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Adiabatic limits of eta and zeta functions of elliptic operators

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Abstract We extend the calculus of adiabatic pseudo-differential operators to study the adiabatic limit behavior of the eta and zeta functions of a differential operator δ, constructed from an elliptic family of operators with base S^1. We show that the regularized values $\eta(\delta_t, 0)$ and $t\zeta(\delta_t, 0)$ have smooth limits as $t \to 0$, and we identify the limits with the holonomy of the determinant bundle, respectively with a residue trace. For invertible families the functions $\eta(\delta_t, s)$ and $t\zeta(\delta_t, s)$ are shown to extend smoothly at $t = 0$. After normalizing with a Gamma factor, the zeta function satisfies in the adiabatic limit an identity reminiscent of the Riemann zeta function, while the eta function converges to the volume of the Bismut-Freed meromorphic family of connection 1-forms.

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1 Introduction

The eigenvalues of elliptic operators on compact manifolds possess miraculous properties. Starting with the foundational work of Minakshisundaram and Pleijel [10], we know for instance that the zeta function

$$s \mapsto \sum_{\mathrm{Spec}(D)\ni\lambda\neq 0} |\lambda|^{-s/2}$$

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of a second order self-adjoint elliptic pseudo-differential operator D on a closed manifold M is analytic in the half-plane $\{\Re(s) > \dim(M)\}$ and extends analytically to $\mathbb{C}$ with simple poles at $\dim(M) - \mathbb{N}$. If D is self-adjoint of order 1, a refinement of the zeta function of D^2 is the eta function of Atiyah, Patodi and Singer [1],

$$s \mapsto \sum_{\mathrm{Spec}(D)\ni\lambda\neq 0} \mathrm{sign}(\lambda)|\lambda|^{-s}.$$

This function is also meromorphic on $\mathbb{C}$ with simple poles at $\dim(M) - \mathbb{N}$.

To grasp the significance of these facts, one must take into account that it is generally impossible to determine all the eigenvalues of a given operator, or to decide which discrete subsets of $\mathbb{R}$ can occur as spectra of self-adjoint elliptic operators. The pole at $s = \dim(M)$ of the zeta function is related to the Weyl asymptotic formula for the eigenvalues, but the next poles are more subtle. Even when one can find explicitly the eigenvalues, there is no obvious reason why the series defining the zeta function should extend analytically to $\mathbb{C}$. In the simplest non-trivial case where M is the circle and $D := i\partial_\theta$ is the angular derivative, the zeta function of the Laplacian D^2 equals the Riemann zeta function up to a factor of 2. Beyond the usual elementary tricks there exists therefore a more fundamental reason for the analytic extension of the classical function $\zeta(s)$, that is the fact that $\mathbb{Z}$ is the spectrum of a differential operator.

One approach to the analysis of $\zeta(s)$ particularly unsuitable for generalizations seems to be the infinite product formula over the primes. Number theorists introduce in this product an additional factor $\Gamma(s/2)$, corresponding to the "prime" 0. Then the normalized Riemann zeta function $\overline{\zeta}(s) := \Gamma(s/2)\zeta(s)$ satisfies the functional equation

$$\overline{\zeta}(1-s) = \pi^{1/2-s}\overline{\zeta}(s).$$

Amazingly enough, we have found in the context of adiabatic limits a functional identity (Theorem 22) in terms of similarly normalized zeta functions. In the same context, the adiabatic limit of the eta function appears to be related to the meromorphic family of connection 1-forms on the determinant line bundle constructed by Bismut and Freed [3] if, again, both are renormalized with appropriate Gamma factors (Theorem 20).

Our initial motivation was to give a simple proof of the holonomy formula of [3] relating the adiabatic limit of the eta invariant to the determinant bundle of a family of Dirac operators on the circle as conjectured by Witten [17]. However, the results turn out to be valid for the normalized eta and zeta functions themselves and not just for their values at $s = 0$ (for comparison, while it is interesting to know that the Riemann zeta function equals $-1/2$ at $s = 0$, it is certainly desirable to say something about it at other points as well). As a corollary we give a formula for the adiabatic limit of the determinant.

This paper treats general elliptic first order differential operators and not only Dirac operators, which is in our view a significant breakthrough.

Indeed, previous works on eta invariants use the geometric construction of the heat kernel and the so-called rescaling technique of Getzler, which is intimately related to the Clifford algebra. These methods are unlikely to extend to the general case, therefore suggesting that Dirac operators enjoy special properties among first-order operators. We show that this is not the case for adiabatic limits of the eta and zeta functions (albeit it is certainly true for statements like the local index theorem).

Our approach is therefore completely different. We construct the calculus $\Psi_{ae}(X;M)$ of extended adiabatic operators, which extends (as the notation shows) the calculus $\Psi_a(N)$ of adiabatic pseudo-differential operators introduced in [6], see also [14] for an approach using differentiable groupoids. The calculus $\Psi_a(N)$ was originally constructed for the study of adiabatic limits in the framework of Melrose's programme for quantizing singular geometric structures [7]. An adiabatic limit means blowing up the metric on the base of a fibration of manifolds by a factor t^{-2} as $t \to 0$. The Laplacian of this metric will stop being elliptic when the parameter t reaches 0. The basic idea is to force this operator to be elliptic at $t = 0$ in an appropriate sense. Instead of heat operators we use complex powers, which are straightforward to construct inside the adiabatic algebra.

A immediate gain of our point of view is replacing the limit as $t \to 0$ by a Laurent-type singularity at the end point of the closed interval $[0, \infty)$. In particular, we deduce, under some invertibility hypothesis, that the eta function has a Taylor expansion in the adiabatic limit, thus we can rule out for instance $\log t$-like terms on a priori grounds.

The regularity at $s = 0$ of the eta function is not used in the proof. This is actually not so surprising, since our formulas are valid (in the sense of Laurent coefficients) also at the poles of the eta function.

Let us describe briefly the contents of the paper. In Section 2 we introduce the basic objects and give an overview of the results. The Dirac operator on the total space of a fibration over the circle is analyzed in Section 3. The determinant line bundle of families over S^1 is reviewed in Section 5, and analytic extensions of zeta-type functions in Section 4. Section 6 deals with the adiabatic algebra and its properties. The main results are stated and proved in Section 7 in the invertible case and in Section 8 in the general case. Finally, in Section 9 we state without proof some related results.

The results were announced in a note in Comptes rendus [12].

Acknowledgements I owe to Richard Melrose the idea of applying the adiabatic algebra to the study of the eta invariant. Jean-Michel Bismut showed to me how to treat the non-invertible case in Section 8. Section 3 is due to Andrei Moroianu (although Proposition 1 may be extracted from [3,2]). The appearance of Gamma functions in the formulas was suggested by Nicuşor Dan. I have also benefited from the advice of Christian Bär, Mattias Dahl and Robert Lauter. Last but not least, I am grateful to the anonymous referee for very pertinent remarks and for detecting many mistakes, both in presentation and in substance.

2 Preliminaries

Let N be a closed Riemannian manifold, E a Hermitian vector bundle over N and δ a first order elliptic differential operator on the sections of E. We define the normalized zeta function of δ by

$$\overline{\zeta}(\delta, s) := \Gamma\left(\frac{s}{2}\right) \sum_{0 \neq \lambda \in \mathrm{Spec}(\delta^*\delta)} \lambda^{-s/2} + \dim\ker(\delta). \tag{1}$$

If δ is self-adjoint, we define similarly its normalized eta function by

$$\overline{\eta}(\delta, s) := \pi^{-\frac{1}{2}} \Gamma\left(\frac{1+s}{2}\right) \sum_{0 \neq \lambda \in \mathrm{Spec}(\delta)} \mathrm{sign}(\lambda)|\lambda|^{-s} + \dim\ker(\delta).$$

These functions are holomorphic in the half-plane $\{\Re(s) > \dim N\}$ and extend analytically to $\mathbb{C}$ with possible simple poles at $\dim N - 2k$, respectively $\dim N - 2k - 1$, for $k \in \mathbb{N}$ (Corollary 4). The definitions differ from the usual zeta and eta functions by some Gamma factors which will turn out to be the key for the validity of our results outside $s = 0$. By the result of [1], $\eta(\delta, s)$ is finite at $s = 0$, hence the regularized value of $\overline{\eta}(\delta, s)$ at $s = 0$ coincides with the refined eta-invariant of Atiyah-Patodi-Singer. We stress however that the results of this paper are independent of this regularity.

Suppose that N is the total space of a locally trivial fibration of compact manifolds $N \xrightarrow{p} M$. We are interested in the restriction of this fibration to a circle in the base, we can therefore assume that $M = S^1$. Let $E = E^+ \oplus E^-$ be a $\mathbb{Z}/2\mathbb{Z}$-graded Hermitian bundle over N and D a family of elliptic differential operators of order 1 over the fibers of p. In other words, for every $x \in S^1$, the operator $D_x : C^\infty(N_x, E^+|_{N_x}) \to C^\infty(N_x, E^-|_{N_x})$ is elliptic and depends smoothly on $x \in S^1$.

We fix a connection in the fibration $N \to S^1$ and a family of metrics g on the fibers extended by 0 on the horizontal distribution. For any vector Y tangent to S^1 we denote by $\tilde{Y}$ its horizontal lift to N. We also fix differential operators $\nabla_{\tilde{Y}}$ on the sections of $E^\pm$ with the property that $(\nabla_{\tilde{Y}} e_1, e_2) + (e_1, \nabla_{\tilde{Y}} e_2) = \tilde{Y}(e_1, e_2)$. Such operators arise for instance as the restriction to horizontal vectors of metric connections in $E^\pm$. Let $\mathrm{tr}(L_{\tilde{Y}} g)$ be the contraction by g of the Lie derivative of the tensor g. Since g is vertical and $\tilde{Y}$ is horizontal, this expression is tensorial in Y. One sees easily that the operators

$$\tilde{\nabla}_Y := \nabla_{\tilde{Y}} + \frac{1}{4}\mathrm{tr}(L_{\tilde{Y}} g)$$

are skew-symmetric. If $|dg|$ is the volume density induced by the metric g then $L_{\tilde{Y}}|dg| = \frac{1}{2}\mathrm{tr}(L_{\tilde{Y}} g)|dg|$.

On S^1 we fix the canonical metric $d\theta^2$ (any other metric is isometric to a multiple of $d\theta^2$). Let ∂_θ be the positively oriented unit vector field on S^1. For $t \in [0, \infty)$ consider the self-adjoint operator

$$\delta_t := \begin{bmatrix} -ti\tilde{\nabla}_{\partial_\theta} & D^* \\ D & ti\tilde{\nabla}_{\partial_\theta} \end{bmatrix}. \tag{2}$$

acting on $C^\infty(N,E)$. Such an operator arises for instance in [3], where the fibers of $N \to S^1$ are even-dimensional and carry a continuous spin structure, $E^\pm$ are the spinor bundles of the fibers and ∇ is the Levi-Civita connection. In that case δ_t is the Dirac operator on N associated to the metric

$$g_t^N := g + \frac{d\theta^2}{t^2}. \tag{3}$$

Note that δ_t is elliptic if and only if $t > 0$; nonetheless, δ_t is an elliptic *adiabatic* family of operators including at the limiting value $t = 0$, in the sense of Section 6.

The purpose of this paper is to relate spectral invariants of the family $\{D_x\}_{x\in S^1}$ on the fibers to those of the operators $\{\delta_t\}_{t\in(0,\infty)}$.

In the case when D is a family of twisted Dirac operators, a conjecture of Witten [17] proved by Bismut-Freed [3] states that $\lim_{t\to 0}\exp(-i\pi\eta(\delta_t))$ exists and equals the holonomy of the determinant line bundle $\det(D)$ over S^1. This turns out to be true for any family D of first order elliptic differential operators, and moreover we prove that $\overline{\eta}(\delta_t)$ is smooth in t (modulo $2\mathbb{Z}$) for $t \in [0,\infty)$. If the family D is invertible, we prove that the function $\overline{\eta}(\delta_t, s)$ admits a Taylor expansion at $t = 0$ with coefficients meromorphic functions on $\mathbb{C}$. The limit as $t \to 0$ is computed explictly in terms of the connection 1-form on the determinant line bundle.

Similar results hold for the zeta function. In the invertible case, we show that the family $t\overline{\zeta}(\delta_t, s)$ admits a Taylor expansion in powers of t with coefficients meromorphic functions on $\mathbb{C}$. We show that $\lim_{t\to 0} t\overline{\zeta}(\delta_t, s)$ equals the average of $\overline{\zeta}(D, s-1)$ over S^1. As corollaries we obtain formulas for the adiabatic limit of $t\zeta(\delta_t, 0)$ and of $t\log\det(\delta_t)$. The later formula takes a simpler form when the manifold N is odd-dimensional. The former is valid for general elliptic families D.

We close this section with a remark about the Gamma functions. The adiabatic limit process involves passing from a $(n+1)$-dimensional manifold to a family of n-dimensional manifolds. From zeta-type functions of operators on N one obtains in the limit a factor

$$f(s) := \int_{-\infty}^{\infty} (1+\tau^2)^{-s} d\tau$$

where τ encodes the missing dimension. We note the identity

$$f(s) = \frac{\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}. \tag{4}$$

3 The lifted Dirac operator

This section computes an important example of lifting an operator from the fibers of a fibration to the total space. This example motivates the definition (2) of the family δ_t.

Let $p : X \to M$ be a fibration of closed manifolds, whose fibers are oriented Spin Riemannian manifolds of dimension $2n$, and assume that there exists a continuous spin structure on the fibers, meaning that TX/M is associated to a principal $Spin_{2n}$ bundle which is a double cover of the orthonormal frame bundle of the fibers. We denote by $\{D_x\}_{x\in M}$ the associated family of Dirac operators. We fix a connection in the fibration p, i.e. a choice of a horizontal distribution in X. This allows us to extend the family of fiberwise metrics to a 2-tensor g on the total space of X. Any metric g^M on M induces by pull-back, together with g, a Riemannian metric on X:

$$g^X = g + p^* g^M.$$

We define a connection ∇ on the vertical sub-bundle TX/M of TX as the vertical projection of the Levi-Civita connection of the metric g^X on X. This depends on the tensor g, but not on the metric g^M on the base. We also denote by ∇ the induced connection on the spinors over the fibers. For all vectors fields Y on M, we define a first-order differential operator $\tilde{\nabla}_Y$ on sections of the spinor bundles by

$$\tilde{\nabla}_Y := \nabla_{\tilde{Y}} + \frac{1}{4}\mathrm{tr}(L_{\tilde{Y}} g). \tag{5}$$

Here tr means the contraction by g and $L_{\tilde{Y}} g$ is the Lie derivative of the tensor g in the direction of the horizontal lift $\tilde{Y}$ of Y. If α is a vertical 1-form, then for all $f \in C^\infty(M)$ we have $L_{\widetilde{fY}}(\alpha) = p^* f L_{\tilde{Y}}(\alpha)$, hence $\mathrm{tr}(L_{\tilde{Y}} g)$ is tensorial in Y. Therefore, $\tilde{\nabla}$ can be thought of as a connection in the (infinite dimensional) vector bundle over M of fiber-wise sections of the spinor bundles, and the correction term ensures that this connection preserves the L^2 metric. It is $\tilde{\nabla}$ that occurs both in the Bismut-Freed connection form and in the lifted Dirac operator.

Let S^1 be a smooth loop embedded in M, and $N := p^{-1}(S^1)$. Fix the trivial (non-bounding) spin structure on S^1. Then N inherits a spin structure, and moreover the spinor bundle is isomorphic to $S^+ \oplus S^-$. The Clifford action of the unit horizontal vector ∂_θ is simply $\mp i$ on $S^\pm$. This is one of the two possible choices, the other one being $\pm i$. We choose this sign in order to get Witten's original sign in the holonomy formula [17].

Let now δ_t be the Dirac operator for the base metric scaled by t^{-2}, i.e. for the metric

$$g_t^N = g + t^{-2} g^M. \tag{6}$$

Proposition 1. *In matrix form, using the splitting $S = S^+ \oplus S^-$, we have*

$$\delta_t = \begin{bmatrix} -ti\tilde{\nabla}_{\partial_\theta} & D^* \\ D & ti\tilde{\nabla}_{\partial_\theta} \end{bmatrix}.$$

Proof: The Levi-Civita connection ∇^t on (N, g_t^N) satisfies the identities

$$\begin{aligned} \nabla^t_U V &= \nabla_U V - \frac{t^2}{2}(L_{\tilde{\partial}_\theta} g)(U,V)\tilde{\partial}_\theta \\ (\nabla^t_{t\tilde{\partial}_\theta} U, V) &= t(\nabla_{\tilde{\partial}_\theta} U, V), \end{aligned}$$

where U, V are vertical vectors and $\tilde{\partial}_\theta$ is the horizontal lift of the unit tangent vector to S^1 for the initial metric g^M. Let ϕ be a section of the spinor bundle $S = S^+ \oplus S^-$. Locally, ϕ is determined by a pair $[P, \sigma]$, where P is a local section of the Spin bundle which sits over a local orthonormal frame $(e_1, \ldots, e_{2n}, t\tilde{\partial}_\theta)$, and σ lives in the standard spinor representation Σ_{2n+1}. For any vector $V \in TN$, we have

$$\begin{aligned}\nabla^t_V(\phi) = [P, V(\sigma)] &+ \frac{1}{2}\sum_{i<j\le 2n} (\nabla^t_V e_i, e_j) c(e_i) c(e_j)\phi \\ &+ \frac{1}{2}\sum_{i\le 2n} (\nabla^t_V e_i, t\tilde{\partial}_\theta) c(e_i) c(t\tilde{\partial}_\theta)\phi.\end{aligned}$$

Recall that $t\tilde{\partial}_\theta$ has length 1 for g^N_t and that $c(t\tilde{\partial}_\theta)$ equals $\mp i$ on $S^\pm$. Therefore

$$\begin{aligned}\delta_t\phi &= \sum_{k=1}^{2n} c(e_k)\left([P, e_k(\sigma)] + \frac{1}{2}\sum_{i<j\le 2n}(\nabla^t_{e_k} e_i, e_j)c(e_i)c(e_j)\phi\right.\\ &\quad \left.+\frac{1}{2}\sum_{i\le 2n}(\nabla^t_{e_k} e_i, t\tilde{\partial}_\theta)c(e_i)c(t\tilde{\partial}_\theta)\phi\right) + c(t\tilde{\partial}_\theta)\nabla^t_{t\tilde{\partial}_\theta}(\phi)\\ &= \begin{bmatrix} -it\nabla_{\tilde{\partial}_\theta} & D^* \\ D & it\nabla_{\tilde{\partial}_\theta}\end{bmatrix}\phi + \frac{1}{2}\sum_{k=1}^{2n}\sum_{i\le 2n}(\nabla^t_{e_k} e_i, t\tilde{\partial}_\theta)c(e_k)c(e_i)c(t\tilde{\partial}_\theta)\phi\\ &= \left(\begin{bmatrix} -it\nabla_{\tilde{\partial}_\theta} & D^* \\ D & it\nabla_{\tilde{\partial}_\theta}\end{bmatrix} - \frac{t}{4}c(t\tilde{\partial}_\theta)\sum_{k,i=1}^{2n}(L_{\tilde{\partial}_\theta} g)(e_k, e_i)c(e_k)c(e_i)\right)\phi\\ &= \left(\begin{bmatrix} -it\nabla_{\tilde{\partial}_\theta} & D^* \\ D & it\nabla_{\tilde{\partial}_\theta}\end{bmatrix} + \frac{t}{4}c(t\tilde{\partial}_\theta)\operatorname{tr}(L_{\tilde{\partial}_\theta} g)\right)\phi.\end{aligned}$$

□

4 Analytic extensions and regularized traces

This section contains a review of the basic results allowing one to define analytic extensions of zeta-type functions. Proofs are included for the benefit of the reader.

Recall that classical pseudo-differential operators on a closed manifold X have a characterization in terms of their Schwartz kernels on X^2. Namely, such a kernel is a distribution ψ on X^2 with *sing supp*(ψ) contained in the diagonal Δ_X; moreover, ψ must be *classical conormal* to Δ_X, in the sense that the Fourier transform in the normal directions of a cut-off of ψ near Δ_X is a classical symbol, i.e. a symbol admitting an asymptotic expansion in homogeneous components with step 1. Since we work with complex powers of operators, we allow classical symbols of any complex orders. By Peetre's Theorem, the differential operators are exactly those pseudo-differential operators with Schwartz kernel supported on the diagonal.

Proposition 2. *Let $P(s)$ be an entire family of classical pseudo-differential operators on X, such that $P(s) \in \Psi^s(X)$. Then* $\mathrm{Tr}(P(s))$, *which is* a priori *defined for $\Re(s) < -\dim X$, extends analytically to $\mathbb{C}$ with possible simple poles at integers $k \geq -\dim X$.*

Proof: For $\Re(s) < -\dim X$, $\mathrm{Tr}(P(s)$ equals the integral of the pointwise trace of the distributional kernel of $P(s)$ over the diagonal. By pulling back to the tangent bundle and then Fourier transforming in the fibers, this becomes the integral on T^*X of the pointwise trace of the full symbol of $P(s)$, which is an entire family $\{a(s)\}_{s\in\mathbb{C}}$ such that $a(s)$ is a classical symbol of order s. The claim follows from the next Lemma. □

Lemma 3. *Let $a(s)$ be an entire family of symbols of order s on a vector bundle V over X. Fix a density $d\mu$ on X and a Hermitian metric g in V. Then $\int_V a(s)d\mu dg$, which is well-defined for $\Re(s) < -\dim V$, extends analytically to $s \in \mathbb{C}$, with at most simple poles at real integers.*

Proof: If $a_k(s,v)$ is an entire family of symbols on V such that for $\|v\| \geq 1$, $a_k(s,v) = h\left(s, \frac{v}{\|v\|}\right)\|v\|^{k+s}$ where h is entire in s, then $\int_V a(s,v)dv$ splits in the integral on the unit ball bundle (which is entire) and the integral on the exterior of the unit ball, which can be computed explicitly in polar coordinates. We denote by $S(V)$ the sphere bundle inside V.

$$\begin{aligned}\int_{|v|\geq 1} a(s,v)dv &= \int_1^\infty \int_{S(V)} h(s,\theta) r^{k+s+\dim V-1} d\theta dr \\ &= -\frac{1}{s+k+\dim V}\int_{S(V)} h(s,\theta)d\theta.\end{aligned}$$

Any family of classical symbols $a(s)$ of order s can be decomposed as $a(s) = a_0(s) + \tilde{a}(s)$, where $a_0(s,v)$ is a family as above for $k=0$, and $\tilde{a}(s)$ is an entire family of symbols of order $s-1$. Then $\int a_0(s,v)dv$ has just one pole at $s = -\dim V$, while $\int \tilde{a}(s,v)dv$ is well-defined for $\Re(s) < -\dim V + 1$. The Lemma follows by iteration of this argument. □

Recall the definition of the Wodzicki residue trace: fix an elliptic first-order positive pseudo-differential operator Q on the m-dimensional manifold X. Then the complex powers of Q form an entire family of classical pseudo-differential operators [16]. By Proposition 2, for any classical pseudo-differential operator $P \in \Psi^{\mathbb{Z}}(X)$ the function $s \mapsto \mathrm{Tr}(Q^{-s}P)$, *a priori* defined for $\Re(s) > m + \deg(P)$, extends analytically to $\mathbb{C}$ with possible simple poles at real integers $k \leq m + \deg P$. The map

$$P \mapsto \mathrm{Res}_{s=0}\mathrm{Tr}(Q^{-s}P) := \mathrm{Tr}_w(P)$$

defines a functional on $\Psi^{\mathbb{Z}}(X)$ which vanishes on $\Psi^{-\dim(X)-1}(X)$. Moreover, as the notation suggests, Tr_w is independent of the choice of Q. From the definition, it follows easily that Tr_w vanishes on commutators (i.e. Tr_w is a trace), and that it is given by the local expression

$$\mathrm{Tr}_w(P) = \int_{S^*X} p_{[-m]} \mathcal{R} \lrcorner \omega^m \tag{7}$$

where $p_{[-m]}$ is the homogeneous component of homogeneity $-m$ in the full symbol of P, S^*X is the sphere bundle of T^*X, $\mathcal{R}$ is the radial vector field on T^*X and ω is the standard symplectic form. As a consequence, the right-hand side of (7) is independent of the quantization defining the full symbol of P.

Corollary 4. *The functions $\overline{\zeta}(\delta, s)$, $\overline{\eta}(\delta, s)$, respectively the family of* 1-*forms defined in* (9)*, admit analytic extensions to $\mathbb{C}$ with possible simple poles at integers* $\dim(N) - 2\mathbb{N}$*, respectively* $\dim(N) - 1 - 2\mathbb{N}$.

Proof: The claim follows directly from Proposition 2 with the exception of possible poles at $\dim N - 2\mathbb{N} - 1$, respectively $\dim N - 2 - 2\mathbb{N}$ and possible additional poles introduced by the Gamma factors. The first type of poles does not arise because of the (anti)-symmetry of polynomial functions. The poles of $\Gamma(s/2)$ occur at $s \in -2\mathbb{N}$. At these values the zeta function is finite, since its residue equals the residue trace of a differential operator and hence vanishes. The function $\Gamma(s + 1/2)$ introduces additional poles at $s \in -2\mathbb{N} - 1$. The eta function is finite at these points, since its residue equals $\mathrm{Tr}_w(\delta^{-2k+1}) = 0$. For the Bismut-Freed connection form, the poles of $\Gamma(s/2+1)$ occur at $s = 2k-2$ for k a negative integer, and one sees that $\mathrm{Tr}((D_U^* D_U)^{-\frac{s}{2}} D_U^{-1} \tilde{\nabla}(D_U))$ is also regular at these points. $\square$

5 The determinant line bundle over S^1

The determinant line bundle $\det(D)$ is a complex line bundle associated to every family of elliptic operators. It has a canonical connection due to Bismut-Freed [3]. In the case where the base is S^1 this bundle is trivial, like every complex vector bundle. Then it is possible to find explicitly the connection form in special trivializations. Using (10) we will get a formula for the holonomy along S^1 without having to deal with open covers; rather, the cover is encoded in the connection form.

Let $\mathcal{E}^{\pm}$ be the infinite-dimensional vector bundles over S^1 whose fiber over $x \in S^1$ is the space of smooth sections in $E^{\pm}$ over N_x. For $\alpha \in \mathbb{R}_+$ let $V_\alpha := \{x \in S^1; \alpha \notin \mathrm{Spec}(D^*D)\}$. For every $x \in S^1$, let $\mathcal{E}_x^{\pm} = \mathcal{E}_{x,\leq\alpha}^{\pm} \oplus \mathcal{E}_{x,>\alpha}^{\pm}$ be the decomposition of $\mathcal{E}_x^{\pm}$ in subspaces spanned by eigensections of D^*D, respectively of DD^*, of eigenvalue smaller or strictly larger than α. Then $\mathcal{E}_{<\alpha}^{\pm}$ form vector bundles over V_α but in general not over S^1. Nevertheless, if we define $\det(D)_\alpha \to V_\alpha$ as the line bundle $\Lambda^{\mathrm{top}}(\mathcal{E}^+_{<\alpha})^* \otimes \Lambda^{\mathrm{top}}(\mathcal{E}^-_{<\alpha})$ then $\det(D)_\alpha$ is isomorphic to $\det(D)_\beta$ over $V_\alpha \cap V_\beta$ via the determinant of the linear isomorphism of finite dimensional vector spaces

$$D : \mathcal{E}^+_{<\alpha,>\beta} \to \mathcal{E}^-_{<\alpha,>\beta}.$$

These isomorphisms clearly fulfill the cochain condition so one obtains a line bundle over S^1 called $\det(D)$.

$\det(D)_\alpha$ inherits the connection $\tilde{\nabla}$ defined in (5). The Bismut-Freed connection on $\det(D)_\alpha$ is defined as

$$\nabla^{bf}{}_\alpha := \tilde{\nabla} + A_\alpha(0)$$

where $A_\alpha(0)$ is the regularized value in $s = 0$ of the meromorphic extension of the family of 1-forms

$$A_\alpha(s) := \Gamma\left(1 + \frac{s}{2}\right) \mathrm{Tr}_{>\alpha}((D^*D)^{-\frac{s}{2}} D^{-1} \tilde{\nabla}(D))$$

which is well defined and analytic for $\Re(s) > n$. One checks easily that $\nabla^{bf}{}_\alpha$ is mapped to $\nabla^{bf}{}_\beta$ under the isomorphism $\det(D)_\alpha \to \det(D)_\beta$, so we finally get a connection ∇^{bf} on $\det(D)$.

Lemma 5. *There exist trivial finite-dimensional vector bundles $U^\pm$ over S^1 and maps of vector bundles $D_{12} : U^+ \to \mathcal{E}^-$, $D_{21} : \mathcal{E}^+ \to U^-$, $D_{22} : U^+ \to U^-$, such that the operator*

$$D_U := \begin{bmatrix} D & D_{12} \\ D_{21} & D_{22} \end{bmatrix} : C^\infty(S^1, \mathcal{E}^+ \oplus U^+) \to C^\infty(S^1, \mathcal{E}^- \oplus U^-) \qquad (8)$$

is invertible; moreover, there exists a finite cover $\mathcal{V} = \{V_{\alpha_1}, \ldots, V_{\alpha_k}\}$ of S^1 such that for $x \in V_\alpha \in \mathcal{V}$, D_U coincides with D on $\mathcal{E}^+_{x,>\alpha}$, and D_U maps $\mathcal{E}^+_{x,<\alpha} \oplus U^+$ onto $\mathcal{E}^-_{x,<\alpha} \oplus U^-$.

Proof: Let $\{V_{\beta_1}, \ldots, V_{\beta_l}\}$ be an open cover of S^1 by sets of the form V_α, and $\{V'_1, \ldots, V'_l\}$ a compact subcover. Let $\{\phi_j\}$ be a partition of unity with $\mathrm{supp}(\phi_j) \subset V_{\beta_j}$ and $\phi_j \equiv 1$ on V'_j. Fix isomorphisms $f_j : \mathcal{E}^+_{<\beta_j} \to \mathbb{C}^{r_j}$ over V_{β_j}. Let $U^- := \mathbb{C}^{\sum r_j}$ and $D_{21} := \sum \phi_j f_j : \mathcal{E}^+ \to U^-$. By construction, $D \oplus D_{21}$ is injective so its finite dimensional cokernel (called U^+) forms a vector bundle over S^1. Let $D_{12} \oplus D_{22} : U^+ \to \mathcal{E}^- \oplus U^-$ be the inclusion map. Then clearly D_U defined by (8) is invertible. Choose the open cover $\mathcal{V}$ such that $\max(\beta_j) < \min(\alpha_i)$. The second statement of the lemma is easily checked. □

We trivialize U^+ and we endow $U^\pm$ with the trivial connections, denoted for simplicity $\tilde{\nabla}$ (so the connections on U^+ and on its image in $\mathcal{E}^- \oplus U^-$ are different).

Proposition 6. *The determinant line bundle with the Bismut-Freed connection is isomorphic to the trivial bundle $\mathbb{C}\times S^1 \to S^1$ with the connection $d+A(0)$, where $A(0)$ is the regularized value in $s=0$ of the meromorphic family of 1-forms*

$$A(D_U, s) := \Gamma\left(1+\frac{s}{2}\right) \operatorname{Tr}((D_U^* D_U)^{-\frac{s}{2}} D_U^{-1} \tilde{\nabla}(D_U)). \tag{9}$$

Proof: Let us compare the connections ${\nabla^{bf}}_{\alpha_j}$ corresponding to the operators D and D_U, for $\alpha_1, \ldots, \alpha_k$ as in Lemma 5. The meromorphic families of 1-forms $A_{\alpha_j}(s)$ coincide by the second part of Lemma 5. The bundles $\det(D)_{\alpha_j}$ and $\det(D_U)_{\alpha_j}$ are canonically isomorphic together with their connections, since $U^{\pm}$ are trivial with trivial connections. Thus $(\det(D), \nabla^{bf}) \simeq (\det(D_U), \nabla^{bf})$. Now $\det(D_U)$ can also be constructed using the cover $\{V_0\}$ because D_U is invertible. For this cover we have $\det(D_U)_0 = \mathbb{C}$ with trivial induced connection $\tilde{\nabla} = d$, and $A_0(s)$ is given by (9). □

In the case where the family D is invertible, one obtains a canonical family of 1-forms $A(D, s)$ by taking $U^{\pm} = 0$.

The Bismut-Freed connection preserves the Quillen metric on $\det(D)$ [15]. In the complex setting, it is just the Hermitian connection associated to that metric. We will call its holonomy around S^1 the holonomy of $\det(D)$. A simple computation shows

$$\operatorname{hol}(\det(D)) = \exp\left(-\int_{S^1} A(0)\right). \tag{10}$$

The results of this section can be adapted to the case of an arbitrary base manifold:

Proposition 7. *Let $D : E^+ \to E^-$ be a family of (classical pseudo-)differential elliptic operators of positive order on the fibers of a fibration $N \to B$ of compact manifolds. There exist finite-dimensional vector bundles $U^{\pm} \to B$ and maps of vector bundles $D_{12} : U^+ \to \mathcal{E}^-$, $D_{21} : \mathcal{E}^+ \to U^-$, $D_{22} : U^+ \to U^-$, such that the operator D_U defined in (8) is invertible. There exists a finite cover $\mathcal{V} = \{V_{\alpha_1}, \ldots, V_{\alpha_k}\}$ of B such that for $x \in V_\alpha \in \mathcal{V}$, D_U coincides with D on $\mathcal{E}^+_{x,>\alpha}$, and D_U maps $\mathcal{E}^+_{x,<\alpha} \oplus U^+$ onto $\mathcal{E}^-_{x,<\alpha} \oplus U^-$. Endow $U^{\pm}$ with connections $\tilde{\nabla}$, then*

$$(\det(D), \nabla^{bf}) \simeq (\mathbb{C}, d + A(D_U, 0)) \otimes \left(\Lambda^{\mathrm{top}}(U^+)^* \otimes \Lambda^{\mathrm{top}}(U^-), \tilde{\nabla}\right)^{-1}.$$

The proof is a simple adaptation of Lemma 5 and Proposition 6. We will not use this statement in the present paper.

6 The extended adiabatic algebra

Adiabatic operators were introduced in [6] for the study of the adiabatic limit of the Hodge cohomology groups. By a different approach using differentiable groupoids a slightly smaller calculus appeared in [14, Example 7]. We refer to [11, 13] for a detailed discussion of the scalar adiabatic algebra; the extension to bundles is straightforward.

In this section we will introduce a larger calculus Ψ_{ae} containing Ψ_a as a subcalculus. This calculus realizes the operators of the type we are interested in and their complex powers *at the limiting value* $t=0$. Therefore, it will suffice to study these limiting operators, which belong to the so-called *suspended* algebra [9].

Let $p: X \to M$ be a fibration of closed manifolds (in this paper, we will use $N \to S^1$), and $E^+, E^- \to X$ two vector bundles. An *adiabatic vector field* is a family $v: [0,\infty) \to \mathcal{V}(X)$, such that $v(0)$ is tangent to the fibers of p. These vector fields form a locally free $C^\infty(X \times [0,\infty))$-module ${}^a\mathcal{V}$, so they are the sections of a vector bundle over $X \times [0,\infty)$ called ${}^aTX$. Let $\tilde{E}^\pm$ denote the lifts of $E^\pm$ to $X \times [0,\infty)$ and U the universal enveloping algebra functor. An adiabatic family of differential operators from E^+ to E^- is an element in $\mathrm{Hom}(\tilde{E}^+, \tilde{E}^-) \otimes_{C^\infty(X\times[0,\infty))} U(C^\infty({}^aTX))$, i.e. a composition of adiabatic vector fields and bundle homomorphisms.

Let $X_a^2 = [X^2 \times [0,\infty); X \times_M X \times \{0\}]$ be the blow-up of the fiber diagonal of p at $t=0$ inside $X^2 \times [0,\infty)$. This means replacing $X \times_M X \times \{0\}$ by the half-sphere bundle of its positive normal bundle, and then gluing along geodesic rays to get the smooth structure. The result is a smooth manifold with corners of codimension 2, and the smooth structure is independent of the metric used to define geodesics.

Denote by β the canonical blow-down map from X_a^2 to $X^2 \times [0,\infty)$. The *lifted diagonal* Δ_a is by definition the closure of $\beta^{-1}(\Delta_X \times (0,\infty))$ inside X_a^2. It is easy to see that $\beta: \Delta_a \to \Delta_X$ is an diffeomorphism. Let $p_1, p_2 : X_a^2 \to X \times [0,\infty)$ be the composition of β with the projections on the first, respectively on the second X factor.

Lemma 8 ([13]). *The interior of the* front face ff_a *of* X_a^2 *introduced by the blow-up is canonically diffeomorphic to* $X \times_M X \times_M {}^aTM_{|t=0}$ *and therefore has a canonical vector bundle structure.*

One can view adiabatic families of differential operators as distributions on $X^2 \times [0,\infty)$ having Schwartz kernels conormal to $\Delta_X \times (0,\infty)$ and supported on it, but, because of the degeneracy, not all such kernels define adiabatic operators. In fact, the Schwartz kernels of adiabatic operators lift to the manifold X_a^2 defined above, and these lifts span *all* the distributions conormal to and supported on Δ_a, and extendable across ff_a. One must make precise the density bundle where these lifted kernels take values; this is $p_2^*\Omega({}^aTX)$, where Ω is the 1-density functor. In light of this fact, we make the following definition.

Definition 9. *An adiabatic pseudo-differential operator is a distributional section in* $E^- \boxtimes (\Omega({}^aTX) \otimes (E^+)^*)$ *over* X_a^2*, classical conormal to* Δ_a*, extendable across the front face and vanishing rapidly to all other boundary faces.*

For $s \in \mathbb{C}$, we denote the space of adiabatic operators of conormality order s by $\Psi_a^s(X, E^+, E^-)$.

In Section 8 we will need a larger calculus that we introduce now. Let $U^\pm \to M$ be vector bundles. Let

$$\begin{aligned} S_{12} &= [X \times M \times [0, \infty); X \times_M M \times \{0\}] \\ S_{21} &= [M \times X \times [0, \infty); M \times_M X \times \{0\}] \end{aligned}$$

be manifolds with corners obtained through the blow-up of "diagonal copies" of X. We call ff_{21}, respectively ff_{12} the new faces introduced by blow-up. Let $\Psi_{12}^{-\infty}(X, U^-, E^+)$ be the space of smooth sections over S_{12} in the bundle $E^+ \boxtimes (\Omega({}^aTM) \otimes (U^-)^*)$, which are rapidly vanishing to the boundary faces other than ff_{12}. We similarly define $\Psi_{21}^{-\infty}(X, E^-, U^+)$ as the space of smooth sections of $p_1^*(U^+) \boxtimes p_2^*(\Omega({}^aTX) \otimes (E^-)^*)$ over S_{21} which are rapidly vanishing at all boundary faces other than ff_{21}. Further, define

$$\Psi_{22}(M, U^+, U^-) := \Psi_a(M, U^+, U^-)$$

to be the space of adiabatic operators corresponding to the identity fibration $M \to M$, and $\Psi_{11}(X, E^+, E^-) := \Psi_a(X, E^+, E^-)$.

Theorem 10. *There exists natural composition maps for adiabatic operators:*

$$\Psi_{ij}^z \circ \Psi_{jk}^w \subset \Psi_{ik}^{z+w}.$$

Proof: In this proof we omit the bundles from the notation. We will construct appropriate *adiabatic triple spaces.* The result will follow from the properties of pull-back, product and push-forward operations on conormal distributions [7,8].

Let us first prove the theorem for the adiabatic algebra $\Psi_a(X)$. Let X_a^3 be the iterated blow-up of X^3,

$$X_a^3 := [X^3 \times [0, \infty); \mathcal{F}_3 \times \{0\}; \mathcal{F}_{12} \times \{0\}, \mathcal{F}_{23} \times \{0\}, \mathcal{F}_{13} \times \{0\}]$$

where

$$\begin{aligned} \mathcal{F}_3 &= \{(x_1, x_2, x_3) \in X^3; p(x_1) = p(x_2) = p(x_3)\}, \\ \mathcal{F}_{ij} &= \{(x_1, x_2, x_3) \in X^3; p(x_i) = p(x_j)\}, \quad i, j = 1, 2, 3 \end{aligned}$$

are fiber diagonals. We claim that there exist p-fibrations $p_{12}, p_{23}, p_{13} : X_a^3 \to X_a^2$, making the diagrams

$$\begin{array}{ccc} X_a^3 & \xrightarrow{\beta} & X^3 \times [0, \infty) \\ \Big\downarrow p_{ij} & & \Big\downarrow p_{ij} \\ X_a^2 & \xrightarrow{\beta} & X^2 \times [0, \infty) \end{array} \tag{11}$$

commute. By symmetry, it is enough to show this for $i = 1, j = 2$. Then p_{12} is defined as the composition of blow-down maps, isomorphisms and projections

$$\begin{aligned}
X_a^3 &= [X^3 \times [0,\infty); \mathcal{F}_3 \times \{0\}; \mathcal{F}_{12} \times \{0\}, \mathcal{F}_{23} \times \{0\}, \mathcal{F}_{13} \times \{0\}] \\
&\to [X^3 \times [0,\infty); \mathcal{F}_3 \times \{0\}, \mathcal{F}_{12} \times \{0\}] \\
&\simeq [X^2 \times X \times [0,\infty); \mathcal{F}_{12} \times \{0\}; \mathcal{F}_3 \times \{0\}] \\
&\to [X^2 \times X \times [0,\infty); \mathcal{F}_{12} \times \{0\}] \\
&\simeq [X^2 \times [0,\infty); X \times_M X \times \{0\}] \times X \\
&\to X_a^2.
\end{aligned}$$

The commutativity of the blow-up [8, Prop. 5.8.1] shows that (11) commutes.

One sees by continuity from the interior that

$$p_{12} : p_{13}^{-1}(\Delta_a) \to X_a^2, \tag{12}$$
$$p_{13} : p_{12}^{-1}(\Delta_a) \cap p_{23}^{-1}(\Delta_a) \to \Delta_a \tag{13}$$

are diffeomorphisms. For $A, B \in \Psi_a(X)$ define

$$A \circ B = (p_{13})_*(p_{12}^* A \cdot p_{23}^* B). \tag{14}$$

Clearly $p_{12}^* A$ is conormal to $p_{12}^{-1}(\Delta_a)$. The product is well-defined, since $p_{12}^* A$ and $p_{23}^* B$ are conormal to transverse submanifolds of X_a^3. The rapid vanishing of adiabatic operators to boundary faces other than ff_a shows that the push-forward is well-defined. Moreover, from (12), (13) and [8, Prop. 6.11.5], the push-forward is again a (classical) conormal distribution to Δ_a.

The composition rule for $\Psi_a(M)$ is a particular case of what we have just shown. The remaining six cases are essentially easier, since at least one of the terms involved is a smooth distribution, but less standard because of the non-symmetry. Let us only show, for instance, how to compose $A \in \Psi_a^z(X)$ with $B \in \Psi_{12}^{-\infty}(X)$. Define

$$S_{112} = [X \times X \times M \times [0,\infty); \mathcal{G}_3 \times \{0\}; \mathcal{G}_{12} \times \{0\}, \mathcal{G}_{23} \times \{0\}, \mathcal{G}_{13} \times \{0\}]$$

where

$$\begin{aligned}
\mathcal{G}_3 &= \{(x_1, x_2, x_3) \in X^2 \times M; p(x_1) = p(x_2) = x_3\}, \\
\mathcal{G}_{12} &= \{(x_1, x_2, x_3) \in X^2 \times M; p(x_1) = p(x_2)\}, \\
\mathcal{G}_{j3} &= \{(x_1, x_2, x_3) \in X^2 \times M; p(x_j) = x_3\}, j = 1, 2.
\end{aligned}$$

There exist p-fibrations $p_{12} : S_{112} \to X_a^2$, $p_{j3} : S_{112} \to S_{12}$, $j = 1, 2$, which commute with the blow-down maps $S_{112} \to X \times X \times M \times [0,\infty)$, $X_a^2 \to X^2 \times [0,\infty)$, $S_{12} \to X \times M \times [0,\infty)$ and the corresponding projections by analogy with (11). Define the composition of A, B by (14). There is no issue with the product now, since B is smooth. The only thing we need to note is that p_{13} maps $p_{12}^{-1}(\Delta_a)$ diffeomorphically onto S_{12}, thus the conormality is "integrated out" through push-forward and the result is smooth. $\square$

The above theorem is yet another materialization of Melrose's program of "microlocalizing boundary fibration structures" [7]. Note that the composition of operators in Ψ_a appears already in [6], and for compactly supported adiabatic operators it appears also in [14]. As a consequence, we get a pseudo-differential calculus $\Psi_{ae}(X, E^{\pm}, U^{\pm})$ formed by matrices

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{ij} belongs to Ψ_{ij}. The algebra $C^{\infty}([0,\infty))$ is central in Ψ_{ae}, so we can restrict adiabatic operators to any given t.

Remark 11. For $t > 0$, $A_{11}(t) \in \Psi(X, E^+, E^-)$, $A_{22}(t) \in \Psi(M, U^+, U^-)$, while the off-diagonal components are smoothing operators

$$A_{21}(t) : C^{-\infty}(M, U^+) \to C^{\infty}(X, E^-),$$
$$A_{12}(t) : C^{-\infty}(X, E^+) \to C^{\infty}(M, U^-).$$

This is an extension of the calculi of pseudo-differential operators on X and M that we call $\Psi_{ex}(X; M)$.

For $E^+ = E^- = E$ and $U^+ = U^- = U$, the space $\mathcal{A} = \Psi_{ae}^{\mathbb{Z}}(X, E, U)$ is an algebra. $\mathcal{A}$ has a natural double filtration $\{\mathcal{A}^{i,k}\}_{i\in\mathbb{Z}, k\in\mathbb{N}}$, increasing in the first and decreasing in the second superscript, i.e. $\mathcal{A}^{i_1,k_1} \subset \mathcal{A}^{i_2,k_2} \Leftrightarrow i_1 \leq i_2$ and $k_1 \geq k_2$. Namely, $\mathcal{A}^{i,k} = t^k \mathcal{A}^i$, where i denotes the conormality order while k indicates the order of vanishing at $t = 0$. This filtrations are compatible with the product on $\mathcal{A}$ by Theorem 10 (for the first filtration) and since $C^{\infty}([0,\infty))$ is central in $\mathcal{A}$ (for the second).

Lemma 12. *The quotient $\mathcal{A}^k / \mathcal{A}^{k-1}$ is canonically isomorphic to*

$$S^{[k]}({}^aT^*X \setminus 0, E) \oplus S^{[k]}({}^aT^*M \setminus 0, U),$$

where $S^{[k]}$ denotes symbols of pure homogeneity k.

Proof: Clearly Ψ_{12} and Ψ_{21} die in the quotient. On $\Psi_a(X)$, the isomorphism is induced by conormal principal symbol short exact sequence

$$0 \to \Psi_a^{k-1}(X, E) \hookrightarrow \Psi_a^k(X, E) \to S^{[k]}(N^*\Delta_a \setminus \{0\}, E) \to 0,$$

keeping in mind that the conormal bundle to Δ_a is ${}^aT^*X$. The second component comes from $\Psi_a(M)$, which is just a special case of the first. $\square$

Melrose [9] introduced a notion of suspension for pseudo-differential operators. If W is any vector bundle over M, one can define $\Psi_{\mathrm{sus}(W)}(X/M)$ as the space of families of pseudo-differential operators over the fibers of the fiber bundle $X \times_M W \to M$, which are translation invariant with respect to W, and whose Schwartz kernels decay rapidly away from the W-diagonal. These operators are identified (via Fourier transform in W) with their *indicial families*, i.e. families over M of parameter-dependent operators on the fibers of $X \to M$, with parameters in W^* having symbolic behavior.

Lemma 13. *The quotient* $\mathcal{A}_\partial := \mathcal{A}^{\mathbb{Z},0}/\mathcal{A}^{\mathbb{Z},1}$ *is canonically isomorphic to the algebra of matrices* $(a_{ij})_{1\leq i,j\leq 2}$ *such that*

1. $a_{11} \in \Psi_{\mathrm{sus}({}^aT^*M_{|t=0})}(X/M, E)$. *The suspending variables* τ *live naturally on* ${}^aT^*M_{|t=0}$*, which is canonically isomorphic to* T^*M.
2. $a_{22} \in S^{\mathbb{Z}}({}^aT^*M_{|t=0}, U)$*, the space of classical symbols on the vector bundle* ${}^aT^*M_{|t=0}$*, with values in* $U \otimes U^*$.
3. $a_{12} \in \mathcal{S}(X \times_M T^*M, E \boxtimes U^*)$, $a_{21} \in \mathcal{S}(T^*M \times_M X, U \boxtimes E^*)$*, where* $\mathcal{S}$ *denotes smooth sections rapidly vanishing at infinity.*

Proof: Consider the restriction of an adiabatic operator A to ff_a. This operation is well-defined, since the conormality locus Δ_a of A is transversal to ff_a, and A is extendable across ff_a. The interior of ff_a is canonically identified with the total space of the vector bundle $N(\Delta_a)|_{\Delta_a\cap \mathrm{ff}_a}$, and $A|_{\mathrm{ff}_a}$ vanishes rapidly at infinity. This means exactly that $A|_{\mathrm{ff}_a} \in \Psi_{\mathrm{sus}({}^aT^*M_{|t=0})}(X/M, E)$. Thus we can Fourier-transform $A|_{\mathrm{ff}_a}$ in each fiber, and we define the *normal operator* of A by

$$\mathcal{N}(A) := \mathcal{F}(A|_{\mathrm{ff}_a}).$$

Note that the density factor in $A|_{\mathrm{ff}_a}$ is used in the Fourier transform.

The second statement is a particular case of the first (note that in the case when the fiber is 0-dimensional, the indicial operator of suspended operators is just a classical symbol).

For the third statement, note that $\mathrm{ff}_{12}^\circ \simeq X \times_M TM$ and $\mathrm{ff}_{21}^\circ \simeq TM \times_M X$. The isomorphism of the lemma is again given by restriction to the front face followed by Fourier transform. □

The normal operator introduced in the above lemma is surjective, multiplicative, and plays the role of a "principal boundary symbol". Modulo choices, this symbol can be extended to a full boundary symbol q with values in the formal series algebra $\mathcal{A}_\partial[[t]]$.

Lemma 14. *The* boundary algebra $\mathcal{A}^{\mathbb{Z},\mathbb{N}}/\mathcal{A}^{\mathbb{Z},\infty}$ *is isomorphic as a vector space to* $\mathcal{A}_\partial[[t]]$.

Proof: The choices involved in the definition of q are a Riemannian metric on M, a connection in the fibration $X \to M$ and the restriction of a covariant derivative in $E \to X$ to horizontal vectors. Let us start with $\Psi_a(X, E)$. We use an analog of the so-called right quantization, i.e. a map

$$\begin{aligned} X \times_M X \times_M {}^aTM &\to X^2 \times [0, \infty) \\ (x_m, y_m, v_m(t)) &\mapsto (x_m, \tilde{c}_{v_m(t)}(y_m)(1), t), \end{aligned}$$

where x_m, y_m are points in X over $m \in M$, v is an adiabatic vector field, c_v is the geodesic starting at m in the direction of v and $\tilde{c}(y_m)$ is the horizontal lift of c starting at y_m. This map lifts to

$$X \times_M X \times_M {}^aTM \xrightarrow{\imath} X_a^2 \tag{15}$$

(note that the restriction of $\imath$ to $\{t = 0\}$ is just the canonical identification $X \times_M X \times_M {}^aTM_{|t=0} \simeq \mathrm{ff}_a^\circ$ from Lemma 8). The map q is defined by pulling back adiabatic operators to $X\times_M X\times_M {}^aTM$ via $\imath$, taking their Taylor series at $t=0$ and then Fourier transforming in the fibers of ${}^aTM_{t=0}$. We need the connection in horizontal directions in order to trivialize the pull-back of E over each fiber of ${}^aTM\to M$, so that we can compute Fourier transform.

The above definition of q can be specialized to $\Psi_a(M,U)$. For $A\in\Psi_{12}$ we proceed similarly: first, lift the map

$$\begin{aligned} X\times_M {}^aTM &\to X\times M\times[0,\infty)\\ (x_m,v_m(t)) &\mapsto (x_m,c_{v_m(t)}(1),t)\end{aligned}$$

to a map $\imath: X\times_M {}^aTM\to S_{12}$. Let $q(A)$ be the Fourier transform of the Taylor series of $\imath^*(A)$ at $t=0$. Note that each Taylor coefficient of $\imath^*(A)$ is an Euclidean density on the fibers of $X\times_M {}^aTM\to X$, with values in $E\boxtimes U^*$; the density factor is used in the Fourier transform. The definition of q on Ψ_{21} is analogous. $\square$

Choose local coordinates x_j in M, and let τ_j be the dual coordinates in T^*M. Then

$$q(it\nabla_{\tilde\partial_{x_j}})=\tau_j \tag{16}$$

is a constant series in t, where $\tilde\partial_{x_j}$ is the horizontal lift of ∂_{x_j} and ∇ is the connection used in the construction of q. If D is a family of differential operators on the fibers of ϕ viewed as an adiabatic operator constant in t then

$$q(D)=D.$$

Lemma 15. *The product structure on $\mathcal{A}^{\mathbb{Z},0}/\mathcal{A}^{\mathbb{Z},\infty}$ induced by the linear isomorphism q is a deformation with parameter t of the suspended (fiber-wise) product:*

$$A(x,\tau)*B(x,\tau)=AB+it\sum_{j=1}^{\dim M}\frac{\partial A}{\partial\tau_j}\nabla_{\tilde\partial_{x_j}}(B)+O(t^2). \tag{17}$$

Proof: The product $*$ is given by a sequence of bi-differential operators with polynomial coefficients. Thus, these coefficients are determined from *differential* adiabatic operators. Both ∂_{τ_j} and $\nabla_{\tilde\partial_{x_j}}$ are derivations on $\mathcal{A}_\partial$ so it is enough to prove (17) on a set of generators. Such generators are for instance families in t of fiberwise differential operators on X/M, and $it\nabla_{\tilde\partial_{x_j}}$. For these generators (17) is true without the error term. $\square$

Let $Q\in\mathcal{A}^{1,0}$ be an adiabatic operator such that $Q(t)$ is elliptic, self-adjoint and positive for all $t>0$ and $\mathcal{N}(Q)$ is a elliptic self-adjoint positive family of suspended operators. We call then Q invertible and positive.

Lemma 16. *[13] Let $Q\in\mathcal{A}^{1,0}$ be an invertible positive adiabatic operator. Then there exists the entire family $\{Q^s\}_{s\in\mathbb{C}}$ of complex powers of Q such that $Q^s\in\mathcal{A}^{s,0}$.*

Proof: Follow the proof of [4] with the boundary symbol and the conormal principal symbol replacing the principal symbol map. □

The trace function $t \mapsto \mathrm{Tr}(A(t))$ of an extended adiabatic operator $A \in \Psi_{ae}^{\alpha,0}(X)$ is smooth for $t > 0$, provided $\Re(\alpha) < -\dim X$. The behavior at $t = 0$ of this function is controllable, and thus provides us with our main tool to study adiabatic limits.

Proposition 17. *Let A be an extended adiabatic operator in $\Psi_{ae}^{\alpha,0}(X)$ with $\Re(\alpha) < -\dim X$. Then the function $t^{\dim M}\mathrm{Tr}(A(t))$ is smooth on $[0,\infty)$. Moreover,*

$$\mathrm{Tr}(A(t)) \sim (2\pi t)^{-\dim M} \int_{T^*M} \mathrm{Tr}(q(A))\omega^{\dim M}, \tag{18}$$

where ω is the standard symplectic form on T^*M, and Tr *in the right-hand side is the fiberwise trace of the formal series of families of suspended operators $q(A)$.*

Proof: For fixed $t > 0$, the trace of $A(t)$ is the integral of its distributional kernel over the diagonal. We cut off this kernel away from the diagonal and from the front face, pull it back through the map (15) and Fourier transform it in the fibers of ${}^{a}TM$. Recall that for the identity fibration $M \to M$ there exist two canonical bundle maps $I, J : {}^{a}TM \to TM \times [0,\infty)$. Namely, I is the canonical inclusion of adiabatic vector fields inside all vector fields, and J is the isomorphism given by $v(t) \mapsto v(t)/t$. We denote by the same letters the duals maps. Then $J^*(I_*\omega) = \omega/t$. The integral of the restriction of the kernel to the diagonal becomes (after Fourier transform) the integral on the total space, with a density factor $(I_*\omega/2\pi)^{\dim M}$. When we go back to T^*M via J^*, this becomes $(\omega/(2\pi t))^{\dim M}$. Therefore

$$\mathrm{Tr}(A(t)) = (2\pi t)^{-\dim M} \int_{T^*M} \mathrm{Tr}(\mathcal{F}(\imath^* A(t)))\omega^{\dim M}.$$

From Lemma 14 we know that $\mathcal{F}(\imath^* A(t))$ is smooth at $t = 0$, with Taylor series $q(A)$. The integrand is absolutely integrable for $\Re(\alpha) < -\dim X$, so the result follows by Lebesgue's dominated convergence theorem. □

We need to apply this result to the meromorphic extension of $\mathrm{Tr}(A(t))$. For any meromorphic function f and $z \in \mathbb{C}$, $k \in \mathbb{Z}$ let $c_{k,z}(f)$ be the coefficient of $(s-z)^k$ in the Laurent expansion of f near z.

Proposition 18. *Let $A(s,t)$ be an entire family of operators in $\mathcal{A}$, such that $A(s) \in \mathcal{A}^{s,0}$. Then for each fixed $k \in \mathbb{Z}$ and $z \in \mathbb{C}$, the function*

$$[0,\infty) \ni t \mapsto t^{\dim M} c_{k,z}(\mathrm{Tr}(A(t)))$$

is smooth, and its Taylor expansion at $t = 0$ is given by

$$(2\pi)^{-\dim M} c_{k,z}\left(\int_{T^*M} \mathrm{Tr}(q(A))\omega^{\dim M}\right).$$

Proof: It follows from the proof of Lemma 3 that if $a(s,t)$ is a smooth family of entire families of symbols, then the Laurent development of $\int_V a(s,t)d\mu dg$ around any complex point s depends smoothly on t. In particular, if $a(s,t) \sim \sum_{j=0}^{\infty} t^j a_j(s)$ is the Taylor expansion at $t=0$ of $a(s,t)$ then $c_{k,z}(a(t)) \sim \sum_{j=0}^{\infty} t^j c_{k,z}(a_j)$. We apply this to the full adiabatic symbols $a(s,t)$ of the family $A(s,t)$, which consist of a symbol on ${}^aT^*X$ and a symbol on ${}^aT^*M$. The result follows from (18) by noting that $a_j(s)$ is the full symbol of $q(A)_{[j]}$, where ${}_{[j]}$ denotes the coefficient of t^j in a formal series. $\square$

7 The adiabatic limit: the invertible case

Throughout this section we assume that the family D is invertible. With this assumption we only need to use the adiabatic algebra $\Psi_a(N)$ and not the extended calculus. As a first application of the formalism from Section 6, we reprove a result from [2].

Proposition 19. *Assume that the family D is invertible. Then there exists $\epsilon > 0$ such that the operator δ_t defined in (2) is invertible for all $0 < t < \epsilon$.*

Proof: In the suspended algebra $\mathcal{A}_\partial = \Psi_a^{\mathbb{Z},0}(N)/\Psi_a^{\mathbb{Z},1}(N)$, the operator

$$\mathcal{N}(\delta_t^2) = \begin{bmatrix} \tau^2 + D^*D & 0 \\ 0 & \tau^2 + DD^* \end{bmatrix}$$

is invertible [9] since D is assumed to be invertible, so $\mathcal{N}(\delta_t)$ is also invertible. Moreover, $\sigma_1(\delta_t) \in C^\infty({}^aS^*N, \mathrm{End}(E))$ is invertible. Clearly, $\sigma_1(\mathcal{N}(\delta_t)) = \sigma_1(\delta_t)_{|t=0}$. Let $B_0 \in \Psi_a^{-1,0}(N,E)$ be such that $\mathcal{N}(B_0) = \mathcal{N}(\delta_t)^{-1}$ and $\sigma_{-1}(B_0) = \sigma_1(\delta_t)^{-1}$. Thus $\delta_t B_0 = I - R_0$, with $R_0 \in \Psi_a^{-1,1}(N)$. Let $Q \in \Psi_a^{-1,0}(N,E)$ be an adiabatic operator which realizes the asymptotic sum $B_0(I + R_0 + R_0^2 + \ldots)$. Then $R(t) := \delta_t Q - I \in \Psi_a^{-\infty,\infty}$. This ideal is canonically isomorphic to the algebra of smooth families (indexed by $[0,\infty)$) of smoothing operators on N, which are rapidly vanishing at $t=0$. Thus, the norm of $R(t)$ as a bounded operator on $L^2(N)$ tends to 0, which implies that $\delta_t Q = I + R(t)$ is invertible for sufficiently small t. $\square$

From our choice of adiabatic right quantization, it follows that in the algebra $\Psi_a^{\mathbb{Z},0}(N)/\Psi_a^{\mathbb{Z},\infty}(N) \cong \mathcal{A}_\delta[[t]]$, we have

$$q(\delta_t) = \begin{bmatrix} -\tau & D^* \\ D & \tau \end{bmatrix}, \tag{19}$$

$$q(\delta_t^2) = \begin{bmatrix} D^*D + \tau^2 & -it\tilde{\nabla}_{\partial_\theta} D^* \\ it\tilde{\nabla}_{\partial_\theta} D & DD^* + \tau^2 \end{bmatrix}. \tag{20}$$

Let $f : [0,\infty) \times \mathbb{C} \to \mathbb{C}$ be a family of meromorphic functions on $\mathbb{C}$ indexed by t. We say that f is a *smooth family of meromorphic functions* if for every $z \in \mathbb{C}$ and every $k \in \mathbb{Z}$, the Laurent coefficient $t \mapsto c_{k,z}(f(t,\cdot))$ of $(s-z)^k$ in $f(s,t)$ is a smooth function of t.

Theorem 20. *If the family D is invertible then the family of eta functions $\overline{\eta}(\delta_t, s)$ –* a priori *defined for $t > 0$ – extends to a smooth family for $t \in [0, \epsilon)$. Moreover,*

$$\lim_{t\to 0} \overline{\eta}(\delta_t, s) = \frac{1}{i\pi} \int_{S^1} A(D, s), \tag{21}$$

where $A(D, s)$ is the Bismut-Freed family of 1-forms defined by (9).

Proof: First, δ_t is invertible and so (2) becomes

$$\overline{\eta}(\delta_t, s) = \pi^{-1/2} \Gamma\left(\frac{1+s}{2}\right) \mathrm{Tr}((\delta_t^2)^{-\frac{s+1}{2}} \delta_t).$$

From Proposition 18 it follows that $t\overline{\eta}(\delta_t, s)$ is a smooth family of meromorphic functions. Further, by (19) and (20) we have

$$\overline{\eta}(\delta_t, s) \sim_{t\to 0} \frac{\Gamma\left(\frac{1+s}{2}\right)}{\sqrt{\pi}} \frac{1}{2\pi t} \int_{T^*S^1} \mathrm{Tr}\left(\begin{bmatrix} D^*D + \tau^2 & -it\tilde{\nabla}_{\partial_\theta} D^* \\ it\tilde{\nabla}_{\partial_\theta} D & DD^* + \tau^2 \end{bmatrix}^{-\frac{s+1}{2}} \right.$$
$$\left. \begin{bmatrix} -\tau & D^* \\ D & \tau \end{bmatrix}\right) d\tau d\theta. \tag{22}$$

where all products and complex powers are in the sense of the product (17). For the singular term, i.e. the coefficient of t^{-1}, this implies

$$\lim_{t\to 0} t\overline{\eta}(\delta_t, s) = \frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}} \int_{T^*S^1} \mathrm{Tr}\left(-\tau(D^*D + \tau^2)^{-\frac{s+1}{2}}\right)$$
$$+ \mathrm{Tr}\left(\tau(DD^* + \tau^2)^{-\frac{s+1}{2}}\right) d\tau d\theta$$

and this integral vanishes because it is the integral of an odd function of τ. We compute now the limit of $\overline{\eta}(\delta_t, s)$ as t tends to 0. Since the matrices of operators $\begin{bmatrix} D^*D + \tau^2 & 0 \\ 0 & DD^* + \tau^2 \end{bmatrix}$ and $\begin{bmatrix} -\tau & D^* \\ D & \tau \end{bmatrix}$ commute modulo t, the constant term in t from (22) becomes

$$\begin{aligned}
\lim_{t\to 0} \overline{\eta}(\delta_t, s) \;=\; & \frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}} \int_{T^*S^1} \mathrm{Tr}\left(\begin{bmatrix} D^*D + \tau^2 & 0 \\ 0 & DD^* + \tau^2 \end{bmatrix}^{-\frac{s+1}{2}}_{[1]} \right. \\
& \left. \begin{bmatrix} -\tau & D^* \\ D & \tau \end{bmatrix}\right) d\tau d\theta \\
& + \frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}} \int_{T^*S^1} -\frac{s+1}{2} \mathrm{Tr}\left(\begin{bmatrix} D^*D + \tau^2 & 0 \\ 0 & DD^* + \tau^2 \end{bmatrix}^{-\frac{s+3}{2}} \right. \\
& \left. \begin{bmatrix} 0 & -i\tilde{\nabla}_{\partial_\theta} D^* \\ i\tilde{\nabla}_{\partial_\theta} D & 0 \end{bmatrix} \begin{bmatrix} -\tau & D^* \\ D & \tau \end{bmatrix}\right) d\tau d\theta \\
=:\; & I(s) + J(s)
\end{aligned} \tag{23}$$

where the subscript $_{[1]}$ denotes the coefficient of t^1 in a formal series. Since the product involved in the complex powers is the deformed product (17), the term $I(s)$ does not vanish directly. In fact, it is impossible to compute the integrand before taking the trace. Nevertheless, the trace can be computed by assuming formally that the operators involved commute. Thus, $I(s)$ has two contributions coming from the diagonal, namely

$$\frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}}\frac{s+1}{2}\frac{s+3}{2}\int_{T^*S^1} -\tau^2 \mathrm{Tr}\left((D^*D+\tau^2)^{-\frac{s+5}{2}} i\tilde{\nabla}_{\partial_\theta}(D^*D)\right) d\tau d\theta, \tag{24}$$

respectively

$$\frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}}\frac{s+1}{2}\frac{s+3}{2}\int_{T^*S^1} \tau^2 \mathrm{Tr}\left((DD^*+\tau^2)^{-\frac{s+5}{2}} i\tilde{\nabla}_{\partial_\theta}(DD^*)\right) d\tau d\theta. \tag{25}$$

These factors cancel each other before integration; alternately, each of them equals 0 after integration; for instance, the quantity (24) equals

$$\frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}}\frac{s+1}{2}\int_{T^*S^1} i\tau^2\partial_\theta \mathrm{Tr}\left((D^*D+\tau^2)^{-\frac{s+3}{2}}\right) d\tau d\theta = 0$$

so $I(s)=0$ for large $\Re(s)$. By unique continuation,

$$I(s)\equiv 0. \tag{26}$$

We focus now on the second term $J(s)$. It is given by

$$\begin{aligned} J(s) = -\frac{s+1}{2}\frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}}\int_{T^*S^1} \Big[&\mathrm{Tr}\left((D^*D+\tau^2)^{-\frac{s+3}{2}}(-i\tilde{\nabla}_{\partial_\theta}D^*)D\right) \\ &+\mathrm{Tr}\left((DD^*+\tau^2)^{-\frac{s+3}{2}}(i\tilde{\nabla}_{\partial_\theta}D)D^*\right)\Big] d\tau d\theta. \end{aligned} \tag{27}$$

Notice that

$$\begin{aligned} -\frac{s+1}{2}&\Big[\mathrm{Tr}\left((D^*D+\tau^2)^{-\frac{s+3}{2}}(\tilde{\nabla}_{\partial_\theta}D^*)D\right) \\ &+\mathrm{Tr}\left((DD^*+\tau^2)^{-\frac{s+3}{2}}(\tilde{\nabla}_{\partial_\theta}D)D^*\right)\Big] \\ &= \mathrm{Tr}\left(\tilde{\nabla}_{\partial_\theta}(D^*D+\tau^2)^{-\frac{s+1}{2}}\right) \\ &= \partial_\theta \mathrm{Tr}\left((D^*D+\tau^2)^{-\frac{s+1}{2}}\right) \end{aligned}$$

so the *sum* of the two terms in (27) vanishes. Thus

$$J(s) = -i\frac{\Gamma\left(\frac{3+s}{2}\right)}{\pi^{3/2}}\int_{T^*S^1}\mathrm{Tr}\left((DD^*+\tau^2)^{-\frac{s+3}{2}}(\tilde{\nabla}_{\partial_\theta}D)D^*\right) d\tau d\theta \tag{28}$$

$$\begin{aligned} &= -i\frac{\Gamma\left(\frac{3+s}{2}\right)}{\pi^{3/2}} f\left(\frac{s+3}{2}\right)\int_{S^1}\mathrm{Tr}\left((DD^*)^{-\frac{s}{2}-1}(\tilde{\nabla}_{\partial_\theta}D)D^*\right) d\theta \\ &= \frac{\Gamma\left(\frac{s}{2}+1\right)}{i\pi}\int_{S^1}\mathrm{Tr}\left((D^*D)^{-\frac{s}{2}}D^{-1}\tilde{\nabla}_{\partial_\theta}(D)\right) d\theta \end{aligned} \tag{29}$$

where $f(s)$ is the function from (4). We have used the commutation formula $(D^*D)^s D^* = D^*(DD^*)^s$. In conclusion, (26) and (27) imply (21). $\square$

This result (for $s = 0$) is used in the index formula of [5] to relate the boundary term with the adiabatic limit of the eta invariant.

Corollary 21. *The limit of* $\exp(-i\pi\overline{\eta}(\delta_t))$ *exists and equals the holonomy of* $\det(D)$*.*

Proof: By specializing to the constant coefficient $\overline{\eta}(\delta_t)$ of $\overline{\eta}(\delta_t, s)$ at $s = 0$, we get

$$\lim_{t\to 0} e^{-i\pi\overline{\eta}(\delta_t)} = e^{-\int_{S^1} A(D,0)}.$$

The right-hand side equals the holonomy along the circle of the connection $d + A(D, 0)$ in the trivial bundle. $\square$

We turn now to the zeta function. Recall that if the family D is invertible then the operator δ_t defined in (2) is also invertible for small enough $t > 0$ (Proposition 19).

Theorem 22. *Assume that the family of operators D is invertible. Then the family of meromorphic functions $t\overline{\zeta}(\delta_t, s)$ extends smoothly to $t = 0$. Moreover,*

$$\lim_{t\to 0} t\overline{\zeta}(\delta_t, s) = \frac{1}{\sqrt{\pi}} \int_{S^1} \overline{\zeta}(D, s-1) d\theta, \tag{30}$$

while

$$\frac{d(t\overline{\zeta}(\delta_t, s))}{dt}|_{t=0} = 0.$$

Proof: As for the eta function, by Proposition 18, (19) and (20) we have

$$\begin{aligned} t\overline{\zeta}(\delta_t, s) &= t\Gamma\left(\frac{s}{2}\right) \mathrm{Tr}\left((\delta_t^2)^{-\frac{s}{2}}\right) \\ &\sim_{t\to 0} \Gamma\left(\frac{s}{2}\right) \frac{1}{2\pi} \int_{T^*S^1} \mathrm{Tr} \begin{bmatrix} D^*D + \tau^2 & -it\tilde{\nabla}_{\tilde{\partial}_\theta} D^* \\ it\tilde{\nabla}_{\tilde{\partial}_\theta} D & DD^* + \tau^2 \end{bmatrix}^{-\frac{s}{2}} d\tau d\theta. \end{aligned} \tag{31}$$

The leading term in this expression is

$$\begin{aligned} &\pi^{-1}\Gamma\left(\frac{s}{2}\right) \int_{T^*S^1} \mathrm{Tr}(D^*D + \tau^2)^{-s/2} d\tau d\theta \\ &= \pi^{-1}\Gamma\left(\frac{s}{2}\right) f\left(\frac{s}{2}\right) \int_{S^1} \mathrm{Tr}(D^*D)^{-\frac{s-1}{2}} d\theta. \end{aligned}$$

Together with (4), this implies (30). It is clear that the off-diagonal terms in $q(\delta_t^2)$ contribute to (31) only modulo t^2. It follows that the coefficient of t in (31) is a sum of two terms coming from the diagonal. These terms are similar to (24) and (25), only that they are odd in τ and thus both equal 0 even before integrating over S^1. $\square$

We can deduce from this result the adiabatic limit behavior of the usual quantities $\zeta(\delta_t, 0)$ and of $\det(\delta_t) := e^{-\zeta'(\delta_t,0)}$. Let $\mathrm{Tr}_\zeta|D|$ denote the regularized value $\zeta(D, -1)$.

Corollary 23. *The functions $t\zeta(\delta_t, 0)$ and $t \log\det(\delta_t)$ extend smoothly to $t = 0$. Moreover,*

$$\lim_{t\to 0} t\zeta(\delta_t, 0) = -\int_{S^1} \mathrm{Tr}_w |D| d\theta \tag{32}$$

$$\lim_{t\to 0} t\log\det(\delta_t) = \int_{S^1} \left(-\mathrm{Tr}_\zeta |D| + (2\log 2 - 2)\mathrm{Tr}_w |D|\right) d\theta. \tag{33}$$

Proof: In terms of ζ, formula (30) reads

$$\lim_{t\to 0} t\Gamma\left(\frac{s}{2}\right)\zeta(\delta_t, s) = \pi^{-1/2}\Gamma\left(\frac{s-1}{2}\right)\int_{S^1}\zeta(D, s-1)d\theta. \tag{34}$$

Recall that

$$\Gamma(0) = -\gamma, \quad \Gamma'\left(-\frac{1}{2}\right) = 2\pi^{1/2}(\gamma + 2\log 2 - 2),$$

where γ is the Euler constant, and $\Gamma(0)$ is the finite part of $\Gamma(s)$ at $s = 0$. Thus

$$\begin{aligned}
&\lim_{t\to 0} t\left(\frac{2}{s} - \gamma\right)\left(\zeta(\delta_t, 0) + s(\zeta)'(\delta_t, 0)\right)\\
&\equiv \frac{1}{\sqrt{\pi}}\left(-2\pi^{1/2} + \pi^{1/2}(\gamma + 2\log 2 - 2)s\right)\\
&\left(\frac{1}{s}\int_{S^1}\mathrm{Tr}_w(D^*D)^{1/2}d\theta + \int_{S^1}\zeta(D, -1)d\theta\right) \pmod{s}.
\end{aligned}$$

Identifying the coefficients of s^{-1} and s^0 we obtain (32) and (33). □

An interesting formula appears in odd dimensions:

Corollary 24. *Assume that N is odd-dimensional. Then*

$$\lim_{t\to 0}\det(\delta_t)^t = e^{-\int_{S^1}\mathrm{Tr}_\zeta(|D|)d\theta}.$$

Proof: In this case $\zeta(\delta_t, 0)$ must vanish and by (32) this implies that $\int_{S^1}\mathrm{Tr}_w(D^*D)^{\frac{1}{2}}d\theta = 0$. The result follows from Corollary 23.

8 The adiabatic limit: the non-invertible case

Consider an elliptic family D as in Section 7, without the invertibility hypothesis. Then δ_t is also non-invertible in general. One could define the complex powers of δ_t^2 as being 0 on the null-space of δ_t. This works fine for a fixed value of t. However, the function $(0,\infty)\times\mathbb{C}\ni(\lambda, s)\mapsto\lambda^s$ does not converge to 0 as λ tends to 0, and this means that the family $(\delta_t^2)^s$ defined in this way is discontinuous at values of t where an eigenvalue crosses 0. This turns out to be a major difficulty when trying to extend the analysis from the previous section to the family δ_t. The extended adiabatic algebra proves to be the right tool to overcome this problem.

Let U^+, U^- be trivial finite-dimensional complex vector bundles over S^1, such that there exists an invertible family of operators D_U extending D by the formula (8). As in Section 5, we extend $\tilde{\nabla}_{\partial_\theta}$ trivially on $C^\infty(S^1, U^\pm)$. Let ϕ be a Schwartz function on $\mathbb{R}$ such that $\phi(0) = 1$. Then

$$Q := \begin{bmatrix} D & D_{12}\phi(\tau) \\ D_{21}\phi(\tau) & D_{22}\phi(\tau) \end{bmatrix} \tag{35}$$

belongs to $\Psi_{ae}^{1,0}(N; E^+ \oplus U^+, E^- \oplus U^-)/\Psi_{ae}^{1,1}(N; E^+ \oplus U^+, E^- \oplus U^-)$. There exists $R_t \in \Psi_{ae}^{-\infty}(N)$ such that $q(R_t) = \begin{bmatrix} 0 & D_{12}\phi(\tau) \\ D_{21}\phi(\tau) & D_{22}\phi(\tau) \end{bmatrix}$ is constant in t. By abuse of notation we write D instead of $\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$. Then $q(R_t + D) = Q$. Let $E := E^+ \oplus E^-$, $U := U^+ \oplus U^-$ and

$$d_t = \begin{bmatrix} -ti\tilde{\nabla}_{\partial_\theta} & D^* + R_t^* \\ D + R_t & ti\tilde{\nabla}_{\partial_\theta} \end{bmatrix} \in \Psi_{ae}^{1,0}(N, E, U). \tag{36}$$

If we replace D by Q, the identities (19), (20) become

$$q(d_t) = \begin{bmatrix} -\tau & Q^* \\ Q & \tau \end{bmatrix}, \tag{37}$$

$$q(d_t^2) = \begin{bmatrix} Q^*Q + \tau^2 + it\partial_\tau(Q^*)\tilde{\nabla}_{\partial_\theta}(Q) & -it\tilde{\nabla}_{\partial_\theta}Q^* \\ it\tilde{\nabla}_{\partial_\theta}Q & QQ^* + \tau^2 + it\partial_\tau(Q)\tilde{\nabla}_{\partial_\theta}(Q^*) \end{bmatrix} \tag{38}$$

In particular, the boundary symbol $\mathcal{N}(d_t^2) = \begin{bmatrix} Q^*Q + \tau^2 & 0 \\ 0 & QQ^* + \tau^2 \end{bmatrix}$ is invertible inside $\mathcal{A}_\partial(N, E, U)$ since $Q(0)$ is invertible. Proposition 19 shows that d_t is invertible for small enough t. Formulas (23), (28) from Section 7 have analogs with δ_t replaced with d_t and D with Q. Such results are, of course, uninteresting, since d_t and Q depend on the choice of $U^\pm$ and of the Schwartz function ϕ, and moreover $I(s)$, $J(s)$ cannot be simplified any further. However, there is one instance when these results can be linked to δ_t and D, namely when we restrict our attention to the constant coefficient in s at $s = 0$. The reason is that the values at $s = 0$ of the eta and zeta functions of the operators δ_t and d_t are closely related.

Lemma 25. *Let d be a symmetric elliptic operator of positive order in the algebra $\Psi_{ex}(X; M)$ defined in Remark 11. Let $\mu \mapsto R_\mu \in \Psi_{ex}^{-\infty}(X; M)$ be a 1-parameter family of symmetric smoothing operators. Then $\overline{\eta}(d + R_\mu)$ is independent of μ modulo $2\mathbb{Z}$.*

Proof: We first show that $\overline{\eta}(d + R_\mu)$ is constant in μ as long as $d + R_\mu$ remains invertible. Write $d_\mu := d + R_\mu$. Then

$$\partial_\mu \mathrm{Tr}\left((d_\mu^2)^{-\frac{s+1}{2}} d_\mu\right) = \mathrm{Tr}\left((d_\mu^2)^{-\frac{s+3}{2}} \left(-\frac{s+1}{2}\right)(\partial_\mu(R_\mu)d_\mu + d_\mu\partial_\mu(R_\mu))d_\mu\right.$$

$$
\begin{aligned}
&+(d_\mu^2)^{-\frac{s+1}{2}}\partial_\mu(R_\mu)\Big) \\
&= -s\mathrm{Tr}\left((d_\mu^2)^{-\frac{s+1}{2}}\partial_\mu(R_\mu)\right).
\end{aligned}
$$

The last expression is the trace of an entire family of smoothing operators so it is entire in s, and it vanishes at $s=0$ because of the s factor.

Let μ_0 be a point where 0 is an eigenvalue of d_{μ_0}. Choose $\alpha \in \mathbb{R}$ with $\pm\alpha \notin \mathrm{Spec}(d_{\mu_0})$. Then for μ close to μ_0, $\pm\alpha \notin \mathrm{Spec}(d_\mu)$. Moreover, if $P_\alpha(\mu)$ is the projection on the (finite-dimensional) span of eigenspaces of d_μ of eigenvalue between $\pm\alpha$, then the map $\mu \mapsto P_\alpha(\mu) \in \Psi_{ex}^{-\infty}(X;M)$ is smooth in a neighborhood of μ_0. Clearly, in this neighborhood $\overline{\eta}(d_\mu) - \overline{\eta}(d_\mu + P_\alpha) \in 2\mathbb{Z}$, and $\overline{\eta}(d_\mu + P_\alpha(\mu))$ is constant in μ by what we proved above since $d_\mu + P_\alpha(\mu)$ is invertible. □

Lemma 26. *Let d_t be the extended adiabatic operator defined in* (36) *and δ_t the family of differential operators defined in* (2). *For all $t>0$,*

$$
\begin{aligned}
&\overline{\eta}(d_t) \equiv \overline{\eta}(\delta_t) + \mathrm{index}(D) \pmod{2\mathbb{Z}} \\
&\zeta(d_t) = \zeta(\delta_t).
\end{aligned}
$$

Proof: For a fixed $t>0$ consider the operator

$$
q_t = \begin{bmatrix} \delta_t & 0 & 0 \\ 0 & -it\partial_\theta & 0 \\ 0 & 0 & it\partial_\theta \end{bmatrix} \in \Psi_{ex}(X;M;E,U)
$$

acting on $C^\infty(N, E^+ \oplus E^-) \oplus C^\infty(S^1, U^+) \oplus C^\infty(S^1, U^-)$. Note that q_t, d_t act on the same space but are written in basis which differ by a permutation. Moreover $d_t - q_t \in \Psi_{ex}^{-\infty}(X;M)$ so by Lemma 25, $\overline{\eta}(q_t) - \overline{\eta}(d_t) \in 2\mathbb{Z}$. On the other hand, $\overline{\eta}(q_t,s) = \overline{\eta}(\delta_t,s) + (\dim U^- - \dim U^+)\overline{\eta}(it\partial_\theta,s)$. The result follows by noting that $\overline{\eta}(it\partial_\theta,s) = 1$ and $\dim U^- - \dim U^+ = \mathrm{index}(D)$.

Similarly, for the zeta function we see in the notation of Lemma 25 that

$$
\begin{aligned}
\partial_\mu \mathrm{Tr}\left((d_\mu^2)^{-\frac{s}{2}}\right) &= -\frac{s}{2}\mathrm{Tr}\left((d_\mu^2)^{-\frac{s}{2}-1}(\partial_\mu(R_\mu)d_\mu + d_\mu\partial_\mu(R_\mu))\right) \\
&= -s\mathrm{Tr}\left((d_\mu^2)^{-\frac{s}{2}-1}d_\mu\partial_\mu(R_\mu)\right)
\end{aligned}
$$

equals s times an entire function, so it vanishes at $s=0$. Secondly, there are no jumps in $\zeta(d_\mu)$ when eigenvalues of d_μ cross 0 (on finite parts of the spectrum, ζ at $s=0$ is just the number of eigenvalues). Thirdly, the function $\zeta(it\partial_\theta,s)$ vanishes at $s=0$. Indeed, this is equivalent to the fact that for the Riemann zeta function,

$$
\zeta(0) = -\frac{1}{2}.
$$

□

We can now prove our main result for non-invertible families. The first formula was proved in [3] in the particular case when D is a family of compatible Dirac operators. In the second formula $|D|$ is defined only on the orthogonal complement of the null-space of D.

Theorem 27. *Let D be a family of elliptic differential operators of order* 1 *over S^1, and δ_t the operator defined by* (2). *Then* $\exp(-i\pi\overline{\eta}(\delta_t))$ *and* $t\zeta(\delta_t)$ *extend smoothly to $t = 0$. Moreover,*

$$\lim_{t\to 0} e^{-i\pi\overline{\eta}(\delta_t)} = (-1)^{\mathrm{index}(D)}\mathrm{hol}(\det(D)) \tag{39}$$

$$\lim_{t\to 0} t\zeta(\delta_t, 0) = -\int_{S^1} \mathrm{Tr}_w |D| d\theta \tag{40}$$

and

$$\lim_{t\to 0}\left(\zeta(\delta_t, 0) + \frac{1}{t}\int_{S^1} \mathrm{Tr}_w |D| d\theta\right) = 0. \tag{41}$$

Proof: Working in the extended adiabatic algebra, we will see as in Section 7 that the families of meromorphic functions $\overline{\eta}(d_t, s)$ and $t\zeta(d_t, s)$ are smooth down to $t = 0$. Our strategy is to compute the asymptotics at $t = 0$ of the regularized value at $s = 0$ of these functions, and then to use Lemma 26 to deduce the asymptotics of $\exp(-i\pi\overline{\eta}(\delta_t))$ and $t\zeta(\delta_t, 0)$.

The eta invariant. From Proposition 18 and the definition of the eta function we write

$$\overline{\eta}(d_t, s) \sim_{t\to 0} \frac{\Gamma\left(\frac{1+s}{2}\right)}{\sqrt{\pi}} \frac{1}{2\pi t}\int_{T^*S^1} \mathrm{Tr}\left(q(d_t^2)^{-\frac{s+1}{2}} * q(d_t)\right) d\tau d\theta. \tag{42}$$

where the products and complex powers are in the sense of the product (17). Using (37), (38) we get for the coefficient of t^{-1}

$$\begin{aligned}\lim_{t\to 0} t\overline{\eta}(d_t, s) = \frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}} \int_{T^*S^1} &\mathrm{Tr}\left(-\tau(Q^*Q + \tau^2)^{-\frac{s+1}{2}}\right) \\ &+\mathrm{Tr}\left(\tau(QQ^* + \tau^2)^{-\frac{s+1}{2}}\right) d\tau d\theta.\end{aligned}$$

Decomposing the trace over a basis of eigensections of Q^*Q, QQ^* we see that the above terms cancel each other (both can be made zero by choosing ϕ to be even in τ). By analogy with (23) we write

$$\begin{aligned}\lim_{t\to 0} \overline{\eta}(d_t, s) &= \frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}} \int_{T^*S^1} \mathrm{Tr}\left(\left(q(d_t^2)^{-\frac{s+1}{2}}\right)_{[1]} \begin{bmatrix} -\tau & 0 \\ 0 & \tau \end{bmatrix}\right) d\tau d\theta \\ &\quad + \frac{\Gamma\left(\frac{1+s}{2}\right)}{2\pi^{3/2}} \int_{T^*S^1} -\frac{s+1}{2}\mathrm{Tr}\left(\begin{bmatrix} Q^*Q + \tau^2 & 0 \\ 0 & QQ^* + \tau^2 \end{bmatrix}^{-\frac{s+3}{2}}\right. \\ &\quad \left.\begin{bmatrix} 0 & -i\tilde{\nabla}_{\partial_\theta} Q^* \\ i\tilde{\nabla}_{\partial_\theta} Q & 0 \end{bmatrix} \begin{bmatrix} -\tau & Q^* \\ Q & \tau \end{bmatrix}\right) d\tau d\theta \\ &=: I(s) + J(s).\end{aligned} \tag{43}$$

By integration by parts with respect to θ we get (see (28))

$$J(s) = -i\frac{\Gamma\left(\frac{3+s}{2}\right)}{\pi^{3/2}} \int_{T^*S^1} \operatorname{Tr}\left((QQ^* + \tau^2)^{-\frac{s+3}{2}} (\tilde{\nabla}_{\partial_\theta} Q) Q^*\right) d\tau d\theta. \qquad (44)$$

However, Q depends on τ since it involves the function $\phi(\tau)$, so the analog of (29) and the explicit decomposition of $I(s)$ in (24), (25) fail.

Let $\mathcal{V} = \{V_{\alpha_1}, \dots, V_{\alpha_k}\}$ be the cover of S^1 from Lemma 5, and $\{W_{\alpha_j}\}$ a partition of S^1 in intervals such that $W_{\alpha_j} \subset V_{\alpha_j}$. We split the integral over S^1 from the definition (43) of $I(s)$, respectively from the formula (44) for $J(s)$, in the sum of the integrals over W_{α_j}:

$$I(s) = \sum_{j=1}^{k} I_{\alpha_j}(s), \quad J(s) = \sum_{j=1}^{k} J_{\alpha_j}(s)$$

(it is worth mentioning that integration by parts in θ was done *before* this splitting). Notice that over W_{α_j} the operators $Q(\tau)$, τ and $\partial_\tau Q$ preserve the decomposition

$$\mathcal{E} \oplus U = \mathcal{E}_{>\alpha_j} \oplus \left(\mathcal{E}_{<\alpha_j} \oplus U\right)$$

while $\partial_\theta Q$ does not necessarily do so. Nevertheless, the form (17) of the product shows that over W_{α_j} we can compute I_{α_j}, respectively J_{α_j}, by projecting Q onto and taking the trace on $\mathcal{E}_{>\alpha_j}$, resp. $\mathcal{E}_{<\alpha_j} \oplus U$:

$$I_{\alpha_j}(s) = I^{>}_{\alpha_j}(s) + I^{<}_{\alpha_j}(s), \quad J_{\alpha_j}(s) = J^{>}_{\alpha_j}(s) + J^{<}_{\alpha_j}(s). \qquad (45)$$

One virtue of this decomposition is that $Q_{>\alpha_j}$, and so also $I^{>}_{\alpha_j}(s)$ and $J^{>}_{\alpha_j}(s)$, do not depend on ϕ anymore. Thus as in Theorem 20, $I^{>}_{\alpha_j}(s)$ has two components (see (24), (25)) which cancel each other (they are also exact forms in θ but we cannot deduce that their integral vanishes individually), while for $J^{>}_{\alpha_j}(s)$, (44) implies the analog of (29)

$$J^{>}_{\alpha_j}(s) = \frac{1}{i\pi} \int_{W_{\alpha_j}} A(D, s)_{>\alpha_j}.$$

A second feature of (45) is that $I^{<}_{\alpha_j}(s) + J^{<}_{\alpha_j}(s)$ involves only traces of finite-dimensional linear endomorphisms. More precisely, $I^{<}_{\alpha_j}(s)$, $J^{<}_{\alpha_j}(s)$ are given by the first term in (23), respectively by (28), where the integral is computed over $W_{\alpha_j} \times \mathbb{R}$, D is replaced by $Q_{<\alpha_j}$ (here we note that $Q_{<\alpha_j}(\theta, \tau)$ is a classical symbol on $W_{\alpha_j} \times \mathbb{R}$, invertible for $\tau = 0$), and hence the trace Tr becomes the endomorphism trace tr. Let us examine what happens when we modify the function ϕ, of course with the restriction that $\phi(0) > 0$. Lemma 25 shows that $\overline{\eta}(d_t)$ (hence also its limit as $t \to 0$) is unaffected. We have seen that $I^{>}_{\alpha_j}(s)$ and $J^{>}_{\alpha_j}(s)$ do not involve ϕ. We deduce that

$$L^{<}(s) := \sum_{j=1}^{k} I^{<}_{\alpha_j}(s) + J^{<}_{\alpha_j}(s)$$

is also independent of ϕ at $s=0$, as long as ϕ is a Schwartz function. However, $L^<(s)$ makes sense and is meromorphic in s for any ϕ a classical symbol of order 0 on $\mathbb{R}$. Indeed, for such ϕ all operators involved in $I^<_{\alpha_j}(s)+J^<_{\alpha_j}(s)$ are still families of classical symbols on $\mathbb{R}$ indexed by W_{α_j}, so Lemma 3 applies.

Let us compute $L^<(s)$ with ϕ replaced by 1. Then the corresponding Q_{α_j} are τ-free. The same argument as above for $I^>_{\alpha_j}(s)$ and $J^>_{\alpha_j}(s)$ (which was explained in Theorem 20) shows that

$$I^<_{\alpha_j}(s)=0, \quad J^<_{\alpha_j}(s)=\frac{1}{i\pi}\int_{W_{\alpha_j}} A(D,s)_{<\alpha_j}.$$

We justify below this substitution (this completes the proof of (39)).

Both $I^<_{\alpha_j}(s)$ and $J^<_{\alpha_j}(s)$ are continuous with respect to variations of the Schwartz function ϕ. We define a deformation for $0<\mu\leq 1$ by $\phi_\mu(\tau):=\phi(\tau\mu)$. Note that $\phi_1=\phi$ and $\phi_\mu\to 1$ pointwise as $\mu\to 0$. Moreover, $|\phi_\mu(\tau)|\leq 1$ while $|\partial_\tau\phi_\mu(\tau)|<C(1+|\tau|)^{-1}$, uniformly in μ. We aim to find uniform L^1 upper bounds for the integrands in $I^<_{\alpha_j}(s)(\mu)$ and $J^<_{\alpha_j}(s)(\mu)$. First, we note

$$\begin{aligned}
\|Q_{<\alpha_j}\| &\leq C\\
\|\partial_\theta(Q_{<\alpha_j})\| &\leq C\\
\|\partial_\tau(Q_{<\alpha_j})\| &\leq C(1+|\tau|)^{-1}\\
\|(Q_{<\alpha_j}Q^*_{<\alpha_j}+\tau^2)^{-s}\| &\leq (\psi(\tau)+\tau^2)^{-\Re(s)}
\end{aligned}$$

where the last inequality holds for $\Re(s)\geq 0$ with some compactly supported non-negative function ψ with $\psi(0)>0$, independent of μ. We will use the inequality

$$|\mathrm{tr}(A)|\leq l\|A\|$$

for an endomorphism A of an l-dimensional vector space. The integrand in $J^<_{\alpha_j}(s)(\mu)$ was defined using (44). Using the above bounds we get

$$|\mathrm{tr}\left((Q_{<\alpha_j}Q^*_{<\alpha_j}+\tau^2)^{-\frac{s+3}{2}}(\tilde\nabla_{\partial_\theta}Q_{<\alpha_j})Q^*_{<\alpha_j}\right)|\leq C_j(\psi(\tau)+\tau^2)^{-\frac{\Re(s)+3}{2}}$$

from which we retain that the integrand in $J^<_{\alpha_j}(s)(\mu)$ is bounded (uniformly in μ) by an L^1 function of τ for all s with $\Re(s)>-2$, in particular for $s=0$.

The integrand in $I^<_{\alpha_j}(s)(\mu)$ admits a similar bound. We show this for the term coming from the upper left corner, the other one being entirely similar. Using (38) we first isolate the term coming from $it\partial_\tau(Q_{<\alpha_j})\tilde\nabla_{\partial_\theta}(Q^*_{<\alpha_j})$. This term is bounded uniformly in τ for $\Re(s)>-3$ by

$$C_j(s)(\psi(\tau)+\tau^2)^{-\frac{\Re(s)+3}{2}}(1+|\tau|)^{-1}\tau,$$

which is L^1 for $\Re(s)>-2$ as before. Finally, the remaining term coming from $\left((Q_{<\alpha_j}Q^*_{<\alpha_j}+\tau^2)^{-\frac{s+1}{2}}\right)_{[1]}\tau$, though inexplicit, is bounded for $\Re(s)>-5$ by

$$C_j(s)(\psi(\tau)+\tau^2)^{-\frac{\Re(s)+5}{2}}\tau^2.$$

In conclusion, for $\Re(s) > -2$ by Lebesgue dominated convergence

$$L^{<}(s)(\mu) \to_{\mu\to 0} L^{<}(s)(0) = \frac{1}{i\pi}\sum_j \int_{W_{\alpha_j}} A(D,s)_{<\alpha_j}.$$

But $L^{<}(0)(\mu)$ is constant in μ, and this ends the proof of (39).

The regularized value $\zeta(0)$. The zeta function of d_t is treated as in Theorem 22. We get as before formula (31) with D replaced by Q:

$$\begin{aligned} t\zeta(d_t,s) &= t\mathrm{Tr}\left((d_t^2)^{-\frac{s}{2}}\right) \\ &\sim_{t\to 0} \frac{1}{2\pi}\int_{T^*S^1} \mathrm{Tr}(q(d_t^2)^{-\frac{s}{2}})d\tau d\theta. \end{aligned}$$

The leading term in this expression is

$$\pi^{-1}\int_{T^*S^1} \mathrm{Tr}(Q^*Q+\tau^2)^{-s/2}d\tau d\theta.$$

It is clear that the off-diagonal terms in $q(d_t^2)$ contribute to the trace only modulo t^2. It follows that the sub-leading term (the coefficient of t in $t\overline{\zeta}(d_t,s)$) comes only from the two diagonal terms in $q(d_t^2)$. If we choose ϕ to be even in τ then these terms are odd in τ, and thus their integral with respect to τ vanishes. This proves (41) modulo (40).

Let us identify $\mathrm{Tr}\left((Q^*Q+\tau^2)^{-\frac{s}{2}}\right)$. As for the eta function, we split the trace according to large or small eigenvalues of Q^*Q. The large eigenvalues part does not depend on ϕ since $Q_{>\alpha_j} = D_{>\alpha_j}$; we compute easily

$$\int_{T^*W_{\alpha_j}} \mathrm{Tr}(Q^*_{>\alpha_j}Q_{>\alpha_j}+\tau^2)^{-s/2}d\tau d\theta = f\left(\frac{s}{2}\right)\int_{W_{\alpha_j}} \mathrm{Tr}(D^*_{>\alpha_j}D_{>\alpha_j})^{-\frac{s-1}{2}}d\theta.$$

At $s=0$ we get $-\int_{W_{\alpha_j}} \mathrm{Tr}_w|D_{>\alpha_j}|d\theta$ since $sf(s/2)_{|s=0} = -\pi$. Remark that $\mathrm{Tr}_w|D_{>\alpha_j}| = \mathrm{Tr}_w|D|$ since Tr_w vanishes on finite rank operators. We claim next that the meromorphic function with simple poles

$$K(s) := \int_{\mathbb{R}} \mathrm{Tr}(Q^*_{<\alpha_j}Q_{<\alpha_j}+\tau^2)^{-s/2}d\tau$$

vanishes at $s=0$, which implies that the contribution of small eigenvalues is null. Indeed, let $A(\tau) := Q^*_{<\alpha_j}Q_{<\alpha_j}$. Clearly the matrix A satisfies $\|A(\tau)\| \leq C$, $\|A'(\tau)\| \leq C(1+\tau^2)^{-1}$ since $\phi(\tau)$ is Schwartz. Integration by parts (for large $\Re(s)$) shows

$$\begin{aligned} K(s) &= \int_{\mathbb{R}} \tau'\mathrm{Tr}(A(\tau)+\tau^2)^{-s/2}d\tau \\ &= s/2\int_{\mathbb{R}} \mathrm{Tr}(\tau(2\tau+A'(\tau))(A(\tau)+\tau^2)^{-s/2-1}d\tau \\ &= sK(s) + s\int_{\mathbb{R}} \mathrm{Tr}((\tau A'(\tau)/2 - A(\tau))(A(\tau)+\tau^2)^{-s/2-1}d\tau. \end{aligned}$$

By unique continuation the identity holds for all $s \in \mathbb{C}$. For $\Re(s) \geq -1$ the last integral converges absolutely, so because of the s factor we deduce that $(1-s)K(s)$ is regular and vanishes at $s=0$, which is to say that $K(0)=0$.

Thus at $s=0$ we deduce (40), (41) for d_t in lieu of δ_t. But by Lemma 26 we know that $\zeta(\delta_t, 0) = \zeta(d_t, 0)$. □

The crucial point in the above proof is reducing the analysis to a finite number of eigenvalues and the corresponding eigenspaces. Indeed, the use of the function $\phi \equiv 1$ in (35) instead of a Schwartz function leads to an operator in the extended adiabatic algebra if and only if the fibration $X \to M$ is the identity fibration $M \to M$.

Note that the sign in formula (39) disappears if we change the definition of δ_t to be consistent with the choice of the bounding spin structure on S^1.

Also, when the dimension of N is even, the eta invariant is a $\pmod{2\mathbb{Z}}$ homotopy invariant of elliptic operators and moreover $\mathrm{index}(D) = 0$ because the fibers are odd-dimensional. Thus for every $t > 0$ we get:

$$e^{-i\pi\overline{\eta}(\delta_t)} = \mathrm{hol}(\det(D)).$$

This identity can also be deduced from the results of [3].

We have deliberately avoided mentioning the regularity of the eta function at $s=0$ since it plays no role in the proof. In light of this regularity however, the first part of the theorem generalizes the Witten-Bismut-Freed holonomy theorem to any family of elliptic first-order differential operators.

9 Closing remarks

The adiabatic algebra is a powerful tool for studying degenerate families of operators such as δ_t. Compared to previous results in this direction, one needs to make only minimal assumptions about the family D, while obtaining significant regularity in the adiabatic limit for free from the adiabatic formalism. The smoothness of the eta and zeta functions in the adiabatic limit is by no means obvious; it can actually fail if D is non-invertible even if δ_t is invertible for all t. For example, consider the operator on S^1

$$\delta_t := \begin{bmatrix} -ti\partial_\theta - t\alpha & 0 \\ 0 & ti\partial_\theta + t\alpha \end{bmatrix}.$$

For general α and s, the expansion of $\eta(\delta_t, s)$ and $\zeta(\delta_t, s)$ at $t=0$ will contain $\log t$ terms. The reason is that δ_t is not invertible as an adiabatic operator.

One can consider other elliptic adiabatic operators constructed from the family D, for instance

$$P_t := t\tilde{\nabla}_{\partial_\theta} + D$$

in the case where D is self-adjoint. If the fibers of $N \to S^1$ are spin, N is even-dimensional and D is the family of Dirac operators on the fibers then

P_t is the chiral Dirac operator for the metric (3) on N. We can apply the previous analysis to $\zeta(P_t,0)$. Formula (40) and Theorems 22, 23 hold for P_t if we divide the right-hand side by 2.

In the particular case where $N=S^1$ is the identity fibration and $D=1$, Theorem 22 says that the function $t\sum_{k=-\infty}^{\infty}(t^2k^2+1)^{-s/2}$, which is well-defined for $(s,t)\in\{\Re(s)>1\}\times(0,\infty)$, extends as a meromorphic function in $s\in\mathbb{C}$ and smooth in $t\in[0,\infty)$. Moreover, for all $s\in\mathbb{C}$,

$$\lim_{t\to 0} t\sum_{k=-\infty}^{\infty}(t^2k^2+1)^{-s/2}=\pi^{1/2}\frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

This limit of meromorphic functions can be checked directly for $\Re(s)>1$, but seems hard to prove for general s by elementary methods.

It is evident that $\operatorname{index}(P_t)=\zeta(P_t,0)-\zeta(P_t^*,0)$. The left-hand side is constant in t, in particular

$$\operatorname{index}(P_t)=\lim_{t\to 0}(\zeta(P_t,0)-\zeta(P_t^*,0)).$$

Using the adiabatic algebra methods developed in this paper, we obtain

$$\begin{aligned}\zeta(P_t,0)&\sim_{t\to 0}-\frac{1}{2t}\int_{S^1}\operatorname{Tr}_w(D^2)^{\frac12}d\theta+\frac14\int_{S^1}\operatorname{Tr}_w\left((D^2)^{-\frac12}\tilde\nabla_{\partial_\theta}(D)\right)d\theta,\\ \zeta(P_t^*,0)&\sim_{t\to 0}-\frac{1}{2t}\int_{S^1}\operatorname{Tr}_w(D^2)^{\frac12}d\theta-\frac14\int_{S^1}\operatorname{Tr}_w\left((D^2)^{-\frac12}\tilde\nabla_{\partial_\theta}(D)\right)d\theta\end{aligned}$$

which imply

$$\operatorname{index}(P_t)=\frac12\int_{S^1}\operatorname{Tr}_w\left((D^2)^{-\frac12}\tilde\nabla_{\partial_\theta}(D)\right)d\theta. \tag{46}$$

Formally, the right-hand side of (46) is $\frac12\int_{S^1}\operatorname{Tr}_w(\tilde\nabla_{\partial_\theta}(\log|D|))d\theta$, and it is easily proved to equal the net flow of eigenvalues through 0 around the circle. We are therefore led to a purely analytical proof of the identity

$$\operatorname{index}(P_t)=\operatorname{sf}_{S^1}(D) \tag{47}$$

where sf denotes the spectral flow [1].

In the spirit of [9] we can interpret the right-hand side of (46) as the integral on S^1 of a closed 1-form defined on S^1 (or more generally on the basis M of the family D) by

$$\operatorname{var}(\eta):=\operatorname{Tr}_w\left((D^2)^{-\frac12}\tilde\nabla(D)\right).$$

In [9] Melrose defined the analogous quantity in the more general setting of 1-suspended operators, and proved subsequently the extension of (47) to that setting. Our methods can be used to reprove Melrose's result for an elliptic adiabatic family in $\Psi_a(N)$.

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