MAT 303: Calculus IV with Applications
Fall 2016

Practice problems for Midterm 2
Solutions

Problem 1:

a) Find the general solution of the ODE $y'' + 4y = 4\cos(2t)$.

b) Make a sketch of y_p vs. t, where $y_p(t)$ denotes the particular solution found in part a). What is the pseudo-period of the oscillation and the time varying amplitude?

Solution. The characteristic equation for the homogeneous ODE is $r^2 + 4 = 0$, which has solutions $r = \pm 2i$. The homogeneous solution is $y_h(t) = C_1 \cos(2t) + C_2 \sin(2t)$. We look for particular solutions $y_p(t) = t(A\cos(2t) + B\sin(2t))$. We compute

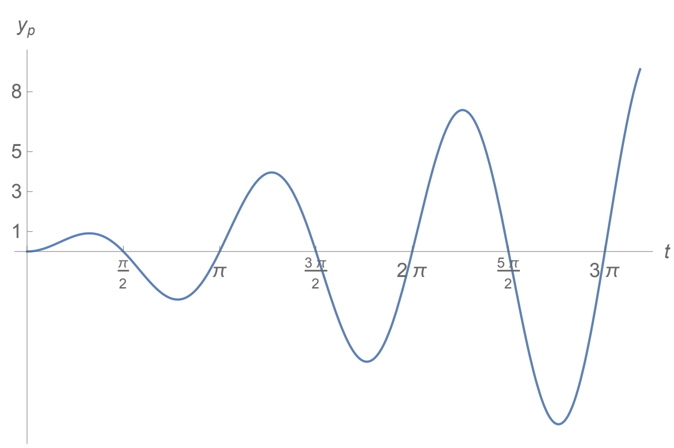
$$y_p'(t) = A\cos(2t) + B\sin(2t) + 2t(-A\sin(2t) + B\cos(2t))$$

$$y_p''(t) = -4A\sin(2t) + 4B\cos(2t) - 4t(A\cos(2t) + B\sin(2t)).$$

Plugging these in the initial ODE we find

$$y_p'' + 4y_p = -4A\sin(2t) + 4B\cos(2t) = 4\cos(2t),$$

which gives $A = 0$ and $B = 1$. Hence a particular solution is $y_p(t) = t\sin(2t)$. The amplitude is $A(t) = t$, the frequency is $\omega = 2$, so the period is $T = \dfrac{2\pi}{\omega} = \pi$. The general solution is $y = y_h + y_p = C_1\cos(2t) + (C_2 + t)\sin(2t)$.



$\square$

Problem 2: Consider the 4th order ODE $y^{(4)} + 4y'' = f(x)$.

a) Obtain the homogeneous solution.

b) For each case given below, give the general form of the particular solution using the method of undetermined coefficients. Do not evaluate the coefficients.

1. $f(x) = 5 + 8x^3$
2. $f(x) = x\sin(5x)$
3. $f(x) = \cos(2x)$
4. $f(x) = 2\sin^2(x)$

SOLUTION.

a) The characteristic equation is $r^4 + 4r^2 = 0$, which has roots $r = 0$ (repeated root of order 2) and $r = \pm 2i$. The homogeneous solution is

$$y_h(x) = C_1 + C_2 x + C_3\cos(2x) + C_4\sin(2x).$$

b)
1. $y_p = x^2(a_0 + a_1 x + a_2 x^2 + a_3 x^3)$
2. $y_p = (a_0 + a_1 x)\cos(5x) + (b_0 + b_1 x)\sin(5x)$
3. $y_p = x(A\cos(2x) + B\sin(2x))$
4. Note that $2\sin^2(x) = 1 - \cos(2x)$, hence $y_p = a_0 x^2 + x(A\cos(2x) + B\sin(2x))$.

□

Problem 3: Consider the boundary value problem (BVP):

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + \lambda y = 0, \quad 1 < t < e, \quad y(1) = \frac{dy}{dt}(e) = 0.$$

a) Find all positive values of $\lambda \in (0,\infty)$ such that the BVP has a nontrivial solution.

b) Determine a nontrivial solution corresponding to each of the values of λ found in part a).

c) For what values of $\lambda \in (0,\infty)$ does the BVP admit a unique solution? What is that solution.

SOLUTION. We make the change of variables $x = \ln(t)$. Note that $\ln(1) = 0$ and $\ln(e) = 1$. The equivalent BVP is

$$\frac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < 1, \quad y(0) = y'(1) = 0.$$

a) Let $\lambda > 0$. The general solution is $y = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$. We have $y(0) = c_1 = 0$ and $y'(1) = -c_2\sqrt{\lambda}\cos(\sqrt{\lambda}) = 0$. This gives $\sqrt{\lambda} = \dfrac{(2n-1)\pi}{2}$, $n = 1, 2, \ldots$.

b) $y = c_2 \sin\left(\frac{(2n-1)\pi x}{2}\right) = c_2 \sin\left(\frac{(2n-1)\pi}{2}\ln(t)\right)$, $n = 1, 2, \ldots$.

c) For $\lambda \neq \frac{(2n-1)\pi}{2}$, $n = 1, 2, \ldots$, the unique solution is $y = 0$.

$\square$

Problem 4: Consider the ODE

$$t^2 y'' + t y' + \lambda y = 0, t > 0. \tag{1}$$

a) For $\lambda = 4$, find two solutions of (1), calculate their Wronskian and thus deduce that they form a fundamental set of solutions.

b) Verify your answer for the Wronskian using Abel's Theorem and a convenient initial condition from part a).

c) Solve the eigenvalue problem (1) on $1 < t < e$, subject to $y(1) = y'(e) = 0$, that is find all values of λ such that the boundary value problem has a nontrivial solution and in that case determine the latter.

SOLUTION.

a) For $\lambda = 4$, the equation becomes $t^2 y'' + t y' + \lambda y = 0, t > 0$. We make a change of variables $x = \ln(t)$ and obtain the ODE $y'' + 4y = 0$. The fundamental solutions are $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$ or $y_1(t) = \cos(2\ln(t))$ and $y_2(t) = \sin(2\ln(t))$. By differentiating with respect to t, we find $y_1'(t) = -\frac{2}{t}\sin(2\ln(t))$ and $y_2'(t) = \frac{2}{t}\cos(2\ln(t))$. For $t > 0$, the Wronskian is

$$W(y_1, y_2) = \frac{2}{t}\cos^2(2\ln(t)) + \frac{2}{t}\sin^2(2\ln(t)) = \frac{2}{t}.$$

Clearly $W \neq 0$ so y_1, and y_2 are linearly independent and form a fundamental set of solutions.

b) We put the original ODE in the form

$$y'' + \frac{1}{t}y' + \frac{4}{t^2}y = 0, \; t > 0.$$

By Abel's theorem we get $W = C\exp\left(-\int \frac{1}{t}dt\right) = C\exp(-\ln(t)) = \frac{C}{t}$. From part a), $W(1) = 2$, which gives $C = 2$.

c) By making a change of variables $x = \ln(t)$, we have to solve the eigenvalue problem

$$y'' + 4y = 0, \quad y(0) = 0, \quad y'(1) = 0.$$

We find eigenvalues $\lambda_n = \left(\frac{(2n-1)\pi}{2}\right)^2$, for $n = 1, 2, \ldots$ and corresponding eigenfunctions $y_n(x) = \sin\left(\frac{(2n-1)\pi x}{2}\right)$, $n = 1, 2, \ldots$.

$\square$

Problem 5: Find the general solution of the system

$$\begin{aligned} x_1' &= 4x_1 + x_2 + x_3 \\ x_2' &= x_1 + 4x_2 + x_3 \\ x_3' &= 4x_1 + x_2 + 4x_3. \end{aligned}$$

SOLUTION. The system can be written as $X' = AX$, where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}.$$

The characteristic polynomial of A is

$$\begin{aligned} \begin{vmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} &= \begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} \\ &= (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (6-\lambda)(3-\lambda)^2. \end{aligned}$$

The eigenvalues are $\lambda_1 = 6$ (of algebraic multiplicity 1) and $\lambda_2 = 3$ (of algebraic multiplicity 2). The eigenvectors for the eigenvalue $\lambda_2 = 3$ are given by the equation $(A - 3I_3)v = 0$. We write

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and obtain $v_1 + v_2 + v_3 = 0$, hence $v_3 = -v_1 - v_2$. Thus

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ -v_1 - v_2 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

The geometric multiplicity is 2. Two linearly independent eigenvectors of $\lambda_2 = 3$ are

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

The eigenvectors for $\lambda_1 = 6$ are solutions of

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We find the following system of equations:

$$\begin{aligned} -2v_1 + v_2 + v_3 &= 0 \\ v_1 - 2v_2 + v_3 &= 0 \\ v_1 + v_2 - 2v_3 &= 0 \end{aligned}$$

The third equation is redundant. Subtracting the second equation from the first we get $-3v_1 + 3v_2 = 0$, so $v_1 = v_2$. Substituting this in the first equation yields $v_3 = v_1$. It follows that

$$w_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda_1 = 6$. The general solution for the given system of equations is

$$x(t) = c_1 e^{3t} w_1 + c_2 e^{3t} w_2 + c_3 w_3 e^{6t}.$$

$\square$

Problem 6: Consider the differential equation

$$x^2 y'' + xy' - 9y = 0, \quad x > 0.$$

We know that $y_1(x) = x^3$ is a solution to this ODE. Use the method of reduction of order to find a second solution y_2. Show that y_1 and y_2 are linearly independent.

Solution. Substitute $y = vx^3$ in the given equation and simplify. We get the differential equation $xv'' + 7v' = 0$, which is separable. We write $\frac{v''}{v'} = -\frac{7}{x}$ and integrate. This gives $\ln v' = -7\ln x + \ln A$, which yields $v' = \frac{A}{x^7}$ and finally $v(x) = -\frac{A}{6x^6} + B$. With $A = -6$ and $B = 0$ we get $v(x) = \frac{1}{x^6}$, so $y_2(x) = \frac{1}{x^3}$.

To show linear independence, assume that $ax^3 + b\frac{1}{x^3} = 0$ for all $x > 0$. This is equivalent to $ax^6 + b = 0$. When $x = 1$ we get $a + b = 0$. When $x = 2$ we get $64a + b = 0$, so the only values of a and b for which both conditions are satisfied is $a = b = 0$. In conclusion, y_1 and y_2 are two linearly independent solutions. $\square$

Problem 7: Find the critical value of λ in which bifurcations occur in the system

$$\dot{x} = 1 + \lambda x + x^2.$$

Sketch the phase portrait for various values of λ and the bifurcation diagram. Classify the bifurcation.

Solution. The critical points c_1 and c_2 of the system verify $1 + \lambda x + x^2 = 0$, so

$$c_{1,2} = \frac{-\lambda \pm \sqrt{\lambda^2 - 4}}{2}.$$

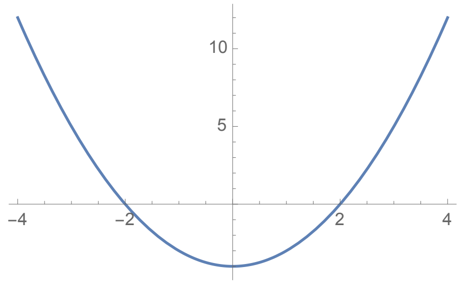


Figure 1: The graph of $\lambda^2 - 4$.

We have three cases to consider. First, suppose $\lambda^2 = 4$. Then $\lambda = \pm 2$. For $\lambda = 2$, the system has one critical point $c = -\dfrac{\lambda}{2} = -1$, which is semi-stable, since $f(x) = 1 + 2x + x^2 = (1+x)^2 \geq 0$ for all x. Similarly, for $\lambda = -2$, the system has one critical point $c = -\dfrac{\lambda}{2} = 1$, which is semi-stable, since $f(x) = 1 - 2x + x^2 = (1-x)^2 \geq 0$ for all x.

If $\lambda^2 < 4$, then $-2 < \lambda < 2$ and there are no critical points.

If $\lambda^2 > 4$, then $\lambda > 2$ or $\lambda < -2$. The system has two distinct critical points:

$$\begin{aligned} c_1 &= \frac{-\lambda - \sqrt{\lambda^2 - 4}}{2} \quad \text{(stable)} \\ c_2 &= \frac{-\lambda + \sqrt{\lambda^2 - 4}}{2} \quad \text{(unstable)} \end{aligned}$$

The function $f(x) = 1 + \lambda x + x^2$ is positive when $x < c_1$ or $x > c_2$, and negative when $c_1 < x < c_2$.

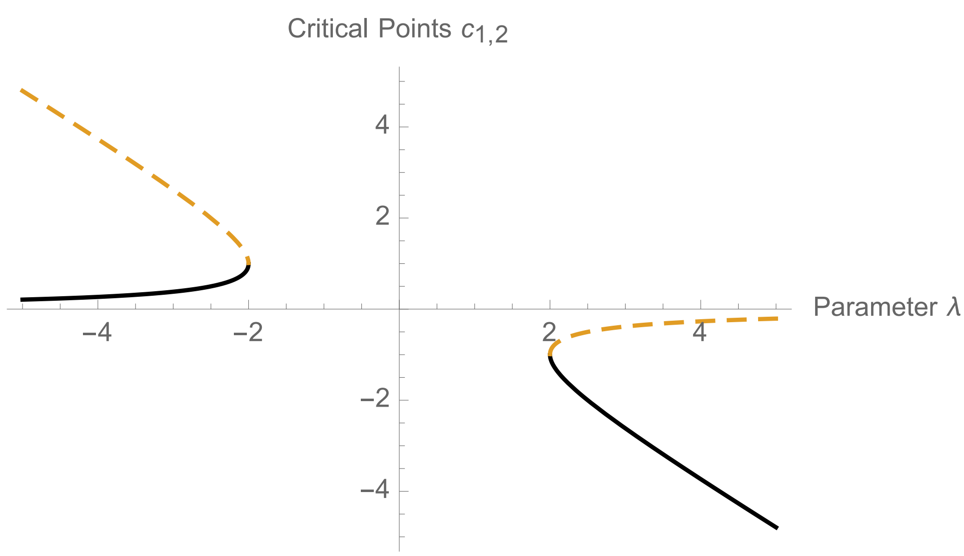


Figure 2: The bifurcation diagram.

The system undergoes a saddle-node bifurcation. □