MAT 324 – Real Analysis
Fall 2014

Final Exam – December 12, 2014

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator or an electronic device.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	**SCORE**
1	
2	
3	
4	
5	
6	
7	
8	
TOTAL	

Problem 1: (20 points) Determine whether the following statements are true or false. No further explanation is necessary.

(1) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

The outer measure is countably subadditive.

(2) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

All Lebesgue measurable sets are also Borel sets.

(3) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

For any subset $A \subset [0,1]$ the characteristic function $\mathbb{1}_A$ defined by $\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$ is measurable.

(4) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

$f \in L^1(\mathbb{R})$ if and only if $|f| \in L^1(\mathbb{R})$.

(5) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

It is possible to define an inner product on the space $L^4[0,1]$ with the norm $\|\cdot\|_4$.

(6) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

There exist functions which belong to $L^4(0,1)$ but not to $L^2(0,1)$.

(7) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

If $f, g \in L^1(\mathbb{R})$ then $h \in L^1(\mathbb{R}^2)$, where $h(x,y) = f(x)g(y)$.

(8) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

Let $\mu(E) = \displaystyle\int_E e^{-\pi x^2} dx$ for any Borel subset $E \subset \mathbb{R}$. Then $\mu(\mathbb{R}) = 1$ and $\mu \ll m$.

(9) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

Let μ, ν and λ be σ-finite measures. If $\mu \ll \nu$ and $\nu \ll \lambda$ then $\mu \ll \lambda$.

(10) $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

Let $\lambda_1, \lambda_2, \mu$ be σ-finite measures on a σ-field $\mathcal{F}$. If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$ then $\lambda_1 \perp \lambda_2$.

Problem 2: (12 points) Consider the set $K = \{\frac{1}{n} \mid n \geq 1\}$ and the function

$$f(x) = \begin{cases} x^2 & \text{if } x \notin K \\ 1 & \text{if } x \in K. \end{cases}$$

Explain why f is measurable and compute $\displaystyle\int_{[0,1]-K} f\, dm$.

Problem 3: (10 points) Let $E \subset [0,1]$ be a measurable set such that for any interval $(a,b) \subset [0,1]$ we have

$$m(E \cap (a,b)) \geq \frac{1}{2}(b-a).$$

Show that $m(E) = 1$.

Problem 4: (12 points) Let $f \in L^1(0,1)$. Compute the following limit if it exists or explain why it does not exist:

$$\lim_{k\to\infty} \int_0^1 k \, \ln\left(1 + \frac{|f(x)|}{k^2}\right) \, dx.$$

Problem 5: (12 points) Compute

$$\int_{(0,\pi)\times(0,\infty)} xy\sin(x)e^{-xy^2}\,dxdy$$

and explain how Fubini's theorem is used.

Problem 6: (12 points) Consider the measurable space $([0,1], \mathcal{F})$, where $\mathcal{F} = \mathcal{B}_{[0,1]}$ is the σ-algebra of Borel subsets of $[0,1]$. Let μ be a σ-finite measure on $\mathcal{F}$ with $\mu \ll m$. Let ν be the counting measure on $\mathcal{F}$, that is $\nu(E) =$ number of elements in E if E is finite and $\nu(E) = \infty$ otherwise. Let C be the Cantor middle-thirds set from $[0,1]$.

a) Explain why $D = C \times C$ belongs to the product σ-algebra $\mathcal{F} \times \mathcal{F}$.

b) Compute $\displaystyle\int_0^1 \int_0^1 \mathbb{1}_D(x,y)\, d\mu(x) d\nu(y)$. Recall that $\mathbb{1}_D(x,y) = \begin{cases} 1 & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D. \end{cases}$

c) Compute $\displaystyle\int_0^1 \int_0^1 \mathbb{1}_D(x,y) d\nu(y) d\mu(x)$. Does this example contradict Fubini's theorem?

Problem 7: (12 points) Suppose μ is a σ-finite measure on $([0,1],\mathcal{F})$ and E_1, E_2 are two measurable subsets of $[0,1]$. Define ν by

$$\nu(E) = \frac{1}{4}\mu(E \cap E_1) + \frac{3}{4}\mu(E \cap E_2), \text{ for all } \; E \in \mathcal{F}.$$

a) Compute $\nu(E_1 \cap E_2)$.

b) Show that $\nu \ll \mu$.

c) Find the Radon-Nikodym derivative $\dfrac{d\nu}{d\mu}$.

Problem 8: (10 points) Let $f \in L^2(0, \infty) \cap L^5(0, \infty)$. Show that $f \in L^3(0, \infty)$.

– Scratch paper –