MAT244H5F – Differential Equations I
FALL 2018
ASSIGNMENT 3

Due Wednesday, **November 14**, in TUT for TUT0101, TUT0102, TUT0103, TUT0104.
Due Friday, **November 16**, in TUT for TUT0105, TUT0106.

Problem 1: Consider the 4th order ODE $y^{(4)} + 4y'' = f(x)$.

a) Obtain the homogeneous solution y_h.

b) For each case given below, give the general form of the particular solution y_p using the method of undetermined coefficients. Do not evaluate the coefficients.

      1. $f(x) = 5 + 8x^3$ 2. $f(x) = x\sin(5x)$

      3. $f(x) = \cos(2x)$ 4. $f(x) = 2\sin^2(x)$

Problem 2: An undamped forced oscillator is described by the differential equation

$$y'' + \omega_0^2 y = F_0 \cos(\omega t)$$

with initial conditions $y(0) = 0$ and $y'(0) = 0$.

a) Find the solution of this initial value problem when $\omega \neq \omega_0$.

b) Find the solution for the resonant case $(\omega = \omega_0)$.

Problem 3: A certain vibrating system with damping is modeled by the differential equation $u'' + 2u' + 4u = 0$.

a) Find the general solution to this equation.

b) Is the system underdamped, overdamped or critically damped?

c) Suppose that the damping were changed, keeping the mass and the spring constant the same, until the system became critically damped. Write the differential equation which models this critically damped system. You are not asked to solve it.

d) Determine the steady state (long time) solution to $u'' + 2u' + 4u = \cos(3t)$.

Problem 4: Consider the ordinary differential equation

$$t^2 y'' + 2ty' - 12y = 0.$$

a) Verify that $y_1(t) = t^3$ is a solution of the given equation.

b) Use the method of reduction of order and find a second solution $y_2(t) = v(t)y_1(t)$ of the given equation. Compute the Wronskian $W(y_1, y_2)$ and use it to conclude that y_1 and y_2 are two linearly independent solutions.

c) Write the general solution of this equation.

Problem 5: Consider the differential equation

$$t^2(t+3)y''' - 3t(t+2)y'' + 6(1+t)y' - 6y = 0, \quad t > 0$$

Assume that a solution of this equation is already known, $y_1(t) = t^2$.

a) Use the method of reduction of order outlined below to find a fundamental set of solutions for the differential equation (that is, three linearly independent solutions).

b) Check that the solutions obtained at part a) are linearly independent (**Hint:** either use the definition of linear independence, or compute the Wronskian).

c) Find the general solution of the differential equation.

> **Reduction of order method for third order linear ODEs:** if $y_1(x)$ is a solution of the differential equation
>
> $$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0$$
>
> then $y_2(x) = v(x)y_1(x)$ is a new solution of the differential equation above, provided that $v(x)$ satisfies the following second order equation in $v'(x)$:
>
> $$y_1 v''' + (3y_1' + p_1 y_1)v'' + (3y_1'' + 2p_1 y_1' + p_2 y_1)v' = 0.$$

Problem 6: Consider the linear systems of differential equations

$$x' = x - 2y, \quad y' = 2x - 3y.$$

a) Sketch the direction field for the linear system.

(write StreamPlot[{x − 2y, 2x−3y}] in WolframAlpha).

a) Use the method of elimination to find a second order linear differential equation that is satisfied by $x(t)$.

b) Find particular solutions $x(t)$ and $y(t)$ such that $x(0) = 1$ and $y(0) = 2$.

Problem 7: Consider the linear systems of differential equations

$$x' = x - 2y, \qquad y' = 2x - 3y.$$

a) Rewrite the system as $X' = AX$, where A is a 2-by-2 matrix and $X = \begin{bmatrix} x \\ y \end{bmatrix}$.

b) Find the eigenvalues and associated eigenvectors for the matrix A.

c) Using the eigenvalue method find the general solution $X(t)$ with $X(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Problem 8: Consider the homogeneous system of differential equations:

$$x' = Ax, \quad \text{where} \quad A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 3 \end{pmatrix}.$$

a) Find the eigenvalues of matrix A and determine their algebraic and geometric multiplicities.

b) Find the general solution of the system of differential equations $x' = Ax$.

Problem 9: Find the general solution of the system

$$
\begin{aligned}
x_1' &= 4x_1 + x_2 + x_3 \\
x_2' &= x_1 + 4x_2 + x_3 \\
x_3' &= x_1 + x_2 + 4x_3.
\end{aligned}
$$