K-Theory of suspended pseudo-differential operators

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Abstract. We compute the K-Theory groups of Melrose's algebra of 1-suspended pseudo-differential operators. The boundary map in the six-term long exact sequence turns out to be related to both the eta invariant of Melrose and to the index of elliptic operators. The proof is based on a new identity between the formal trace and the Wodzicki residue trace on the suspended algebra.

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1. Introduction

The algebra $\Psi^{\mathbb{Z}}_{\mathrm{sus}}(M,E)$ of 1-suspended pseudo-differential operators was introduced by Richard Melrose in [9]. This algebra and its higher dimensional cousins play a pervasive role in the analysis of algebras of pseudo-differential operators constructed in the framework of Melrose's program [10], as they are the natural receptacles of the operator-valued symbol maps describing the behavior near the boundary. See for instance our recent work [5].

In fact, the algebra $\Psi^{\mathbb{Z}}_{\mathrm{sus}}(M,E)$ is quite interesting in its own right. In sharp contrast to the usual algebra of pseudo-differential operators $\Psi^{\mathbb{Z}}(M,E)$, the suspended algebra possesses a trace functional extending by regularization the trace from operators of order $-\infty$. This trace allows one to define an invariant of invertible suspended operators which extends the eta invariant of Atiyah, Patodi and Singer [1] for Dirac operators. Moreover, a symbolic version of this trace functional, called the formal trace, gives an interpretation of the spectral flow as a $\mathbb{Z}$-valued 1-cocycle on the space of elliptic suspended operators.

The first result of this paper (Corollary 4) is a description of the relationship between the regularized trace, the formal trace, and the analog of the Wodzicki residue trace that exists for very general pseudo-differential operator algebras.

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The eta invariant is known to be an odd-dimensional analog of the index of elliptic operators. By looking at the K-Theory of the algebra of 1-suspended operators of order 0 we have found that in the suspended setting the analogy holds beyond a formal resemblance. Our second result (Theorem 8) describes the boundary map

$$\delta : K_0(C(S^*_{\text{sus}}M) \to K_{-1}(C_0(\mathbb{R}, \mathtt{k}))$$

both in terms of the eta-invariant of suspended operators, and of the index of some un-suspended operators. To explain this result in the simplest case, let D be the chiral Dirac operator on the closed even-dimensional spin manifold M. Then the Dirac operator on $\mathbb{R} \times M$

$$A := \begin{bmatrix} -i\frac{\partial}{\partial x} & D^* \\ D & i\frac{\partial}{\partial x} \end{bmatrix}$$

is an elliptic self-adjoint suspended operator. We construct from this operator a projector inside the suspended symbol algebra and show that its image through the boundary map δ equals the index of D.

The idea of the proof of Theorem 8 is to describe the boundary map in terms of the suspended residue trace, which is symbolic (Lemma 1). To do this we use Corollary 4 to go from eta invariants and the formal trace of Melrose [9] to the residue trace. On the other hand, the residue trace is by definition the residue at $s = 0$ of the trace of a certain entire family of suspended operators, and we can compute explicitly this residue as an index by expanding the trace over the eigensections of D^*D.

As a corollary the boundary map is surjective, and this determines completely the K-Theory groups of $\Psi^0_{\text{sus}}(M, E)$. In Section 6 we discuss an alternate way of proving Theorem 8 based on the results of [11].

The K-Theory of algebras of pseudo-differential operators has been computed in several other instances; see [11, 8, 2].

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2. Basics of the suspended algebra

We recall the definitions and basic results of [9]. Let M be a closed smooth manifold of dimension n and $E \to M$ a vector bundle. Then $\Psi^s_{\mathrm{sus}}(M, E)$ is defined as the space of those classical pseudo-differential operators P of order $s \in \mathbb{C}$ on sections of the pull-back of E to $\mathbb{R} \times M$ with the following properties:

- P is translation-invariant with respect to $\mathbb{R}$; this is equivalent to the fact that the Schwartz kernel $p(x_1, x_2, y_1, y_2)$ of P depends only on $(x_1 - x_2, y_1, y_2)$. Thus we write $p(x_1 - x_2, y_1, y_2)$ instead.

- The kernel $p(x, y_1, y_2)$ decays rapidly as $|x|$ tends to infinity. This distributional kernel has singular support concentrated at $\{x = 0, y_1 = y_2\}$, thus it is smooth as $|x| \to \infty$.

These two conditions ensure that $\Psi^{\mathbb{Z}}_{\mathrm{sus}}(M, E)$ is an algebra with product given by composition of operators. When $E = \mathbb{C}$ we will write simply $\Psi^{\mathbb{Z}}_{\mathrm{sus}}(M)$.

After a partial Fourier transform in the variable x, a suspended operator becomes a family of classical pseudo-differential operators on M with real parameter τ. In this description, composition of suspended operators corresponds to pointwise multiplication of families. Note however that τ is a symbolic variable, thus the inclusion $\Psi^{\mathbb{Z}}_{\mathrm{sus}}(M, E) \hookrightarrow C^\infty(\mathbb{R}, \Psi^{\mathbb{Z}}(M, E))$ is far from being an equality. Namely, an operator $A(\tau)$ belongs to $\Psi^{\mathbb{Z}}_{\mathrm{sus}}(M, E)$ if and only if its symbol $a(y, \mu, \tau)$ is a classical symbol in the joint variable (τ, μ), i.e. on the bundle $T^*M \oplus \mathbb{R}$. In the sequel we view suspended operators as families of classical operators on M with this property (the so-called associated *indicial family* in the terminology from [9]). Similar algebras of pseudo-differential operators have been considered by other authors under the name *parameter-dependent pseudo-differential operators* [13, 15].

3. Definition of the traces

3.1. The regularized trace

There exist quite a few trace functionals on the ideal $\Psi^{-\infty}_{\mathrm{sus}}(M, E)$ (although the trace as operators on $C^\infty_c(\mathbb{R} \times M)$ is always infinite). One distinguished choice is the trace at $x = 0$. After Fourier transform and renormalization, this is just

$$A \mapsto \overline{\mathrm{Tr}}(A) := \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{Tr}(A(\tau)) d\tau. \tag{1}$$

It is not surprising that (1) extends by the same formula to $A \in \Psi^s_{\mathrm{sus}}(M,E)$ for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) < -n-1$. One striking result of [9] is that this trace extends canonically to a trace functional on the full suspended algebra $\Psi^{\mathbb{Z}}_{\mathrm{sus}}(M,E)$. For comparison, the analogous statement for pseudo-differential operators is false: there exists no extension of the trace functional from smoothing operators to a trace functional on $\Psi^{\mathbb{Z}}(M,E)$.

The extension depends on the symbolic nature of the parameter τ. Let $A(\tau) \in \Psi^k_{\mathrm{sus}}(M,E)$. Denote by ∂_τ the partial derivative operator with respect to τ. Then $\partial^j_\tau A(\tau) \in \Psi^{k-j}_{\mathrm{sus}}(M,E)$. For $k-j < -n$, this implies that $\partial^j_\tau A(\tau)$ is of trace class for every fixed τ. Moreover $\mathrm{Tr}\left(\partial^j_\tau A(\tau)\right)$ is a classical symbol of order $k-j+n$ on $\mathbb{R}$ [9, Lemma 1]. To see this, let $a(y,\mu,\tau) \in S^k(T^*M \oplus \mathbb{R})$ be the full symbol of A in a fixed quantization, i.e. the Fourier transform in the normal direction of a cut-off of the Schwartz kernel of A near the diagonal. The quantization commutes with the derivative with respect to τ, so the symbol of $\partial^j_\tau A(\tau)$ is $\partial^j_\tau a(y,\mu,\tau)$, which is a symbol of order $k-j$. Now $\mathrm{Tr}\left(\partial^j_\tau A(\tau)\right)$ equals the integral of the pointwise trace of the Schwartz kernel of A over the diagonal in M^2, or in terms of Fourier transforms,

$$\mathrm{Tr}\left(\partial^j_\tau A(\tau)\right) = \frac{1}{(2\pi)^n}\int_{T^*M} \partial^j_\tau \mathrm{tr}(a(y,\mu,\tau))d\mu dy. \qquad (2)$$

The integral is convergent for $k-j < -n$. The claim now follows from the fact that the integral with respect to n of the symbolic variables of a classical symbol of order $k-j$ is a classical symbol of order $k-j+n$ in the remaining variable(s).

Let I_τ be the operator of integration, $f \mapsto \mathrm{I}_\tau(f) : (\tau \mapsto \int_0^\tau f(\nu)d\nu)$. Then for $f \in C^\infty(\mathbb{R})$ we have $\mathrm{I}_\tau\partial_\tau(f) \equiv f$ modulo constant functions, and $\partial_\tau \mathrm{I}_\tau(f) = f$.

For $k \in \mathbb{Z}$ denote by $S^k_{\log}(\mathbb{R})$ the space of those smooth functions f admitting an asymptotic expansion at $\pm\infty$ of the form

$$f(\tau) \sim_{\tau\to\pm\infty} \sum_{j=-\infty}^{k} \alpha_j^\pm |\tau|^j + \sum_{j=0}^{k} \beta_j^\pm |\tau|^j \log|\tau|.$$

We call such functions log-polyhomogeneous symbols, although they may differ from other objects with the same name in the literature. We have

$$\mathrm{I}_\tau(S^k_{\log}(\mathbb{R})) \subset S^{k+1}_{\log}(\mathbb{R}), \quad \partial_\tau(S^k_{\log}(\mathbb{R})) \subset S^{k-1}_{\log}(\mathbb{R}).$$

For $A \in \Psi^k_{\mathrm{sus}}(M,E)$ and $j > k+n$, define

$$H_j(A) := \mathrm{I}^j_\tau \mathrm{Tr}\left(\partial^j_\tau A\right).$$

Clearly $H_j(A) - H_{j+1}(A)$ is a polynomial in τ. It follows that for any $j > k + n$, $\mathrm{I}_\tau H_j(A)$ is a log-polyhomogeneous symbol on $\mathbb{R}$ and is independent of j up to a polynomial function. Thus

$$(\mathrm{I}_\tau H_j(A))(\tau) - (\mathrm{I}_\tau H_j(A))(-\tau)$$

has a log-polyhomogeneous asymptotic expansion as $\tau \to +\infty$ which is independent of j up to an odd polynomial. One defines $\overline{\mathrm{Tr}}(A)$ as $(2\pi)^{-1}$ times the constant term $\alpha_0^+ - \alpha_0^-$ of this asymptotic expansion, which is independent of j altogether. It is easy to see (for instance taking $j = 0$) that $\overline{\mathrm{Tr}}$ extends the trace (1) for suspended operators of order $-\infty$. Moreover, it is a trace, in the sense that it vanishes on commutators in the suspended algebra. This follows from the fact that the L^2 trace on operators of sufficiently small order on M has this property: if $A \in \Psi^a(M)$, $B \in \Psi^b(M)$ and $a + b < -n$ then $\mathrm{Tr}([A, B]) = 0$.

3.2. The formal trace

The second trace introduced by Melrose uses the fact that ∂_τ is an exterior derivation on the algebra $\Psi^{\mathbb{Z}}_{\mathrm{sus}}(M, E)$. Thus $\widetilde{\mathrm{Tr}} := \overline{\mathrm{Tr}} \circ \partial_\tau$ is again a trace. This functional vanishes on the ideal $\Psi^{-\infty}_{\mathrm{sus}}(M, E)$ by (1); it actually vanishes on $\Psi^{-n-1}_{\mathrm{sus}}(M, E)$ so it only depends on a finite number of symbol asymptotic terms. From the definition we see immediately that $(2\pi)\widetilde{\mathrm{Tr}}(A)$ is the constant term in the asymptotic expansion as $\tau \to \infty$ of $H_j(A)(\tau) - H_j(A)(-\tau)$, for j sufficiently large.

3.3. The Wodzicki residue

The third important trace functional on $\Psi^{\mathbb{Z}}_{\mathrm{sus}}(M, E)$ (first analyzed in [12]) is an analog of the Wodzicki residue trace. Fix a Riemannian metric on M and a metric in E. Let Q be any elliptic self-adjoint and positive suspended operator of order 1. Then the complex powers Q^s form an analytic family of suspended operators of order s; moreover, for any $A \in \Psi^k_{\mathrm{sus}}(M, E)$, the function $s \mapsto \overline{\mathrm{Tr}}(Q^{-s}A)$ is analytic for $\mathrm{Re}(s) > k + n + 1$ and extends to a meromorphic function on $\mathbb{C}$ with possible simple poles at $k + n + 1, k + n, k + n - 1, \ldots$. We denote the analytic extension by the same symbol. Note that this brings a clash of notations: $\overline{\mathrm{Tr}}(Q^{-s}A)$ is defined when $\mathrm{Re}(s) > k+n+1$ or when $k - s$ is integer; the values at these integer points might be different from those of the analytic extension of $\overline{\mathrm{Tr}}(Q^{-s}A)$, which in fact typically has poles there. We define now

$$\mathrm{Tr}_w(A) := \mathrm{Res}_{s=0} \overline{\mathrm{Tr}}(Q^{-s}A).$$

Clearly $\mathrm{Tr}_w(A) = 0$ unless A is of integer order. Let $S^*_{\mathrm{sus}}M$ be the sphere bundle in the vector bundle $T^*_{\mathrm{sus}}M := T^*M \oplus \mathbb{R}$. The volume form on $S^*_{\rm sus}M$ is the contraction of $\omega^n \wedge d\tau$ by the radial vector field $\mathcal{R}$ of the bundle $T^*_{\rm sus}M$, where ω is the canonical symplectic form on T^*M (recall that on the total space of any vector bundle there exists a canonical vector field $\mathcal{R}$ tangent to the fibers defined by $\mathcal{R}_v := v$). Let $a(y,\mu,\tau) \sim \sum_{i=0}^\infty a_{k-i}(y,\mu,\tau)$ be the asymptotic expansion of the full symbol of A in some fixed quantization.

LEMMA 1.

$$\mathrm{Tr}_w(A) = \frac{1}{(2\pi)^{n+1}} \int_{S^*_{\rm sus}M} \mathrm{tr}(a_{-n-1}) \mathcal{R} \lrcorner \omega^n \wedge d\tau. \tag{3}$$

Proof. Completely analogous to the corresponding statement in the un-suspended case.

As the notation suggests, Tr_w is therefore independent of the choice of Q.

We close this section with two useful properties of the operations H_j.

LEMMA 2. *For $A \in \Psi^k_{\rm sus}(M,E)$ and $j > k+n+1$, we have*

$$H_j(\tau A) = \tau H_j(A) + \mathbb{C}[\tau], \quad H_j(\partial_\tau A) = \partial_\tau H_j(A) + \mathbb{C}[\tau].$$

Proof. The second identity follows readily from the commutation rule of I_τ and ∂_τ. We note that for any $g \in C^\infty(\mathbb{R})$ and $i \geq 1$, $l \in \mathbb{Z}$ we have

$$\begin{aligned} \mathrm{I}^i_\tau(\tau\partial_\tau g + lg) &= \mathrm{I}^i_\tau(\partial_\tau(\tau g) + (l-1)g) \\ &= \mathrm{I}^{i-1}_\tau(\tau g + (l-1)\mathrm{I}_\tau g) + \mathbb{C}[\tau]. \end{aligned}$$

Let $f(\tau) := \mathrm{Tr}\left(\partial^{j-1}_\tau A(\tau)\right)$. Applying the above remark inductively with $l = i$, $g = \mathrm{I}^{j-i}_\tau f$ and i running from j to 1 we obtain

$$\begin{aligned} H_j(\tau A) &= \mathrm{I}^j_\tau(\tau\partial_\tau f + jf) \\ &= \tau \mathrm{I}^{j-1}_\tau f + \mathbb{C}[\tau] \\ &= \tau H_{j-1}(A) + \mathbb{C}[\tau]. \end{aligned}$$

Use now the fact that $H_j(A) - H_{j-1}(A)$ is a polynomial.

4. Relations between traces on $\Psi^{\mathbb{Z}}_{\mathrm{sus}}(M,E)$

PROPOSITION 3. *Let $A \in \Psi^k_{\mathrm{sus}}(M,E)$. Then*

$$\begin{aligned}\widetilde{\mathrm{Tr}}(A) &= \frac{1}{(2\pi)^{n+1}} \lim_{r\to\infty}\left(\int_{B_r(T^*M)\times\{1\}} \mathrm{tr}(a_{-n})\,\omega^n\right.\\ &\quad \left.- \int_{B_r(T^*M)\times\{-1\}} \mathrm{tr}(a_{-n})\omega^n\right) \qquad (4)\\ \mathrm{Tr}_w(A) &= \frac{1}{(2\pi)^{n+1}}\left(\int_{T^*M\times\{1\}} \mathrm{tr}(a_{-n-1})\omega^n\right.\\ &\quad \left.+ \int_{T^*M\times\{-1\}} \mathrm{tr}(a_{-n-1})\omega^n\right) \qquad (5)\end{aligned}$$

Proof. The first statement was proved in [9, Proposition 6]. It is enough to assume that the full symbol $a(y,\mu,\tau)$ of A is homogeneous of homogeneity l outside the unit ball, with $-n \le l \le k$. For $l=-n$ it is noted in [9, Proposition 6] that each individual limit in the right-hand side diverges logarithmically and the divergences cancel each-other. We give here another proof of the case $l \neq -n$, since we cannot follow the proof from [9] in that case.

In local coordinates (y,μ,τ) the radial vector field $\mathcal{R}$ takes the form $\mathcal{R} = \sum_{i=1}^n \mu_i\partial_{\mu_i} + \tau\partial_\tau$. The assumption on a implies that $\mathcal{R}\mathrm{tr}(a) = l\mathrm{tr}(a) + C_c^\infty(T^*M\oplus\mathbb{R})$. Note that

$$H_j(A) = \mathrm{I}_\tau^j \mathrm{Tr}\left(\partial_\tau^j A\right) = \frac{1}{(2\pi)^n}\mathrm{I}_\tau^j \int_{T^*M} \partial_\tau^j \mathrm{tr}(a(y,\mu,\tau))d\mu dy$$

where the integral is well-defined for $j > l+n$. By Lemma 2

$$\begin{aligned}\tau\partial_\tau H_j(A) &= H_j(\tau\partial_\tau A) + \mathbb{C}[\tau]\\ &= \frac{1}{(2\pi)^n}\mathrm{I}_\tau^j\int_{T^*M} \partial_\tau^j(\mathcal{R}(\mathrm{tr}(a(y,\mu,\tau)))d\mu dy\\ &\quad -\frac{1}{(2\pi)^n}\mathrm{I}_\tau^j\int_{T^*M}\partial_\tau^j\sum_{i=1}^n \mu_i\partial_{\mu_i}\mathrm{tr}(a(y,\mu,\tau))d\mu dy + \mathbb{C}[\tau]\\ &= (l+n)H_j(A) + \mathbb{C}[\tau] + C_c^\infty(\mathbb{R}). \qquad (6)\end{aligned}$$

In the last equality we have integrated by parts with respect to μ_i, or more sophisticatedly we used the fact that the adjoint of the differential operator $\mathcal{R}-\tau\partial_\tau$ on T^*M with respect to the volume form ω^n is $-(\mathcal{R}-\tau\partial_\tau)-n$.

Now $H_j(A)$ is a log-polyhomogeneous symbol:

$$H_j(A) \sim_{\tau\to\pm\infty} \sum_{i=-\infty}^{l+n} \alpha_i^\pm|\tau|^i + \sum_{i=0}^{l+n}\beta_i^\pm|\tau|^i\log|\tau|$$

so

$$\tau\partial_\tau H_j(A) \sim_{\tau\to\pm\infty} \sum_{i=-\infty}^{l+n} (i\alpha_i^\pm + \beta_i^\pm)|\tau|^i + \sum_{i=0}^{l+n} i\beta_i^\pm |\tau|^i \log|\tau|.$$

Substituting in (6) and identifying coefficients we get $i\beta_i^\pm = (l+n)\beta_i^\pm$, and for even i,

$$i(\alpha_i^+ - \alpha_i^-) + (\beta_i^+ - \beta_i^-) = (l+n)(\alpha_i^+ - \alpha_i^-)$$

(we anti-symmetrized in τ to kill in even degrees the unknown polynomial term from (6)) . In particular if $l \neq -n$ the first identity gives $\beta_0^\pm = 0$ and so from the second we get $\alpha_0^+ - \alpha_0^- = 0$.

The second statement follows from Lemma 1. For $r > 1$, let

$$V_r := B_r(T^*M) \times [-1,1] \setminus B^*_{\mathrm{sus}}M$$

where $B^*_{\mathrm{sus}}M$ is the unit ball in the vector bundle $T^*_{\mathrm{sus}}M$. Note that

$$\partial V_r = S^*_{\mathrm{sus}}M \bigcup S_r(T^*M) \times [-1,1] \bigcup B_r(T^*M) \times \{\pm 1\}.$$

The form $\mathrm{tr}(a_{-n-1})\omega^n \wedge d\tau$ is homogeneous of order 0 on V_r, or equivalently $L_{\mathcal{R}}(\mathrm{tr}(a_{-n-1})\omega^n \wedge d\tau) = 0$. By Stokes' formula, (3) can be written as the integral of $\mathrm{tr}(a_{-n-1})\mathcal{R}\lrcorner\omega^n \wedge d\tau$ on the complement of $S^*_{\mathrm{sus}}M$ in the boundary of V_r, with the induced orientation. First examine the component $S_r(T^*M) \times [-1,1]$. By homogeneity,

$$\begin{aligned}&\int_{S_r(T^*M)\times[-1,1]} \mathrm{tr}(a_{-n-1})\mathcal{R}\lrcorner\omega^n \wedge d\tau\\ &\quad = \int_{S_1(T^*M)\times[-1/r,1/r]} \mathrm{tr}(a_{-n-1})\mathcal{R}\lrcorner\omega^n \wedge d\tau.\end{aligned}$$

This shrinks towards 0 as r tends to ∞. The remaining part of the boundary is $B_r(T^*M)\times\{\pm1\}$, with orientation ±1. On $B_r(T^*M)\times\{\tau\}$, the form $\mathcal{R}\lrcorner\omega^n \wedge d\tau$ equals $\tau\omega^n$. Thus

$$\begin{aligned}\mathrm{Tr}_w(A) \;=\; &\frac{1}{(2\pi)^{n+1}} \lim_{r\to\infty} \left(\int_{B_r(T^*M)\times\{1\}} \mathrm{tr}(a_{-n-1})\omega^n \right.\\ &\left. + \int_{B_r(T^*M)\times\{-1\}} \mathrm{tr}(a_{-n-1})\omega^n \right).\end{aligned}$$

Now (5) follows from this, since $\mathrm{tr}(a_{-n-1})\omega^n$ is absolutely integrable on $T^*M \times \{\pm1\}$ (notice that

$$|a_{-n-1}(y,\mu,\tau)| < C(1+|\mu|^2+|\tau|^2)^{-\frac{n+1}{2}} < C|\tau|^{-n-1}).$$

COROLLARY 4.

$$\mathrm{Tr}_w = \widetilde{\mathrm{Tr}} \circ \mu_\tau,$$

where μ_τ *is the multiplication operator by the suspended operator* $\tau \in \Psi^1_{\mathrm{sus}}(M, E)$.

Proof. On $T^*M \times \{\tau\}$ we have $(\tau a)_{-n} = \tau a_{-n-1}$. The result follows from Proposition 3.

We claim now that $\mathrm{Tr}_w(\partial_\tau A) = 0$ for all $A \in \Psi_{\mathrm{sus}}(M, E)$. Indeed,

$$\mathrm{Tr}_w(\partial_\tau A) = \mathrm{Res}_{s=0} \int_{\mathbb{R}} \left(\partial_\tau(\mathrm{Tr}(Q^{-s}A)) - \mathrm{Tr}(\partial_\tau(Q^{-s})A)\right) d\tau.$$

Now $\int_{\mathbb{R}} \partial_\tau(\mathrm{Tr}(Q^{-s}A))$ vanishes for $\mathrm{Re}(s) >> 0$, so it vanishes identically on $\mathbb{C}$, in particular its residue at $s = 0$ is null. On the other hand, $\partial_\tau(Q^{-s}) = sB(s)$ for some entire family $B(s)$ of suspended operators of order s. It is routine to see that $\int_{\mathbb{R}} \mathrm{Tr}(B(s)A)d\tau$ has an analytic extension to $\mathbb{C}$ with at most simple poles, so the extra factor of s will kill the residue at $s = 0$.

Using Corollary 4 we see that $\widetilde{\mathrm{Tr}} \circ \mu_\tau \circ \partial_\tau = 0$, or again $\overline{\mathrm{Tr}} \circ \partial_\tau \circ \mu_\tau \circ \partial_\tau = 0$. The first identity means exactly that $\widetilde{\mathrm{Tr}}$ is translation-invariant.

5. K-Theory of suspended operators

The eta invariant of invertible suspended pseudo-differential operators was defined by Melrose [9] as

$$\eta(A) := \frac{2}{i}\overline{\mathrm{Tr}}(A^{-1}\partial_\tau A). \tag{7}$$

Although we will not use this fact, recall that for any first-order invertible differential operator D on M, $\eta(i\tau + D)$ coincides with the Atiyah-Patodi-Singer eta invariant of D. This was proved in [9] for admissible Dirac operators and extended to the general case in [7]. We claim that the Melrose eta invariant and the three traces on $\Psi^{\mathbb{Z}}_{\mathrm{sus}}(M)$ analyzed in the previous sections are intimately related to the K-Theory of the suspended algebra. We look at the principal symbol short exact sequence

$$0 \to \Psi^{-1}_{\mathrm{sus}}(M) \to \Psi^0_{\mathrm{sus}}(M) \to C^\infty(S^*_{\mathrm{sus}}M) \to 0 \tag{8}$$

and complete it with respect to the norm topology of bounded operators on $L^2(M \otimes \mathbb{R})$. The result is the short exact sequence of C^*-algebras

$$0 \to C_0(\mathbb{R}, \mathtt{k}) \to \overline{\Psi^0_{\mathrm{sus}}(M)} \to C(S^*_{\mathrm{sus}}M) \to 0. \tag{9}$$

Here $\mathtt{k}$ denotes the ideal of compact operators on $L^2(M)$, and C_0 are the continuous functions vanishing at infinity with values in $\mathtt{k}$, i.e. a function $f \in C_0(\mathbb{R}, \mathtt{k})$ satisfies $\lim_{|\tau|\to\infty} \|f(\tau)\|_{\mathcal{L}(L^2(M))} = 0$.

The algebra $\Psi^0_{\mathrm{sus}}(M)$ is a Fréchet Ψ^*-algebra in the sense of Gramsch [3]. Indeed, the only nontrivial condition to be checked is spectral invariance, in other words that suspended operators which are invertible inside bounded operators are in fact invertible as suspended operators; this follows by a standard argument as in [4, Theorem 5.4] for instance, see also [14]. Since $\Psi^0_{\mathrm{sus}}(M)$ is a Ψ^*-algebra it follows that it is closed under holomorphic functional calculus hence the K-Theory of the algebras in (8) and of their closure (9) inside $\mathcal{L}(L^2(M \otimes \mathbb{R}))$ coincide.

It is the K-Theory of these completed algebras that we compute effectively. In fact, it is already clear that we will compute $K_*(\overline{\Psi^0_{\mathrm{sus}}(M)})$ using the six-term long exact sequence in K-Theory associated to (9), since we see immediately what the K-Theory groups of the first and the third algebras are. Namely

$$\begin{aligned} K_0(C_0(\mathbb{R}, \mathtt{k})) &\cong 0 \\ K_{-1}(C_0(\mathbb{R}, \mathtt{k})) &\cong \mathbb{Z} \\ K_*(C(S^*_{\mathrm{sus}}M)) &\cong K^*(S^*_{\mathrm{sus}}M), \end{aligned}$$

where the second isomorphism is given by the winding number viewed as a map from $\pi_1((1 + M(\mathtt{k}))^{-1})$ into $\mathbb{Z}$ (here $M(\mathtt{k})$ denotes the space of infinite matrices with entries in $\mathtt{k}$, all but a finite number of which are 0, and is not related to the manifold M). Thus the only element to determine in this long exact sequence is the boundary map

$$\delta : K_0(C(S^*_{\mathrm{sus}}M)) \to K_{-1}(C_0(\mathbb{R}, \mathtt{k})). \tag{10}$$

REMARK 5. *For the analogous question in the setting of the algebra $\Psi^0(M)$, the only interesting map is the boundary*

$$d : K^1(S^*M) \to K_0(\mathtt{k}) \cong \mathbb{Z},$$

which is just the index (of a pseudo-differential operator on M with principal symbol homotopic to a given invertible matrix-coefficient function on S^*M).

Let $D : C^\infty(M, E^+) \to C^\infty(M, E^-)$ be an elliptic differential operator of order 1. Note that for any non-negative Schwartz function ϕ such that $\phi(0) > 0$, we have $\phi(\tau)P_{\ker(D)} \in \Psi^{-\infty}_{\mathrm{sus}}(M, E^+, E^-)$ where $P_{\ker(D)}$ denotes the orthogonal projection on $\ker(D)$ in $L^2(M, E^+)$. Let $j \in \mathbb{Z}$ be such that $E^+ \oplus E^-$ is a direct summand inside the trivial bundle $\mathbb{C}^j$. We associate to D a projector in $M_j(\Psi^0_{\rm sus}(M)/\Psi^{-\infty}_{\rm sus}(M))$ as follows: first construct the suspended operator

$$A := \begin{bmatrix} -\tau & D^* \\ D & \tau \end{bmatrix}. \tag{11}$$

Notice that A is elliptic, symmetric and

$$\widetilde{A^2} := \begin{bmatrix} \tau^2 + D^*D + \phi(\tau)P_{\ker(D)} & 0 \\ 0 & \tau^2 + DD^* + \phi(\tau)P_{\ker(D^*)} \end{bmatrix}$$

is a positive invertible suspended operator and $\widetilde{A^2} - A^2 \in \Psi^{-\infty}_{\rm sus}(M)$. Let $Q := (\widetilde{A^2})^{1/2}$. Clearly Q commutes with A modulo $\Psi^{-\infty}_{\rm sus}(M, E^+ \oplus E^-)$ so $(Q^{-1}A)^2 \simeq 1$ modulo $\Psi^{-\infty}_{\rm sus}(M, E^+ \oplus E^-)$. Therefore

$$P := \frac{1 - Q^{-1}A}{2}$$

is a projector modulo $\Psi^{-\infty}_{\rm sus}(M, E^+ \oplus E^-)$. Add to P the identity $I_{E^\perp}$ of a complement of $E^+ \oplus E^-$ inside $\mathbb{C}^j$ and define the projector $[P]$ as the class of $\tilde{P} := \begin{bmatrix} P & 0 \\ 0 & I_{E^\perp} \end{bmatrix}$ modulo $\Psi^{-\infty}_{\rm sus}(M, \mathbb{C}^j)$.

PROPOSITION 6. *The image of the K-Theory class of the projector $[P]$ through the boundary map (10) is* $\delta([P]) = \mathrm{index}(D)$.

Proof. The image of the projector $[P]$ through the boundary map δ is by definition the class of $\exp(2i\pi\tilde{P})$ (i.e. its winding number) inside $\pi_1(1+M_j(\mathtt{k}))$. But in fact $\exp(2i\pi\tilde{P})$ belongs to $1+\Psi^{-\infty}_{\rm sus}(M, \mathbb{C}^j)$ since $\tilde{P}$ is a projector modulo $\Psi^{-\infty}_{\rm sus}(M, \mathbb{C}^j)$. For such loops the winding number is identified to half the eta-invariant (7) by [9, Proposition 8, (5.3) and (9.1)] and is given by the formula

$$\begin{aligned} w\left(e^{2i\pi\tilde{P}}\right) &= \frac{1}{2}\eta(e^{2i\pi\tilde{P}}) \\ &= \frac{1}{i}\overline{\mathrm{Tr}}(e^{-2i\pi\tilde{P}}\partial_\tau\left(e^{2i\pi\tilde{P}}\right)) \\ &= \frac{1}{i}\overline{\mathrm{Tr}}(e^{-2i\pi P}\partial_\tau\left(e^{2i\pi P}\right)). \end{aligned}$$

Since $\overline{\mathrm{Tr}}$ is a trace on $\Psi_{\rm sus}(M)$ and $t \mapsto \exp(itP)$ is a group morphism, it follows that

$$\overline{\mathrm{Tr}}(e^{-2i\pi P}\partial_\tau\left(e^{2i\pi P}\right)) = 2\pi i\overline{\mathrm{Tr}}(\partial_\tau(P))$$

and so

$$\delta([P]) = 2\pi\overline{\mathrm{Tr}}(\partial_\tau(P)) = -\pi\overline{\mathrm{Tr}}(\partial_\tau(Q^{-1}A)) = -\pi\widetilde{\mathrm{Tr}}(Q^{-1}A).$$

Let

$$\begin{aligned} Q_1 &:= (\tau^2 + D^*D + \phi(\tau)P_{\ker(D)})^{1/2} \\ Q_2 &:= (\tau^2 + DD^* + \phi(\tau)P_{\ker(D^*)})^{1/2}. \end{aligned}$$

Now $\widetilde{\mathrm{Tr}}(Q^{-1}A) = \widetilde{\mathrm{Tr}}((Q^{-1}A)_{11} + (Q^{-1}A)_{22})$ (in terms of the writing of $Q^{-1}A$ as a 2×2 matrix of operators) so

$$\frac{1}{\pi}\delta([P]) = \widetilde{\mathrm{Tr}}(Q_1^{-1}\tau) - \widetilde{\mathrm{Tr}}(Q_2^{-1}\tau).$$

Applying now Corollary 4 we write this as $\mathrm{Tr}_w(Q_1^{-1}) - \mathrm{Tr}_w(Q_2^{-1})$. In other words we proved

$$\begin{aligned} 2\delta([P]) &= \mathrm{Res}_{s=0}\overline{\mathrm{Tr}}(Q_1^{-s-1}) - \mathrm{Res}_{s=0}\overline{\mathrm{Tr}}(Q_2^{-s-1}) \\ &= \mathrm{Res}_{s=0}\int_{\mathbb{R}}(\mathrm{Tr}(Q_1^{-s-1}) - \mathrm{Tr}(Q_2^{-s-1}))d\tau. \end{aligned}$$

For each fixed τ we expand the traces in terms of eigensections of D^*D and DD^*:

$$\begin{aligned} \mathrm{Tr}(Q_1^{-s-1}) &= \sum_{\lambda\in\mathrm{Spec}(D^*D)\setminus\{0\}}(\tau^2+\lambda)^{-\frac{s+1}{2}} + \sum_{\lambda\in\ker(D)}(\tau^2+\phi(\tau))^{-\frac{s+1}{2}} \\ \mathrm{Tr}(Q_2^{-s-1}) &= \sum_{\lambda\in\mathrm{Spec}(DD^*)\setminus\{0\}}(\tau^2+\lambda)^{-\frac{s+1}{2}} + \sum_{\lambda\in\ker(D^*)}(\tau^2+\phi(\tau))^{-\frac{s+1}{2}} \end{aligned}$$

where multiplicities are taken into account. It is well-known that

$$\mathrm{Spec}(D^*D)\setminus\{0\} = \mathrm{Spec}(DD^*)\setminus\{0\}$$

so the terms corresponding to non-zero eigenvalues cancel each-other. Thus

$$\mathrm{Tr}(Q_1(\tau)^{-s-1}) - \mathrm{Tr}(Q_2(\tau)^{-s-1}) = \mathrm{index}(D)(\tau^2+\phi(\tau))^{-\frac{s+1}{2}}. \tag{12}$$

We are left with

$$2\delta([P]) = \mathrm{index}(D)\cdot\mathrm{Res}_{s=0}\int_{\mathbb{R}}(\tau^2+\phi(\tau))^{-\frac{s+1}{2}}d\tau.$$

The proposition follows from the next lemma.

LEMMA 7. *For all non-negative Schwartz functions ϕ with $\phi(0) > 0$ we have*

$$\mathrm{Res}_{s=0}\int_{\mathbb{R}}(\tau^2+\phi(\tau))^{-\frac{s+1}{2}}d\tau = 2.$$

Proof. $(\tau^2 + \phi(\tau))^{-\frac{s+1}{2}}$ is a classical symbol of order $-s-1$, with asymptotic expansion

$$|\tau|^{-s-1} + |\tau|^{-s} S^{-2}(\mathbb{R})$$

where $S^{-2}(\mathbb{R})$ denotes the space of classical symbols of order -2 on $\mathbb{R}$. Let ψ be a smooth function such that $\psi(\tau) = 0$ for $|\tau| < 1/2$ and $\psi(\tau) = 1$ for $|\tau| \geq 1$. Then

$$(\tau^2 + \phi(\tau))^{-\frac{s+1}{2}} = \psi(\tau)|\tau|^{-s-1} + |\tau|^{-s} r(\tau),$$

for some $r(\tau) \in S^{-2}(\mathbb{R})$. First,

$$\int_{\mathbb{R}} |\tau|^{-s} r(\tau) d\tau$$

is absolutely integrable, hence holomorphic in s, for $\mathrm{Re}(s) > -1$. We decompose then the remaining term $\int_{\mathbb{R}} \psi(\tau)|\tau|^{-s-1} d\tau$ in the integral on the interval $[-1, 1]$ (which is entire in s) and the integral on $\mathbb{R} \setminus [-1, 1]$, which is computed explicitly and gives $\frac{2}{s}$.

We want to conclude that the boundary map δ is surjective. Indeed, if there existed a first-order differential operator on M of index 1 then surjectivity would be a consequence of Proposition 6. However such an operator does not always exist, for instance if $\dim(M)$ is odd. Thus we must extend Proposition 6 to pseudo-differential operators.

Let $D \in \Psi^1_{\mathrm{sus}}(M, E^+, E^-)$ be a suspended operator of order 1 which restricts to an elliptic operator on M for $\tau = 0$. It is fairly obvious (see [9, (3.2)]) that $\sigma_1(D(\tau))(x, \xi) = \sigma_1(D)(x, \xi, 0)$ for all $x, \xi \in T^*M$, $\tau \in \mathbb{R}$, in other words the principal symbol of $D(\tau)$ is independent of τ. Thus $D(\tau)$ is also elliptic for all $\tau \in \mathbb{R}$. Assume moreover that for $|\tau| < \epsilon$ the null-space of $D(\tau)$ as an operator on $C^\infty(M, E^+, E^-)$ is of constant dimension. Then the null-space of $D^*(\tau)$ is also of constant dimension for $|\tau| < \epsilon$ by the homotopy invariance of the index. Choose ϕ to be a non-negative smooth function on $\mathbb{R}$ supported in $[-\epsilon, \epsilon]$. It follows that $\phi(\tau) P_{\ker(D(\tau))}$, $\phi(\tau) P_{\ker(D^*(\tau))}$ are suspended operators of order $-\infty$. Let A be defined by (11) on $E^+ \oplus E^-$. Set

$$\widetilde{A^2} := \begin{bmatrix} \tau^2 + D^*D + \phi(\tau) P_{\ker(D(\tau))} & 0 \\ 0 & \tau^2 + DD^* + \phi(\tau) P_{\ker(D^*(\tau))} \end{bmatrix}.$$

Thus $\widetilde{A^2}$ is an invertible operator in $\Psi^2_{\mathrm{sus}}(M, E^+ \oplus E^-)$. Let

$$Q := (\widetilde{A^2})^{1/2}, \quad P := \frac{1 - Q^{-1}A}{2}.$$

Then P is a projector modulo $\Psi_{\mathrm{sus}}^{-\infty}(M, E^+ \oplus E^-)$ as before, and we add to it the identity of a complement bundle to make it into a matrix-valued projector $[P]$.

THEOREM 8. *Let $[P]$ be the above projector. Then the image of its class in $K_0(S^*_{\mathrm{sus}}M)$ through the boundary map (10) is*

$$\delta([P]) = \mathrm{index}(D).$$

Proof. Basically unchanged from the proof of Proposition 6 in the particular case of differential operators. Write as before

$$2\delta([P]) = \mathrm{Res}_{s=0} \int_{\mathbb{R}} (\mathrm{Tr}(Q_1^{-s-1}) - \mathrm{Tr}(Q_2^{-s-1}))d\tau$$

where

$$\begin{aligned} Q_1 &:= (\tau^2 + D^*(\tau)D(\tau) + \phi(\tau)P_{\ker(D(\tau))})^{1/2} \\ Q_2 &:= (\tau^2 + D(\tau)D^*(\tau) + \phi(\tau)P_{\ker(D^*(\tau))})^{1/2}. \end{aligned}$$

The difference is that now we must decompose the traces of Q_1^{-s-1}, Q_2^{-s-1} over the spectrum of $D(\tau)^*D(\tau)$, resp. $D(\tau)D(\tau)^*$ which varies with $\tau \in \mathbb{R}$. However, this does not affect the validity of the crucial identity (12) by the homotopy invariance of the index. Lemma 7 finishes the proof.

Note the similarity of Theorem 8 with [6, Proposition 3], which holds however only for differential operators. It would be interesting to investigate whether our results have an extension to the more general framework of differentiable groupoids.

COROLLARY 9. *The map $\delta : K_0(C(S^*_{\mathrm{sus}}M)) \to K_{-1}(C_0(\mathbb{R}, \mathtt{k}))$ is surjective.*

Proof. Choose a matrix-valued elliptic pseudo-differential operator in $\Psi^1(M, \mathbb{C}^N)$ of index 1. Such an operator always exists as a consequence of Fedosov's formula (see [4, Theorem 5.18]). We can extend its symbol $\sigma(y, \mu)$ to a smooth 1-symbol $\tilde{\sigma}(y, \mu, \tau)$ on $T^*_{\mathrm{sus}}M$ simply by extending $\sigma(y, \mu)|_{S^*M}$ to $S^*_{\mathrm{sus}}M$ arbitrarily and then extending by homogeneity 1 to $T^*_{\mathrm{sus}}M \setminus \{0\}$. Choose a suspended operator $D \in \Psi^1_{\mathrm{sus}}(M, \mathbb{C}^N)$ with symbol $\tilde{\sigma}$. From the discussion preceding Theorem 8 it follows that $D(\tau) \in \Psi^1(M, \mathbb{C}^N)$ is elliptic for all fixed τ. By homotopy invariance, the index of $D(\tau)$ is 1 for all $\tau \in \mathbb{R}$. By adding a perturbation in $\Psi^{-\infty}_{\mathrm{sus}}(M)$ we can moreover make $\ker(D(\tau))$ of constant dimension 1 for $|\tau| < \epsilon$. Applying Theorem 8 to this perturbed D implies the result.

In conclusion $K_0(\Psi^0_{\mathrm{sus}}(M)) = \ker(\delta)$, $K_1(\Psi^0_{\mathrm{sus}}(M)) = K^1(S^*_{\mathrm{sus}}M)$.

6. An alternate approach to Theorem 8

Corollary 9 implies the computation in [11, Corollary 6] of the K-Theory of $\Psi^0_{\mathrm{sus}}(M)$. In fact, Theorem 8 is equivalent to [11, Theorem 8] which states that for all $a \in K^0(S^*_{\mathrm{sus}}M)$

$$\delta(a) = (-1)^n \int_{S^*_{\mathrm{sus}}M} \mathrm{Ch}(a) p^* \mathcal{T}(M), \tag{13}$$

where Ch is the Chern character, $\mathcal{T}(M)$ is the Todd genus of $TM \otimes \mathbb{C}$ and $p : S^*_{\mathrm{sus}}M \to M$ is the projection. Indeed, the Atiyah-Singer index formula suggests a strong link between (13) and Theorem 8.

We sketch the proof below. Let a denote the class in $K^0(S^*_{\mathrm{sus}}M)$ of the projector P from Theorem 8. Let $U \in C^\infty(S^*_{\mathrm{sus}}M, p^*E^+, p^*E^-)$ be the principal symbol of D. The class a is the class of the vector bundle F on $S^*_{\mathrm{sus}}M$ defined by the span of eigenvectors of positive eigenvalue of the self-adjoint invertible endomorphism of $E^+ \oplus E^-$

$$\begin{bmatrix} -\tau & U^* \\ U & \tau \end{bmatrix}.$$

The bundle F is the set of all vectors of the form

$$(v, \left((\tau^2 + UU^*)^{1/2} - \tau\right)^{-1} Uv) \in p^*E^+ \oplus p^*E^- \tag{14}$$

or equivalently

$$\left((\tau^2 + U^*U)^{1/2} + \tau\right)^{-1} U^*w, w) \in p^*E^+ \oplus p^*E^-. \tag{15}$$

The projection onto the first, respectively onto the second factor is an isomorphism from $F|_{\{\tau<1\}}$ to $p^*E^+|_{\{\tau<1\}}$, respectively from $F|_{\{\tau>-1\}}$ to $p^*E^-|_{\{\tau>-1\}}$. Thus F is obtained by glueing p^*E^+ and p^*E^- over $\{\tau = 0\}$ via the composite isomorphism $(UU^*)^{-1/2}U$. This isomorphism is homotopic to the symbol U, so F (hence the class a) is just the index bundle of D. Theorem 8 follows now from the classical index formula of Atiyah-Singer and (13). Going backwards along this argument proves (13) from Theorem 8 using the Atiyah-Singer formula and a bit of topological K-Theory.

This approach is quite unsatisfactory. Indeed, the Atiyah-Singer index theorem has been used twice in opposite directions, once to identify the analytic index with the right-hand side of (13), and a second time to prove (13) (the starting point in the proof of [11, Theorem 8] is exactly the index theorem; the rest of the argument uses the algebra of b-pseudodifferential operators and the K-Theory of its norm completion, computed in the body of [11]). It seems that for our limited purposes the approach of [11] is unnecessarily complicated and moreover it obscures the intrinsic properties of the algebra $\Psi_{\mathrm{sus}}(M)$. In light of our results the "higher indices" from [11] become nothing else than usual indices of elliptic operators, at least when the dimension of the suspension is 1.

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