-2)\times$*integration along the fibers of* ${}^aS^*X_{|t=0} \to M$.

Proof: From Remark 7.2.1, it is enough to prove the claim for class $[a]$ represented by some $a \in T_jHH_k(S_a(X)) \cap im(\alpha)$. Such a class can be represented by a chain $A \in T_jF_{k-n-m}C_k(S_a(X))$. By H-unitality, we can find $\tilde{A} \in T_jC_k(\Psi_a(X))$ a pre-image of A, such that $b\tilde{A} \in C_{k-1}(I_a)$. Since $\delta_0(a) = 0$, it follows that there exists $\gamma \in T_jC_k(I_a)$ such that $b\tilde{A} - b\gamma \in T_{j+1}$. By replacing $\tilde{A}$ with $\tilde{A} - \gamma$, we can assume that $\delta(a) = [b\tilde{A}]$ is represented by a chain in T_{j+1}. Let A^j, A^{j+1} be the components of $\tilde{A}$ in $C_k(\mathcal{V}^\psi \otimes \mathcal{L})_j$, respectively $C_k(\mathcal{V}^\psi \otimes \mathcal{L})_{j+1}$, where the subscript denotes t-degree. Using Lemma 7.1.10, we see that

$$\begin{aligned} \delta_1(a) &= \int_{{}^aT^*M_{|t=0}/M} *_a^M K(b\tilde{A})_{j+1} = \left(\int_{{}^aT^*M_{|t=0}/M} *_a^M K(Q^z b\tilde{A})_{j+1}\right)_{|z=0} \\ &= \left(\int_{{}^aT^*M_{|t=0}/M} *_a^M K(bQ^z\tilde{A})_{j+1}\right)_{|z=0} \end{aligned} \tag{7.7}$$

$$+\left(\int_{{}^aT^*M_{|t=0}/M} *_a^M K(zQ^z e_{D_z}\tilde{A})_{j+1}\right)_{|z=0}. \quad (7.8)$$

We shall prove that (7.7) vanishes and (7.8) equals the desired expression. In (7.7), we can replace $\tilde{A}$ by $A^j + A^{j+1}$ since we are looking only at the part of degree $j+1$ in t. Let $b = b_0 + b_1 + \ldots$ be the expansion of b according to the t-degree. By Proposition 5.3.5, $K(b_0 Q^z A^{j+1})$ vanishes for small $Re(z)$, so the analytic continuation of the corresponding part in (7.7) is 0 at $z = 0$.

Proposition 7.2.4 *$b_0(A^j) = 0$ implies that $\int_{{}^aT^*M_{|t=0}/M} *_a^M K(b_1 Q^z A^j)$ tends to 0 in $\Lambda^{2n-k+1}(M)/d\Lambda^{2n-k}(M)$ as $z \to 0$.*

Proof: This would be clear if we knew that $b_0(Q^z A^j) = 0$. Indeed, $K(b_1 Q^z A^j)$ would be equal in this case to $k^{-1} d_v(K(Q^z A^j))$, when both sides are defined. This happens for $Re(z) < 0$ (compare with Proposition 4.0.10). The explanation for the multiplicative factor k is that our analogue to *HKR* is anti-symmetrized. In the no-fiber case, this condition holds by commutativity of the vertical algebra $\mathcal{V}^\psi(M)$ (Definition 2.2.9).

Lemma 7.2.5 *If A^j is b_0-exact, $A^j = b_0 C$, where $C \in T_j F_{k-n-m} C_k(\Psi_a(X))$, then $\int_{{}^aT^*M_{|t=0}/M} *_a^M K(b_1 Q^z A^j)_{j+1}$ vanishes at $z = 0$.*

Proof: $K(b_1 Q^z A^j)_{j+1} = K(b_1 Q^z b_0 C)_{j+1} = K(b_1 b_0 Q^z C + b_1 z Q^z e_{D_z} C)$. The last term belongs to $F_{k-m-n-2+z} C_{h-1}$ and is a multiple of z, hence like in Proposition 7.1.2, its integral vanishes at $z = 0$. From $b^2 = 0$, we deduce $b_1 b_0 = -b_0 b_1$. Since $b_1 Q^z C \in T_{j+1}$, it follows from Proposition 5.3.5 that $\int_{{}^aT^*M_{|t=0}/M} *_a^M K(b_0 b_1 Q^z C)_{j+1} = 0$ for $Re(z)$ small. By analytic continuation, this holds for all z. □

Write $A^j = A_0 \otimes \ldots \otimes A_k$. Since $b_0(A^j) = 0$, we can assume that $\forall i$, $A_0 \otimes \ldots \otimes A_i A_{i+1} \otimes \ldots \otimes A_k = 0$ (see Lemma 2.4.3). The difference is an exact chain like in Lemma 7.2.5, so it can be ignored. We claim that

$$\int_{{}^aT^*M_{|t=0}/M} *_a^M K(b_1 Q^z A^j) - k d_v \int_{{}^aT^*M_{|t=0}/M} *_a^M (K(Q^z A^j)) = o(z). \quad (7.9)$$

Since both terms are local in the base and independent of ∇ by the same argument as Lemma 7.1.3, it is enough to prove the claim in local coordinates for the trivial connection. We have

$$\begin{aligned} & d_v \int_{{}^aT^*M_{|t=0}/M} *_\phi^M (K(Q^z A^j)) \\ = & \sum \int_{{}^aT^*M_{|t=0}/M} \epsilon(I) Tr_V(\partial_{x_i}(Q^z A_0 \partial_{\xi_{i_1}} A_1 \ldots \partial_{\xi_i} A_s \ldots \partial_{\xi_{i_{k-1}}} A_k)) \\ & d\xi_1 \ldots d\xi_n d\hat{x}_{i_1} \ldots d\hat{x}_{i_k}. \end{aligned} \quad (7.10)$$

The essential ingredient in the proof of (7.9) is the following

Lemma 7.2.6 *Assume that for* $i = 0, \ldots, k$, $A_0 \otimes \ldots \otimes A_i A_{i+1} \otimes \ldots \otimes A_k = 0$ *as above, where the multiplication is in the vertical sense. Then for* $Re(z) < 0$,

$$\int_{^aT^*M_{|t=0}/M} Tr_V(Q^z A_0 \partial_{\xi_{i_1}} A_1 \ldots \partial_{\xi_{i_k}} A_k) \tag{7.11}$$
$$= \; -\int_{^aT^*M_{|t=0}/M} Tr_V(\partial_{\xi_{i_k}}(Q^z A_0)\partial_{\xi_{i_1}} A_1 \ldots \partial_{\xi_{i_{k-1}}} A_{k-1} A_k) + o(z).$$

Proof: In writing the integral, we drop the density factor $d\xi_1 \ldots d\xi_n$. The hypothesis implies that $A_k Q^z A_0 \otimes A_1 \otimes \ldots \otimes A_{k-1} = [A_k, Q^z] A_0 \otimes A_1 \otimes \ldots \otimes A_{k-1}$, has order $k - n - m + z - 1$. This implies that

$$\int_{^aT^*M_{|t=0}/M} Tr_V(\partial_{\xi_{i_k}}(A_k Q^z A_0)\partial_{\xi_{i_1}} A_1 \ldots \partial_{\xi_{i_{k-1}}} A_{k-1})$$

is well-defined and holomorphic for $Re(z) < 1$. In particular, it is holomorphic around $z = 0$, and since $A_k A_0 \otimes \ldots \otimes A_{k-1} = 0$, it vanishes at $z = 0$, hence it is a multiple of z. Expand $\partial_{\xi_{i_k}}(A_k Q^z A_0)$ and commute A_k and $\partial_{\xi_{i_k}} A_k$ to the right inside the trace (possible because the total order is $z - n - m - 1 < -m$). This finishes the proof. $\Box$

Using this lemma, the proof proceeds like the proof of the fact that $d_1 = k d_v$, stated in the beginning of the proof of Proposition 7.2.4. Note that in (7.11) we can shift $\partial_{\xi_{i_{k-1}}}$ one step to the right, by the same argument as in Lemma 7.2.6, with no extra error. By repeated applications of these operations, the left-hand side of (7.10) is equal to

$$k \sum \int_{^aT^*M_{|t=0}/M} \epsilon(I) Tr_V \left(\partial_{x_i}(Q^z A_0 \partial_{\xi_{i_1}} A_1 \ldots \partial_{\xi_{i_{k-1}}} A_{k-1} \partial_{\xi_i} A_k)\right) + o(z).$$

We will show that this equals

$$k d_v \int_{^aT^*M_{|t=0}/M} *^M_\phi (K(Q^z A^j)) \pmod{o(z)}.$$

First, drop the factor of k. We distinguish three types of terms: i) terms where ∂_{x_i} differentiates $Q^z A_0$; ii) terms where ∂_{x_i} differentiates one of the middle factors, and iii) terms containing $\partial_{x_i}\partial_{\xi_i} A_k$. The terms of type i) contribute to the Poisson bracket $\{A_k, A_0\}$. By integration by parts with respect to ξ_i , type iii) terms give us the remaining part of $\{A_k, A_0\}$ and terms of the type $\ldots \partial_{\xi_i}\partial_{\xi_{i_s}} A_s \ldots \partial_{x_i} A_k$, that we declare as being of type iv). We claim that, modulo $o(z)$, the terms of type ii) make up

$$\sum_{s<p} (-1)^{s+p} \ldots (\partial_{x_i} A_s \partial_{\xi_{i_s}} A_{s+1} + \partial_{\xi_{i_s}} A_s \partial_{x_i} A_{s+1}) \ldots \partial_{\xi_i}\partial_{\xi_{i_{p-1}}} A_p \ldots .$$

Summing over s, we get half of $\ldots d_v\{A_p, A_{p+1}\} \ldots$. The other half arises from the type iv) terms. Let us illustrate this on a type ii) term: let $Y_i(s)$ be the type ii) term with the derivative ∂_{x_i} on position s. If $s = k-1$, then we get part of $\ldots d\{A_{k-1}, A_k\}$.

Otherwise,

$$\begin{aligned} Y_i(s) &= \int_{{}^aT^*M_{|t=0}/M} Tr_V\left(Q^z A_0 \partial_{\xi_{i_1}} A_1 \dots \partial_{x_i}\partial_{\xi_{i_s}} A_s \dots \partial_{\xi_i} A_k\right) \\ &= \int_{{}^aT^*M_{|t=0}/M} Tr_V(Q^z A_0 \partial_{\xi_{i_1}} A_1 \dots \partial_{x_i}\partial_{\xi_{i_s}} A_s \dots \\ &\quad (-A_{k-1}\partial_{\xi_{k-1}}\partial_{\xi_i} A_k - \partial_{\xi_{k-1}}\partial_{\xi_i} A_{k-1} A_k - \partial_{\xi_i} A_{k-1} \partial_{\xi_{k-1}} A_k)). \end{aligned}$$

Like in the lemma, the middle term is of order $o(z)$. The last term brings ∂_{ξ_i} one step to the left, and we either get part of $\dots d\{A_{k-2}, A_{k-1}\}$, or we continue the process. As for the first term, summing it over s at step p will yield

$$\int_{{}^aT^*M_{|t=0}/M} Tr_V(Q^z A_0 \partial_{x_i} A_1 \dots \partial_{x_i}\partial_{\xi_{i_s}} A_s \dots).$$

By integration by parts, this is equal to a part of $\int_{{}^aF\to M} Tr_V(\{Q^z A_0, A_1\}\dots)$ plus $\int_{{}^aF\to M} Tr_V(Q^z A_0 \partial x_i \partial \xi_i A_1 \dots)$. This last term cancels with the similar one from the type iv) terms.

Remark 7.2.7 *Formula (7.9) is certainly correct if we can commute Q^z modulo T_j, as explained in the beginning of the proof. All we have to do is to make sure that non-commutativity only affects the result by a factor of order $o(z)$.*

This finishes the proof of the proposition. □

Note that the second term in (7.7) is by definition $R_{j+1}(e_{D_Q}\tilde{A})$. From Remark 7.1.13, this is a dt-free form of t-degree $j+k-n$. From Propositions 3.2.6 and 7.1.12,

$$R(e_{D_Q}\tilde{A}) = -2\int_{{}^aS^*X_{|t=0}/M} \imath_{\mathcal{R}}(r^{-1}dr \wedge [a]_{E_2}) = -2\int_{{}^aS^*X_{|t=0}/M} [a]_{E_2}.$$

We have assumed that $a \in T_j$. We can also ask that $[a]_{E_2}$ be of pure homogeneity with respect to L_D. This homogeneity must be at least $j+k-n$, and if it is bigger, then $[a]_{E_2}$ is the class of some $a' \in T_{j+1}$. So we can also assume that $[a]_{E_2}$ has homogeneity $j+k-n$. Using again Remark 7.1.13, we conclude that $R_{j+1}(e_{D_Q}\tilde{A}) = R(e_{D_Q}\tilde{A}) = R(e_{d_Q}a)$. This ends the proof of Theorem 7.2.3. □

Recall the decomposition of $T_i/T_\infty HH_k(S_a(X))$ as eigenspaces of $L_{\frac{d}{dt}}$. Let a be of pure type with respect to this decomposition. Assume a is dt-free. Since $*_\phi$ increases the t degree by at most $k-n$, it follows that $[a] \in t^{i+h-n}H^{2n+2m-k}(S^*X \times S^1)$ with $h \leq k$ Hence $L_{\frac{d}{dt}}$ acts on $[a]$ by multiplication with $i+h-n$. Assume that $\delta_1(a) = 0$. This means that $\delta(a) \in T_{i+s}HH_{k-1}(I_a)$ with $s \geq 2$. Assume s is minimal with this property. Let $\delta(a)_s$ be the image of $\delta(a)$ in $T_{i+s}/T_{i+s+1}HH_{k-1}(I_a) \cong E_2^{i+s,k-1-i-s}(I_a)$. $L_{\frac{d}{dt}}$ acts with eigenvalue $i+s+k-1-n$ on this space. But from $h \leq k$ and $s \geq 2$ we get $i+h-n < i+s+k-1-n$. Since δ and $L_{\frac{d}{dt}}$ commute, we get the following:

Proposition 7.2.8 *δ_1 is the only significant part of δ. In other words, if $\delta_1(a) = 0$ for some $a \in T_i HH_k(S_a(X))$, then there exists some $\tilde{a}$ with $a - \tilde{a} \in T_{i+1}HH_k(S_a(X))$ and $\delta(\tilde{a}) = 0$.*

□

Theorem 7.2.3 and Proposition 7.2.8 completely characterize $\delta : HH(S_a(X)) \to HH(I_a(X))$.

7.3 Traces on the adiabatic algebras

From the long exact sequence of the sequence (6.3), the ideals $I_a(X)$ and $\Psi_a^{-\infty}(X)$ have the same homology, except in dimensions 0 and 1. Consider the following map:

$$Tr : \Psi_a^{-\infty}(X) \to C^\infty[0,\infty)[t^{-1}], \quad A \longmapsto (\tau \mapsto \int_{\Delta_a \cap \{t=\tau\}} A_{|\Delta_a \cap \{t=\tau\}}).$$

Lemma 7.3.1 *The map Tr generates $HH^0(\Psi_a^{-\infty}(X))$ over $C^\infty[0,\infty)[t^{-1}]$.*

Proof: Let first $A \in \dot{C}^\infty([0,\infty), \Psi^{-\infty}(X)) \cong \dot{\Psi}_a^{-\infty}(X)$ be a rapidly vanishing family of smoothing operators. For any $t > 0$, the map $A \mapsto Tr(A)(t)$ is a trace. Choose any local embedding of TX in X^2. Cut-off A near the diagonal and pull it back to TX as a compactly supported section in the fiber density bundle. The pull-back of the fiber density bundle to the 0-section of TX is just the density bundle $\Omega(TX)$. In local coordinates, and pulling back via the canonical map ϕ_a, we get

$$Tr(A) = \int_{T^*X} \hat{A}\omega^n = \int_{{}^aT^*X} \hat{A}\omega_a^n,$$

where ω, ω_a are the symplectic form, respectively the adiabatic symplectic form. This formula extends to $\Psi_a^{-\infty}(X)$, and is an extension of the map

$$I_a(X) \ni A \longmapsto \int_{{}^aT^*M_{|t=0}} Tr_v(A)\omega_a^n = \int_{{}^aT^*M_{|t=0}} *_a K(A) \in \mathcal{L},$$

which, by Theorem 5.4.10 and Propositions 5.4.9 and 4.0.13, is a generator over $\mathcal{L}$ of $HH^0(I_a(X))$. This implies that the boundary map $HH_1(I_a(X)) \to HH_0(\dot{\Psi}_a^{-\infty}(X))$ vanishes. □

We note here that for $t > 0$, $\omega_a = (\phi_a^{-1})^*(\omega)$. In local coordinates, this is a polynomial in t^{-1} of degree 1. It follows that ω_a^n belongs to $T_{-n}\Lambda^2({}^aT^*X)$.

Recall now that e_D is an isomorphism from $im(\beta) \subset HH_1(I_a(X))$ to $HH_0(I_a(X))$. A similar statement is valid for $\dot{\Psi}_a^{-\infty}(X)$. From this, it follows that $Tr \circ e_D$ is a Hochschild cochain which generates $im(\beta) \subset HH_1(\Psi_a^{-\infty}(X))$, and that the other boundary map of (6.3) also vanishes.

Since we have the explicit $C^\infty[0,\infty)[t^{-1}]$-valued cochain Tr on $\Psi_a^{-\infty}(X)$, we can imitate the construction of the residue trace to get explicit cocycles on $\Psi_a(X)$.

Proposition 7.3.2 *The maps*

$$\begin{array}{rl} \Psi_a(X) \to C^\infty[0,\infty)[t^{-1}], & A \mapsto Res_{z=0}Tr(Q^z A) \\ C_1(\Psi_a(X)) \to C^\infty[0,\infty)[t^{-1}], & A \otimes B \mapsto Res_{z=0}Tr(Q^z D(B)A) \end{array}$$

are cocycles on $\Psi_a(X)$.

Proof: Since e_D is a map on homology, it is enough to prove the statement for the first map. We have seen that the residue is well-defined. We want to show that it vanishes on commutators. From (1.1),

$$\begin{aligned} Res_{z=0}Tr(Q^z[A,B]) &= Res_{z=0}Tr([Q^z A, B] + zQ^z \frac{Q^{-z}BQ^z - B}{z} A) \\ &= 0, \end{aligned}$$

since for small $Re(z)$, the first term is the trace of a commutator of operators of order less than $-n-m$, and the second term has no residue at zero since we have a multiple of z inside the residue. □

7.4 Properties of the residue functional

We can extend Proposition 7.1.14 to arbitrary orders. We first treat the semi-classical limit case. Recall that the subscript i means taking the part in T_i/T_{i+1}.

Proposition 7.4.1 *If $A \in T_i C_k(S_a(M))$ is a boundary, then $R_i(A)$ is d_v-exact.*

Proof: The spectral sequence ${}^T E_2$, which computes $HH(S_a(M))$ using the T_i filtration, degenerates at E_2, so we can assume $A = b(P)$ with $P \in T_{i-1}C_{k+1}(S_a(M))$. What makes the no-fiber case special is that $(Q^z P)_{i-1}$ is b-closed in the vertical algebra, because that algebra is now commutative. The projection $\Psi_a \to S_a$ is surjective; let $\tilde{P}$ be a pre-image of P. Using Lemma 7.1.10, we get:

$$\begin{aligned} R_i(A) &= R_i(bP) = Res_{z=0} \int_{{}^aT^*M_{|t=0}/M} *_a^M (K(Q^z b\tilde{P}))_i \\ &= Res_{z=0} \int_{{}^aT^*M_{|t=0}/M} *_a^M (K(b(Q^z \tilde{P}))_i \\ &\quad + Res_{z=0} \int_{{}^aT^*M_{|t=0}/M} *_a^M (K(zQ^z e_{Q^z} \tilde{P}))_i. \end{aligned}$$

By Proposition 7.1.5 and because of the z factor, the last term vanishes since $e_{Q^z}\tilde{P} \in T_i$ is entire. As for the first term, we claim it is exact.

Lemma 7.4.2 *For $Re(z)$ sufficiently small, we have*

$$\int_{{}^aT^*M_{|t=0}/M} *_a^M K(b(Q^z P))_i = d_v \left(\int_{{}^aT^*M_{|t=0}/M} *_a^M (K(Q^z P))_{i-1} \right). \tag{7.12}$$

Proof: We notice that Q^zP survives at ${}^TE_1^{i,k-i}(\Psi_a^{z+Z}(M))$ Then (7.12) is the computation of $d_1[Q^zP]$ in the spectral sequence of the symbol algebra of order $z+\mathbb{Z}$. The identity $d_1 = *_a^{-1}d_v*_a$ (see (4.3)) is purely formal, hence it holds in any generality. The proof of Proposition 7.2.4 applies to prove this identity. $\square$

By the deRham theorem, the space of exact forms is closed. A meromorphic family of forms, which takes exact values for z in an open set, must have exact residues. This finishes the proof of Proposition 7.4.1. $\square$

From Theorem 3.1.10 and Proposition 7.1.14, it follows that every Hochschild class in $T_iHH(S_a(X))$ has a representative a such that $R_i(a)$ is closed. Together with Proposition 7.4.1, this proves:

Corollary 7.4.3 *R_i descends as a map*

$$R : T_iHH_k(S_a(M)) \longrightarrow H^{n-k+j}(M) \otimes \Lambda^j\mathcal{L}.$$

$\square$

Proposition 7.4.1 is a model for the general case. The main difference is that multiplication by Q^z destroys the cycle property.

Proposition 7.4.4 *Let $A \in T_iC_k(\Psi_a(X))$. Assume that $bA \in T_{i+1}C_{k-1}(\Psi_a(X))$. Then $R_{i+1}(bA) = dR_i(A)$, where d is the deRham differential on M.*

Proof: Q^zA_i is not a cycle in the vertical algebra, but, since $b(A_i) = 0$, it has the property that $b_0(Q^zA_i) = o(z)$. The proof of Proposition 7.2.4 goes through, with the following modification: replace congruences $\pmod{o(z)}$ by equalities of residues at $z = 0$. $\square$

It follows that if $A \in T_iC_k(\Psi_a(X))$ is Hochschild closed, then $R_i(A)$ is deRham closed. Let now $A \in T_iF_jC_k(S_a(X)) \cap im(\alpha)$. Assume that A is b-exact and that $j > k - n - m$. Then, there exists some $P \in T_{i-1}C_{k+1}(S_a(X))$ such that $bP - A \in T_iF_{j-1}C_k(S_a(X))$. This is a restatement of Proposition 3.3.7. By Proposition 7.4.4, $R_i(A) = R_i(A - bP)$ modulo exact forms. Inductively, $R_i(A) = R_i(A')$, where $A' \in T_iF_{k-n-m}C_k(\Psi_a(X))$. From Proposition 7.1.14, $R_i(A')$ is exact. As a result of these statements, we get the following

Theorem 7.4.5 *R_i descends as a map on the Hochschild homology of adiabatic symbols with values in the cohomology of M (with twisted coefficients):*

$$T_iHH_k(S_a(X)) \xrightarrow{R_i} H^{n-k}(M, \Omega M)t^{i+k-n} \oplus H^{n-h+1}(M, \Omega M)t^{i+k-n-2}dt.$$

$\square$

Appendix A

Degeneracy without derivations

Here we prove the degeneracy result 4.0.16 without using the derivation D, while based on the analysis of the E_2 terms computed in Proposition 4.0.13. This approach illustrates an explicit cycle in $C_n(A^\partial(M))$.

By H-unitality, the short exact sequence (4.2) induces a long exact sequence in the ${}^TE_1$ terms of the algebras in (4.1), computed in Proposition 4.0.9. This is easily seen to split in short exact sequences. With the differential d_1, these become short exact sequences of complexes, hence yielding long exact sequences of the ${}^TE_2$ terms computed in Proposition 4.0.13.

In this chapter, we will use the notation $\imath$ and p for all the maps induced by the maps in (4.1) in homology, spectral sequences, etc. Recall the notations ${}^FE$ and ${}^TE$ for the spectral sequences for $(C_\bullet(S_a^\partial(M)), b)$ with respect to the two filtrations, by sum of orders and by sum of powers of t, respectively.

A.1 The E_2 terms

Proposition A.1.1 *The spectral sequence ${}^FE(S_a^\partial(M))$ degenerates at ${}^FE_2$.*

Proof: We shall give a "derivation-free" proof. Consider the canonical algebra projection $S_a(M) \xrightarrow{\tau} S_a^\partial(M)$. By direct computation, τ descends at the ${}^FE_1$ level as the surjection

$$\Lambda^*({}^aT^*M \setminus \{0\})[t^{-1}] \longrightarrow \Lambda^*({}^aT^*M_{|t=0} \setminus \{0\}) \otimes \Lambda^*\mathcal{L}.$$

From Proposition 3.1.9 it follows that the differential d_1^a on $S_a^\partial(M)$ is also conjugate to d_v by the isomorphism $*_a^M$. It follows that

$$ {}^FE_2^{i,j}(S_a^\partial(M)) \cong \begin{cases} H^{n-i}({}^aS^*M_{|t=0} \times S^1) \otimes \mathcal{L} & \text{if } j = n \\ H^{n-i}({}^aS^*M_{|t=0} \times S^1) \otimes \mathcal{L}dt & \text{if } j = n+1 \\ 0 & \text{otherwise.} \end{cases} \tag{A.1}$$

Comparing with Theorem 3.1.10, we see that the map τ induces a surjection at the ${}^FE_2$ level. From A.1, it follows that $d_s = 0$ for $s \geq 3$. Hence, it is enough to prove that d_2 is zero on ${}^FE_2(S_a(M))$. We claim that d_2 is zero for the algebra $\dot{C}^\infty([0,\infty))S_a(M)$. This algebra is isomorphic to the algebra of rapidly vanishing families of symbols on

M. The isomorphism is described in Remark 2.1.8. Since this last algebra is the tensor product of the symbol algebra of M with $\dot{C}^\infty([0,\infty))$, d_2 is indeed zero by the similar statement of [2]. The map d_2 on $S_a(M)$ is $C^\infty([0,\infty))$-linear. Since it is zero for $t>0$, it must vanish for all t. $\square$

The convergence of the spectral sequence ${}^F E(S_a^\partial(M))$ is somewhat tautological. We only need to keep the t degrees uniformly bounded below.

Proposition A.1.2 *The spectral sequence ${}^T E(S_a^\partial(M))$ degenerates at ${}^T E_2$ and is convergent.*

Proof: Note that since $\mathcal{L}$ is central in $S_a^\partial(M)$, it follows that the Hochschild homology $HH(S_a^\partial(M))$ has a natural $HH(\mathcal{L})$-module structure. In particular, it is a $\mathcal{L}$-vector space. Using the convergence of ${}^F E(S_a^\partial(M))$ and formulas (4.6) and (A.1), we make the following dimension count

$$
\begin{aligned}
&\sum_k \dim_{\mathcal{L}}(HH_k(S_a^\partial(M))) = \sum_{i,k} \dim_{\mathcal{L}} {}^F E_2^{i,k-i}(S_a^\partial(M)) \\
&\quad = \sum_k (\dim_{\mathbb{C}} H^{2n-k}({}^a S^* M_{|t=0} \times S^1) + \dim_{\mathbb{C}} H^{2n-k+1}({}^a S^* M_{|t=0} \times S^1)) \\
&\quad = \sum_k \dim_{\mathbb{C}} {}^T E_2^{j,k}(S_a^\partial(M)) \geq \sum_k \dim_{\mathbb{C}} {}^T E_\infty^{i,k}(S_a^\partial(M)) \\
&\quad \geq \sum_k \dim_{\mathbb{C}}(T_j/T_{j+1}(HH_k(S_a^\partial(M))) = \sum_k \dim_{\mathcal{L}}(HH_k(S_a^\partial(M))).
\end{aligned}
$$

Here j can be any fixed integer. This shows that all inequalities must be equalities, which at the same time proves the degeneracy and convergence of ${}^T E(S_a^\partial(M))$. $\square$

A.2 The long exact sequence of E_2 terms

After conjugation by $*_a$, the short exact sequence (4.2) splits in copies of

$$
0 \to (\Lambda^*_{\mathcal{S}}({}^a T^* M_{|t=0}), d) \to (\Lambda^*_{\mathcal{S}^{\mathbb{Z}}}({}^a T^* M_{|t=0}), d) \to (\Lambda^*_{S}({}^a T^* M_{|t=0} \setminus \{0\}), d) \to 0 \quad \text{(A.2)}
$$

according to the t and dt degree, where d is the deRham differential on $\Lambda^*({}^a T^* M_{|t=0})$.

Proposition A.2.1 *If $\chi(M)=0$, then the boundary map δ of the long exact sequence deduced from (A.2) is surjective. If $\chi(M)\neq 0$, then* $\operatorname{coker}(\delta) \cong H^n_{\mathcal{S}}({}^a T^* M_{|t=0}), \ker p \cong H^n(M) \subset H^n_{\mathcal{S}^{\mathbb{Z}}}({}^a T^* M_{|t=0})$*, and the map* $\operatorname{coker}(\delta) \xrightarrow{\imath} \ker p$ *is the isomorphism which takes the Thom class of ${}^a T^* M_{|t=0}$ to the Euler class of M.*

Proof: There exists a canonical isomorphism

$$
T^* M \to {}^a T^* M_{|t=0}, \quad \text{(A.3)}
$$

induced by the bundle map $T^*M \times [0,\infty) \xrightarrow{\phi/t} {}^{\sigma}T^*M$. Since d is invariant by pull-back, this isomorphism induces a canonical isomorphism of complexes between the short exact sequence (A.2) and

$$0 \to (\Lambda^*_{\mathcal{S}}(T^*M), d) \xrightarrow{\imath} (\Lambda^*_{S^{\mathbb{Z}}}(T^*M), d) \xrightarrow{p} (\Lambda^*_S(T^*M \setminus \{0\}), d) \to 0. \tag{A.4}$$

As in Lemma 3.1.11, we notice that $h(\Lambda^*_S(T^*M \setminus \{0\}), d)$ lives only in homogeneity 0. Choose a metric on T^*M and let $r = \|\xi\|$ be the radial function on the fibers. Then, as before,

$$h_k(\Lambda^*_{(0)}(T^*M \setminus \{0\}), d) \cong H^k(S^*M) \oplus H^{k-1}(S^*M) \otimes r^{-1}dr. \tag{A.5}$$

The boundary map is obtained as follows: let ψ be a cut-off function, $\psi(x) \equiv 0$ for $x < 1/3$ and $\psi \equiv 1$ for $x > 2/3$. Let $\alpha \in \Lambda^*_{(0)}(T^*M \setminus \{0\})$ represent the class $[\alpha] \in h_k(\Lambda^*_{(0)}(T^*M \setminus \{0\}), d)$. Then $\psi(r)\alpha \in \Lambda^*_{S^{\mathbb{Z}}}(T^*M)$ maps to α through the surjection p in (A.4), and

$$\delta[\alpha] = [d(\psi(r)\alpha)] = [\psi'(r)dr \wedge \alpha]. \tag{A.6}$$

Hence, using the isomorphism (A.5), δ vanishes on $H^{k-1}(S^*M) \otimes r^{-1}dr$.

Lemma A.2.2 *On $H^k(S^*M)$, δ coincides with the boundary map in the cohomology sequence of the pair (B^*M, S^*M).*

Proof: First, notice that there exists a natural isomorphism

$$H^k_{\mathcal{S}}(T^*M) \cong H^k(B^*M, S^*M).$$

Then note that the boundary map in the cohomology sequence of the pair is given exactly by the same formula as (A.6). □

By dimension reasons, the only map

$$H^k(B^*M, S^*M) \longrightarrow H^k(B^*M) \cong H^k(M)$$

in the cohomology sequence of the pair which can be non-zero is that for $k = n$. In this case, the map takes a generator of $H^n(B^*M, S^*M)$, the Thom class of the bundle, to its restriction to the 0 section, i.e. the Euler class. This vanishes if and only if $\chi(M) = 0$.

Thus, when $\chi(M) \neq 0$, the Thom class generates the cokernel of δ in $H^*_{\mathcal{S}}(T^*M)$. Also in that case, it follows that the image of the fundamental class of M under the pull-back map to $H^n_{\mathcal{S}}(T^*M)$ is non-zero and a generator of $\ker p$. □

Proposition A.2.3 *If $\chi(M) = 0$, then*

$$\dim_{\mathcal{L}} E_2(S^{\partial}_a(M)) = \dim_{\mathcal{L}} E_2(A^{\partial}(M)) + \dim_{\mathcal{L}} E_2(I_a(M)). \tag{A.7}$$

If $\chi(M) \neq 0$*, then*

$$\dim_{\mathcal{L}} E_2(S_a^\partial(M)) \;=\; \dim_{\mathcal{L}} E_2(A^\partial(M)) + \dim_{\mathcal{L}} E_2(I_a(M)) - 4. \qquad \text{(A.8)}$$

Proof: The E_2 terms of the spectral sequences for the Hochschild complexes of the algebras in the sequence (4.1) have a natural structure of $\mathcal{L}$-vector spaces. We have seen that, after conjugation by $*_\phi + t^{-1}dt \wedge \imath_{\mathcal{M}} *_\phi$, the short exact sequence of E_1 terms is the direct sum of two copies of the sequence of complexes (A.2), tensored with $\mathcal{L}$. The result follows from Proposition A.2.1. Note that if $\chi(M) \neq 0$, the difference between (A.7) and (A.8) arises from the four one-dimensional spaces $H^n(M)$, $dt \otimes H^n(M)$, $H^n_{\mathcal{S}}({}^a T^*M_{|t=0})$ and $dt \otimes H^n_{\mathcal{S}}({}^a T^*M_{|t=0})$. □

A.3 The case $\chi(M) = 0$

In this section, assume that the Euler characteristic of the base M vanishes.

Proposition A.3.1

$$\begin{aligned}\sum_k \dim_{\mathcal{L}} HH_k(S_a^\partial(M)) \;&\leq\; \sum_k \dim_{\mathcal{L}} HH_k(A^\partial(M)) \\ &\quad + \sum_k \dim_{\mathcal{L}} HH_k(I_a(M)). \qquad \text{(A.9)}\end{aligned}$$

Proof: By the result of [18], the short exact sequence (4.1) induces a long exact sequence in homology. Indeed, we have proved in Proposition 6.2.1 that $I_a(M)$ is H-unital. Then (A.9) is true for any fixed k. □

Theorem A.3.2 *Assuming* $\chi(M) = 0$*, the spectral sequences for* $I_a(M)$ *and* $A^\partial(M)$ *collapse at* E_2 *and are convergent.*

Proof: Use formulas (A.7) and (A.9) and Proposition A.1.2:

$$\begin{aligned}\dim_{\mathcal{L}} E_2(S_a^\partial(M)) \;&=\; \dim_{\mathcal{L}} E_\infty(S_a^\partial(M)) = \dim_{\mathcal{L}} HH(S_a^\partial(M)) \\ &\leq\; \dim_{\mathcal{L}} HH(A^\partial(M)) + \dim_{\mathcal{L}} HH(I_a(M)) \\ &\leq\; \dim_{\mathcal{L}} E_\infty(A^\partial(M)) + \dim_{\mathcal{L}} E_\infty(I_a(M)) \\ &\leq\; \dim_{\mathcal{L}} E_2(A^\partial(M)) + \dim_{\mathcal{L}} E_2(I_a(M)) \\ &=\; \dim_{\mathcal{L}} E_2(S_a^\partial(M))\end{aligned}$$

Hence, all inequalities must, in fact, be equalities. The assertions follow. □

A.4 Two distinguished elements in homology

In this section, assume that $\chi(M) \neq 0$. This implies that M is orientable. Let $(U, (x_i)_{i=1,\dots,n})$ be a local coordinate system on M. Let $\phi \in C_c^\infty(U)$ be a positive test function supported in U, such that

$$\int_U \phi(x) dx_1 \wedge \ldots \wedge dx_n = 1. \tag{A.10}$$

Let $\psi \in C_c^\infty(U)$ be a cut-off function, so that $\psi \equiv 1$ on $\mathrm{supp}(\phi)$. For $i = 1, \ldots, n$, define $\phi_i(x) = \psi(x) x_i$. Then $\phi_i \in C^\infty(M)$.

Definition A.4.1

$$\begin{aligned} a_1 &= \phi \otimes (\phi_1 \wedge \ldots \wedge \phi_n) \in C_n(A^\partial(M)); \\ a_2 &= \phi \otimes (t \wedge \phi_1 \wedge \ldots \wedge \phi_n) \in C_{n+1}(A^\partial(M)). \end{aligned}$$

Proposition A.4.2 *The chains a_1, a_2 are cycles and are non-zero at E_2.*

Proof: Since $C^\infty(M) \otimes \mathcal{L}$ is a commutative subalgebra of $A^\partial(M)$, we know that a_1, a_2 are b-closed, where b is the Hochschild boundary map. Their representatives at E_1 are $[a_1] = \phi d\phi_1 \wedge \ldots d\phi_n = \phi dx_1 \wedge \ldots dx_n$, and $[a_2] = \phi dt \wedge d\phi_1 \wedge \ldots d\phi_n = \phi dt \wedge dx_1 \wedge \ldots dx_n$. From (A.10), it follows that $[a_1], [a_2]$ have volume 1, respectively dt, hence are representatives for $H^n(M)$ and $dt \otimes H^n(M)$. As noted at the end of the proof of Proposition A.2.3, in the case where $\chi(M) \neq 0$, $H^n(M)$ and $dt \otimes H^n(M)$ inject canonically into $E_2(A^\partial(M))$. □

Proposition A.4.3 *The homology classes $[a_1], [a_2]$ map to 0 under p.*

Proof: Since a_1, a_2 are already purely homogeneous, we will write a_1, a_2 instead of $p(a_1), p(a_2)$. We want to prove that a_1, a_2 are exact in $C_\bullet(S_a^\partial(M), b)$. We have seen above that a_1, a_2 are representatives at E_2 for generators of $H^n(M)$ and $dt \otimes H^n(M)$. In the case where $\chi(M) \neq 0$, these two spaces span the kernel of the map

$$E_2(A^\partial(M)) \xrightarrow{p} E_2(S_a^\partial(M)).$$

So, at least, $[a_1]_{E_2^{0,n}(S_a^\partial(M))} = 0$ and $[a_2]_{E_2^{1,n+1}(S_a^\partial(M))} = 0$. Combining with (A.1) and with Proposition A.1.2, we see that $a_2 = 0$ in homology.

To prove that a_1 is exact, note that a_1 is a well-defined element of $C_\bullet(S_a(M))$. Using Remark 2.1.8, view a_1 as a chain in the algebra of time-dependent symbols on M. But a_1 does not depend on t, so it is, in fact, a chain on the subalgebra $\Psi(M)/\Psi^{-\infty}(M)$ of *constant* families of symbols. As above, $[a_1]$ represents the zero class in $E_2^{0,n}(\Psi(M)/\Psi^{-\infty}(M))$. By the result of [2], it follows that $[a_1]$ also represents the zero class in $HH_n(\Psi(M)/\Psi^{-\infty}(M))$. Now reverse the process: let $\gamma \in C_{n+1}(\Psi(M)/\Psi^{-\infty}(M))$ be a primitive. View γ as a time dependent family of symbols, and localize it in t. It follows that for $t > 0$, the class of a_1 in $HH_n(S_a(M))$ is locally zero. By continuity, it is zero for all t. Using the surjection τ from Proposition A.1.1, we conclude that a_1 is zero in $HH_n(S_a^\partial(M))$. □

A.5 The case $\chi(M) \neq 0$

Theorem A.5.1 *Assuming that $\chi(M) \neq 0$, the spectral sequence for $I_a(M)$ collapses at E_2. Both spectral sequences for $A^\partial(M)$ and for $I_a(M)$ are convergent.*

Proof: By assumption, we can use the results of Section A.4. By Proposition A.4.2, we know that a_1, a_2 define homology classes. Let $q \in \{0,1,2\}$ be the number of these classes which are nonzero at E_∞. For each of the $2-q$ classes that vanish, there must exist an element in E_2 mapping to it under some differential d_s. This implies

$$\dim_{\mathcal{L}} E_2(A^\partial(M)) \geq \dim_{\mathcal{L}} E_\infty(A^\partial(M)) + 2(2-q). \tag{A.11}$$

By Proposition A.4.3,

$$\dim_{\mathcal{L}} \ker(HH(A^\partial(M)) \xrightarrow{p} HH(S_a^\partial(M))) \geq q, \tag{A.12}$$

which implies by exactness

$$\dim_{\mathcal{L}} \operatorname{coker}(HH(S_a^\partial(M)) \xrightarrow{\delta} HH(I_a(M)) \geq q. \tag{A.13}$$

Combining (A.12) and (A.13), we get an inequality similar to (A.9):

$$\begin{aligned} \dim_{\mathcal{L}} HH(S_a^\partial(M)) \;\leq\; & \dim_{\mathcal{L}} HH(A^\partial(M)) \\ & + \dim_{\mathcal{L}} HH(I_a(M)) - 2q. \end{aligned} \tag{A.14}$$

Using (A.11) and (A.14), we get:

$$\begin{aligned} \dim_{\mathcal{L}} E_2(S_a^\partial(M)) &= \dim_{\mathcal{L}} E_\infty(S_a^\partial(M)) = \dim_{\mathcal{L}} HH(S_a^\partial(M)) \\ &\leq \dim_{\mathcal{L}} HH(A^\partial(M)) + \dim_{\mathcal{L}} HH(I_a(M)) - 2q \\ &\leq \dim_{\mathcal{L}} E_\infty(A^\partial(M)) + \dim_{\mathcal{L}} E_\infty(I_a(M)) - 2q \\ &\leq \dim_{\mathcal{L}} E_2(A^\partial(M)) + \dim_{\mathcal{L}} E_2(I_a(M)) + 4 \\ &= \dim_{\mathcal{L}} E_2(S_a^\partial(M)). \end{aligned}$$

We conclude that all the inequalities above must be equalities. The theorem follows. □

Remark A.5.2 *The main difference from Theorem A.3.2 is that we do could not prove that the classes $[a_1], [a_2]$ are non-zero in homology. Of course, by Theorem 5.4.10 and Corollary 4.0.16, they* are *non-zero.*

Appendix B

A Čech complex adapted to Hochschild homology

Let M be a C^∞ manifold and A a sheaf of algebras over M. Let $\mathcal{A}$ be a $\mathcal{L}$-filtered algebra such that $G\mathcal{A} \cong A \otimes \mathcal{L}$, e.g. a star algebra on a Poisson manifold. Let $\mathcal{U}$ be a good cover of M.

The Hochschild-Čech complex $C_{\check{H}}(\mathcal{A}, \mathcal{U})$ is the following second quadrant homology double complex:

Definition B.0.3

$$C^{i,k}_{\check{H}}(\mathcal{A}, \mathcal{U}) = \check{C}^{-i}(C_k(\mathcal{A}), \mathcal{U}).$$

The augmented complex $\bar{C}_{\check{H}}(\mathcal{A}, \mathcal{U})$ *is obtained from* $C_{\check{H}}(\mathcal{A}, \mathcal{U})$ *by adding the bar resolution of* $\mathcal{A}$ *as the* 1*-column. We shall denote by* $ToT(\bar{C}_{\check{H}}(\mathcal{A}, \mathcal{U}))$ *the total complex of this double complex, defined as the* direct product *rather than the direct sum, of the diagonals of* $\bar{C}_{\check{H}}(\mathcal{A}, \mathcal{U})$.

Theorem B.0.4 *Assume that the following conditions hold:*

1. *There is an acyclic sheaf* $\mathcal{F}$ *and a sheaf isomorphism* $HH(A) \cong \mathcal{F}$*;*
2. A *and* $G\mathcal{A} \cong A \otimes \mathcal{L}$ *are* H*-unital.*
3. $\bigcap F_i\mathcal{A} = \emptyset$, $\bigcup F_i\mathcal{A} = \mathcal{A}$ *and*

$$\varinjlim_{i \to -\infty} \varprojlim_{j \to \infty} F_i\mathcal{A}/F_j\mathcal{A} \cong \mathcal{A};$$

Then

$$h(ToT(\bar{C}_{\check{H}}(\mathcal{A}, \mathcal{U}))) \cong 0.$$

Proof: Filter the complex $\bar{C}_{\check{H}}(\mathcal{A}, \mathcal{U})$ by the total t-degree. This filtration is compatible with the Čech and Hochschild boundary maps, so we can form a spectral sequence, denoted ${}^T E(\mathcal{A})$.

Lemma B.0.5 ${}^T E_1(\mathcal{A}) \cong 0$.

Proof: By the assumption on the associated graded algebra of $\mathcal{A}$, we have an isomorphism of complexes

$$({}^T E_0(\mathcal{A}), d_0) \cong (\bar{C}_{\check{H}}(A \otimes \mathcal{L}, \mathcal{U}), b + (-1)^i \check{d}).$$

Hence we want to show that the second complex is acyclic. Consider the filtration by columns of this double complex. It induces a spectral sequence abutting at the homology of $ToT(\bar{C}_{\check{H}}(A \otimes \mathcal{L}, \mathcal{U}))$.

Lemma B.0.6 *The E_2 term of this new spectral sequence vanishes.*

Proof: Use the Künneth formula of [12]. By assumption,

$$\begin{aligned} E_1((C_{\check{H}}(A \otimes \mathcal{L}, \mathcal{U}), b + (-1)^i \check{d}) &\cong \check{C}^{\text{aug}}(HH(A \otimes \mathcal{L})) \\ &\cong \check{C}^{\text{aug}}(\mathcal{F}) \otimes HH(\mathcal{L}). \end{aligned}$$

We see that d_1 is just the Čech differential, because all the above isomorphisms are compatible with restrictions. By assumption, $\mathcal{F}$ is acyclic, hence E_2 vanishes. □

Lemma B.0.7 *The spectral sequence from Lemma B.0.6 converges.*

Proof: Let a be an element in ${}^T E_1(\mathcal{A})$. There exists a standard way of constructing a formal sum c such that $a = d_0(c)$, that we will omit. By the definition of the ToT complex as a direct product, this c is automatically sumable. □

This finishes the proof of Lemma B.0.5. □

We want c from the proof of Lemma B.0.7 to be of degree bounded below. This is indeed possible: if γ is 0 at E_2, then $\exists \mu$ such that $d_1[\mu] = [\gamma]$ and $\deg(\mu) \geq \min(\deg(\gamma), 0)$. If γ is 0 at E_1, then there exists μ such that $d_0[\mu] = [\gamma]$ and $\deg(\mu) \geq \deg(\gamma)$. Now, for every $a \in {}^T E_1(\mathcal{A})$, the same diagram chasing process will produce some formal sum c such that $\deg(c) \geq \min(\deg(a), 0)$ and $(b + (-1)^i \check{d})(c) = a$. Again using the fact that ToT is defined as a direct sum and condition 3 of the theorem, we see that c is sumable, hence $[a] = 0$. This proves that ${}^T E(\mathcal{A})$ converges. By Lemma B.0.5, the theorem follows. □

Corollary B.0.8 *The boundary map in the augmentation complex,*

$$C_\bullet(\mathcal{A}) \to ToT_{\bullet -1}(C_{\check{H}}(\mathcal{A}, \mathcal{U})), \quad a \longmapsto b(a) + (-1)^\bullet \check{d}(a)$$

is a quasi-isomorphism.

□

B.1 The Čech complex of the adiabatic algebras

Choose a good cover $\mathcal{U}$ of M. By Propositions 5.4.9, 3.1.1 and 6.2.1, the algebras $S_a(X)$ and $I_a(X)$ satisfy the hypothesis of theorem B.0.4, . It follows that we can compute their Hochschild homology by using the Hochschild-Čech complex.

Proposition B.1.1 *$HH(\Psi_a(X)/\dot{\Psi}_a^{-\infty}(X))$ can be computed by using the Hochschild-Čech complex.*

Proof: The short exact sequence 4.1 induces maps of augmented Hochschild-Čech complexes. This can be thought of as a double complex with three columns, each of them being itself a double complex. Take the filtration induced by the columns of the three (augmented) double complexes. H-unitality of $I_a(X))$ implies that E_1 of the induced spectral sequence vanishes. This means that the ToT homology of the triple complex vanishes. Theorem 6.4.1 shows that there exists a long exact sequence relating the homologies of the three augmented Hochschild-Čech complexes. The result follows by Theorem B.0.4. $\square$

Appendix C

Cyclic homology of the adiabatic algebras

The periodic cyclic homology of the various algebras discussed above can be computed without any technical difficulty. It seems that in the symbol algebra case, the particular $*$-product structure is completely irrelevant in the computation. The same is true for the smoothing part, where the only part that matters is the $\pmod t$, i.e. commutative, multiplication.

Proposition C.0.2

$$HC^{per}_{\bullet}(S_a(X)) \cong \bigoplus_{k\in Z} H^{\bullet+2k}(T^*X \times S^1) \oplus \bigoplus_{k\in Z} H^{\bullet+2k-1}(T^*X \times S^1) \otimes \frac{dt}{t}. \qquad \text{(C.1)}$$

Proof: Compute $HC^{per}(S_a(X))$ as the homology of a double (b, B) complex, by using the order filtration. The E_1 term of the resulting spectral sequence is just the homology of the cyclic bi-complex of the graded associated algebra $S({}^aT^*X)[t^{-1}]$ defined in Proposition 2.2.7. This turns out to be the deRham cohomology of the algebra $S({}^aT^*X)[t^{-1}]$, i.e. the right-hand side of (C.1). This cohomology only lives in homogeneity 0, as in Lemma 3.1.11. Hence the spectral sequence degenerates at E_1. The convergence of this spectral sequence for the ToT complex is automatic. $\square$

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