MAT244H5F – Differential Equations I

Fall 2018

Assignment 2

Due Wednesday, **October 17**, in TUT for TUT0101, TUT0102, TUT0103, TUT0104.
Due Friday, **October 19**, in TUT for TUT0105, TUT0106.

Problem 1: Find the general solution of the differential equation $3xy^2y' = 3x^4 + y^3$. Is this a Bernoulli equation?

Problem 2: Find the general solution of the differential equation $x^2y' = xy + x^2e^{y/x}$ by first writing the equation in normal form and then making the substitution $v = y/x$. Write the differential equation for v and solve it.

Problem 3: The growth of cancerous tumors can be modeled by the Gompertz law

$$\frac{dN}{dt} = -aN\log(bN),$$

where $N(t)$ is proportional to the number of cells in the tumor and $a, b > 0$ are biological parameters. Find the critical points of this model and classify them as *stable* or *unstable*. Draw a typical trajectory of the solution $N(t)$ which starts at $N(0) = \dfrac{1}{2b}$. Sketch the vector field for $a = b = 2$ in the rectangle $-3 \leq t \leq 3$, $-3 \leq y \leq 3$.

Problem 4: The fish and game department in a certain state is planning to issue hunting permits to control the deer population (one deer per permit). It is known that if the deer population falls below a certain level m, the deer will become extinct. It is also known that if the deer population rises above the carrying capacity M, the population will decrease back to M through disease and malnutrition. Consider the following model for the growth rate of the deer population as a function of time:

$$\frac{dP}{dt} = rP(M - P)(P - m),$$

where P is the deer population and $r > 0$ is a constant of proportionality. Assume $M > m > 0$.

a) Determine the equilibrium points of this model and classify each one as stable or unstable. Draw the phase line and sketch several graphs of solutions.

b) About how many permits should be issued if there are $M + 1$ deers? What if there are $m - 1$ deers?

Problem 5: Solve the simplified logistic equation

$$\frac{dy}{dx} = y(y-1), \quad \text{with} \;\; y(0) = \frac{2}{3}.$$

What would be the general solution if $y(0) = 1$?

Problem 6: Consider the following initial value problem

$$\frac{dy}{dt} = 3 + 2t - y, \qquad y(0) = 1.$$

a) Estimate $y(1)$ using Euler's method with stepsize $h = 0.25$.

b) Find $y(t)$ and the exact value of $y(1)$. Is the answer from part a) an overestimate or an underestimate?

Problem 7: Consider the second order differential equation $x^2y'' + xy' = 0$ for $x > 0$. Make the substitution $v = \ln(x)$ and write a differential equation for v. Find the general solution $y(x)$ of the initial equation.

Problem 8: Find the solution of the initial value problem and sketch the graph of the solution for $-1 \leq x \leq 1$.

a) $y'' - 3y' + 2y = 0, \qquad y(0) = 0, \qquad y'(0) = 5$

b) $y'' + 6y' + 13y = 0, \qquad y(0) = 2, \qquad y'(0) = 0$

c) $y'' + 2y' + y = 0, \qquad y(0) = 2, \qquad y'(0) = -1$

Comment: To plot the graph of the solution it may be useful to use WolframAlpha. For example, to plot the graph of the function x^2 on the interval $[-1, 1]$ write **Plot**[{x^2}, {x, −1, 1}].

Problem 9: Find a particular solution y_p of the differential equation:

a) $y'' + 4y = 4\sin(2t)$

b) $y'' - 4y = 4e^{3t}$

Problem 10: Find the general solution of the differential equation:

a) $y^{(4)} - y = 0$

b) $y^{(4)} - 8y'' + 16y = 0$

c) $y^{(4)} + 2y^{(3)} + 3y'' + 2y' + y = 0$ (*Hint:* Expand $(r^2 + r + 1)^2$)

d) $y^{(3)} + y' - 10y = 0$

Comment: To solve an equation such as $x^2 - 2 = 0$ in WolframAlpha write **Solve**[x^2−2==0, x] or **Roots**[x^2−2], or **NSolve**[x^2−2==0, x] to solve the equation numerically (useful for equations which are not polynomial). To factor the polynomial $x^2 - 2$ try **Factor**[x^2−2] or **Simplify**[x^2−2].