MAT 351 Differential Equations: Dynamics & Chaos

Spring 2016

Assignment 7

Due Thursday, **May 5**, in class.

Problem 1: Consider the system

$$\begin{aligned}\dot{x} &= 10(y-x)\\ \dot{y} &= 28x - y + xz\\ \dot{z} &= xy - \frac{8}{3}z.\end{aligned}$$

a) Consider $E = 28x^2 + 10y^2 + 10(z-56)^2$. Show that $E > 0$ and $\dot{E} > 0$ in the region

$$R = \left\{ 28x^2 + y^2 + \frac{8}{3}(z-28)^2 < \frac{8}{3}28^2 \;\text{ and }\; x > 0, y > 0, z > 0 \right\}.$$

What does this tell us about the points in the region R? The region R is the part of the solid ellipsoid $28x^2 + y^2 + \frac{8}{3}(z-28)^2 < \frac{8}{3}28^2$ where x, y, and z are all positive.

b) (Extra Credit - 5p) Show that this system is **not chaotic** in the region where x, y, and z are all positive.

Hint: It is enough to show that most solutions tend to ∞ in forward time. Note that this is not the Lorenz system: in the equation for $\dot{y}$ we have $+xz$ instead of $-xz$.

Problem 2: Determine whether $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ has sensitive dependence on initial conditions. Is the map f transitive?

Problem 3: Consider the *tent map* $T : [0,1] \to [0,1]$ defined by

$$T(x) = \begin{cases} 2x & \text{if } \; 0 \le x < \frac{1}{2} \\ 2 - 2x & \text{if } \; \frac{1}{2} \le x \le 1 \end{cases}$$

(a) Sketch the graphs of T, T^2 and T^3. What does the graph of T^n look like?

(b) Use the graph of T^n to conclude that T has exactly 2^n periodic points of period n. These points do not necessarily have least period n, but are fixed by T^n.

(c) (Extra Credit - 3p) The tent map is chaotic. In this exercise, you are asked to prove that the set of all periodic points of T is dense in $[0,1]$.

Problem 4: Consider the logistic map $G : [0,1] \to [0,1]$ defined by $G(x) = 4x(1-x)$.

(a) Prove that G is topologically conjugate to the tent map $T : [0,1] \to [0,1]$,

$$T(x) = \begin{cases} 2x & \text{if} \quad 0 \le x < \frac{1}{2} \\ 2 - 2x & \text{if} \quad \frac{1}{2} \le x \le 1 \end{cases}$$

You need to verify that there exists a homeomorphism $h : [0,1] \to [0,1]$ such that $h(G(x)) = T(h(x))$. *Hint:* Consider $h(x) = (1 - \cos(\pi x))/2$.

b) Use the previous problem to conclude that the logistic map G is chaotic.