Stony Brook University

NAME: Midterm Solutions

RECITATION #: ____

MAT 303 – Fall 2016 – Midterm 1

October 5, 2016

INSTRUCTIONS – PLEASE READ

- Please turn off your cell phone and put it away.
- Please write your name and your section number right now.
- This is a closed book exam. You are NOT allowed to use a calculator or any other electronic device or aid.
- Show your work. To receive full credit, your answers must be neatly written and logically organized. If you need more space, write on the back side of the preceding sheet, but be sure to label your work clearly. You do not need to simplify your answers unless explicitly instructed to do so.
- Academic integrity is expected of all Stony Brook University students at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1.	
2.	
3.	
4.	
Total	

LEC 01	MWF	12:00pm-12:53pm	Harriman Hall 137	Raluca Tanase
R01	W	10:00am-10:53am	Library W4535	Dyi-Shing Ou
R02	F	1:00pm- 1:53pm	Library E4330	Timothy Ryan
R03	Tu	5:30pm- 6:23pm	Library W4525	Timothy Ryan
R04	W	7:00pm- 7:53pm	Library W4525	Alexandra Viktorova
R05	M	5:30pm- 6:23pm	Lgt Engr Lab 152	Jiasheng Teh

Problem 1. (25 points) Find the solution to the given initial value problem:

a) $\frac{dy}{dt} - 2y = -2e^t, \quad y(0) = 1.$

Sol. This is a linear equation. An integrating factor is $\mu(t) = e^{\int -2dt} = e^{-2t}$

$$\underbrace{e^{-2t}\frac{dy}{dt} - 2e^{-2t}y}\ \cancel{\frac{dy}{dt}} = -2e^{-t}$$

$$\frac{d}{dt}\left(y\cdot e^{-2t}\right) = -2e^{-t} \Rightarrow y\cdot e^{-2t} = \int -2e^{-t}dt = 2e^{-t}+c$$

$$\Rightarrow y(t) = e^{2t}(2e^{-t}+c) = 2e^t + Ce^{2t}$$

$$y(0) = 1 \Rightarrow 2+c = 1 \Rightarrow c = -1$$

$$\Rightarrow y(t) = 2e^t - e^{2t}$$

b) $\frac{dy}{dt} + y = e^t y^3, \quad y(0) = -1.$

Sol. This is a Bernoulli equation with exponent $n=3$.

if $y \neq 0$ we can divide by y^3

$$\frac{1}{y^3}\frac{dy}{dt} + \frac{1}{y^2} = e^t$$

Do the substitution $v = y^{1-3} = \frac{1}{y^2}$

$$\frac{dv}{dt} = \frac{d}{dt}\left(\frac{1}{y^2}\right) = -\frac{2}{y^3}\frac{dy}{dt}$$

$\Rightarrow \frac{dv}{dt} - 2v = -2e^t$, which is exactly the equation from part (a).

$$\Rightarrow v(t) = 2e^t + Ce^{2t}$$

$$v = y^{-2} \Rightarrow y^2 = \frac{1}{v} = \frac{1}{2e^t + Ce^{2t}}$$

$$y(0) = -1$$

$$\Rightarrow \frac{1}{2+c} = 1 \Rightarrow 2+c = 1 \Rightarrow c = -1$$

$$y^2 = \frac{1}{2e^t - e^{2t}} \Rightarrow y(t) = -\frac{1}{\sqrt{2e^t - e^{2t}}},$$

which is well defined if

$$2e^t > e^{2t}$$
$$\ln(2e^t) > \ln(e^{2t})$$
$$(\ln 2) + t > 2t$$
$$\ln 2 > t$$

Since $y(0)$ is negative, we select the negative sol.

Problem 2. (25 points) Determine whether the following equation is exact. If it is exact, find its solutions.

$$(2xy-1)\frac{dy}{dx} = 3x^2 - y^2$$

Solution: $3x^2 - y^2 - (2xy-1)\frac{dy}{dx} = 0$

$M(x,y) = 3x^2 - y^2$
$N(x,y) = -2xy + 1$ $\Rightarrow$ $\frac{\partial M}{\partial y}(x,y) = -2y = \frac{\partial N}{\partial x}(x,y)$ $\Rightarrow$ the equation is exact.

There exists a function $F(x,y)$ such that $\begin{cases} \frac{\partial F}{\partial x} = M \\ \frac{\partial F}{\partial y} = N \end{cases}$

$F(x,y) = \int \frac{\partial F}{\partial x}(x,y)\,dx = \int (3x^2 - y^2)\,dx = x^3 - xy^2 + h(y)$

$\frac{\partial F}{\partial y}(x,y) = -2xy + h'(y)$
$\frac{\partial F}{\partial y}(x,y) = N(x,y) = -2xy + 1$ $\Bigg| \Rightarrow h'(y) = 1 \Rightarrow h(y) = y + c_1$

$F(x,y) = x^3 - xy^2 + y + c_1$

The solution set of the differential equation is $F(x,y) = C$, that is

$x^3 - xy^2 + y = C$, where C is any real constant.

Problem 3. (25 points)

a) Find all values of a and b for which the initial value problem

$$x\frac{dy}{dx} = y, \quad y(a) = b$$

has (i) a unique local solution, (ii) no solution, or (iii) infinitely many local solutions.

Sol. $\left.\begin{array}{l} F(x,y) = \frac{y}{x} \\ \frac{\partial F}{\partial y}(x,y) = \frac{1}{x} \end{array}\right\}$ continuous when $x \neq 0$. $\Rightarrow$ by the existence & uniqueness theorem, for any initial condition $y(a)=b$ with $a \neq 0$, there exists (locally around $x=a$) a unique solution $y(x)$ of the initial value problem.

If $a=0$, then the theorem does not apply, so we need to solve the diff. eq. first. Using part b) we know that the general solution is $y(x) = C\cdot x$, so we conclude the following:

If $a=0$ and $b \neq 0$, there is no solution as $\left.\begin{array}{l} y(x) = Cx \\ y(0) = b \end{array}\right\} \Rightarrow C\cdot 0 = b \Rightarrow b=0$ (contradiction)

If $a=0$ and $b=0$, the IVP has infinitely many solutions. Every solution $y(x) = Cx$ satisfies the initial condition $y(0)=0$.

b) Solve the differential equation $x\dfrac{dy}{dx} = y$.

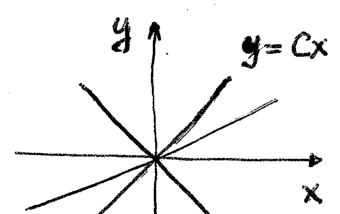
Sol. This is a separable equation with a singular solution $y=0$.

When $y \neq 0$ we write $\frac{1}{y}dy = \frac{1}{x}dx \Rightarrow \int \frac{1}{y}dy = \int \frac{1}{x}dx$

$\Rightarrow \ln|y| = \ln|x| + C_1 \Rightarrow |y| = |x|e^{C_1}$

$\Rightarrow y = x\cdot C$, where $C = \pm e^{C_1}$ is any random constant.

General solution: $y(x) = C\cdot x$



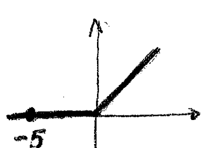
c) How many global solutions can you define which satisfy the initial condition $y(-5) = 0$?

Sol. We first solve $\left\{\begin{array}{l} y(x) = C\cdot x \\ y(-5) = 0 \end{array}\right.$ and get $y(x) = 0$. Since $-5 \neq 0$, by part a) we know that the solution $y(x)=0$ is locally unique arountd the point $x=-5$.

This solution is also globally unique, because we cannot glue it to another solution to obtain a differentiable function. Any glueing defines a function of the form

$$y_c(x) = \left\{\begin{array}{l} 0, \; x<0 \\ Cx, \; x \geq 0 \end{array}\right.$$

. However y_c is not differentiable at 0 unless $C=0$.



Graph of $y_c(x)$

Problem 4. (25 points) Consider the following model for the growth rate of the deer population as a function of time:

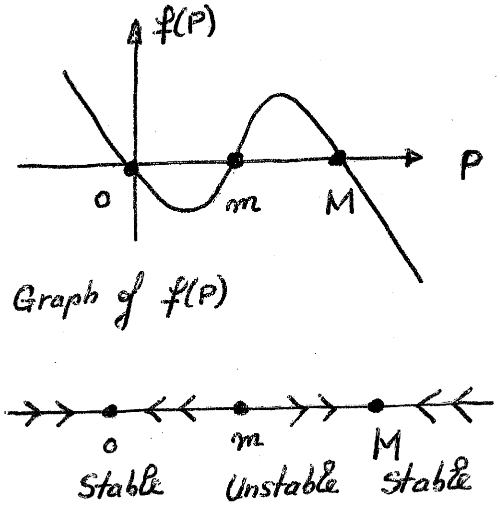
$$\frac{dP}{dt} = P(M-P)(P-m),$$

where P is the deer population and $M > m > 0$ are positive constants. It is known that if the deer population rises above the carrying capacity M, the population will decrease back to M through disease and malnutrition.

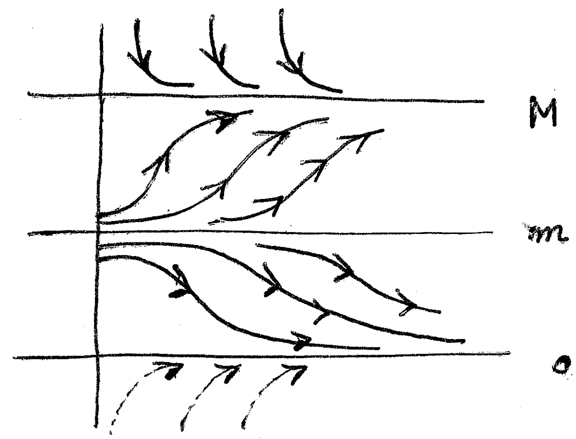
a) Determine the equilibrium points of this model and classify each one as stable or unstable. Sketch several graphs of solutions of the differential equation.

Sol. Define $f(P) = P(M-P)(P-m)$

Critical points: $f(P) = 0 \Rightarrow P = 0$, or $P = M$, or $P = m$



Graph of $f(P)$



Typical Solution Curves

b) What happens to the deer population in the long run if the initial population $P(0)$ is between 0 and M? (You are not required to solve the differential equation).

If $P(0) = 0, m, M$ then the population does not change.

If $0 < P(0) < m$ then the population will become extinct.

If $m < P(0) < M$ then the population will increase to the carrying capacity M.