MAT324: Real Analysis – Fall 2016
Assignment 5

Due Thursday, **October 20**, in class.

Problem 1: Suppose $\int_E f\,dm = \int_E g\,dm$ for every measurable set $E \in \mathcal{M}$. Show that $f = g$ almost everywhere.

Problem 2: Suppose $(f_n)_{n\geq 1}$ is a sequence of non-negative measurable functions on $E \in \mathcal{M}$. If f_n decreases to f almost everywhere and $\int_E f_1 dm < \infty$, then show that

$$\lim_{n\to\infty} \int_E f_n dm = \int_E f dm.$$

Hint: Look at the sequence $g_n = f_1 - f_n$.

Problem 3: Suppose $(f_n)_{n\geq 1}$ is a sequence of non-negative measurable functions. Show that

$$\int \sum_{n=1}^{\infty} f_n dm = \sum_{n=1}^{\infty} \int f_n dm.$$

Problem 4: Compute the following limits if they exist and justify the calculations:

a) $\displaystyle\lim_{n\to\infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx$

b) $\displaystyle\lim_{n\to\infty} \int_0^\infty \frac{n^2 x e^{-n^2x^2}}{1+x} dx.$

c) $\displaystyle\lim_{n\to\infty} \int_1^\infty \frac{n^2 x e^{-n^2x^2}}{1+\sqrt[n]{x}} dx.$

Problem 5: Suppose $E \in \mathcal{M}$. Let (g_n) be a sequence of integrable functions which converges a.e. to an integrable function g. Let (f_n) be a sequence of measurable functions which converge a.e. to a measurable function f. Suppose further that $|f_n| \leq g_n$ a.e. on E for all $n \geq 1$. Show that if $\int_E g\,dm = \lim_{n\to\infty} \int_E g_n\,dm$, then $\int_E f\,dm = \lim_{n\to\infty} \int_E f_n\,dm$.
Hint: Rework the proof of the Dominated Convergence Theorem.

Problem 6: Let $E \in \mathcal{M}$. Let (f_n) be a sequence of integrable functions which converges a.e. to an integrable function f. Show that $\int_E |f_n - f|\,dm \to 0$ as $n \to \infty$ if and only if $\int_E |f_n|\,dm \to \int_E |f|\,dm$ as $n \to \infty$.

Problem 7: Consider two functions $f, g : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if} \quad x = \frac{p}{q} \in \mathbb{Q}, \text{ where } p \text{ and } q \text{ are relatively prime} \\ 0 & \text{if} \quad x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

and

$$g(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

Show that f is Riemann integrable on $[0,1]$, but g is not Riemann integrable on $[0,1]$.

Problem 8: Consider the function $f : [0,\infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if} \quad x > 0 \\ 1 & \text{if} \quad x = 0. \end{cases}$$

Show that f has an improper Riemann integral over the interval $[0,\infty)$, but f is not Lebesgue integrable.