MAT324: Real Analysis – Fall 2016
Assignment 8 – Solutions

Problem 1: Suppose $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$. Prove that f also belongs to $L^3(\mathbb{R})$.

Solution. Notice that $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ implies $|f| \in L^2(\mathbb{R})$, and $f^2 \in L^2(\mathbb{R})$. By Hölder's inequality,

$$\left| \int_{\mathbb{R}} |f| f^2 dx \right| \leq \left(\int_{\mathbb{R}} |f|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |f^2|^2 dx \right)^{\frac{1}{2}} < \infty$$

Hence $f \in L^3(\mathbb{R})$. □

Problem 2: Determine if the following functions belong to $L^\infty(\mathbb{R})$.

a) $f(x) = \frac{1}{x^2}\chi_{(0,n]}$ for some $n > 0$.

b) $f(x) = \frac{1}{\sqrt{x}}\chi_{[n,n^2]}$ for some $n > 0$.

Solution.

a) f is not essentially bounded. Given $M > 0$, for any $\epsilon > 0$, we have $x = \frac{1}{\sqrt{M+\epsilon}}$. Then $f(x) = M + \epsilon > M$, implying that

$$m(\{x \in \mathbb{R} | |f(x)| > M\} \geq m\left(\left(0, \frac{1}{M}\right)\right) = \frac{1}{M} > 0.$$

b) f is bounded by $\frac{1}{\sqrt{n}}$ □

Problem 3: Consider the function

$$f(x,y) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Consider the sets $E_k = \{(x,y) \in [0,1] \times [0,1] : f(x,y) \in [k, k+1)\}$. Consider non-negative simple functions $\varphi_n = \sum_{k=0}^{n} k\chi_{E_k}$ for $k \geq 1$, and let $\varphi = \sum_{k=0}^{\infty} k\chi_{E_k}$. Using the definition of the integral, compute $\int_{[0,1]\times[0,1]} \varphi_n \, dm_2$ and $\int_{[0,1]\times[0,1]} \varphi \, dm_2$. Deduce that $f \notin L^1([0,1] \times [0,1])$.

Solution. See Example 6.15 and Exercise 6.1 in the textbook. □

Problem 4: Consider the measure spaces $(X, \mathcal{F}_1, \mu)$ and $(Y, \mathcal{F}_2, \nu)$ where $X = Y = [0,1]$, $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{B}_{[0,1]}$ is the σ-algebra of Borel subsets of $[0,1]$. Let μ be the Lebesgue measure on $\mathcal{F}_1$

and ν be the counting measure on $\mathcal{F}_2$, that is $\nu(E)$ = number of elements in E if E is finite and $\nu(E) = \infty$ otherwise. Let $D = \{(x, y) \mid x = y\}$ and consider

$$D_n = \bigcup_{k=1}^{n} \left(\left[\frac{k-1}{n}, \frac{k}{n} \right] \times \left[\frac{k-1}{n}, \frac{k}{n} \right] \right)$$

a) Show that $D = \bigcap_{n=1}^{\infty} D_n$ and that $D \in \mathcal{F}_1 \times \mathcal{F}_2$.

b) Compute $\int_0^1 \int_0^1 \chi_D(x, y)\, d\mu(x) d\nu(y)$ and $\int_0^1 \int_0^1 \chi_D(x, y) d\nu(y) d\mu(x)$ and show that they are not equal.

Recall that χ_D is the characteristic function of the set D and $\chi_D(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$

Note: This problem does not contradict Theorem 6.12 since ν is not σ-finite.

SOLUTION.

a) Notice that if $(x, y) \in D_n$, then $|x - y| \leq \frac{1}{n}$. In particular, if $(x, y) \in \bigcap_{n=1}^{\infty} D_n$, then $|x - y| \leq \frac{1}{n}$, $\forall\ n \in \mathbb{N}$, hence $x = y \Rightarrow (x, y) \in D$. The other inclusion is trivial. Since D can be expressed as countable intersection of countable unions of products, it belongs to the product σ-algebra.

b)

$$\begin{aligned}
\int_0^1 \int_0^1 \chi_D(x, y)\, d\mu(x) d\nu(y) &= \int_0^1 \left[\int_0^1 \chi_D(x, y) d\mu(x) \right] d\nu(y) \\
&= \int_0^1 0\ d\nu(y) \\
&= 0
\end{aligned}$$

The second equality follows from the fact that for fixed y, $\chi_D(x, y)$ is 0 Lebesgue a.e. On the other hand,

$$\begin{aligned}
\int_0^1 \int_0^1 \chi_D(x, y) d\nu(y) d\mu(x) &= \int_0^1 \left[\int_0^1 \chi_D(x, y) d\nu(y) \right] d\mu(x) \\
&= \int_0^1 1\ d\mu(x) \\
&= 1
\end{aligned}$$

□