MAT324: Real Analysis – Fall 2016

Assignment 4 – Solutions

Problem 4: Let $\{f_n\}$ be a sequence of measurable functions defined on $\mathbb{R}$. Show that the sets

$$\begin{aligned} E_1 &= \{x \in \mathbb{R} \mid \lim_{n\to\infty} f_n(x) \text{ exists and is finite}\} \\ E_2 &= \{x \in \mathbb{R} \mid \lim_{n\to\infty} f_n(x) = \infty\} \\ E_3 &= \{x \in \mathbb{R} \mid \lim_{n\to\infty} f_n(x) = -\infty\} \end{aligned}$$

are measurable.

Solution. Theorem 3.5 of the textbook says that if $\{f_n\}$ is a sequence of measurable functions, then the functions $g = \liminf_n f_n$ and $h = \limsup_n f_n$ are measurable.

Notice that $\lim_{n\to\infty} f_n(x) = \infty$, if and only if $\liminf_n f_n(x) = \infty$. Hence,

$$E_2 = \{x | g(x) = \infty\} = \bigcap_{k\in\mathbb{N}} \{x | g(x) > k\}$$

is measurable. Likewise,

$$E_3 = \{x | h(x) = -\infty\} = \bigcap_{k\in\mathbb{N}} \{x | h(x) < -k\}$$

is measurable.

Further, notice that $E_1 = \{x \in | g(x) = h(x)\} \setminus (E_2 \cup E_3)$, hence E_1 is also measurable. □

Problem 2: Let $\mathcal{C} \subset [0,1]$ be the Cantor middle-thirds set. Suppose that $f : [0,1] \to \mathbb{R}$ is defined by $f(x) = 0$ for $x \in \mathcal{C}$ and $f(x) = k$ for all x in each interval of length 3^{-k} which has been removed from $[0,1]$ at the k^{th} step of the construction of the Cantor set. Show that f is measurable and calculate $\int_{[0,1]} f dm$.

Solution. Denote by $f_n : [0,1] \to \mathbb{R}$ the function constructed following way: If $\mathcal{C}_k$ denotes the union of the intervals of lenght 3^{-k} removed in the k-th step of the construction of the Cantor middle-third set, let $f_n(x) = k$ for $x \in \mathcal{C}_k$, and zero elsewhere. Then f_n is a simple function (it only takes $(n+1)$ values). Furthermore, it is easy to see that $f_n \to f$ pointwise, hence f is a measurable function. In addition, the sequence f_n is increasing to f, hence the Monotone Convergence Theorem gives us

$$\int_{[0,1]} f dm = \lim_{n\to\infty} \int_{[0,1]} f_n dm = \lim_{n\to\infty} \left(\sum_{k=1}^{n} k 2^{k-1} 3^{-k} \right) = \frac{1}{3} \lim_{n\to\infty} \left[\sum_{k=1}^{n} k \left(\frac{2}{3}\right)^{k-1} \right]$$

The answer up to this point is fine. With a little more effort, one can get the answer 3. This uses the following relation:

$$\sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2} \quad \text{if } 0 < |x| < 1$$

There are a number of ways one can use to prove this fact, including Riemman sums and Taylor‘s formula. $\square$

Problem 3: Let E be a measurable set. For a function $f : E \to \mathbb{R}$ we define the *positive part* $f^+ : E \to \mathbb{R}$, $f^+(x) = \max(f(x), 0)$, and the *negative part* $f^- : E \to \mathbb{R}$, $f^-(x) = \min(f(x), 0)$. Prove that f is measurable if and only if both f^+ and f^- are measurable.

PROOF. One could directly apply the definition of a measurable function or use Theorem 3.5 for the maximum/minimum of two functions $f(x)$ and $g(x) = 0$. $\square$

Problem 4: Prove that if f is integrable on $\mathbb{R}$ and $\int_E f(x)dm \geq 0$ for every measurable set E, then $f(x) \geq 0$ a.e. x.

SOLUTION. Since f is integrable, it is in particular measurable. Let E be the measurable set $E = \{x|f(x) < 0\}$. By hypothesis, and using monotonicity of the integral

$$0 \leq \int_E f(x)dm \leq \int_E 0dm = 0 \Rightarrow \int_E f(x)dm = 0$$

Notice that $-f$ is a positive function on E, and

$$\int_E (-f(x))dm = 0.$$

Now Theorem 4.4 implies that $-f$ is zero almost everywhere. By the definition of E, this happens if and only if E has zero measure. $\square$

Problem 5: Let E be a measurable set. Suppose $f \geq 0$ and let $E_k = \{x \in E \mid 2^k < f(x) \leq 2^{k+1}\}$ for any integer k. If f is finite almost everywhere, then

$$\bigcup_{k=-\infty}^{\infty} E_k = \{x \in E \mid f(x) > 0\},$$

and the sets E_k are disjoint.

(a) Prove that f is integrable if and only if $\sum\limits_{k=-\infty}^{\infty} 2^k m(E_k) < \infty$.

(b) Let $a > 0$ and consider the function

$$f(x) = \begin{cases} |x|^{-a} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Use part a) to show that f is integrable on $\mathbb{R}$ if and only if $a < 1$.

SOLUTION.

(a) Suppose f is integrable. Since $f(x) > 2^k$ on E_k, we have

$$\int_{E_k} f dm \geq \int_{E_k} 2^k dm = 2^k m(E_k)$$

Therefore, by the comparison test,

$$\sum_{k=-\infty}^{\infty} 2^k m(E_k) \leq \sum_{k=-\infty}^{\infty} \int_{E_k} f dm = \int_{\mathbb{R}} f dm < \infty$$

Next suppose $\sum\limits_{k=-\infty}^{\infty} 2^k m(E_k) < \infty$. Then $2\left(\sum\limits_{k=-\infty}^{\infty} 2^k m(E_k)\right) = \sum\limits_{k=-\infty}^{\infty} 2^{k+1} m(E_k) < \infty$. Since $f(x) \leq 2^{k+1}$ on E_k, we have

$$\int_{E_k} f dm \leq \int_{E_k} 2^{k+1} dm = 2^{k+1} m(E_k).$$

Then

$$\int_{\mathbb{R}} f dm = \sum_{k=-\infty}^{\infty} \int_{E_k} f dm \leq \sum_{k=-\infty}^{\infty} 2^{k+1} m(E_k) < \infty,$$

and f is integrable.

(b) Following part a), we need to find the measure of the sets E_k. If $K \geq 0$, then

$$2^k < |x|^{-a} \leq 2^{k+1},$$

and

$$2^{-k} > |x|^a \geq 2^{-k-1}$$
$$2^{\frac{-k}{a}} > |x| \geq 2^{\frac{-k-1}{a}}$$

Then $m(E_k) = 2 \cdot 2^{\frac{-k-1}{a}}(2^{\frac{1}{a}} - 1)$. If $k < 0$, then $2^k < |x|^{-a} \leq 2^{k+1}$ implies $|x| \geq 1$, hence $m(E_k) = 0$, if $k < 0$. Thus,

$$\sum_{k=-\infty}^{\infty} 2^k m(E_k) = \sum_{k=-0}^{\infty} 2^{k+1} \cdot 2^{\frac{-k-1}{a}}(2^{\frac{1}{a}} - 1)$$
$$\sum_{k=-\infty}^{\infty} 2^k m(E_k) = (2^{\frac{1}{a}} - 1) \sum_{k=0}^{\infty} 2^{\frac{(k+1)(a-1)}{a}}$$
$$\sum_{k=-\infty}^{\infty} 2^k m(E_k) = (2^{\frac{1}{a}} - 1) \sum_{k=0}^{\infty} \left[2^{\frac{(a-1)}{a}}\right]^{(k+1)}$$

Notice that this geometric series converge if and only if $2^{\frac{(a-1)}{a}} < 1$, and this happens if and only if $a < 1$. $\square$