MAT 303: Calculus IV with Applications

Fall 2016

Practice problems for Midterm 2

Problem 1:

a) Find the general solution of the ODE $y'' + 4y = 4\cos(2t)$.

b) Make a sketch of y_p vs. t, where $y_p(t)$ denotes the particular solution found in part a). What is the pseudo-period of the oscillation and the time varying amplitude?

Problem 2: Consider the 4th order ODE $y^{(4)} + 4y'' = f(x)$.

a) Obtain the homogeneous solution.

b) For each case given below, give the general form of the particular solution using the method of undetermined coefficients. Do not evaluate the coefficients.

1. $f(x) = 5 + 8x^3$
2. $f(x) = x\sin(5x)$
3. $f(x) = \cos(2x)$
4. $f(x) = 2\sin^2(x)$

Problem 3: Consider the boundary value problem (BVP):

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + \lambda y = 0, \quad 1 < t < e, \quad y(1) = \frac{dy}{dt}(e) = 0.$$

a) Find all positive values of $\lambda \in (0, \infty)$ such that the BVP has a nontrivial solution.

b) Determine a nontrivial solution corresponding to each of the values of λ found in part a).

c) For what values of $\lambda \in (0, \infty)$ does the BVP admit a unique solution? What is that solution.

Problem 4: Consider the ODE

$$t^2y'' + ty' + \lambda y = 0, t > 0. \tag{1}$$

a) For $\lambda = 4$, find two solutions of (1), calculate their Wronskian and thus deduce that they form a fundamental set of solutions.

b) Verify your answer for the Wronskian using Abel's Theorem and a convenient initial condition from part a).

c) Solve the eigenvalue problem (1) on $1 < t < e$, subject to $y(1) = y'(e) = 0$, that is find all values of λ such that the boundary value problem has a nontrivial solution and in that case determine the latter.

Problem 5: Find the general solution of the system

$$\begin{aligned} x_1' &= 4x_1 + x_2 + x_3 \\ x_2' &= x_1 + 4x_2 + x_3 \\ x_3' &= 4x_1 + x_2 + 4x_3. \end{aligned}$$

Problem 6: Consider the differential equation

$$x^2y'' + xy' - 9y = 0, \quad x > 0.$$

We know that $y_1(x) = x^3$ is a solution to this ODE. Use the method of reduction of order to find a second solution y_2. Show that y_1 and y_2 form a fundamental set of solutions of the differential equation (that is, show that they are linearly independent).

Problem 7: Find the critical value of λ in which bifurcations occur in the system

$$\dot{x} = 1 + \lambda x + x^2.$$

Sketch the phase portrait for various values of λ and the bifurcation diagram. Classify the bifurcation.