MAT 303: Calculus IV with Applications
Fall 2016

Lecture Notes - Friday, December 2

Let A be an $n \times n$ matrix. Let λ_1 be an eigenvalue of A. Let's recall first some definitions:

Algebraic Multiplicity: The algebraic multiplicity of λ_1 is the number of times the factor $\lambda - \lambda_1$ appears in the characteristic polynomial $p(\lambda) = \det(A - \lambda I_n)$.

Geometric Multiplicity: The geometric multiplicity of λ_1 is the maximum number of linearly independent eigenvectors corresponding to the eigenvalue λ_1.

Defect: The defect of the eigenvalue λ_1 is equal to the Algebraic Multiplicity of λ_1 minus its Geometric Multiplicity.

Invertible: An $n \times n$ square matrix S is called invertible if there exists an $n \times n$ matrix S^{-1}, called the inverse of S, such that $S^{-1}S = SS^{-1} = I_n$.

It can be shown that S is invertible if and only if the determinant of S is different from 0. It can also be shown that if S is invertible, then its inverse S^{-1} is unique.

The inverse of a 2×2 matrix: If $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det(S) = ad - bc \neq 0$, then

$$S^{-1} = \frac{1}{\det(S)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

DiagonalizationTheorem : Let A be an $n \times n$ matrix. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A (not necessarily distinct). Assume that all eigenvalues of A have algebraic multiplicity equal to their geometric multiplicity. Then A has n linearly independent eigenvectors $v_1, \ldots, v_n$. If we denote by S the matrix whose columns are the eigenvectors $v_1, \ldots, v_n$, then S is an invertible matrix, and

$$S^{-1}AS = D,$$

where $D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$ is a diagonal matrix whose diagonal entries are precisely the eigenvalues of A. We say that A is a diagonalizable matrix.

Remark: It can be shown that if A has repeated eigenvalues with geometric multiplicity different from the algebraic multiplicity (so A has fewer than n eigenvectors!), then A cannot be diagonalized. It is still possible however to transform A into a nearly diagonal matrix called the Jordan canonical form.

Remark: Matrix multiplication is not commutative in general! Hence $S^{-1}AS$ is not the same as $AS^{-1}S = AI_n = A$!

Jordan Block: Let λ be an eigenvalue of some matrix A. An $m \times m$ matrix B is called a *Jordan block of size m* corresponding to λ if

$$B = \begin{pmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda & 1 & 0 \\ 0 & 0 & \dots & 0 & \lambda & 1 \\ 0 & 0 & \dots & 0 & 0 & \lambda \end{pmatrix}, \tag{1}$$

that is, the entries on the main diagonal of B are all equal to λ, and the entries right above the main diagonal are equal to 1. All the other entries are equal to 0.

The exponential matrix of a Jordan Block: If B has the form from Equation (1), then

$$e^{Bt} = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{3!} & \dots & \frac{t^{m-1}}{(m-1)!} \\ 0 & 1 & t & \frac{t^2}{2} & \dots & \vdots \\ \dots & \dots & \dots & \dots & \dots & \vdots \\ 0 & 0 & \dots & 1 & t & \frac{t^2}{2} \\ 0 & 0 & \dots & 0 & 1 & t \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

Proof. To make the computations easier, let's give the proof for $m = 4$. The Jordan block can be written as $B = \lambda I_4 + C$, where $C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is a nilpotent matrix. The matrix C has the property that $C^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $C^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $C^4 = 0_4$.

The matrix exponential e^{Ct} can be computed as

$$e^{Ct} = I_4 + \sum_{n=1}^{\infty} \frac{C^n t^n}{n!} = I_4 + Ct + C^2 \frac{t^2}{2} + C^3 \frac{t^3}{3!} = \begin{pmatrix} 1 & t & t^2/2 & t^3/3! \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix $D = \lambda I_4$ has the property that $D^n = I_4 \lambda^n$ for every $n > 0$. Therefore

$$e^{Dt} = I_4 + \sum_{n=1}^{\infty} \frac{D^n t^n}{n!} = I_4 + \sum_{n=1}^{\infty} \frac{I_4 \lambda^n t^n}{n!} = I_4 (1 + \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{n!}) = I_4 e^{\lambda t}$$

In conclusion $e^{Bt} = e^{\lambda I_4 t + Ct} = e^{Dt}e^{Ct} = e^{Dt}I_4 e^{\lambda t} = e^{Dt}e^{\lambda t} = e^{\lambda t}\begin{pmatrix} 1 & t & t^2/2 & t^3/3! \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

□

Examples:

a) If $B = (\lambda)$ is a Jordan block of size 1, then $e^{Bt} = e^{\lambda t}(1) = \left(e^{\lambda t}\right)$.

a) If $B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, then $e^{Bt} = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

c) $B = \begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$, $e^{Bt} = e^{5t}\begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{5t} & te^{5t} & t^2e^{5t}/2 & t^3e^{5t}/6 \\ 0 & e^{5t} & te^{5t} & t^2e^{5t}/2 \\ 0 & 0 & e^{5t} & te^{5t} \\ 0 & 0 & 0 & e^{5t} \end{pmatrix}$.

Jordan Canonical Form Theorem :

Let A be any $n \times n$ matrix. There exists an invertible matrix S such that

$$S^{-1}AS = J, \quad \text{where} \quad J = \begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \cdots & \\ 0 & & & J_s \end{pmatrix} \tag{2}$$

J is a block diagonal matrix, where $J_1, J_2, \ldots, J_s$ are Jordan blocks corresponding to the eigenvalues of A, as in Equation (1). The columns of the matrix S are chosen as follows: Each column of S that corresponds to the first column of each Jordan block J_k is an eigenvector v_k of A. The rest of the columns are generalized eigenvectors.

Remarks: An eigenvalue can produce several Jordan blocks (the number of Jordan blocks for one eigenvalue is equal to the geometric multiplicity of the eigenvalue). The size of one Jordan block is equal to the length of a chain of generalized eigenvectors.

Examples: Let A be a 2×2 matrix.

a) If A has a repeated eigenvalue λ_1 with algebraic and geometric multiplicity 2, then its Jordan canonical form is the diagonal matrix $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$. In this case, J has two Jordan blocks, each of size 1.

b) If A has a repeated eigenvalue λ_1 with algebraic multiplicity 2, and geometric multiplicity 1 then its Jordan canonical form is the block diagonal matrix $J = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$. The Jordan matrix has a single Jordan block, of size 2.

Examples: Let A be a 7×7 matrix. Assume that A has eigenvalues λ (with algebraic multiplicity 2 and geometric multiplicity 1) and μ (with algebraic multiplicity 5 and geometric multiplicity 2). Matrix A has 3 linearly independent eigenvectors v_1, w_1, u_1: one corresponding to the eigenvalue λ, and two corresponding to the eigenvalue μ. Assume that after doing some computations, we found 3 chains of generalized eigenvectors

$$\begin{aligned}
\{v_1, v_2\}, &\quad \textit{where} \quad (A-\lambda I_7)v_1 = 0, (A-\lambda I_7)v_2 = v_1 \\
\{w_1, w_2\}, &\quad \textit{where} \quad (A-\mu I_7)w_1 = 0, (A-\mu I_7)w_2 = w_1 \\
\{u_1, u_2, u_3\}, &\quad \textit{where} \quad (A-\mu I_7)u_1 = 0, (A-\mu I_7)u_2 = u_1, (A-\mu I_7)u_3 = u_2
\end{aligned}$$

If we let S be the 7×7 matrix with columns $S = [v_1, v_2, w_1, w_2, u_1, u_2, u_3]$, then

$$S^{-1}AS = \begin{pmatrix}
\lambda & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \mu
\end{pmatrix} \tag{3}$$

This is the Jordan canonical form of A. It has 3 Jordan blocks, one corresponding to the eigenvalue λ, and two corresponding to the eigenvalue μ. The canonical form is unique up to permutations of the Jordan blocks. If we write the columns of S is a different order, for example $S = [v_1, v_2, u_1, u_2, u_3, w_1, w_2]$, we get

$$S^{-1}AS = \begin{pmatrix}
\lambda & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \mu
\end{pmatrix} \tag{4}$$

Applications of the Jordan canonical form: Assume that $S^{-1}AS = J$, where J is the Jordan canonical form.

1. Exponential of a Jordan matrix: If $J_1, \dots J_s$ are the Jordan blocks of J, then

$$e^{Jt} = \begin{pmatrix}
e^{J_1 t} & & & 0 \\
& e^{J_2 t} & & \\
& & \dots & \\
0 & & & e^{J_s t}
\end{pmatrix}.$$

The exponential matrix of each block can then be computed as in Equation (1).

2. Computing powers of the matrix A: It can be easily seen that $A = SJS^{-1}$ and so $A^n = S^{-1}J^nS$ for every $n > 0$.

3. Finding the exponential matrix of A: $e^{At} = Se^{Jt}S^{-1}$. Indeed,

$$e^{At} \quad = \quad \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = \sum_{n=0}^{\infty} \frac{(SJ^nS^{-1})t^n}{n!} = S(\sum_{n=0}^{\infty} \frac{J^n t^n}{n!})S^{-1} = Se^{Jt}S^{-1}.$$

4. Solving the system of differential equations $x' = Ax$: The general solution of the system $x' = Ax$ is given by $x(t) = e^{At}c$, where c is a vector of coefficients. If J is the Jordan canonical form of A, then the solution can also be written as:

$$x(t) = Se^{Jt}S^{-1}c = Se^{Jt}\overline{c},$$

where $\overline{c} = S^{-1}c$ is a new vector of random constants. The matrix $\Phi(t) = Se^{Jt}$ is therefore a fundamental matrix of the system $x' = Ax$. When solving the system $x' = Ax$ we prefer to work with the fundamental matrix $\Phi(t)$ because it does not require finding the inverse of the matrix S, unlike e^{At}.

5. Reducing a system of differential equations Consider the system of differential equations

$$x' = Ax. \tag{5}$$

We do the variable substitution $x = Sy$, which gives $x' = Sy'$ and obtain the equivalent system

$$Sy' = ASy$$

After multiplication by S^{-1}, we end up with

$$y' = S^{-1}ASy = Jy \tag{6}$$

A fundamental matrix of the system $y' = Jy$ is e^{Jt}. The general solution is then $y(t) = e^{Jt}c$. Since the relation between Systems (5) and (6) is given by the substitution $x = Sy$, it follows that the general solution of System (5) is $x(t) = Sy(t) = Se^{Jt}c$. Therefore $\Phi(t) = Se^{Jt}$ is a fundamental matrix for System (5).

Solving a nonhomogeneous system $x' = Ax + f(t)$: As proven in class, using integrating factors, the general solution is given by the Variation of Parameters formula:

$$x(t) = e^{At} \int e^{-At} f(t)dt = \Phi(t) \int \Phi^{-1}(t)f(t)dt,$$

where $\Phi(t)$ is any fundamental matrix of $x' = Ax$.

Example 1: Based on Exercise 5.5.26 from the textbook. We will solve the system $x' = Ax$, where $A = \begin{pmatrix} 5 & -1 & 1 \\ 1 & 3 & 0 \\ -3 & 2 & 1 \end{pmatrix}$ using two different methods.

SOLUTION. The eigenvalues of A are the roots of the characteristic polynomial

$$\begin{aligned}
p(\lambda) &= \begin{vmatrix} 5-\lambda & -1 & 1 \\ 1 & 3-\lambda & 0 \\ -3 & 2 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 5-\lambda & -1 & 1-1 \\ 1 & 3-\lambda & 0+(3-\lambda) \\ -3 & 2 & (1-\lambda)+2 \end{vmatrix} = \begin{vmatrix} 5-\lambda & -1 & 0 \\ 1 & 3-\lambda & 3-\lambda \\ -3 & 2 & 3-\lambda \end{vmatrix} \\
&= \begin{vmatrix} 5-\lambda & -1 & 0 \\ 1-(-3) & (3-\lambda)-2 & (3-\lambda)-(3-\lambda) \\ -3 & 2 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 5-\lambda & -1 & 0 \\ 4 & 1-\lambda & 0 \\ -3 & 2 & 3-\lambda \end{vmatrix} \\
&= (3-\lambda)\begin{vmatrix} 5-\lambda & -1 \\ 4 & 1-\lambda \end{vmatrix} = (3-\lambda)\left((5-\lambda)(1-\lambda)+4\right) = (3-\lambda)(9-6\lambda+\lambda^2) = (3-\lambda)^3.
\end{aligned}$$

Therefore $\lambda = 3$ is the only eigenvalue of A, and it has algebraic multiplicity 3. To find its geometric multiplicity, we solve the equation

$$(A-3I_3)v = 0, \text{ where } v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

This gives

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which gives the system of equations:

$$\begin{aligned}
2a - b + c &= 0 \\
a &= 0 \\
-3a + 2b - 2c &= 0
\end{aligned}$$

The third equation is redundant. The first equation gives $c = b$. Therefore, all eigenvectors corresponding to the eigenvalue $\lambda = 3$ are of the form

$$v = \begin{pmatrix} 0 \\ b \\ b \end{pmatrix} = b\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

All eigenvectors are therefore scalar multiples of the eigenvector $v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and they belong to the line spanned by v_1 in $\mathbb{R}^3$. Since we can find at most one linearly independent eigenvector, it follows that the geometric multiplicity of $\lambda = 3$ is 1.

We need to find two generalized eigenvectors v_2 and v_3, starting from v_1,by successively solving the equations

$$(A-3I_3)v_2 = v_1 \tag{7}$$
$$(A-3I_3)v_3 = v_2. \tag{8}$$

$\{v_1, v_2, v_3\}$ will then be a chain of length 3 for the eigenvalue λ. For Equation (7), we solve

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

and get the system

$$\begin{aligned} 2a - b + c &= 0 \\ a &= 1 \\ -3a + 2b - 2c &= 1 \end{aligned}$$

which reduces to $a = 1$ and $c = b - 2$. Therefore, $v_2 = \begin{pmatrix} 1 \\ b \\ b-2 \end{pmatrix} = b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$. The first term of the sum can be ignored, because it is an eigenvector of A, and we can choose $v_2 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$.

We now solve Equation (8).

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ -3 & 2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix},$$

which gives the system

$$\begin{aligned} 2a - b + c &= 1 \\ a &= 0 \\ -3a + 2b - 2c &= -2 \end{aligned}$$

which is equivalent to $a = 0$ and $c = b+1$. Hence $v_2 = \begin{pmatrix} 0 \\ b \\ b+1 \end{pmatrix} = b \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The first term can be ignored since it is an eigenvector of A, so we choose the second generalized eigenvector to be $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

We have obtained three linearly independent vectors, forming a chain of length 3:

$$\{v_1, v_2, v_3\} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Now we go back to solving the system $x' = Ax$.

Method 1: Three linearly independent solutions can be found by setting

$$\begin{aligned}
x_1(t) &= e^{3t}v_1 = e^{3t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\
x_2(t) &= e^{3t}(v_2 + tv_1) = e^{3t}\begin{pmatrix} 1 \\ t \\ -2+t \end{pmatrix} \\
x_3(t) &= e^{3t}(v_3 + tv_2 + \frac{t^2}{2}v_1) = e^{3t}\begin{pmatrix} t \\ t^2/2 \\ 1-2t+t^2/2 \end{pmatrix}
\end{aligned}$$

The general solution is $x(t) = c_1x_1(t) + c_2x_2(t) + c_3x_3(t)$, where c_1, c_2, c_3 are real numbers. In expanded form this gives:

$$x(t) = e^{3t}\begin{pmatrix} c_2 + c_3t \\ c_1 + c_2t + c_3\frac{t^2}{2} \\ c_1 + c_2(t-2) + c_3(\frac{t^2}{2} - 2t + 1) \end{pmatrix}.$$

Method 2: Consider the matrix $S = [v_1, v_2, v_3] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -2 & 1 \end{pmatrix}$.

Since A has a repeated eigenvalue $\lambda = 3$, with algebraic multiplicity 3, and geometric multiplicity 1, the Jordan canonical form J of A will contain a single Jordan block:

$$J = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Then $e^{Jt} = e^{\lambda t}\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$, so a fundamental matrix for $x' = Ax$ is given by

$$\Phi(t) = Se^{Jt} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -2 & 1 \end{pmatrix} e^{\lambda t}\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = e^{\lambda t}\begin{pmatrix} 0 & 1 & t \\ 1 & t & \frac{t^2}{2} \\ 1 & t-2 & \frac{t^2}{2} - 2t + 1 \end{pmatrix}.$$

The general solution of $x' = Ax$ is given by

$$x' = \Phi(t)c = \Phi(t)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = e^{\lambda t}\begin{pmatrix} c_2 + c_3t \\ c_1 + c_2t + c_3\frac{t^2}{2} \\ c_1 + c_2(t-2) + c_3(\frac{t^2}{2} - 2t + 1) \end{pmatrix}$$

Example 2: Based on Exercise 5.5.16 in the textbook. Solve the system $x' = Ax$, where $A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -2 & -3 \\ 2 & 3 & 4 \end{pmatrix}$, in two different ways.

SOLUTION. The characteristic polynomial of A is

$$p(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -2 & -2-\lambda & -3 \\ 2 & 3 & 4-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} -2-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = (1-\lambda)((-2-\lambda)(4-\lambda)+9) = (1-\lambda)^3.$$

The matrix A has a single eigenvalue $\lambda = 1$, with algebraic multiplicity 3. To find the eigenvectors corresponding to the eigenvalue $\lambda = 1$ we solve the system $(A - I_3)v = 0$:

$$\begin{pmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies 2a + 3b + 3c = 0 \implies c = -\frac{2}{3}a - b$$

$$\implies v = \begin{pmatrix} a \\ b \\ -\frac{2}{3}a - b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ -\frac{2}{3} \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

where a and b are random constants. Because it's more convenient to work with integer numbers, we can also write

$$v = a \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

where $a := a/3$ and b are any random constants. All eigenvectors belong to a plane in $\mathbb{R}^3$, spanned by the linearly independent eigenvectors

$$v = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

hence the eigenvalue $\lambda = 1$ has geometric multiplicity 2.

We need to find one more generalized eigenvector u, starting from either v or w.

Starting from v: Find u such that $(A - I_3)u = v$. This condition implies

$$\begin{pmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \implies 0a + 0b + 0c = 3 \implies 0 = 3,$$

which is a contradiction, hence there does not exist any generalized eigenvector u that solves $(A - I_3)u = v$. It means that $\{v\}$ is a chain of length 1.

Starting from w: Find u such that $(A - I_3)u = w$. This condition implies

$$\begin{pmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \implies 2a + 3b + 3c = -1 \implies c = -\frac{1}{3} - \frac{2}{3}a - b$$

$$\implies v = \begin{pmatrix} a \\ b \\ -\frac{1}{3} - \frac{2}{3}a - b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ -\frac{2}{3} \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{3} \end{pmatrix}. \tag{9}$$

The first two terms can be ignored, as they are eigenvectors of the matrix A. A generalized eigenvector is given by $u = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{3} \end{pmatrix}$. Notice that we cannot scale u by any factor we want as this vector does not appear with a random constant in front in the formula given in Equation 9. We could pick another generalized eigenvector with integer entries by setting say $a = 3$, $b = 0$ in Equation 9, but we will choose to work with the generalized eigenvector that we have already selected.

The set $\{w, u\}$ is a chain of length 2. This is a maximal chain starting from the eigenvector w (if we tried to continue the chain by finding a vector z such that $(A - I_3)z = u$, we would get a contradiction).

Together, $\{v, w, u\}$ are a linearly independent set formed with eigenvectors and generalized eigenvectors of the matrix A.

Method 1: Three linearly independent solutions can be found by setting

$$\begin{aligned} x_1(t) &= e^t v = e^t \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \\ x_2(t) &= e^t w = e^t \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \\ x_3(t) &= e^t (u + tw) = e^t \begin{pmatrix} 0 \\ t \\ -\frac{1}{3} - t \end{pmatrix} \end{aligned}$$

The general solution is $x(t) = c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t)$, where c_1, c_2, c_3 are real numbers. In expanded form this gives:

$$x(t) = e^t \begin{pmatrix} 3c_1 \\ c_2 + c_3 t \\ -2c_1 + -c_2 + c_3(-\frac{1}{3} - t) \end{pmatrix}.$$

Method 2: Consider the matrix $S = [v, w, u] = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & -\frac{1}{3} \end{pmatrix}$.

Since A has a repeated eigenvalue $\lambda = 1$, with algebraic multiplicity 3, and geometric multiplicity 2, the Jordan canonical form J of A will contain two Jordan blocks, as follows: the first Jordan block is of size 1, corresponding to the chain formed by $\{v\}$, whereas the second Jordan block corresponds to the chain $\{w, u\}$ and has size 2. The Jordan canonical form of the matrix A is therefore

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Then $e^{Jt} = e^{\lambda t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$, so a fundamental matrix for $x' = Ax$ is given by

$$\Phi(t) = Se^{Jt} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & -\frac{1}{3} \end{pmatrix} e^{\lambda t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = e^{\lambda t} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & t \\ -2 & -1 & -t - \frac{1}{3} \end{pmatrix}.$$

The general solution of $x' = Ax$ is given by

$$x' = \Phi(t)c = \Phi(t) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = e^{\lambda t} \begin{pmatrix} 3c_1 \\ c_2 + c_3 t \\ -2c_1 - c_2 + c_3(-t - \frac{1}{3}) \end{pmatrix}.$$