

**MAT244H5F – Differential Equations I**  
FALL 2018  
ASSIGNMENT 1

Due Wednesday, **September 26**, in TUT for TUT0101, TUT0102, TUT0103, TUT0104.  
Due Friday, **September 28**, in TUT for TUT0105, TUT0106.

**Problem 1:** Determine whether the given differential equation is *homogeneous*, *linear*, or *separable*. If it is separable, find the solution  $y(x)$ .

- a)  $y' + y^2 \sin(x) = 0$
- b)  $xy' = \sqrt{1 - y^2}$
- c)  $(x^2 + 1)y' + 3xy = 6x$

**Problem 2:** In each of the cases below, determine whether the *existence* of at least one solution of the initial value problem is guaranteed by Theorem 1 from Section 1.3 and if so, whether its *uniqueness* is guaranteed.

- a)  $\frac{dy}{dx} = 2x^2y^2, y(1) = -1.$
- b)  $y\frac{dy}{dx} = x - 1, y(1) = 0.$

**Problem 3:**

- a) Show that there are an infinite number of solutions to the initial value problem  $y' = y^{2/3}, y(0) = 0.$
- b) Show that there is a unique solution to the initial value problem  $y' = y^{3/2}, y(0) = 0.$

*Hint:* It may be useful to look at Problems 1.3.27 and 1.3.29 from the textbook.

**Problem 4:** Solve the initial value problem

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2}, \quad y(0) = -1.1$$

and sketch the solution (use the code on the next page). Sketch the slope field in the square  $-2 < x < 2, -2 < y < 2.$  What is the long term behavior of the solution as  $x \rightarrow \infty$ ?

Suppose that the initial value is  $y(0.25) = 0.75.$  Does the solution exist for all  $x \in (-2, 2),$  or for a smaller interval? Can there be two solutions with this initial value?

### Example code in Mathematica/WolframAlpha

To visualize the vector field it is easier to type the following code in [www.wolframalpha.com](http://www.wolframalpha.com):

```
StreamPlot[{1, x^2/(1 - y^2)}, {x, -2, 2}, {y, -2, 2}]
```

To visualize a solution starting with  $y(0)=-1.1$  type the following (the solution is plotted in red):

```
StreamPlot[{1, x^2/(1 - y^2)}, {x, -2, 2}, {y, -2, 2}, StreamPoints -> {{{{0, -1.1}, Red}, Automatic}}]
```

To visualize two solutions starting with  $y(0)=-1.1$ , respectively  $y(0.1)=0.75$  type the following (one solution is plotted in red, and the other in green):

```
StreamPlot[{1, x^2/(1 - y^2)}, {x, -2, 2}, {y, -2, 2},  
StreamPoints -> {{{{0, -1.1}, Red}, {{0.25, 0.75}, Green}, Automatic}}]
```

**Problem 5:** Find the *general solution* to the differential equation and the *particular solution* with the given initial condition.

$$x \frac{dy}{dx} - 3y = x^4, \quad y(1) = 1, \quad x > 0.$$

**Problem 6:** Find the *general solution* to the differential equation and the *particular solution* with the given initial condition.

$$y' = (1 - y) \cos(x), \quad y(\pi) = 2.$$

**Problem 7:** An important tool in archeological research is radiocarbon dating, developed by the American chemist Willard F. Libby (who was awarded the Nobel prize in chemistry in 1960). This is a means of determining the age of certain wood and plant remains, and hence of animal or human bones or artifacts found buried at the same levels. Radiocarbon dating is based on the fact that some wood or plant remains contain residual amounts of carbon-14, a radioactive isotope of carbon. This isotope is accumulated during the lifetime of the plant and begins to decay at its death. Since the half-life of carbon-14 is long (approximately 5730 years), measurable amounts of carbon-14 remain after many thousands of years. If even a tiny fraction of the original amount of carbon-14 is still present, then by appropriate laboratory measurements the proportion of the original amount of carbon-14 that remains can be accurately determined. In other words, if  $Q(t)$  is the amount of carbon-14 at time  $t$  and  $Q_0$  is the original amount, then the ratio  $Q(t)/Q_0$  can be determined, as long as this quantity is not too small. Present measurement techniques permit the use of this method for time periods of 50,000 years or more.

- a) Assuming that  $Q$  satisfies the differential equation  $Q' = -rQ$ , determine the decay constant  $r$  for carbon-14.
- b) Find an expression for  $Q(t)$  at any time  $t$ , if  $Q(0) = Q_0$ .
- c) Suppose that certain remains are discovered in which the current residual amount of carbon-14 is 20% of the original amount. Determine the age of these remains.

**Problem 8:** A young person with no initial capital invests  $k$  dollars per year at an annual rate of return  $r$ . Assume that investments are made continuously and that the return is compounded continuously.

- a) Determine the sum  $S(t)$  accumulated at any time  $t$ .
- b) If  $r = 7.5$ , determine  $k$  so that \$1 million will be available for retirement in 40 years.
- c) If  $k = \$2000/\text{year}$ , determine the return rate  $r$  that must be obtained to have \$1 million available in 40 years.

**Problem 9:** Determine whether the following equation is *exact*. If it is exact, find the solution.

- a)  $2xy \frac{dy}{dx} = 3x^2 - y^2$ .
- b)  $(3xy + y^2) + (x^2 + xy)y' = 0$ .
- c)  $\left(x^3 + \frac{y}{x}\right) dx + (y^2 + \ln x) dy = 0$ .

**Problem 10:** The equation in Problem 6, part b) is not exact, however, we can multiply the original equation by the integrating factor  $\mu(x, y)$  to obtain an *exact* differential equation of the form

$$(3xy + y^2)\mu(x, y) + (x^2 + xy)\mu(x, y)y' = 0.$$

Find an integrating factor  $\mu(x, y)$  and solve this differential equation.