(b) If the substances P and Q are the same, then $p = q$ and Eq. (i) is replaced by

$$dx/dt = \alpha(p - x)^2. \tag{ii}$$

If $x(0) = 0$, determine the limiting value of $x(t)$ as $t \to \infty$ without solving the differential equation. Then solve the initial value problem and determine $x(t)$ for any t.

2.6 Exact Equations and Integrating Factors

For first order equations there are a number of integration methods that are applicable to various classes of problems. The most important of these are linear equations and separable equations, which we have discussed previously. Here, we consider a class of equations known as exact equations for which there is also a well-defined method of solution. Keep in mind, however, that those first order equations that can be solved by elementary integration methods are rather special; most first order equations cannot be solved in this way.

EXAMPLE 1

Solve the differential equation

$$2x + y^2 + 2xyy' = 0. \tag{1}$$

The equation is neither linear nor separable, so the methods suitable for those types of equations are not applicable here. However, observe that the function $\psi(x, y) = x^2 + xy^2$ has the property that

$$2x + y^2 = \frac{\partial \psi}{\partial x}, \qquad 2xy = \frac{\partial \psi}{\partial y}. \tag{2}$$

Therefore, the differential equation can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Assuming that y is a function of x, we can use the chain rule to write the left side of Eq. (3) as $d\psi(x, y)/dx$. Then Eq. (3) has the form

$$\frac{d\psi}{dx}(x, y) = \frac{d}{dx}(x^2 + xy^2) = 0. \tag{4}$$

By integrating Eq. (4) we obtain

$$\psi(x, y) = x^2 + xy^2 = c, \tag{5}$$

where c is an arbitrary constant. The level curves of $\psi(x, y)$ are the integral curves of Eq. (1). Solutions of Eq. (1) are defined implicitly by Eq. (5).

In solving Eq. (1) the key step was the recognition that there is a function ψ that satisfies Eqs. (2). More generally, let the differential equation

$$M(x, y) + N(x, y)y' = 0 \tag{6}$$

be given. Suppose that we can identify a function $\psi(x, y)$ such that

$$\frac{\partial\psi}{\partial x}(x, y) = M(x, y), \qquad \frac{\partial\psi}{\partial y}(x, y) = N(x, y), \tag{7}$$

and such that $\psi(x, y) = c$ defines $y = \phi(x)$ implicitly as a differentiable function of x. Then

$$M(x, y) + N(x, y)y' = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi[x, \phi(x)]$$

and the differential equation (6) becomes

$$\frac{d}{dx}\psi[x, \phi(x)] = 0. \tag{8}$$

In this case Eq. (6) is said to be an **exact** differential equation. Solutions of Eq. (6), or the equivalent Eq. (8), are given implicitly by

$$\psi(x, y) = c, \tag{9}$$

where c is an arbitrary constant.

In Example 1 it was relatively easy to see that the differential equation was exact and, in fact, easy to find its solution, at least implicitly, by recognizing the required function ψ. For more complicated equations it may not be possible to do this so easily. How can we tell whether a given equation is exact, and if it is, how can we find the function $\psi(x, y)$? The following theorem answers the first question, and its proof provides a way of answering the second.

Theorem 2.6.1 Let the functions M, N, M_y, and N_x, where subscripts denote partial derivatives, be continuous in the rectangular[17] region R: $\alpha < x < \beta, \gamma < y < \delta$. Then Eq. (6)

$$M(x, y) + N(x, y)y' = 0$$

is an exact differential equation in R if and only if

$$M_y(x, y) = N_x(x, y) \tag{10}$$

at each point of R. That is, there exists a function ψ satisfying Eqs. (7),

$$\psi_x(x, y) = M(x, y), \qquad \psi_y(x, y) = N(x, y),$$

if and only if M and N satisfy Eq. (10).

[17]It is not essential that the region be rectangular, only that it be simply connected. In two dimensions this means that the region has no holes in its interior. Thus, for example, rectangular or circular regions are simply connected, but an annular region is not. More details can be found in most books on advanced calculus.

The proof of this theorem has two parts. First, we show that if there is a function ψ such that Eqs. (7) are true, then it follows that Eq. (10) is satisfied. Computing M_y and N_x from Eqs. (7), we obtain

$$M_y(x,y) = \psi_{xy}(x,y), \qquad N_x(x,y) = \psi_{yx}(x,y). \tag{11}$$

Since M_y and N_x are continuous, it follows that ψ_{xy} and ψ_{yx} are also continuous. This guarantees their equality, and Eq. (10) is valid.

We now show that if M and N satisfy Eq. (10), then Eq. (6) is exact. The proof involves the construction of a function ψ satisfying Eqs. (7)

$$\psi_x(x,y) = M(x,y), \qquad \psi_y(x,y) = N(x,y).$$

We begin by integrating the first of Eqs. (7) with respect to x, holding y constant. We obtain

$$\psi(x,y) = Q(x,y) + h(y), \tag{12}$$

where $Q(x,y)$ is any differentiable function such that $\partial Q(x,y)/\partial x = M(x,y)$. For example, we might choose

$$Q(x,y) = \int_{x_0}^{x} M(s,y)\,ds, \tag{13}$$

where x_0 is some specified constant in $\alpha < x_0 < \beta$. The function h in Eq. (12) is an arbitrary differentiable function of y, playing the role of the arbitrary constant. Now we must show that it is always possible to choose $h(y)$ so that the second of Eqs. (7) is satisfied—that is, $\psi_y = N$. By differentiating Eq. (12) with respect to y and setting the result equal to $N(x,y)$, we obtain

$$\psi_y(x,y) = \frac{\partial Q}{\partial y}(x,y) + h'(y) = N(x,y).$$

Then, solving for $h'(y)$, we have

$$h'(y) = N(x,y) - \frac{\partial Q}{\partial y}(x,y). \tag{14}$$

In order for us to determine $h(y)$ from Eq. (14), the right side of Eq. (14), despite its appearance, must be a function of y only. One way to show that this is true is to show that its derivative with respect to x is zero. Thus we differentiate the right side of Eq. (14) with respect to x, obtaining

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial}{\partial x}\frac{\partial Q}{\partial y}(x,y). \tag{15}$$

By interchanging the order of differentiation in the second term of Eq. (15), we have

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial}{\partial y}\frac{\partial Q}{\partial x}(x,y),$$

or, since $\partial Q/\partial x = M$,

$$\frac{\partial N}{\partial x}(x,y) - \frac{\partial M}{\partial y}(x,y),$$

which is zero on account of Eq. (10). Hence, despite its apparent form, the right side of Eq. (14) does not, in fact, depend on x. Then we find $h(y)$ by integrating Eq. (14), and upon substituting this function in Eq. (12), we obtain the required function $\psi(x, y)$. This completes the proof of Theorem 2.6.1.

It is possible to obtain an explicit expression for $\psi(x, y)$ in terms of integrals (see Problem 17), but in solving specific exact equations, it is usually simpler and easier just to repeat the procedure used in the preceding proof. That is, integrate $\psi_x = M$ with respect to x, including an arbitrary function of $h(y)$ instead of an arbitrary constant, and then differentiate the result with respect to y and set it equal to N. Finally, use this last equation to solve for $h(y)$. The next example illustrates this procedure.

EXAMPLE 2

Solve the differential equation

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0. \tag{16}$$

By calculating M_y and N_x, we find that

$$M_y(x, y) = \cos x + 2xe^y = N_x(x, y),$$

so the given equation is exact. Thus there is a $\psi(x, y)$ such that

$$\psi_x(x, y) = y\cos x + 2xe^y,$$
$$\psi_y(x, y) = \sin x + x^2e^y - 1.$$

Integrating the first of these equations, we obtain

$$\psi(x, y) = y\sin x + x^2e^y + h(y). \tag{17}$$

Setting $\psi_y = N$ gives

$$\psi_y(x, y) = \sin x + x^2e^y + h'(y) = \sin x + x^2e^y - 1.$$

Thus $h'(y) = -1$ and $h(y) = -y$. The constant of integration can be omitted since any solution of the preceding differential equation is satisfactory; we do not require the most general one. Substituting for $h(y)$ in Eq. (17) gives

$$\psi(x, y) = y\sin x + x^2e^y - y.$$

Hence solutions of Eq. (16) are given implicitly by

$$y\sin x + x^2e^y - y = c. \tag{18}$$

EXAMPLE 3

Solve the differential equation

$$(3xy + y^2) + (x^2 + xy)y' = 0. \tag{19}$$

We have

$$M_y(x, y) = 3x + 2y, \qquad N_x(x, y) = 2x + y;$$

since $M_y \neq N_x$, the given equation is not exact. To see that it cannot be solved by the procedure described above, let us seek a function ψ such that

$$\psi_x(x, y) = 3xy + y^2, \qquad \psi_y(x, y) = x^2 + xy. \tag{20}$$

Integrating the first of Eqs. (20) gives

$$\psi(x, y) = \tfrac{3}{2}x^2y + xy^2 + h(y), \tag{21}$$

where h is an arbitrary function of y only. To try to satisfy the second of Eqs. (20), we compute ψ_y from Eq. (21) and set it equal to N, obtaining

$$\tfrac{3}{2}x^2 + 2xy + h'(y) = x^2 + xy$$

or

$$h'(y) = -\tfrac{1}{2}x^2 - xy. \tag{22}$$

Since the right side of Eq. (22) depends on x as well as y, it is impossible to solve Eq. (22) for $h(y)$. Thus there is no $\psi(x, y)$ satisfying both of Eqs. (20).

Integrating Factors. It is sometimes possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor. Recall that this is the procedure that we used in solving linear equations in Section 2.1. To investigate the possibility of implementing this idea more generally, let us multiply the equation

$$M(x, y) + N(x, y)y' = 0 \tag{23}$$

by a function μ and then try to choose μ so that the resulting equation

$$\mu(x, y)M(x, y) + \mu(x, y)N(x, y)y' = 0 \tag{24}$$

is exact. By Theorem 2.6.1, Eq. (24) is exact if and only if

$$(\mu M)_y = (\mu N)_x. \tag{25}$$

Since M and N are given functions, Eq. (25) states that the integrating factor μ must satisfy the first order partial differential equation

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0. \tag{26}$$

If a function μ satisfying Eq. (26) can be found, then Eq. (24) will be exact. The solution of Eq. (24) can then be obtained by the method described in the first part of this section. The solution found in this way also satisfies Eq. (23), since the integrating factor μ can be canceled out of Eq. (24).

A partial differential equation of the form (26) may have more than one solution; if this is the case, any such solution may be used as an integrating factor of Eq. (23). This possible nonuniqueness of the integrating factor is illustrated in Example 4.

Unfortunately, Eq. (26), which determines the integrating factor μ, is ordinarily at least as hard to solve as the original equation (23). Therefore, although in principle integrating factors are powerful tools for solving differential equations, in practice they can be found only in special cases. The most important situations in which simple integrating factors can be found occur when μ is a function of only one of the variables x or y, instead of both.

Let us determine conditions on M and N so that Eq. (23) has an integrating factor μ that depends on x only. If we assume that μ is a function of x only, then the partial derivative μ_x reduces to the ordinary derivative $d\mu/dx$ and $\mu_y = 0$. Making these substitutions in Eq. (26), we find that

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu. \tag{27}$$

If $(M_y - N_x)/N$ is a function of x only, then there is an integrating factor μ that also depends only on x; further, $\mu(x)$ can be found by solving Eq. (27), which is both linear and separable.

A similar procedure can be used to determine a condition under which Eq. (23) has an integrating factor depending only on y; see Problem 23.

EXAMPLE 4

Find an integrating factor for the equation

$$(3xy + y^2) + (x^2 + xy)y' = 0 \tag{19}$$

and then solve the equation.

In Example 3 we showed that this equation is not exact. Let us determine whether it has an integrating factor that depends on x only. On computing the quantity $(M_y - N_x)/N$, we find that

$$\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{1}{x}. \tag{28}$$

Thus there is an integrating factor μ that is a function of x only, and it satisfies the differential equation

$$\frac{d\mu}{dx} = \frac{\mu}{x}. \tag{29}$$

Hence

$$\mu(x) = x. \tag{30}$$

Multiplying Eq. (19) by this integrating factor, we obtain

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0. \tag{31}$$

Equation (31) is exact, since

$$\frac{\partial}{\partial y}(3x^2y + xy^2) = 3x^2 + 2xy = \frac{\partial}{\partial x}(x^3 + x^2y).$$

Thus there is a function ψ such that

$$\psi_x(x,y) = 3x^2y + xy^2, \qquad \psi_y(x,y) = x^3 + x^2y. \tag{32}$$

Integrating the first of Eqs. (32), we obtain

$$\psi(x,y) = x^3y + \tfrac{1}{2}x^2y^2 + h(y).$$

Substituting this expression for $\psi(x,y)$ in the second of Eqs. (32), we find that

$$x^3 + x^2y + h'(y) = x^3 + x^2y,$$

so $h'(y) = 0$ and $h(y)$ is a constant. Thus the solutions of Eq. (31), and hence of Eq. (19), are given implicitly by

$$x^3y + \tfrac{1}{2}x^2y^2 = c. \tag{33}$$

Solutions may also be found in explicit form since Eq. (33) is quadratic in y.

You may also verify that a second integrating factor for Eq. (19) is

$$\mu(x,y) = \frac{1}{xy(2x + y)}$$

and that the same solution is obtained, though with much greater difficulty, if this integrating factor is used (see Problem 32).

PROBLEMS

Determine whether each of the equations in Problems 1 through 12 is exact. If it is exact, find the solution.

1. $(2x+3)+(2y-2)y'=0$
2. $(2x+4y)+(2x-2y)y'=0$
3. $(3x^2-2xy+2)+(6y^2-x^2+3)y'=0$
4. $(2xy^2+2y)+(2x^2y+2x)y'=0$
5. $\dfrac{dy}{dx}=-\dfrac{ax+by}{bx+cy}$
6. $\dfrac{dy}{dx}=-\dfrac{ax-by}{bx-cy}$
7. $(e^x\sin y-2y\sin x)+(e^x\cos y+2\cos x)y'=0$
8. $(e^x\sin y+3y)-(3x-e^x\sin y)y'=0$
9. $(ye^{xy}\cos 2x-2e^{xy}\sin 2x+2x)+(xe^{xy}\cos 2x-3)y'=0$
10. $(y/x+6x)+(\ln x-2)y'=0, \quad x>0$
11. $(x\ln y+xy)+(y\ln x+xy)y'=0; \quad x>0, \quad y>0$
12. $\dfrac{x}{(x^2+y^2)^{3/2}}+\dfrac{y}{(x^2+y^2)^{3/2}}\dfrac{dy}{dx}=0$

In each of Problems 13 and 14, solve the given initial value problem and determine at least approximately where the solution is valid.

13. $(2x-y)+(2y-x)y'=0, \quad y(1)=3$
14. $(9x^2+y-1)-(4y-x)y'=0, \quad y(1)=0$

In each of Problems 15 and 16, find the value of b for which the given equation is exact, and then solve it using that value of b.

15. $(xy^2+bx^2y)+(x+y)x^2y'=0$
16. $(ye^{2xy}+x)+bxe^{2xy}y'=0$
17. Assume that Eq. (6) meets the requirements of Theorem 2.6.1 in a rectangle R and is therefore exact. Show that a possible function $\psi(x,y)$ is

$$\psi(x,y)=\int_{x_0}^{x}M(s,y_0)\,ds+\int_{y_0}^{y}N(x,t)\,dt,$$

where (x_0,y_0) is a point in R.

18. Show that any separable equation

$$M(x)+N(y)y'=0$$

is also exact.

In each of Problems 19 through 22, show that the given equation is not exact but becomes exact when multiplied by the given integrating factor. Then solve the equation.

19. $x^2y^3+x(1+y^2)y'=0, \quad \mu(x,y)=1/xy^3$
20. $\left(\dfrac{\sin y}{y}-2e^{-x}\sin x\right)+\left(\dfrac{\cos y+2e^{-x}\cos x}{y}\right)y'=0, \quad \mu(x,y)=ye^x$
21. $y+(2x-ye^y)y'=0, \quad \mu(x,y)=y$
22. $(x+2)\sin y+(x\cos y)y'=0, \quad \mu(x,y)=xe^x$
23. Show that if $(N_x-M_y)/M=Q$, where Q is a function of y only, then the differential equation

$$M+Ny'=0$$

has an integrating factor of the form

$$\mu(y)=\exp\int Q(y)\,dy.$$

24. Show that if $(N_x - M_y)/(xM - yN) = R$, where R depends on the quantity xy only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form $\mu(xy)$. Find a general formula for this integrating factor.

In each of Problems 25 through 31, find an integrating factor and solve the given equation.

25. $(3x^2y + 2xy + y^3) + (x^2 + y^2)y' = 0$
26. $y' = e^{2x} + y - 1$
27. $1 + (x/y - \sin y)y' = 0$
28. $y + (2xy - e^{-2y})y' = 0$
29. $e^x + (e^x \cot y + 2y \csc y)y' = 0$
30. $[4(x^3/y^2) + (3/y)] + [3(x/y^2) + 4y]y' = 0$
31. $\left(3x + \dfrac{6}{y}\right) + \left(\dfrac{x^2}{y} + 3\dfrac{y}{x}\right)\dfrac{dy}{dx} = 0$

    Hint: See Problem 24.
32. Solve the differential equation

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

using the integrating factor $\mu(x, y) = [xy(2x + y)]^{-1}$. Verify that the solution is the same as that obtained in Example 4 with a different integrating factor.

2.7 Numerical Approximations: Euler's Method

Recall two important facts about the first order initial value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0. \tag{1}$$

First, if f and $\partial f/\partial y$ are continuous, then the initial value problem (1) has a unique solution $y = \phi(t)$ in some interval surrounding the initial point $t = t_0$. Second, it is usually not possible to find the solution ϕ by symbolic manipulations of the differential equation. Up to now we have considered the main exceptions to the latter statement: differential equations that are linear, separable, or exact, or that can be transformed into one of these types. Nevertheless, it remains true that solutions of the vast majority of first order initial value problems cannot be found by analytical means, such as those considered in the first part of this chapter.

Therefore, it is important to be able to approach the problem in other ways. As we have already seen, one of these ways is to draw a direction field for the differential equation (which does not involve solving the equation) and then to visualize the behavior of solutions from the direction field. This has the advantage of being a relatively simple process, even for complicated differential equations. However, it does not lend itself to quantitative computations or comparisons, and this is often a critical shortcoming.

For example, Figure 2.7.1 shows a direction field for the differential equation

$$\frac{dy}{dt} = 3 - 2t - 0.5y. \tag{2}$$