Midterm #2 — April 12, 2013, 10:00 to 10:53 AM

Name: Solution Key

Circle your recitation:

R01 (Claudio · Fri) R02 (Xuan · Wed) R03 (Claudio · Mon)

- **You have a maximum of 53 minutes.** This is a closed-book, closed-notes exam. No calculators or other electronic aids are allowed.
- Read each question carefully. Show your work and justify your answers for full credit. You do not need to simplify your answers unless instructed to do so.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

Grading

1	/30
2	/15
3	/20
4	/15
5	/20
Total	/100

1. (*30 points*) Find the general solution to each of the following differential equations:

(a) (*10 points*) $y'' - 4y' + 3y = 0$

Solution: The characteristic equation of this homogeneous, constant-coefficient DE is $r^2 - 4r + 3 = 0$, which factors as $(r-1)(r-3) = 0$. Therefore, the roots are $r = 1$ and $r = 3$, so the general solution is $y = c_1e^x + c_2e^{3x}$.

(b) (*10 points*) $y'' - 4y' + 4y = 0$

Solution: The characteristic equation of this DE is $r^2 - 4r + 4 = 0$, which factors as $(r-2)^2 = 0$. Therefore, the only root is a double root $r = 2$, so the general solution is $y = c_1e^{2x} + c_2xe^{2x}$.

(c) (*10 points*) $y'' - 4y' + 5y = 0$

Solution: The characteristic equation of this DE is $r^2 - 4r + 5 = 0$, which has discriminant $4^2 - 4(1)(5) = -4$, and therefore has complex roots. By the quadratic formula, these roots are $r = \frac{1}{2}(4 \pm \sqrt{-4}) = 2 \pm i$, so the general solution is $y = c_1e^{2x}\cos x + c_2e^{2x}\sin x$.

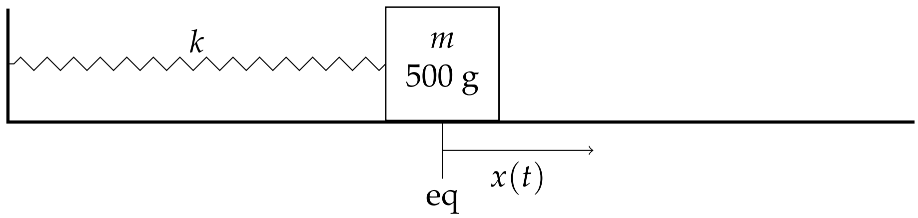
2. (*15 points*) Two solutions to the DE $y'' - 3y' - 10y = 0$ are $y_1 = e^{5x}$ and $y_2 = e^{-2x}$. Find a solution to this DE satisfying the initial conditions $y(0) = 7$ and $y'(0) = 7$.

Solution: Since this DE is second-order, homogeneous, and linear, and since y_1 and y_2 are clearly linearly independent, $y(x) = c_1e^{5x} + c_2e^{-2x}$ is the general solution to this DE. We then match it and its derivative $y'(x) = 5c_1e^{5x} - 2c_2e^{-2x}$ to the initial conditions at $x = 0$. We obtain the linear system

$$c_1 + c_2 = 7, \qquad 5c_1 - 2c_2 = 7.$$

Then $c_2 = 7 - c_1$, so $5c_1 - 14 + 2c_1 = 7$, and $c_1 = 21/7 = 3$. Hence, $c_2 = 7 - 3 = 4$, so $y(x) = 3e^{5x} + 4e^{-2x}$ solves this IVP.

3. (*20 points*) A 500-gram test mass m is attached to a spring of unknown spring constant k and allowed to settle into its equilibrium position, as shown:



The mass is struck sharply at time $t = 0$, and the resulting displacement measured to be

$$x(t) = 0.25e^{-3t}\sin 5t \qquad (x \text{ in meters, } t \text{ in seconds})$$

(a) (*5 points*) Is this system underdamped, critically damped, or overdamped? Explain.

Solution: Since the displacement of the mass in this free system takes the form of a function $e^{-at}\sin bt$, which contains a sinusoidal factor, the system is underdamped.

(b) (*10 points*) Find the spring constant k of the spring and the damping constant c resulting from natural friction in the system. Include appropriate units.

Solution: We reconstruct the differential equation $mx'' + cx' + kx = 0$ governing the motion. First, since the motion is a multiple of $e^{-3t}\sin 5t$, this DE has complex roots $r = -3 \pm 5i$. Therefore, its characteristic equation is a multiple of $(r+3-5i)(r+3+5i) = (r+3)^2 + 5^2 = r^2 + 6r + 34 = 0$. The coefficient on the r^2-term must be $m = 1/2$ (in mks units), though, so we multiply through by this to obtain $\frac{1}{2}r^2 + 3r + 17$. Hence, $c = 3$ N-s/m, and $k = 17$ N/m.

Many students tried to use the relation $\omega_0 = \sqrt{k/m}$ to determine k, but since damping is present in the system, ω_0 does not come directly from the frequency of the sine factor in the displacement, and therefore is not 5 rad/s. The frequency in the displacement is instead ω_1, and $\omega_0^2 = \omega_1^2 + p^2$, where $p = 3$ is the exponent in the displacement's exponential factor e^{-pt}.

(c) (*5 points*) Suppose we then apply a periodic force $f(t) = 10\cos\omega t$ (in newtons) to the spring-mass system, where we may vary the forcing frequency ω. Find the value of ω at which the system exhibits practical resonance, or explain why it does not occur.

Solution: We recall that, as a function of ω and the parameters m, c, k, and F_0, the amplitude of the steady-state solution is

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.$$

(We can arrive at this solution by assuming a steady periodic displacement of the form $A\cos\omega t + B\sin\omega t$; see pp. 219–220 for details.) By minimizing the denominator, we can show that, if $c^2 < 2km$, $C(\omega)$ has a maximum at $\omega_m = \sqrt{\frac{2km - c^2}{2m^2}}$. Since $c^2 = 3^2 = 9$ and $2km = 2(17)(\frac{1}{2}) = 17$, c is small enough for practical resonance to occur at

$$\omega_m = \sqrt{\frac{2km - c^2}{2m^2}} = \sqrt{\frac{17 - 9}{2(1/2)^2}} = \sqrt{16} = 4\frac{\text{rad}}{\text{s}}.$$

4. (*15 points*) Find a particular solution to the nonhomogeneous DE

$$y^{(3)} - 6y'' + 12y' = 40e^{2x} - 24.$$

Solution: From the form of the forcing term $40e^{2x} - 24$, we expect the particular solution to contain terms corresponding to the roots $r = 2$ (from the e^{2x}) and $r = 0$ (from the constant term). Before we construct our guess, though, we must also check whether these roots also contribute to the complementary solution. The characteristic equation for the associated homogeneous DE is

$$r^3 - 6r^2 + 12r = r(r^2 - 6r + 12) = 0,$$

which then has $r = 0$ as a single root. We check for overlap with $r = 2$: since $(2)^2 - 6(2) + 12 = 4 - 12 + 12 = 4 \neq 0$, 2 is not a root. Hence, to avoid overlaps on $r = 0$, we guess a particular solution of the form

$$y = Ae^{2x} + Bx.$$

(Notice we have shifted only the constant term by a power of x, and not the non-overlapping e^{2x} term.) Then $y' = 2Ae^{2x} + B$, $y'' = 4Ae^{2x}$, and $y''' = 8Ae^{2x}$, which we plug into the nonhomogeneous DE:

$$8Ae^{2x} - 6(4Ae^{2x}) + 12(2Ae^{2x} + B) = 40e^{2x} - 24.$$

Simplifying, $8Ae^{2x} + 12B = 40e^{2x} - 24$, so $8A = 40$ and $12B = -24$. Then $A = 5$ and $B = -2$, so a particular solution is

$$y = 5e^{2x} - 2x.$$

5. (*20 points*) Let $y_1(x) = x$ and $y_2(x) = x^4$.

(a) (*10 points*) Show that y_1 and y_2 are both solutions to the DE $x^2y'' - 4xy' + 4y = 0$.

Solution: We compute the first and second derivatives of these functions:

$$y_1' = 1, \quad y_1'' = 0, \quad y_2' = 4x^3, \quad y_2'' = 12x^2.$$

Plugging these into the homogeneous DE,

$$\begin{aligned} x^2y_1'' - 4xy_1' + 4y_1 &= x^2(0) - 4x(1) + 4(x) = -4x + 4x = 0, \\ x^2y_2'' - 4xy_2' + 4y_2 &= x^2(12x^2) - 4x(4x^2) + 4(x^4) = (12 - 16 + 4)x^4 = 0. \end{aligned}$$

(b) (*5 points*) Show that y_1 and y_2 are linearly independent functions on the entire real line.

Solution: We compute the Wronskian $W(x)$ of y_1 and y_2:

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^4 \\ 1 & 4x^3 \end{vmatrix} = 4x^4 - x^4 = 3x^4.$$

Since this function is nonzero on the real line (in fact, everywhere but $x = 0$), these functions are linearly independent.

(c) (*5 points*) Find the general solution to the nonhomogeneous DE

$$x^2y'' - 4xy' + 4y = 6x^4 - 9x.$$

Solution: We use variation of parameters to solve this nonhomogeneous DE. First, we write the *normalized* version of the DE, dividing by x^2 to obtain

$$y'' - \frac{4}{x}y' + \frac{4}{x^2}y = 6x^2 - 9x^{-1}.$$

Then $f(x) = 6x^2 - 9x^{-1}$, which we include in the formula for variation of parameters:

$$\begin{aligned}
y &= -y_1 \int \frac{y_2(x)f(x)}{W(x)}\,dx + y_2 \int \frac{y_1(x)f(x)}{W(x)}\,dx \\
&= -x \int \frac{x^4(6x^2 - 9x^{-1})}{3x^4}\,dx + x^4 \int \frac{x(6x^2 - 9x^{-1})}{3x^4}\,dx \\
&= -x \int 2x^2 - \frac{3}{x}\,dx + x^4 \int \frac{2}{x} - \frac{3}{x^4}\,dx \\
&= -x \left(\frac{2}{3}x^3 - 3\ln x\right) + x^4(2\ln x + x^{-3}) \\
&= 3x\ln x - \frac{2}{3}x^4 + 2x^4 \ln x + x.
\end{aligned}$$

The general solution to the homogeneous equation is $y_c = c_1y_1 + c_2y_2 = c_1x + c_2x^4$, so adding this to the particular solution above produces the general solution to the nonhomogeneous equation:

$$y = c_1x + c_2x^4 + 3x\ln x - \frac{2}{3}x^4 + 2x^4\ln x + x = C_1x + C_2x^4 + 3x\ln x + 2x^4\ln x,$$

where $C_1 = c_1 + 1$ and $C_2 = c_2 - \frac{2}{3}$ are new arbitrary constants that absorb the y_1 and y_2 terms coming from the integration. (We also observe that if we had included the constants of integration, we would have obtained the general solution directly from the variation of parameters formula.)