MAT324: Real Analysis – Fall 2016

Assignment 10 – Solutions

Problem 1: Let λ_1, λ_2 and μ be measures on a measurable space $(X, \mathcal{F})$. Show that if $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$ then $(\lambda_1 + \lambda_2) \ll \mu$.

Solution. Suppose $E \in \mathcal{F}$ is such that $\mu(E) = 0$. Since $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, $\lambda_1(E) = \lambda_2(E) = 0$, so $(\lambda_1 + \lambda_2)(E) = 0$, thus $\lambda_1 + \lambda_2 \ll \mu$. $\square$

Problem 2: Let $X = [0, 1]$ with Lebesgue measure and consider probability measures μ and ν given by densities f and g as follows

$$\nu(E) = \int_E f\, dm \quad \text{and} \quad \mu(E) = \int_E g\, dm,$$

for every measurable subset $E \subset [0, 1]$. Suppose $f(x), g(x) > 0$ for every $x \in [0, 1]$. Is ν absolutely continuous with respect to μ (that is $\nu \ll \mu$)? If it is, determine the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$. Is μ absolutely continuous with respect to ν (that is $\mu \ll \nu$)?

Solution. Let E be a Lebesgue measurable set and let m denote the Lebesgue measure. Notice that since $f > 0$, $\mu(E) = \int_E f dm = 0$ iff $m(E) = 0$. In particular, if $\mu(E) = 0$, then $\nu(E) = \int_E g dm = 0$, for $m(E) = 0$, so $\nu \ll \mu$. The argument to show that $\mu \ll \nu$ is similar. Furthermore, since $\nu \ll \mu \ll m$, by proposition 7.7, (ii) in the textbook,

$$\frac{d\nu}{dm} = \frac{d\nu}{d\mu}\frac{d\mu}{dm},$$

so $\frac{d\nu}{d\mu} = \frac{f}{g}$. $\square$

Problem 3: Suppose μ is a σ-finite measure on $([0, 1], \mathcal{F})$ and $E_1, E_2, \ldots, E_{2014}$ are measurable subsets of $[0, 1]$. Define ν on $\mathcal{F}$ by $\nu(E) = \sum\limits_{k=1}^{2014} \mu(E \cap E_k)$. Show that $\nu \ll \mu$ and find the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$.

Solution. Notice that

$$\mu(E \cap E_k) = \int_E \chi_{E_k} dm,$$

So for each measure ν_k, defined by $\nu_k(E) = \mu(E \cap E_k)$, we have $\nu_k \ll m$. By problem 2, $\nu = \sum_{k=1}^{2014} \nu_k \ll m$. In addition, from $\nu_k(E) = \int_E \chi_{E_k} dm$, we know that $\frac{d\nu_k}{dm} = \chi_{E_k}$. By proposition 7.7 (i) in the textbook, $\frac{d\nu}{dm} = \sum_{k=1}^{2014} \chi_{E_k}$. $\square$

Problem 4: Let $\lambda_1, \lambda_2, \mu$ be measures on a σ-algebra $\mathcal{F}$. Show that

a) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$ then $(\lambda_1 + \lambda_2) \perp \mu$.

b) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$ then $\lambda_2 \perp \lambda_1$.

SOLUTION.

a) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$ then there exist disjoint sets A_i, B_i, $i = 1, 2$, such that $A_i \cup B_i = X$, $\mu(A_i) = 0$, $\lambda_i(B_i) = 0$. Then the sets $A = A_1 \cup A_2$ and $B = B_1 \cap B_2$ satisfy $X = A \cup B$, $A \cap B = \emptyset$, and

$$\mu(A) \leq \mu(A_1) + \mu(A_2) = 0$$
$$(\lambda_1 + \lambda_2)(B) \leq \lambda_1(B_1) + \lambda_2(B_2) = 0$$

This implies that $(\lambda_1 + \lambda_2) \perp \mu$.

b) Suppose $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$. Let $A, B \in \mathcal{F}$ be disjoint sets, $A \cup B = X$, and $\lambda_2(A) = 0$, $\mu(B) = 0$. Since $\lambda_1 \ll \mu$, we also have $\lambda_1(B) = 0$, so $\lambda_2 \perp \lambda_1$. □

Problem 5: For a point x, define the Dirac measure δ_x to be

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

For a fixed set B define the Lebesgue measure restricted to B by $m_B(A) = m(A \cap B)$. Let $\mu = \delta_1 + m_{[2,4]}$ and $\nu = \delta_0 + m_{(1,2)}$. Show that $\nu \perp \mu$.

SOLUTION. μ and ν are concentrated on disjoint sets, namely $\{1\} \cup [2, 4]$ and $\{0\} \cup (1, 2)$, respectively, so they are mutually singular. □