MAT 351 Differential Equations: Dynamics & Chaos
SPRING 2016

ASSIGNMENT 5

Due Thursday, **March 24**, in class.

Problem 1: Show that $(0, \pi)$ is a nonlinear center for the system $\dot{x} = \sin(y)$, $\dot{y} = \sin(x)$. *Hint:* Find an energy function $E(x, y)$ of the form $\alpha \sin(x) + \beta \cos(y)$, $\alpha \cos(x) + \beta \sin(y)$, $\alpha \sin(x) + \beta \sin(y)$, or $\alpha \cos(x) + \beta \cos(y)$ and show that this has a local min or max at $(0, \pi)$.

Problem 2: For each of the following systems, decide whether it is a gradient system. If so, find $V(x, y)$ and sketch the phase portrait. On a separate graph, sketch the equipotentials $V(x, y) =$ constant. If the system is not a gradient system, explain why not and go on to the next question.

a) $\dot{x} = y + x^2 y, \quad \dot{y} = -x + 2xy$

b) $\dot{x} = 2x, \quad \dot{y} = 8y$

c) $\dot{x} = -2xe^{x^2+y^2}, \quad \dot{y} = -2ye^{x^2+y^2}$

Problem 3: Consider the nonlinear system

$$\begin{aligned} \dot{x} &= -y + x(x^2 + y^2) \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \\ \dot{y} &= x + y(x^2 + y^2) \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \end{aligned}$$

a) Use polar coordinates $x = r\cos(\theta)$ and $y = r\sin(\theta)$ and show that the system becomes

$$\begin{aligned} \dot{r} &= r^3 \sin\left(\frac{1}{r}\right) \\ \dot{\theta} &= 1 \end{aligned}$$

b) Observe that if $\dot{r} = 0$ then $r = 0$ or $r = \dfrac{1}{n\pi}$ for $n = 1, 2, 3, \ldots$. The latter corresponds to closed orbits of radius $r = \dfrac{1}{n\pi}$, which are limit cycles. In this exercise you have to show that these cycles are stable for even n and unstable for odd n.

Hint: Consider $h(t) = \dfrac{1}{r(t)} - n\pi$, where $|h(t)|$ is much smaller than $n\pi$. Substitute this into the equation for r and show that $\dot{h} = -\dfrac{1}{n\pi + h}(-1)^n \sin(h)$. Then show that $\dot{h} > 0$ or $\dot{h} < 0$ when n is even. What does this tell us about $\dot{r}$? Use this information to show stability. Treat the case when n is odd similarly.

c) (Extra Credit - 2p) We demonstrated the existence of infinitely many nested limit cycles in part b). Make a sketch of the phase portrait and include at least three cycles.

Problem 4: Apply Bendixon's negative criterion or Dulac's criterion to show that there are no periodic solutions to:

a) $\dot{x} = -x + y^2, \quad \dot{y} = -y^3 + x^2$

b) $\dot{x} = -2xe^{x^2+y^2}, \quad \dot{y} = -2ye^{x^2+y^2}$

c) $\dot{x} = y, \quad \dot{y} = a_1 x + a_2 y + a_3 x^2 + a_4 y^2$, where a_1, a_2, a_3, a_4 are nonzero constants.

Problem 5: Consider the nonlinear system

$$\begin{aligned} \dot{x} &= x - y - x(x^2 + 5y^2) \\ \dot{y} &= x + y - y(x^2 + y^2) \end{aligned}$$

a) Classify the fixed point at the origin.

b) Rewrite the system in polar coordinates ($x = r\cos(\theta)$ and $y = r\sin(\theta)$).

c) Prove that the system has a limit cycle in the annular region $\dfrac{1}{\sqrt{2}} - \epsilon < r < 1 + \epsilon$. Here ϵ is a small enough positive number (e.g. $\epsilon = 0.05$).

Hint: Apply the Poincaré-Bendixson Theorem.