# MAT244H5F – Differential Equations I

Fall 2018

Assignment 4

Due Wednesday, **November 28**, in TUT for TUT0101, TUT0102, TUT0103, TUT0104.
Due Friday, **November 30**, in TUT for TUT0105, TUT0106.

**Problem 1:** Find the fundamental matrix for the following differential equation

$$x' = Ax, \text{ where } A = \begin{pmatrix} 2 & -5 \\ -4 & -2 \end{pmatrix}.$$

Find a solution satisfying the initial condition $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

**Problem 2:** Find the fundamental matrix for the following differential equation

$$x' = Ax, \text{ where } A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

Find a solution satisfying the initial condition $x(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

**Problem 3:** Consider the non-homogeneous system of differential equations:

$$x' = Ax + f(t), \text{ where } A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad f(t) = t\begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Find a particular solution of the form $x_p(t) = e^t\mathbf{u} + t\mathbf{v} + \mathbf{w}$, where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are constant vectors. Substitute $x_p$ in the equation and determine algebraic equations for $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

**Problem 4:** Consider the non-homogeneous system of differential equations:

$$x' = Ax + f(t), \text{ where } A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad f(t) = \begin{pmatrix} te^t \\ e^t \end{pmatrix}.$$

a) Write the matrix $A = I_2 + B + C$, where $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ are nilpotent matrices. Use this information to compute the exponential $e^{At}$.

b) Find a particular solution of the system.
*Hint:* The inverse of the matrix $e^{At}$ is $e^{-At}$, obtained by replacing $t$ with $-t$ in a).

**Problem 5:** Rewrite each system in the form $x' = Ax$, for some matrix $A$. You are **not** asked to find the general solution. Sketch the vector field and some typical trajectories for the following linear systems. Determine whether the fixed point $(0,0)$ is stable, asymptotically stable, or unstable. Classify the fixed point as sink, saddle, source, or center.

a) $x' = x$, $y' = x + y$

b) $x' = -x + y$, $y' = -5x + y$

c) $x' = -3x + 2y$, $y' = x - 2y$.

**Problem 6:** For each of the following systems, find the equilibria points, compute the Jacobian matrix at each equilibrium point and find its eigenvalues. Use this information to classify the equilibria points (sink, saddle, source) and sketch some neighboring trajectories.

a) $x' = x - y$, $y' = x^2 - 4$

b) $x' = \sin(y)$, $y' = xy + x - 1$ in the region $-2 < x < 2$ and $-2 < y < 2$.

**Problem 7:** Two species of fish that compete with each other for food, but do not prey on each other, are bluegill and redear. Suppose that a pond is stocked with bluegill and redear, and let $x(t)$ and $y(t)$ be the populations of bluegill and redear, respectively, at time $t$. Let $B \geq 1$ be the the carrying capacity of the pond for bluegill (in the absence of redear) and $R \geq 1$ the carrying capacity for redear (in the absence of bluegill). Suppose that the competition is modeled by the equations:

$$\begin{aligned}\frac{dx}{dt} &= \epsilon_1 x \left(1 - \frac{1}{B}x - \frac{\gamma_1}{B}y\right) \\ \frac{dy}{dt} &= \epsilon_2 y \left(1 - \frac{1}{R}y - \frac{\gamma_2}{R}x\right)\end{aligned}$$

where $\epsilon_1, \epsilon_2, \gamma_1, \gamma_2 > 0$ are constants.

a) Suppose that $\gamma_1 > \frac{B}{R}$ and $\frac{R}{B} > \gamma_2$. Show that the only equilibrium populations in the pond are no fish, no redear, or no bluegill.

b) Suppose that $\frac{B}{R} > \gamma_1$ and $\frac{R}{B} > \gamma_2$. Show that there is an equilibrium point $(x^*, y^*)$ at which both species can coexist (that is $x^* > 0$ and $y^* > 0$). Find the critical point $(x^*, y^*)$ in terms of $B, R, \gamma_1, \gamma_2$ and compute the Jacobian matrix at $(x^*, y^*)$. Classify $(x^*, y^*)$ as sink, source, or saddle and determine whether it is asymptotically stable, stable, or unstable.

> We remark that, by fishing, it is possible to reduce the population of bluegill to such a level that they will die out. Fishing only for bluegill has the effect of reducing $B$ at such a level that $\frac{B}{R} < \gamma_1$. We are then in the situation of part a) where we don't have an equilibrium at which both species can coexist.

c) Suppose that $B = 200, R = 300, \gamma_1 = 0.5, \gamma_2 = 1, \epsilon_1 = 0.2$ and $\epsilon_2 = 0.3$. Analyze the phase portrait from Figure 2 and determine whether the species can coexist if at a given time $t_0$ we have $x(t_0) = 40$ and $y(t_0) = 50$. What are the populations of bluegill and redear in the lake after a large period of time?

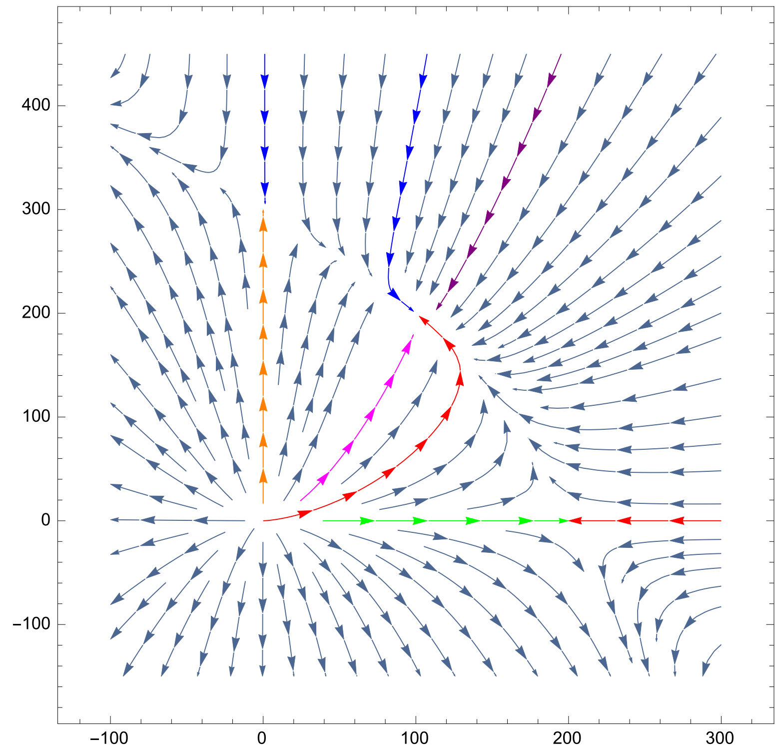


**Figure 1:** Direction field and phase plane portrait for the competing species of fish. Several typical trajectories are sketched in different colors. Note that there are 4 equilibria points, but only one where both species coexist.

Mathematica code for Figure 2

```
StreamPlot[{0.2*x (1 - 1/200 x - 0.5/200 y), 0.3*y (1 - 1/300 y - 1/300 x)},
        {x, -100, 300}, {y, -150, 450},
    StreamPoints -> {{{{50, 20}, Red}, {{85, 219}, Blue}, {{150, 300}, Purple},
    {{45, 50}, Magenta}, {{0, 50}, Orange}, {{1, 400}, Blue}, {{50, 0}, Green},
    {{250, 0}, Red}, Automatic}}, PlotRange -> Full]
```

**Problem 8:** Sir Alan Hodgkin and Sir Andrew Huxley studied the excitation and transmission of neural impulses and were awarded the Nobel Prize in Physiology or Medicine in 1963. Consider the following system of differential equations inspired by the Hodgkin-Huxley equations:

$$\begin{aligned} x' &= -y \\ y' &= -0.8y - x(x-1)(x-2) \end{aligned}$$

The phase portrait and some typical trajectories are sketched below. Determine the critical points. Using **only** the direction field and phase plane portrait, discuss the type and stability of each critical point.

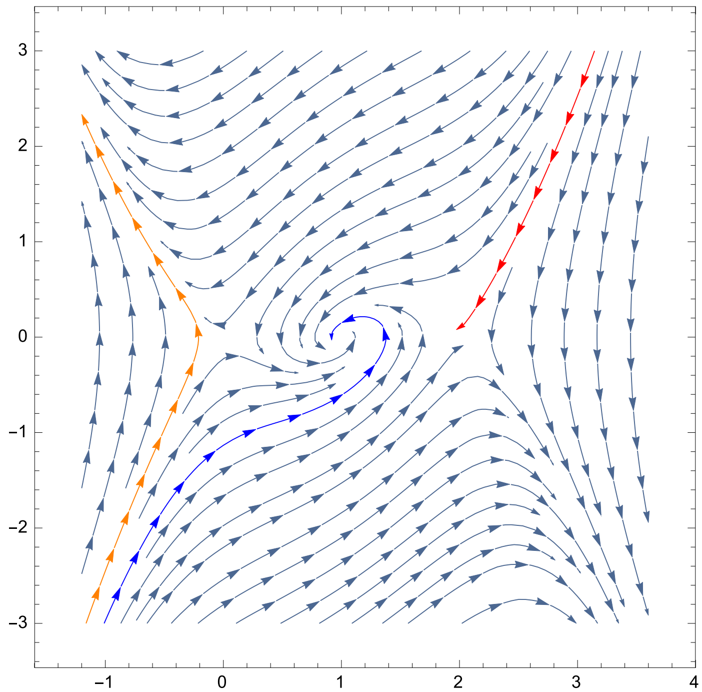


**Figure 2:** Direction field and phase plane portrait for Problem 8.