H_2 optimal control for a wide class of discrete-time linear stochastic systems

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Abstract

In this paper the problem of H_2-control of a discrete-time linear system subject to Markovian jumping and independent random perturbations is considered. Several kinds of H_2 types of performance criteria (often calls H_2-norms) are introduced and characterized via solutions of some suitable linear equations on the spaces of symmetric matrices. The purpose of such performance criteria is to provide a measure of the effect of additive white noise perturbations over an output of the controlled system. Different aspects specific to the discrete-time framework are revealed. The problem of optimization of H_2-norms is solved under the assumption that full state vector is available for measurements. One shows that among all stabilizing controllers of higher dimension, the best performance is achieved by a zero order controller. The corresponding feedback gain of the optimal controller is constructed based on the stabilizing solution of a system of discrete-time generalized Riccati equations. The case of discrete-time linear stochastic systems with coefficients depending upon the states both at time t and at time $t-1$ of the Markov chain, is also considered.

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1 Introduction

The problem of optimal control associated to a deterministic or stochastic controlled system subject to some white noise perturbations has a long history. For the stochastic framework we refer to [1]-[11], [23],[25]-[35]. A natural performance index for a such optimization problem is provide by the limit for t tends to infinity of the mean square (second moment) of a suitable output of the closed-loop system. The value of a such performance criteria is expressed in terms

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of observability Gramian of the closed-loop system. In [13] was observed that the same formula based on the observability Gramian corresponds to the state space setting of the H_2-norm of a linear time invariant deterministic system. So, in the literature by H_2-control problems associated to a deterministic or stochastic time invariant or time varying controlled system one understands any control problem asking the minimization of a quadratic cost functional over the trajectories of the closed-loop system subject to additive white noise.

In the case of time-varying linear stochastic systems described by Ito differential equations the H_2 optimization problem was solved in [14] for the finite dimensional case and in [11] for the infinite dimensional case. In the case of continuous-time linear stochastic systems subject to Markovian jumping the H_2-optimization problem was considered in [7] and [22]. In [17, 18] the H_2-optimization problem was investigated in the case of continuous time linear stochastic systems subject to both multiplicative and additive white noise and Markovian switching.

For the discrete-time framework the H_2 optimization problem was considered in [24] and [25] for systems with independent random perturbations and in [6, 9] and [10] for the systems affected by Markovian jumping. In [10] a convincing motivation for the applicability of H_2-optimization problem for discrete-time systems with Markovian jumping is given.

In the present paper we consider the problem of H_2 optimal control for a wide class of discrete time linear stochastic systems. We refer to linear stochastic systems subject to Markovian jumping and independent random perturbations. Our goal is to provide an unified approach of this optimization problem and to reveal the aspects specific to the discrete time as well to the presence of a Markov chain in the coefficients of the system. For a such discrete time linear stochastic system we introduce 3 types of H_2 performances (H_2- norms). We prove that under some additional assumptions these performance criteria can be expressed using the solutions of some linear equations on certain space of symmetric matrices. Since the usual H_2 performances associated to a discrete time linear stochastic system with Markovian jumping are strongly dependent upon the initial distribution of the Markov chain we proposed a new performance criteria not depending upon the initial distributions of the Markov chain. Concerning the problem of optimal control with respect to the H_2 performances we restrict our attention to the case of full state measurements. This is due to the length limitation of the paper. For the considered optimization problem we show that among all stabilizing controllers of higher dimension, the best performance is achieved by a zero order controller. That is a state feedback. It is the same state feedback which solves the linear quadratic optimization problem (the standard regulator problem) for this class of discrete time linear stochastic systems (see [20]. In the paper, special attention is paid to the case of discrete-time controlled systems with coefficients depending upon the state both at time t and a time t-1 of the Markov chain. We consider that a such class of systems provides a good mathematical model in the case when some delays in the transmission of the data can arise either on the channel from sensors to controller or from controllers to actuators.

The outline of the paper is:

Section 2 contains a detailed description of the mathematical model of the controlled systems under consideration in the paper. Also the definitions of three H_2 norms are introduced and the optimization problems which we want to solve are stated. In section 3 we give formulae of the those three H_2 norms defined in section 2. The obtained formulae are based on solutions of some suitable linear equations on the certain spaces of symmetric matrices. In the last part of this section several robustness issues concerning the H_2-norms of discrete time linear systems with Markovian jumping are discussed. We feel that such issues were less discussed in the existing papers in the field. Section 4 contains the solution of the H_2 optimization problem under the assumption that the full state vector is available for measurements. The proofs of several auxiliary results involved in Section 3 are collected in a Appendix. Also a brief discussion of the problem of the existence of the stabilizing solution of a discrete time stochastic generalized Riccati equation can be found in the last part of the Appendix.

2 H_2 norms of discrete-time linear stochastic systems

2.1 Model setting

Consider the discrete-time linear system (G) described by:

$$(G): \begin{cases} x(t+1) = (A_0(\eta_t) + \sum\limits_{k=1}^{r} w_k(t)A_k(\eta_t))x(t) + B_v(\eta_t)v(t) \\ z(t) = C(\eta_t)x(t), t \in \mathbf{Z}_+. \end{cases} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $z(t) \in \mathbf{R}^{n_z}$ a controlled output, $\{w_k(t)\}_{t\geq 0}, 1 \leq k \leq r$, are sequences of random variables and $\{v(t)\}_{t\geq 0}$ is a sequence of m_v-dimensional random vectors on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$, while $\{\eta_t\}_{t\geq 0}$ is a homogenous Markov chain with the set of the states $\mathcal{D} = \{1, 2, ..., N\}$ and the transition probability matrix P. This means that for each $t \geq 0$ we have

$$\mathcal{P}\{\eta_{t+1} = j|\mathcal{G}_t\} = \mathcal{P}\{\eta_{t+1} = j|\eta_t\} = p(\eta_t, j) \tag{2}$$

for all $j \in \mathcal{D}$, where $p(i,j)$ are the elements of the $N \times N$ stochastic matrix P and $\mathcal{G}_t = \sigma(\eta_0, \eta_1, ..., \eta_t)$ (the smallest σ-algebra generated by the random variables η_s, $0 \leq s \leq t$). For more details concerning Markov chains we refer to [12].

In (1), $A_k(i) \in \mathbf{R}^{n\times n}, B_v(i) \in \mathbf{R}^{n\times m_v}$, $C(i) \in \mathbf{R}^{n\times n_z}$ are given matrices.

Z_+ stands for the set of nonnegative integers.

Throughout this paper the following assumptions are made:

$\mathbf{H}_1$) If $w(t) = (w_1(t), w_2(t), ..., w_r(t))^T$ then $\{w(t)\}_{t\geq 0}$ is a sequence of independent random vectors with the following properties:

$$E[w(t)] = 0, E[w(t)w^T(t)] = I_r, t \geq 0,$$

I_r being the identity matrix of size r.

$\mathbf{H}_2$) The stochastic processes $\{w(t)\}_{t\geq 0}$ and $\{\eta(t)\}_{t\geq 0}$ are independent.

$\mathbf{H}_3$) $\{v(t)\}_{t\geq 0}$ is a sequence of independent random vectors with the properties:

$$E[v(t)] = 0, E[v(t)v^T(t)] = I_{m_v}, t \geq 0$$

and $\{v(t)\}_{t\geq 0}$ is independent of stochastic processes $\{w(t)\}_{t\geq 0}$ and $\{\eta_t\}_{t\geq 0}$.

Throughout the paper, the superscript T stands for the transpose of a matrix or a vector, while $E[\cdot]$ stands for the expectation.

Related to the Markov chain η_t, we define $\pi_t(i) = \mathcal{P}(\eta_t = i), i \in \mathcal{D}, t \geq 0$ and $\pi_t = (\pi_t(1), \pi_t(2), ..., \pi_t(N))$ is known as the distribution of the random variable η_t.

The sequence $\{\pi_t\}_{t\geq 0}$ solves the forward linear equation

$$\pi_{t+1} = \pi_t P. \tag{3}$$

For each $t \geq 0$ we introduce the following subset of $\mathcal{D}$:

$$\mathcal{D}_t = \{i \in \mathcal{D}, \pi_t(i) > 0\}. \tag{4}$$

From (3), one obtains that for a Markov chain is possible to have $\mathcal{D} \setminus \mathcal{D}_t \neq \phi$ for some $t \geq 1$, even if $\mathcal{D}_0 = \mathcal{D}$.

Let $A(t) = A_0(\eta_t) + \sum\limits_{k=1}^{r} w_k(t) A_k(\eta_t), t \geq 0$.

Set $\Phi(t, s) = \begin{cases} A(t-1)A(t-2)...A(s), & if \quad t \geq s+1 \\ I_n, & if \quad \text{t=s.} \end{cases}$

If $x(t, t_0, x_0)$ is the solution of (1) with the initial value $x(t, t_0, x_0) = x_0$ then we have the following representation formula:

$$x(t, t_0, x_0) = \Phi(t, t_0)x_0 + \sum_{l=t_0}^{t-1} \Phi(t, l+1) B_v(\eta_l) v(l) \tag{5}$$

for all $t \geq t_0 + 1$.

We have

$$x(t, t_0, x_0) = \Phi(t, t_0)x_0 + x_0(t, t_0) \tag{6}$$

with $x_0(t, t_0) = x(t, t_0, 0) = \sum\limits_{l=t_0}^{t-1} \Phi(t, l+1) B_v(\eta_l) v(l)$.

The corresponding output is

$$z(t, t_0, x_0) = C(\eta_t)\Phi(t, t_0)x_0 + z_0(t, t_0) \tag{7}$$

where $z_0(t, t_0) = C(\eta_t) x_0(t, t_0)$.

In (7) $C(\eta_t)\Phi(t, t_0)x_0$ is the transitory component of the output signal while $z_0(t, t_0)$ is the answer of the system determined by the exogenous noise $v(t)$.

2.2 H_2 type norms

The linear system obtained from (1) is:

$$x(t+1) = (A_0(\eta_t) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t)) x(t). \tag{8}$$

We recall that the zero state equilibrium of (8) is exponentially stable in mean square (ESMS) for shortness, if there exist $\beta \geq 1, q \in (0, 1)$ such that

$$E[|\Phi(t, 0)x_0|^2] \leq \beta q^t |x_0|^2, t \geq 0 \tag{9}$$

for all $x_0 \in \mathbf{R}^n$ and for every initial distribution π_0 of the Markov chain.

For other equivalent definitions of (ESMS) related to linear systems of type (8) we refer to [16].

Under the assumption that the zero state equilibrium of (8) is (ESMS) we introduce the following performance criteria associated to the system (1):

$$\|G\|_2 = (\lim_{l\to\infty} \frac{1}{l}\sum_{t=0}^{l} E[|z(t,0,x_0)|])^{\frac{1}{2}} \tag{10}$$

$$\tilde{|}|G\tilde{|}|_2 = (\lim_{l\to\infty} \frac{1}{l}\sum_{t=0}^{l}\sum_{i\in\mathcal{D}_0} E[|z(t,0,x_0)|^2/\eta_0 = i])^{\frac{1}{2}} \tag{11}$$

$$|||G|||_2 = (\lim_{t\to\infty} E[|z(t,s,x_0)|^2])^{\frac{1}{2}}. \tag{12}$$

Since in the deterministic framework (i.e. $\mathcal{D} = \{1\}$ and $A_k(1) = 0, 1 \le k \le r$), the right hand side of (10)-(12) provides the state space characterization of the H_2 norm of a linear time invariant deterministic system, we shall preserve the same terminology in this general framework of stochastic systems (1). That is why we shall call H_2-norms the cost functionals introduced by (10)-(12).

Having in mind (7) and (9) one can see that the transitory component of the output $z(t,s,x_0)$ do not influence the performances (10)-(12). Explicit formulae for the performances (10)-(12) will be derived in section 3.

2.3 H_2 optimization

Consider the discrete time controlled stochastic system (G) described by:

$$(G): \begin{cases} x(t+1) = [A_0(\eta_t) + \sum_{k=1}^{r} w_k(t)A_k(\eta_t)]x(t) + [B_0(\eta_t) + \sum\limits_{k=1}^{r} w_k(t)B_k(\eta_t)]u(t) + B_v(\eta_t)v(t) \\ y(t) = x(t) \\ z(t) = C_z(\eta_t)x(t) + D_z(\eta_t)u(t) \end{cases} \tag{13}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ is the control input, $y(t) \in \mathbf{R}^n$ is the vector of the measurements, $z(t) \in \mathbf{R}^{n_z}$ is the controlled output and $\{w_k(t)\}_{t\ge0}, 1 \le k \le r$, $\{\eta_t\}_{t\ge0}$, $\{v(t)\}_{t\ge0}$ are as before and verify $\mathbf{H_1}) - \mathbf{H_3}$). It is assumed that the whole state vector is available for the measurements. The coefficients $A_k(i), B_k(i), 0 \le k \le r$, $B_v(i), C_z(i), D_z(i), i \in \mathcal{D}$ are constant matrices of appropriate dimensions.

To control the systems of type (13) we consider dynamic controllers of the form:

$$(G_c): \begin{cases} x_c(t+1) = [A_{c0}(\eta_t) + \sum\limits_{k=1}^{r} w_k(t)A_{ck}(\eta_t)]x_c(t) + (B_{c0}(\eta_t) + \sum_{k=1}^{r} w_k(t)B_{ck}(\eta_t))u_c(t) \\ y_c(t) = C_c(\eta_t)x_c(t) + F_c(\eta_t)u_c(t), \end{cases} \tag{14}$$

$t \ge 0$, where $x_c \in \mathbf{R}^{n_c}$ is the vector of the states of the controller, $u_c(t) \in \mathbf{R}^n$ is the vector of the inputs of the controller and $y_c(t) \in \mathbf{R}^m$ is the output of the controller. The integer n_c often known as the order of the controller is not prefixed. It will be determined together with the matrices $A_{ck}(i), B_{ck}(i), C_c(i), F_c(i)$. If $n_c = 0$ the controller (G_c) reduces to a feedback gain $y_c(t) = F_c(\eta_t)u_c(t)$.

Coupling a controller (G_c) of type (14) to a system (G) of type (13) taking $u_c(t) = y(t), u(t) = y_c(t)$ one obtains the following closed loop system:

$$(G_{cl}) : \begin{cases} x_{cl}(t+1) = [A_{0_{cl}}(\eta_t) + \sum\limits_{k=1}^{r} w_k(t)A_{kcl}(\eta_t)]x_{cl}(t) + B_{vcl}(\eta_t)v(t) \\ z_{cl}(t) = C_{cl}(\eta_t)x_{cl}(t), t \geq 0 \end{cases} \tag{15}$$

where $x_{cl}(t) = \begin{pmatrix} x^T(t) & x_c^T(t) \end{pmatrix}^T$,

$$A_{kcl}(i) = \begin{pmatrix} A_k(i) + B_k(i)F_c(i) & B_k(i)C_c(i) \\ B_{ck}(i) & A_{ck}(i) \end{pmatrix}, 0 \leq k \leq r, \quad B_{vcl}(i) = \begin{pmatrix} B_v(i) \\ 0 \end{pmatrix}$$
$$C_{cl}(i) = \begin{pmatrix} C_z(i) + D_z(i)F_c(i) & D_z(i)C_c(i) \end{pmatrix}. \tag{16}$$

Definition 2.1. We say that a controller (G_c) of type (14) is a stabilizing controller for the system G of type (13) if the zero state equilibrium of the linear system

$$x_{cl}(t+1) = (A_{0cl}(\eta_t) + \sum_{k=1}^{r} w_k(t)A_{kcl}(\eta_t))x_{cl}(t)$$

is exponentially stable in mean square.

In the sequel we shall denote $\mathcal{K}_s(G)$ the class of stabilizing controllers for a given system (G) of type (13). Now we are in position to state the optimization problems associated to a system (13):

$\mathbf{OP_1}$. Find an admissible controller $\tilde{G}_c$ such that the corresponding closed-loop system $\tilde{G}_{cl}$ satisfy

$$||\tilde{G}_{cl}||_2 = \min_{G_c \in \mathcal{K}_s(G)} ||G_{cl}||_2.$$

$\mathbf{OP_2}$. Find an admissible controller $\tilde{G}_c$ such that the corresponding closed-loop system $\tilde{G}_{cl}$ satisfy $\tilde{||}\tilde{G}_{cl}\tilde{||}_2 = \min\limits_{G_c \in \mathcal{K}_s(G)} \tilde{||}G_{cl}\tilde{||}_2$.

$\mathbf{OP_3}$. Find an admissible controller $\tilde{G}_c$ such that the corresponding closed-loop system $\tilde{G}_{cl}$ satisfy $|||\tilde{G}_{cl}|||_2 = \min\limits_{G_c \in \mathcal{K}_s(G)} |||G_{cl}|||$.

Since in the case $N = 1$ the norm (10)-(11) coincide it follows that for the system subject to independent random perturbations we have only two H_2-optimization problems $\mathbf{OP}_1$ and $\mathbf{OP}_3$, respectively.

2.4 Systems with coefficients depending upon η_t and η_{t-1}

The explicit formulae of the H_2-norms (10)-(12) will be derived as special cases of some corresponding H_2 norms defined for a more general class of discrete-time stochastic systems with coefficients depending upon η_t and η_{t-1}.

Let us consider the discrete-time controlled systems (G) described by:

$$\begin{cases} x(t+1)=[A_0(\eta_t,\eta_{t-1})+\sum\limits_{k=1}^{r}w_k(t)A_k(\eta_t,\eta_{t-1})]x(t)+ \\ +[B_0(\eta_t,\eta_{t-1})+\sum\limits_{k=1}^{r}w_k(t)B_k(\eta_t,\eta_{t-1})]u(t)+B_v(\eta_t,\eta_{t-1})v(t) \\ y(t)=[C_0(\eta_t,\eta_{t-1})+\sum\limits_{k=1}^{r}w_k(t)C_k(\eta_t,\eta_{t-1})]x(t)+D_v(\eta_t,\eta_{t-1})v(t) \\ z(t)=[C_z(\eta_t,\eta_{t-1})x(t)+D_z(\eta_t,\eta_{t-1})u(t), t\geq 1 \end{cases}$$

where $x(t), u(t), y(t), z(t)$ have the same meaning as in the case of the system (13), while $\{w_k(t)\}_{t\geq 0}, \{\eta_t\}_{t\geq 0}, \{v(t)\}_{t\geq 0}$ are stochastic processes which satisfy assumptions $\mathbf{H}_1) - \mathbf{H}_3)$. $A_k(i,j) \in \mathbf{R}^{n\times n}, B_k(i,j) \in \mathbf{R}^{n\times m}, C_k(i,j) \in \mathbf{R}^{n_y\times n}, 0 \leq k \leq r,\ B_v(i,j) \in \mathbf{R}^{n\times m_v}, D_v(i,j) \in \mathbf{R}^{n_y\times m_v}, C_z(i,j) \in \mathbf{R}^{n_z\times n}, D_z(i,j) \in \mathbf{R}^{n_z\times m}, i\jmath \in \mathcal{D}$ are given matrices.

The above systems can be obtained in a natural way from systems of type (13) if a delay in the transmission of the measurements is possible between the sensors and controller. Consider that in (13) an output

$$\tilde{y}(t)=(C_0(\eta_t)+\sum_{k=1}^{r}w_k(t)C_k(\eta_t))x(t)+D_v(\eta_t)v(t) \tag{17}$$

instead of $y(t)=x(t)$.

Let us assume that at instance t, in the system (13), the measurement $\check{y}(t)=\tilde{y}(t-1)$ is introduced in the controller instead of $\tilde{y}(t)$.

Setting $\tilde{x}(t)=\begin{pmatrix} x^T(t) & x^T(t-1) \end{pmatrix}^T$ one obtains the following system derived from (13)with measurement output (17)

$$\begin{aligned} \tilde{x}(t+1) &= (\tilde{A}_0(\eta_t,\eta_{t-1})+\sum_{k=1}^{2r}\tilde{w}_k(t)\tilde{A}_k(\eta_t,\eta_{t-1}))\tilde{x}(t)+ \\ &\quad (\tilde{B}_0(\eta_t,\eta_{t-1})+\sum_{k=1}^{2r}\tilde{w}_k(t)\tilde{B}_k(\eta_t,\eta_{t-1}))u(t)+\tilde{B}_v(\eta_t,\eta_{t-1})\tilde{v}(t). \\ \tilde{y}(t) &= [\tilde{C}_0(\eta_t,\eta_{t-1})+\sum_{k=1}^{2r}\tilde{w}_k(t)\tilde{C}_k(\eta_t,\eta_{t-1})]\tilde{x}(t)+\tilde{D}_v(\eta_t,\eta_{t-1})\tilde{v}(t) \\ z(t) &= \tilde{C}_z(\eta_t,\eta_{t-1})\tilde{x}(t)+\tilde{D}_z(\eta_t,\eta_{t-1})u(t) \end{aligned} \tag{18}$$

where

$$\tilde{A}_0(i,j)=\begin{pmatrix} A_0(i) & 0 \\ I_n & 0 \end{pmatrix}, \tilde{A}_k(i,j)=\begin{pmatrix} A_k(i) & 0 \\ 0 & 0 \end{pmatrix}, 1\leq k\leq r,$$
$$\tilde{A}_k(i,j)=0,\quad r+1\leq k\leq 2r, \tilde{C}_0(i,j)=\begin{pmatrix} 0 & C_0(j) \end{pmatrix}, \tilde{C}_k(i,j)=0, 1\leq k\leq r,$$
$$\tilde{C}_k(i,j)=\begin{pmatrix} 0 & C_{k-r}(j) \end{pmatrix},\qquad r+1\leq k\leq 2r.$$
$$\tilde{B}_k(i,j)=\begin{pmatrix} B_k(i) \\ 0 \end{pmatrix}, 0\leq k\leq r, \tilde{B}_k(i,j)=0\quad r+1\leq k\leq 2r. \tag{19}$$
$$\tilde{B}_v(i,j)=\begin{pmatrix} B_v(i) & 0 \\ 0 & 0 \end{pmatrix}\in\mathbf{R}^{2n\times 2m_v}, \tilde{D}_v(i,j)=\begin{pmatrix} 0 & D_v(j) \end{pmatrix}, \tilde{C}_z(i,j)=\begin{pmatrix} C_z(i) & 0 \end{pmatrix},$$
$$\tilde{D}_z(i,j)=D_z(i), \tilde{w}_k(t)=w_k(t), 1\leq k\leq r, \tilde{w}_k(t)=w_{k-r}(t-1), r+1\leq k\leq 2r,$$
$$\tilde{v}(t)=\begin{pmatrix} v^T(t) & v^T(t-1) \end{pmatrix}^T.$$

To redefine the H_2 norms of type (10)-(12) in the case of systems with coefficients depending upon η_t and η_{t-1} we consider the uncontrolled system:

$$(\mathbf{G}): \begin{cases} x(t+1) = (A_0(\eta_t, \eta_{t-1}) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t, \eta_{t-1}))x(t) + B_v(\eta_t, \eta_{t-1})v(t) \\ z(t) = C(\eta_t, \eta_{t-1})x(t), t \geq 1 \end{cases} \tag{20}$$

As in the case of system (1), $x(t, t_0, x_0), t \geq t_0 \geq 1, x_0 \in \mathbf{R}^n$ stands for the trajectory of (20) with the initial value $x(t_0, t_0, x_0) = x_0$ and $z(t, t_0, x_0) = C(\eta_t, \eta_{t-1})x(t, t_0, x_0)$ is a corresponding output.

The analogous of norms (10)-(12) defined for the system (20) are:

$$\|\mathbf{G}\|_2 = [\lim_{l \to \infty} \frac{1}{l} \sum_{t=1}^{l} E[|z(t, 1, x_0)|^2]]^{\frac{1}{2}} \tag{21}$$

$$\tilde{|}|\mathbf{G}|\tilde{|}_2 = [\lim_{l \to \infty} \frac{1}{l} \sum_{t=1}^{l} \sum_{i \in \mathcal{D}_0} E[|z(t, 1, x_0)|^2 | \eta_0 = i]]^{\frac{1}{2}} \tag{22}$$

$$|||\mathbf{G}|||_2 = [\lim_{t \to \infty} E[|z(t, s, x_0)|^2]]^{\frac{1}{2}}. \tag{23}$$

In the next section we shall show how we can express the right hand side of (21)-(23) in terms of solution of some suitable linear equations. Such linear equations extend to this framework the well known equations of observability Gramian and controllability Gramian from the deterministic framework.

3 The computation of H_2 type norms

3.1 Some preliminaries

Let $\mathcal{S}_n \subset \mathbf{R}^{n \times n}$ be the linear subspace of $n \times n$ real symmetric matrices and $\mathcal{S}_n^N = \mathcal{S}_n \oplus \mathcal{S}_n \oplus ... \oplus \mathcal{S}_n$. We have $X \in \mathcal{S}_n$ iff $X = (X(1), X(2), ..., X(N)), X(i) \in \mathcal{S}_n$. $\mathcal{S}_n^N$ is a real Hilbert space with the inner product:

$$< X, Y >= \sum_{i=1}^{N} Tr[X(i)Y(i)] \tag{24}$$

for arbitrary $X, Y \in \mathcal{S}_n^N$. $Tr[\cdot]$ is the trace operator. Moreover, the Hilbert space $\mathcal{S}_n^N$ is an ordered linear space with respect to the order relation " $\leq$ " induced by the convex cone

$$\mathcal{S}_n^{N+} = \{X \in \mathcal{S}_n^N | X = (X(1), ..., X(N)), X(i) \geq 0, 1 \leq i \leq N\}.$$

Here $X(i) \geq 0$ means that $X(i)$ is positive semidefinite. Together with the norm $|\cdot|_2$ induced by the inner product (24) we consider the norm $|\cdot|_1$ defined as $|X|_1 = \max_{i \in \mathcal{D}} \max\{|\lambda(i)| | \lambda(i) \in \sigma(X(i))\}$, where $\sigma(X(i))$ stands for the set of the eigenvalues of the matrix $X(i)$.

Consider the discrete-time linear system

$$x(t+1) = [A_0(\eta_t, \eta_{t-1}) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t, \eta_{t-1})]x(t) \tag{25}$$

obtained from (20) taking $B_v(i,j)=0$.

Using the matrices $A_k(i,j)$ and the transition probability matrix P we construct the linear operator $\Upsilon : \mathcal{S}_n^N \to \mathcal{S}_n^N$ as $\Upsilon H = (\Upsilon H(1), \Upsilon H(2), ..., \Upsilon H(N))$ with

$$\Upsilon H(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(j,i)A_k(i,j)H(j)A_k^T(i,j) \tag{26}$$

$i \in \mathcal{D}, H \in \mathcal{S}_n^N$. The linear operator Υ defined above will be called the Lyapunov type operator associated to the discrete-time linear system (25).

By direct computation one obtains that the adjoint operator Υ^* with respect to the inner product (24) is given by $\Upsilon^* H = (\Upsilon^* H(1), \Upsilon^* H(2), ..., \Upsilon^* H(N))$,

$$\Upsilon^* H(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)A_k^T(j,i)H(j)A_k(j,i), \quad i \in \mathcal{D} \tag{27}$$

$H \in \mathcal{S}_n^N$. We recall that the zero state equilibrium of (25) is exponentially stable in mean square (ESMS) if there exist $\beta \geq 1, q \in (0,1)$ such that $E[|x(t,1,x_0)|^2] \leq \beta q^{t-1}|x_0|^2$ for all solutions $x(t,x_0)$ of (25). Different equivalent definitions and details concerning the characterization of the concept of ESMS for systems of type (25) can be found in [19].

From Theorem 3.7 in [19] we have:

Proposition 3.1. *Under the assumptions* $\mathbf{H}_1$)-$\mathbf{H}_2$) *the following are equivalent:*

(i) The zero state equilibrium of the system (25) is (ESMS).

(ii) $\rho(\Upsilon) < 1$, $\rho(\cdot)$ *being the spectral radius.*

The above Proposition together with Theorem 3.5 in [15] lead to:

Proposition 3.2. *Assume:*

a) Assumptions $\mathbf{H}_1$)-$\mathbf{H}_2$) *are fulfilled.*

b) The zero state equilibrium of (25) is ESMS.

Then the following hold:

(i) The algebraic equation on $\mathcal{S}_n^N$*:* $Y = \Upsilon Y + H$ *has a unique solution which is given by* $Y = \sum_{t=0}^{\infty} \Upsilon^t H$.

(ii) The algebraic equation on $\mathcal{S}_n^N : X = \Upsilon^* X + H$ *has a unique solution which is given by* $X = \sum_{t=0}^{\infty} (\Upsilon^*)^t H$.

Moreover, if $H \in \mathcal{S}_n^{N+}$ *then the solution* X *and* Y *of the above equations belong to* $\mathcal{S}_n^{N+}$.

Concerning the stochastic matrices we recall the following result proved in [12]:

Proposition 3.3. *If* $P \in \mathbf{R}^{N\times N}$ *is a stochastic matrix then the Cesaro limit* $\lim_{l\to\infty} \frac{1}{l}\sum_{t=0}^{l} P^t$ *is well defined. If*

$$Q = \lim_{l\to\infty} \frac{1}{l}\sum_{t=0}^{l} P^t \tag{28}$$

then Q *is also a stochastic matrix and we have* $QP = PQ = Q$.

Definition 3.1. We say that the stochastic matrix P is a **non-degenerate stochastic matrix** if for each $j \in \mathcal{D}$ there exists $i \in \mathcal{D}$ such that $p(i,j) > 0$.

Based on (3) one can see that $\pi_t(i) > 0$ for each $t \geq 1$ and $1 \leq i \leq N$, if $\pi_0(j) > 0, 1 \leq j \leq N$.

At the end of this subsection we introduce several σ-algebras generated by the stochastic processes $\{w(t)\}_{t\geq 0}$, $\{\eta_t\}_{t\geq 0}$, $\{v(t)\}_{t\geq 0}$.

Thus we denote

$$\mathcal{F}_t = \sigma(w(0), w(1), ..., w(t))$$

$$\mathcal{G}_t = \sigma(\eta_0, \eta_1, ..., \eta_t)$$

$$\hat{\mathcal{F}}_t = \sigma(v(0), v(1), ..., v(t))$$

$$\mathcal{H}_t = \mathcal{F}_t \vee \mathcal{G}_t$$

$$\hat{\mathcal{H}}_t = \mathcal{F}_t \vee \mathcal{G}_t \vee \hat{\mathcal{F}}_t$$

$$\tilde{\mathcal{H}}_t = \hat{\mathcal{H}}_{t-1} \vee \sigma(\eta_t).$$

We recall that if $\mathcal{F}_1$ and $\mathcal{F}_2 \subset \mathcal{F}$ are two σ-algebras then $\mathcal{F}_1 \vee \mathcal{F}_2 \subset \mathcal{F}$ stands for the smallest σ-algebra containing $\mathcal{F}_1$ and $\mathcal{F}_2$.

3.2 The computations of the norm (21) and the norm (22)

We start with the following auxiliary result:

Lemma 3.1. *Under the assumptions* $\mathbf{H}_1$)-$\mathbf{H}_3$) *we have* $E[x^T(t+1)H(\eta_t)x(t+1)|\eta_{s-1}] = E[x^T(t)(\Upsilon^* H)(\eta_{t-1})x(t)|\eta_{s-1}] + \sum_{j=1}^{N} E[Tr[H(j)B_v(j,\eta_{t-1})B_v^T(j,\eta_{t-1})]p(\eta_{t-1},j)|\eta_{s-1}]$, $\forall t \geq s \geq 1$, $H \in \mathcal{S}_n^N$, *where* $x(t) = x(t,s,x_0)$ *is a trajectory of the system (20) starting from* x_0 *at* $t = s$.

Proof. (see Appendix A_1).

Let $\mathcal{A}(t) = A_0(\eta_t, \eta_{t-1}) + \sum_{k=1}^{r} w_k(t)A_k(\eta_t, \eta_{t-1})$. We define $\Theta(t,s) = \mathcal{A}(t-1)\mathcal{A}(t-2)...\mathcal{A}(s)$ if $t > s \geq 1$ and $\Theta(t,s) = I_n$ if $t = s$. $\Theta(t,s)$ will be called the fundamental matrix solution of the system (25).

The solutions of the affine system (20) have the representation

$$x(t,s,x_0) = \Theta(t,s)x_0 + \sum_{l=s}^{t-1} \Theta(t,l+1)B_v(\eta_l,\eta_{l-1})v(l) \tag{29}$$

for all $t \geq s+1, s \geq 1, x_0 \in \mathbf{R}^n$. Often we shall write $x_0(t,s)$ instead of $x(t,s,0)$.

Lemma 3.2. *Under the assumptions* $\mathbf{H}_1$)-$\mathbf{H}_3$) *the following hold:*

(i) $E[x_0(t,s)x_0^T(t,s)] = \sum_{l=s}^{t-1} E[\Theta(t,l+1)B_v(\eta_l,\eta_{l-1})B_v^T(\eta_l,\eta_{l-1})\Theta^T(t,l+1)]$;

(ii) $E[x_0(t,s)x_0^T(t,s)\chi_{\{\eta_{t-1}=j\}}] = \sum_{l=s}^{t-1} E[\Theta(t,l+1)B_v(\eta_l,\eta_{l-1})B_v^T(\eta_l,\eta_{l-1})\Theta^T(t,l+1)\chi_{\{\eta_{t-1}=j\}}]$ *for all* $t > s \geq 1$, *where as usual* χ_M *is the indicator function of the set* $M \in \mathcal{F}$.

Proof. Using (29) for $x_0 = 0$ one compute firstly the conditional expectations

$$E[x_0(t,s)x_0^T(t,s)|\mathcal{H}_{t-1}]$$

and

$$E[x_0(t,s)x_0^T(t,s)\chi_{\{\eta_{t-1}=j\}}|\mathcal{H}_{t-1}].$$

To this end one takes into account that $\Theta(t, l+1), B_v(\eta_l, \eta_{l-1})$ are $\mathcal{H}_{t-1}$ measurable while $v(l)$ are independent of $\mathcal{H}_{t-1}$. Details are omitted.

Remark 3.1. If together with assumptions $\mathbf{H}_1$)-$\mathbf{H}_3$) we assume that the zero state equilibrium of (25) is ESMS then from Lemma 3.2 one obtains that:

$$\sup_{t \geq s \geq 1} E[|x_0(t, s)|^2] \leq \gamma < \infty. \tag{30}$$

On the other hand from the representation formula (29) one deduces that

$$E[|x(t, s, x_0) - x_0(t, s)|^2] \leq \beta q^{t-s}|x|^2 \tag{31}$$

for all $t \geq s \geq 1, x_0 \in \mathbf{R}^n, \beta \geq 1, q \in (0, 1)$.

Combining (30) and (31) we may conclude that

$$\sup_{t \geq s \geq 1} E[|x(t, s, x_0)|^2] \leq \gamma_1(1 + |x_0|^2) \tag{32}$$

for all $x_0 \in \mathbf{R}^n$.

Lemma 3.3. *Assume:*

a) The assumptions $\mathbf{H}_1$)-$\mathbf{H}_3$) *are fulfilled.*

b) The zero state equilibrium of (25) is ESMS.

Under these conditions we have:

$$\lim_{l \to \infty} \frac{1}{l} \sum_{t=1}^{l} E[|C(\eta_t, \eta_{t-1})x(t, 1, x_0)|^2 | \eta_0 = i] = \sum_{i_1, i_2=1}^{N} Tr[B_v^T(i_1, i_2)\tilde{X}(i_1)B_v(i_1, i_2)]p(i_2, i_1)q(i, i_2)$$

for all $x_0 \in \mathbf{R}^n, i \in \mathcal{D}_0, x(t, 1, x_0)$ *is the trajectory of (20) starting from* x_0 *at* $t = 1$, $\tilde{X} = (\tilde{X}(1), \tilde{X}(2), ..., \tilde{X}(N))$ *is the unique solution of the affine equation on* $\mathcal{S}_n^N$

$$X = \Upsilon^* X + \tilde{C} \tag{33}$$

where $\tilde{C} = (\tilde{C}(1), \tilde{C}(2), ..., \tilde{C}(N))$ *with*

$$\tilde{C}(i) = \sum_{j=1}^{N} p(i, j)C^T(j, i)C(j, i) \tag{34}$$

and $q(i, i_2)$ *are the entries of the matrix* Q *introduced by (28).*

Proof. see Appendix A_2.

Now we are in position to prove result which provide explicit formula of the H_2 norms (21)-(22).

Theorem 3.1. *Assume:*

a) The assumptions $\mathbf{H}_1$)-$\mathbf{H}_3$) *are fulfilled.*

b) The zero state equilibrium of (25) is ESMS.

Then: (i)

$$\begin{aligned}(||\mathbf{G}||_2)^2 &= \sum_{i_1, i_2=1}^{N} q^{\pi_0}(i_2)p(i_2, i_1)Tr[\tilde{X}(i_1)B_v(i_1, i_2)B_v^T(i_1, i_2)] \\ &= \sum_{i_1, i_2=1}^{N} p(i_1, i_2)Tr[C(i_2, i_1)Y^{\pi_0}(i_1)C(i_2, i_1)]\end{aligned}$$

(ii)

$$(\tilde{||}\mathbf{G}\tilde{||}_2)^2 = \sum_{i_1,i_2=1}^{N} q^{\mathcal{D}_0}(i_2)p(i_2,i_1)Tr[\tilde{X}(i_1)B_v(i_1,i_2)B_v^T(i_1,i_2)]$$

$$= \sum_{i_1,i_2=1}^{N} p(i_1,i_2)Tr[C(i_2,i_1)Y^{\mathcal{D}_0}(i_1)C(i_2,i_1)]$$

where $\tilde{X} \in \mathcal{S}_n^{N+}$ *is the unique solution of the linear equation (33)-(34) while* $Y^{\pi_0} = (Y^{\pi_0}(1), Y^{\pi_0}(2), ..., Y^{\pi_0}(N)) \in \mathcal{S}_n^{N+}$ *and* $Y^{\mathcal{D}_0} = (Y^{\mathcal{D}_0}(1), Y^{\mathcal{D}_0}(2), ..., Y^{\mathcal{D}_0}(N)) \in \mathcal{S}_n^{N+}$ *respectively are the unique solutions of the linear equations:*

$$Y = \Upsilon Y + B^{\pi_0} \tag{35}$$

and

$$Y = \Upsilon Y + B^{\mathcal{D}_0} \tag{36}$$

respectively, with $B^{\pi_0} = (B^{\pi_0}(1), B^{\pi_0}(2), ..., B^{\pi_0}(M))$,

$$B^{\pi_0}(i) = \sum_{j=1}^{N} q^{\pi_0}(j)p(j,i)B_v(i,j)B_v^T(i,j) \tag{37}$$

and $B^{\mathcal{D}_0} = (B^{\mathcal{D}_0}(1), B^{\mathcal{D}_0}(2), ..., B^{\mathcal{D}_0}(N))$,

$$B^{\mathcal{D}_0}(i) = \sum_{j=1}^{N} q^{\mathcal{D}_0}(j)p(j,i)B_v(i,j)B_v^T(i,j), 1 \le i \le N \tag{38}$$

$q^{\pi_0}(i) = \sum_{j=1}^{N} \pi_0(j)q(j,i)$ *and* $q^{\mathcal{D}_0}(i) = \sum_{j\in\mathcal{D}_0} q(j,i), 1 \le i \le N$.

Proof. We start with the proof of (ii). Directly from the equalities in Lemma 3.3 one obtains that

$$\begin{aligned}(\tilde{||}\mathbf{G}\tilde{||}_2)^2 = \lim_{l\to\infty}\frac{1}{l}\sum_{t=1}^{l}\sum_{i\in\mathcal{D}_0} E[|z(t,1,x_0)|^2|\eta_0 = i] = \\ \sum_{i_1,i_2=1}^{N}\sum_{i\in\mathcal{D}_0} q(i,i_2)p(i_2,i_1)Tr[\tilde{X}(i_1)B_v(i_1,i_2)B_v^T(i_1,i_2)] = \\ \sum_{i_1,i_2=1}^{N} q^{\mathcal{D}_0}(i_2)p(i_2,i_1)Tr[\tilde{X}(i_1)B_v(i_1,i_2)B_v^T(i_1,i_2)]\end{aligned} \tag{39}$$

which confirms the validity of the first equality of (ii).

Further (24) and (38) allow us to write

$$\sum_{i_1,i_2=1}^{N} q^{\mathcal{D}_0}(i_2)p(i_2,i_1)Tr[\tilde{X}(i_1)B_v(i_1,i_2)B_v^T(i_1,i_2)] = \sum_{i_1=1}^{N} Tr[\tilde{X}(i_1)B^{\mathcal{D}_0}(i_1)] =< \tilde{X}, B^{\mathcal{D}_0} > .$$

Using the equation verified by $Y^{\mathcal{D}_0}$ and equality (39) we have:

$$(\tilde{|}|\mathbf{G}\tilde{|}|_2)^2 =< \tilde{X}, Y^{\mathcal{D}_0} > - < \tilde{X}, \Upsilon Y^{\mathcal{D}_0} >=< \tilde{X} - \Upsilon^* \tilde{X}, Y^{\mathcal{D}_0} >=< \tilde{C}, Y^{\mathcal{D}_0} > .$$

Taking into account (24) and (34) we may write finally

$$\tilde{|}|\mathbf{G}\tilde{|}|_2)^2 = \sum_{i_1,i_2=1}^{N} p(i_1,i_2) Tr[C^T(i_2,i_1)C(i_2,i_1)Y^{\mathcal{D}_0}(i_1)]$$

which confirms the second equality of (ii).

To prove (i) we take into account that $E[|z(t,1,x_0)|^2] = \sum_{i\in\mathcal{D}_0} \pi_0(i) E[|z(t,1,x_0)^2|\eta_0 = i]$. Thus, multiplying by $\pi_0(i)$ the equalities proved in Lemma 3.3 and proceeding as in the first part of the proof one obtains that (i) holds and the proof ends.

Using Lemma 3.1 for $H = \tilde{X}$ one can prove:

Proposition 3.4. *Assume:*

a) Assumptions $\mathbf{H}_1$)-$\mathbf{H}_3$) *are fulfilled.*

b) The zero state equilibrium of (25) is ESMS.

c) P is a non-degenerate stochastic matrix.

d) $\pi_0(i) > 0, 1 \le i \le N$.

Under these conditions, the following hold:

(i) $\lim_{l\to\infty} \frac{1}{l} \sum_{t=s}^{s+l-1} \sum_{i=1}^{N} E[|C(\eta_t,\eta_{t-1})x(t,s,x_0)|^2|\eta_{s-1} = i]$

$= \sum_{i_1,i_2=1}^{N} \tilde{q}(i_2)p(i_2,i_1)Tr[\tilde{X}(i_1)B_v(i_1,i_2)B_v^T(i_1,i_2)$

(ii) $\lim_{l\to\infty} \frac{1}{l} \sum_{t=s}^{s+l-1} E[|C(\eta_t,\eta_{t-1})x(t,s,x_0)|^2] = \sum_{i_1,i_2=1}^{N} q^{\pi_0}(i_2)p(i_2,i_1)Tr[\tilde{X}(i_1)B_v(i_1,i_2)B_v^T(i_1,i_2)$, *for every solution $x(t,s,x_0)$ of the system (20) starting from x_0 at $t = s, q^{\pi_0}(i_2)$ is defined as in Theorem 3.1 while $\tilde{q}(i_2) = \sum_{i=1}^{N} q(i,i_2)$.*

To prove the equality in (ii) one uses the fact that $\pi_{s-1}(i) = \sum_{j=1}^{N} \pi_0(j)p^{s-1}(j,i)$, where $p^{s-1}(j,i)$ are the entries of P^{s-1}. The details are omitted.

From Theorem 3.1 one sees that the H_2-norms introduced by (21)-(22) do not depend upon the initial values x_0 of the solutions $x(t,1,x_0)$ of the system (20).

The result stated in the Proposition 3.4 shows that under some additional assumptions the norms (21)-(22) do not depend upon the initial time $t = s$, too.

3.3 The computation of the norm (23)

We start by:

Lemma 3.4.*Assume:*

a) the assumptions $\mathbf{H}_1$)-$\mathbf{H}_3$) *are fulfilled.*

b) the transition probability matrix P is a non-degenerate stochastic matrix

c) $\pi_0(i) > 0, 1 \le i \le N$.

Under these conditions we have:

$$E[x_0(t,s)x_0^T(t,s)\chi_{\{\eta_{t-1}=j\}}] = \sum_{l=s}^{t-1}(\Upsilon^{t-l-1}H_l)(j)$$

where $H_l = (H_l(1), H_l(2), ..., H_l(N)), H_l(i) = \sum\limits_{i_1,i_2=1}^{N} \pi_0(i_1)p^{l-1}(i_1,i_2)p(i_2,i_1)B_v(i_1,i_2)B_v^T(i_1,i_2)$, *with* $p^{l-1}(i_1,i_2)$ *as in Proposition 3.4.*

Proof. see Appendix A.4.

Before to state the next result we introduce an additional assumption:

$\mathbf{H}_4$) The transition probability matrix P has the following property: $\lim\limits_{l\to\infty} P^l$ exists.

Remark 3.2. Under the assumption $\mathbf{H}_4$) if $Q = \lim\limits_{l\to\infty} P^l$ then the matrix Q is the same as that in (28).

Lemma 3.5.*Assume:*

a) the assumptions $\mathbf{H}_1$)-$\mathbf{H}_4$) *are fulfilled.*

b) the zero state equilibrium of the system (25) is ESMS.

c) the transition probability matrix P *is a non-degenerate stochastic matrix.*

d) $\pi_0(i) > 0, i \in \mathcal{D}$.

Under these conditions we have:

$$\lim_{t\to\infty} E[x(t,s,x_0)x^T(t,s,x_0)\chi_{\{\eta_{t-1}=j\}}] = Y^{\pi_0}(j)$$

for all $j \in \mathcal{D}$, *where* $Y^{\pi_0} = (Y^{\pi_0}(1), Y^{\pi_0}(2), ..., Y^{\pi_0}(N)) \in \mathcal{S}_n^{N+}$ *is a unique solution of the linear equation (35), (37).*

Proof. see Appendix A_5.

The main result of this subsection is:

Theorem 3.2. *Under the assumptions of Lemma 3.5 we have the following formula for the* H_2*-norm (23):*

$$(|||\mathbf{G}|||_\mathbf{2}{}^2 = \sum_{i_1,i_2=1}^{N} Tr[C(i_1,i_2)Y^{\pi_0}(i_2)C^T(i_1,i_2)]p(i_2,i_1) =$$

$$\sum_{i_1,i_2=1}^{N} q^{\pi_0}(i_2)p(i_2,i_1)Tr[B_v^T(i_1,i_2)\tilde{X}(i_1)B_v(i_1,i_2)]$$

where $\tilde{X} = (\tilde{X}(1), \tilde{X}(2), ..., \tilde{X}(N)) \in \mathcal{S}_n^{N+}$ *is the unique solution of the linear equation (33)-(34) and* $Y^{\pi_0} = (Y^{\pi_0}(1), Y^{\pi_0}(2), ..., Y^{\pi_0}(N)) \in \mathcal{S}_n^{N+}$ *is the unique solution of the linear equation (35)-(37) and* q^{π_0} *is defined as in Theorem 3.1.*

Proof. Set $x(t) = x(t,s,x_0)$ and $z(t) = z(t,s,x_0), t \geq s \geq 1, x_0 \in \mathbf{R}^n$. Since $x(t)$ is $\hat{\mathcal{H}}_{t-1}$-measurable we may write successively

$$E[|z(t)|^2|\hat{\mathcal{H}}_{t-1}] = E[Tr(C(\eta_t,\eta_{t-1})x(t)x^T(t)C^T(\eta_t,\eta_{t-1}))|\hat{\mathcal{H}}_{t-1}] =$$

$$\sum_{i_1,i_2=1}^{N} E[Tr(C(i_1,i_2)x(t)x^T(t)C^T(i_1,i_2))\chi_{\{\eta_t=i_1\}}\chi_{\{\eta_{t-1}=i_2\}}|\hat{\mathcal{H}}_{t-1}] =$$

$$\sum_{i_1,i_2=1}^{N} Tr[C(i_1,i_2)x(t)x^T(t)C^T(i_1,i_2)]\chi_{\{\eta_{t-1}=i_2\}}E[\chi_{\{\eta_t=i_1\}}|\hat{\mathcal{H}}_{t-1}].$$

Using Corollary A_1 from below with $\hat{\mathcal{H}}_{t-1}$ instead of $\mathcal{H}_t$ we obtain $E[\chi_{\{\eta_t=i_1\}}|\hat{\mathcal{H}}_{t-1}] = p(\eta_{t-1}, i_1)$.

Thus we have

$$E[|z(t)|^2|\hat{\mathcal{H}}_{t-1}] = \sum_{i_1,i_2=1}^{N} p(i_2,i_1)Tr[C(i_1,i_2)x(t)x^T(t)C^T(i_1,i_2)]\chi_{\{\eta_{t-1}=i_2\}}.$$

Taking the expectation in the last equality one gets:

$$E[|z(t)|^2] = \sum_{i_1,i_2=1}^{N} p(i_2,i_1)Tr\{C(i_1,i_2)E[x(t)x^T(t)\chi_{\{\eta_{t-1}=i_2\}}]C^T(i_1,i_2)\}, t \geq s \geq 1, x_0 \in \mathbf{R}^n.$$

Based on Lemma 3.5 we may conclude

$$\lim_{t\to\infty} E[|z(t,s,x_0)|^2] = \sum_{i_1,i_2=1}^{N} p(i_2,i_1)Tr[C(i_1,i_2)Y^{\pi_0}(i_1)C^T(i_1,i_2)], s \geq 1, x_0 \in \mathbf{R}^n.$$

This confirms the validity of the first equality in the statement. The second equality may be proved in the same way as in Theorem 3.1. Thus the proof ends.

3.4 The computation of the H_2-norms for the system of type (1)

The systems described by (1) can be regarded as systems of type (20) in two ways.

First we may transform the system (1) as:

$$(\tilde{\mathbf{G}}) : \begin{cases} \tilde{x}(t+1) = (\tilde{A}_0(\eta_t,\eta_{t-1}) + \sum\limits_{k=1}^{r} w_k(t)\tilde{A}_k(\eta_t,\eta_{t-1}))\tilde{x}(t) + \tilde{B}_v(\eta_t,\eta_{t-1}))v(t) \\ \tilde{z}(t) = \tilde{C}(\eta_t,\eta_{t-1})\tilde{x}(t) \end{cases} \tag{40}$$

$t \geq 1$, where

$$\begin{aligned} \tilde{A}_k(i,j) = A_k(i), 0 \leq k \leq r, \\ \tilde{B}_v(i,j) = B_v(i), \tilde{C}(i,j) = C(i), i,j \in \mathcal{D}. \end{aligned} \tag{41}$$

Also, (1) could be view as system of type (20) as follows:

$$(\hat{\mathbf{G}}) : \begin{cases} \hat{x}(t+1) = [\hat{A}_0(\eta_t,\eta_{t-1}) + \sum\limits_{k=1}^{r} \hat{w}_k(t)\hat{A}_k(\eta_t,\eta_{t-1})]\hat{x}(t) + \hat{B}_v(\eta_t,\eta_{t-1})\hat{v}(t) \\ \hat{z}(t) = \hat{C}(\eta_t,\eta_{t-1})\hat{x}(t), t \geq 1 \end{cases} \tag{42}$$

where

$$\begin{aligned} \hat{A}_k(i,j) = A_k(j), 0 \leq k \leq r, \hat{B}_v(i,j) = B_v(j), \\ \hat{C}(i,j) = C(j), i,j \in \mathcal{D} \\ \hat{x}(t) = x(t-1), \hat{w}_k(t) = w_k(t-1), \hat{v}(t) = v(t-1), t \geq 1. \end{aligned} \tag{43}$$

For each $s \geq 1, x_0 \in \mathbf{R}^n$, let $\tilde{x}(t,s,x_0), \hat{x}(t,s,x_0), x(t,s,x_0)$ be the solutions of (40), (42), (1) respectively, starting from x_0, at $t = s$. It is easy to see that:

$$\tilde{x}(t,s,x_0) = x(t,s,x_0), t \geq s \geq 1, x_0 \in \mathbf{R}^n \tag{44}$$

$$\hat{x}(t,s,x_0) = x(t-1,s-1,x_0), t \geq s \geq 1, x_0 \in \mathbf{R}^n. \tag{45}$$

Further if $\tilde{z}(t,s,x_0) = \tilde{C}(\eta_t,\eta_{t-1})\tilde{x}(t,s,x_0)$, $\hat{z}(t,s,x_0) = \hat{C}(\eta_t,\eta_{t-1})\hat{x}(t,s,x_0)$, $z(t,s,x_0) = C(\eta_t)x(t,s,x_0), t \geq s \geq 1$, then from (41), (43), (44), (45) we have

$$\tilde{z}(t,s,x_0) = z(t,s,x_0), t \geq s \geq 1, x_0 \in \mathbf{R}^n \tag{46}$$

$$\hat{z}(t,s,x_0) = z(t-1,s-1,x_0), t \geq s \geq 1, x_0 \in \mathbf{R}^n. \tag{47}$$

If $\tilde{\Upsilon} : \mathcal{S}_n^N \to \mathcal{S}_n^N, \hat{\Upsilon} : \mathcal{S}_n^N \to \mathcal{S}_n^N$ are the Lyapunov operators associated to system (40), (42), respectively then from (26), (41) and (43) we have:

$$(\tilde{\Upsilon}H)(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(j,i)A_k(i)H(j)A_k^T(i) = (\Lambda H)(i) \tag{48}$$

$$(\hat{\Upsilon}H)(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(j,i)A_k(j)H(j)A_k^T(j) = (\mathcal{L}H)(i) \tag{49}$$

for all $i \in \mathcal{D}, H \in \mathcal{S}_n^N$ where Λ and $\mathcal{L}$ are the Lyapunov type operators associated to the linear system (8) (see [16]).

These two operators play an important role in characterization of the exponential stability in mean square for discrete-time linear systems with independent random perturbations and Markovian jumping.

Using equality (47) and Theorem 3.1 specialized to system $(\hat{\mathbf{G}})$ we obtain:

Theorem 3.3. *Assume:*

a) the assumptions $\mathbf{H}_1) - \mathbf{H}_3)$ *are fulfilled.*

b) the zero state equilibrium of the system (8) is ESMS.

Under these conditions the H_2*-norms of the system (1) defined by (10) and (11) are given by*

(i) $||G||_2^2 = \sum_{i_1,i_2=1}^{N} q^{\pi_0}(i_2)p(i_2,i_1)Tr[\tilde{\mathcal{X}}(i_1)B_v(i_2)B_v^T(i_2)] = \sum_{i=1}^{N} Tr[C(i)\mathcal{Y}^{\pi_0}(i)C^T(i)]$

(ii) $\tilde{||G||}_2^2 = \sum_{i_1,i_2=1}^{N} q^{\mathcal{D}_0}(i_2)p(i_2,i_1)Tr[\tilde{\mathcal{X}}(i_1)B_v(i_2)B_v^T(i_2)] = \sum_{i=1}^{N} Tr[C(i)\mathcal{Y}^{\mathcal{D}_0}(i)C^T(i)]$

where $\tilde{\mathcal{X}} = (\tilde{\mathcal{X}}(1), \tilde{\mathcal{X}}(2), ..., \tilde{\mathcal{X}}(N)) \in \mathcal{S}_n^{N+}$ *is the unique solution of the algebraic equation*

$$\mathcal{X} = \mathcal{L}^*\mathcal{X} + \tilde{\mathcal{C}} \tag{50}$$

where $\tilde{\mathcal{C}} = (\tilde{\mathcal{C}}(1), \tilde{\mathcal{C}}(2), ..., \tilde{\mathcal{C}}(N))$,

$$\tilde{\mathcal{C}}(i) = C^T(i)C(i), i \in \mathcal{D} \tag{51}$$

$\mathcal{Y}^{\pi_0} = (\mathcal{Y}^{\pi_0}(1), \mathcal{Y}^{\pi_0}(2), ..., \mathcal{Y}^{\pi_0}(N)) \in \mathcal{S}_n^{N+}$ *and* $\mathcal{Y}^{\mathcal{D}_0} = (\mathcal{Y}^{\mathcal{D}_0}(1), \mathcal{Y}^{\mathcal{D}_0}(2), ..., \mathcal{Y}^{\mathcal{D}_0}(N))$ $in\mathcal{S}_n^{N+}$ *are the unique solutions of the algebraic equations*

$$\mathcal{Y} = \mathcal{L}\mathcal{Y} + \mathcal{B}^{\pi_0} \tag{52}$$

$$\mathcal{Y} = \mathcal{L}\mathcal{Y} + \mathcal{B}^{\mathcal{D}_0} \tag{53}$$

where $\mathcal{B}^{\pi_0} =)(\mathcal{B}^{\pi_0}(1), \mathcal{B}^{\pi_0}(2), ..., \mathcal{B}^{\pi_0}(N))$,

$$\mathcal{B}^{\pi_0}(i) = \sum_{j=1}^{N} q^{\pi_0}(j)p(j,i)B_v(j)B_v^T(j) \tag{54}$$

and $\mathcal{B}^{\mathcal{D}_0} = (\mathcal{B}^{\mathcal{D}_0}(1), \mathcal{B}^{\mathcal{D}_0}(2), ..., \mathcal{B}^{\mathcal{D}_0}(N))$,

$$\mathcal{B}^{\mathcal{D}_0}(i) = \sum_{j=1}^{N} q^{\mathcal{D}_0}(j)p(j,i)B_v(j)B_v^T(j), i \in \mathcal{D} \tag{55}$$

$q^{\pi_0}(j) = \sum\limits_{i=1}^{N} \pi_0(i)q(i,j)$ *and* $q^{\mathcal{D}_0}(j) = \sum\limits_{i \in \mathcal{D}_0} q(i,j)$.

It must be remarked that if $\mathcal{D}_0 = \mathcal{D}$ then the H_2-norm defined by (11) does not depend upon the initial distribution of the Markov chain.

From Theorem 3.2 we obtain:

Theorem 3.4. *Assume:*

a) Assumptions $\mathbf{H}_1) - \mathbf{H}_4)$ *are fulfilled.*

b) The zero state equilibrium of the system (8) is (ESMS).

c) The transition probability matrix P *is a non-degenerate stochastic matrix.*

d) $\pi_0(i) > 0, 1 \leq i \leq N$.

Under these conditions the H_2*-norm of the system (1) defined by (12) is given by*

$$|||G|||_2^2 = \sum_{j=1}^{N} Tr[C(j)\mathcal{Y}^{\pi_0}(j)C^T(j)] = \sum_{i_1,i_2=1}^{N} q^{\pi_0}(i_2)p(i_2,i_1)Tr[\tilde{\mathcal{X}}(i_1)B_v(i_2)B_v^T(i_2)]$$

where $\mathcal{Y}^{\pi_0}$ *is the unique solution of the equation (52)-(54) while* $\tilde{\mathcal{X}}$ *is the unique solution of the equation (50)-(51) and* q^{π_0} *is defined as before.*

Remark 3.3.

a) In the special case $A_k(i) = 0, 1 \leq k \leq r, i \in \mathcal{D}$ the equality proved in Theorem 3.4 reduces to the one proved in [10]

b)The H_2-norms defined by (10)-(12) in the discrete-time context have analogous in the continuous time framework for linear stochastic systems with multiplicative and additive white noise and Markovian jumping (see [18, 17]).

In the afore mentioned works was shown that the continuous time counterpart of H_2-norms (10) and (12) are well defined under the same assumptions and they coincide. Unlike the continuous time case, in the discrete-time case we proved the well definiteness of H_2-norm defined by (12) under some stronger assumptions than the norm defined by (10). It remains as a challenge for further research to prove the well-possedness of the H_2-norm (12) under weaker assumptions than the ones in Theorem 3.4 from above.

At the end of this subsection we remark that the equality (46) together with Theorem 3.1 and Theorem 3.2 lead to some expressions of the H_2-norms (10)-(12) which do not have a correspondent in the continuous time framework. Thus we have:

Theorem 3.5. *Under the assumptions of Theorem 3.4 the following hold:*

(i) $(||G||_2)^2 = (|||G|||_2)^2 = \sum_{i_1,i_2=1}^{N} q^{\pi_0}(i_2)p(i_2,i_1)Tr[B_v^T(i_1)\overline{\mathcal{X}}(i_1)B_v(i_1)] = \sum_{i_1,i_2=1}^{N} p(i_1,i_2)Tr[C(i_2)\mathcal{Z}^{\pi_0}(i_1)C^T(i_2)]$

(ii) $(\tilde{||}G\tilde{||}_2)^2 = \sum_{i_1,i_2=1}^{N} q^{\mathcal{D}_0}(i_2)p(i_2,i_1)Tr[B_v^T(i_1)\overline{\mathcal{X}}(i_1)B_v(i_1)] = \sum_{i_1,i_2=1}^{N} p(i_1,i_2)Tr[C(i_2)\mathcal{Z}^{\mathcal{D}_0}(i_1)C^T(i_2)]$

where $\overline{\mathcal{X}} = (\overline{\mathcal{X}}(1), \overline{\mathcal{X}}(2), ..., \overline{\mathcal{X}}(N)) \in \mathcal{S}_n^{N+}$ *is the unique solution of the algebraic equation*

$$\mathcal{X} = \Lambda^*\mathcal{X} + \overline{\mathcal{C}} \tag{56}$$

where $\overline{\mathcal{C}} = (\overline{\mathcal{C}}(1), \overline{\mathcal{C}}(2), ..., \overline{\mathcal{C}}(N))$*,*

$$\overline{\mathcal{C}}(i) = \sum_{j=1}^{N} p(i,j)C^T(j)C(j), i \in \mathcal{D} \tag{57}$$

while $\mathcal{Z}^{\pi_0} = (\mathcal{Z}^{\pi_0}(1), \mathcal{Z}^{\pi_0}(2), ..., \mathcal{Z}^{\pi_0}(N))$ *and* $\mathcal{Z}^{\mathcal{D}_0} = (\mathcal{Z}^{\mathcal{D}_0}(1), \mathcal{Z}^{\mathcal{D}_0}(2), ..., \mathcal{Z}^{\mathcal{D}_0}(N))$ *are the unique solutions of the algebraic equations*

$$Z = \Lambda\mathcal{Z} + \mathcal{B}^{\pi_0} \tag{58}$$

$$Z = \Lambda\mathcal{Z} + \mathcal{B}^{\mathcal{D}_0} \tag{59}$$

respectively, with $\mathcal{B}^{\pi_0}$ *and* $\mathcal{B}^{\mathcal{D}_0}$ *are given by (54)-(55).*

Remark 3.4 The result stated in Theorem 3.5 confirms the importance of the consideration of the class of systems with the coefficients depending upon both η_t and η_{t-1}. The study of the H_2 norms for such systems performed in subsection 3.2 and subsection 3.3 allows us to derive new formulae for H_2 norms of system (1). The formulae of H_2 norms derived in Theorem 3.5 are specific to the discrete-time framework; they have not analogous in the continuous time case.

3.5 Some robustness issues

As we can see from Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4, respectively, if $N \geq 2$ the H_2-norms associated to stochastic systems (20), (1) respectively, are strongly dependent upon the initial distributions π_0 of the Markov chain, or upon the subset $\mathcal{D}_0$ of the states i, such that $\mathcal{P}\{\eta_0 = i\} > 0$. Unfortunately, the initial distributions of the Markov chain are not known apriori. To avoid such inconvenient specific to the stochastic systems subject to Markovian jumping, one could made the additional assumption: for each $i \in \mathcal{D}$, $\lim_{t\to\infty} \mathcal{P}\{\eta_t = i\}$ exists and it does not depend upon the initial distribution $\mathcal{P}\{\eta_0 = j\}, j \in \mathcal{D}$.

One can check using (3) that the above assumption is equivalent to the fact that assumption $\mathbf{H}_4$) is fulfilled and additionally the matrix $Q = \lim_{t\to\infty} P^t$ has the property $q(i,j) = q(j), i, j \in \mathcal{D}$.

Another idea to overcome the problems due to the presence of the initial distribution of the Markov chain in the formula of the H_2-norms is to introduce a suitable upper-bound of these norms.

Thus in the case of the system (20) we define

$$(\hat{|}|\mathbf{G}\hat{|}|)_2^2 = \sum_{i_1,i_2=1}^{N} \tilde{q}(i_2)p(i_2,i_1)Tr[B_v^T(i_1,i_2)\tilde{X}(i_1)B_v(i_1,i_2)] \tag{60}$$

where $\tilde{q}(i_2) = \sum_{i_1=1}^{N} q(i_1,i_2)$. We have

$$\begin{aligned} q^{\pi_0}(i_2) &\leq \tilde{q}(i_2) \\ q^{\mathcal{D}_0}(i_2) &\leq \tilde{q}(i_2) \end{aligned} \tag{61}$$

for every initial distribution π_0 and for all subsets $\mathcal{D}_0 \subset \mathcal{D}$. So, under the assumptions of Theorem 3.1 we have:

$$||\mathbf{G}||_2 \leq \hat{|}|\mathbf{G}\hat{|}|_2, \tilde{|}|\mathbf{G}\tilde{|}|_2 \leq \hat{|}|\mathbf{G}\hat{|}|_2. \tag{62}$$

Under the assumptions of Theorem 3.2 also we have

$$|||\mathbf{G}|||_2 \leq \hat{|}|\mathbf{G}\hat{|}|_2. \tag{63}$$

Reasoning as in the proof of Theorem 3.1 we may obtain

$$\begin{aligned} (\hat{|}|\mathbf{G}\hat{|}|_2)^2 &= \sum_{i_1,i_2=1}^{N} \tilde{q}(i_2)p(i_2,i_1)Tr[B_v^T(i_1,i_2)\tilde{X}(i_1)B_v(i_1,i_2)] \\ &= \sum_{i_1,i_2=1}^{N} p(i_1,i_2)Tr[C(i_2,i_1)\tilde{Y}(i_1)C^T(i_2,i_1)] \end{aligned} \tag{64}$$

where $\tilde{X}$ is the solution of (33)-(34), while $\tilde{Y} = (\tilde{Y}(1), \tilde{Y}(2), ..., \tilde{Y}(N)) \in \mathcal{S}_n^{N+}$ is the unique solution of the algebraic equation

$$Y = \Upsilon Y + \tilde{B} \tag{65}$$

with $\tilde{B} = (\tilde{B}(1), \tilde{B}(2), ..., \tilde{B}(N))$,

$$\tilde{B}(i) = \sum_{j=1}^{N} \tilde{q}(j)p(j,i)B_v(i,j)B_v^T(i,j). \tag{66}$$

Using Lemma 3.3 we may prove:

Proposition 3.5. *Under the assumptions in Theorem 3.1*

$$(\hat{|}|\mathbf{G}\hat{|}|_2)^2 = \lim_{l\to\infty} \frac{1}{l} \sum_{t=1}^{l} \sum_{i=1}^{N} E[|z^i(t,1,x_0)|^2]$$

where $z^i(t,1,x_0) = C(\eta_t,\eta_{t-1})x^i(t,1,x_0)$.

$x^i(t,1,x_0), t \geq 1$ being the solution of the system (20) corresponding to the Markov chain with the initial distribution $\mathcal{P}\{\eta_0 = i\} = 1$ and $\mathcal{P}\{\eta_0 = j\} = 0$ if $j \neq i$.

In the case of system (1) the equality (64) becomes:

$$(\hat{||}G\hat{||}_2)^2 = \sum_{i_1,i_2=1}^{N} \tilde{q}(i_2)p(i_2,i_1)Tr[B_v^T(i_2)\tilde{\mathcal{X}}(i_1)B_v(i_2)] = \sum_{i=1}^{N} Tr[C(i)\tilde{\mathcal{Y}}(i)C^T(i)] \tag{67}$$

where $\tilde{\mathcal{X}} = (\tilde{\mathcal{X}}(1), \tilde{\mathcal{X}}(2), ..., \tilde{\mathcal{X}}(N))$ is the unique solution of the equation (50)-(51) and $\tilde{\mathcal{Y}} = (\tilde{\mathcal{Y}}(1), \tilde{\mathcal{Y}}(2), ..., \tilde{\mathcal{Y}}(N)) \in \mathcal{S}_n^{N+}$, is the unique solution of the algebraic equation

$$\mathcal{Y} = \mathcal{L}\mathcal{Y} + \tilde{\mathcal{B}} \tag{68}$$

where $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}(1), \tilde{\mathcal{B}}(2), ..., \tilde{\mathcal{B}}(N))$,

$$\tilde{\mathcal{B}}(i) = \sum_{j=1}^{N} \tilde{q}(j)p(j,i)B_v(j)B_v^T(j) \tag{69}$$

$\tilde{q}(j)$ being as before.

In the process of the designing of a H_2-optimal controller one may add to the list of H_2 performances criteria another one which is asking the minimization of $\hat{||} \cdot \hat{||}_2$ of the closed-loop system.

4 H_2-optimal controllers

In this section we illustrate how the results proved in the previous section can be used to solve the H_2-optimization problems stated before.

In the first part of this section we focus our attention in minimizing the H_2 performance criteria associated to system (13). The general case when only an output of type (17) is available for measurements will be considered in a future paper.

To have an unified approach of the four optimization problems we want to solve we introduce the notation $|| \cdot ||_{2.\mu}, \mu \in \{1,2,3,4\}$ as $|| \cdot ||_{21}$ instead of $|| \cdot ||_2$ defined by (10), $|| \cdot ||_{22}$ instead of $\tilde{||} \cdot \tilde{||}_2$, defined by (11), $|| \cdot ||_{23}$ instead of $||| \cdot |||_2$ defined by (12) and $|| \cdot ||_{24}$ instead of $\hat{||} \cdot \hat{||}_2$ defined by (67).

From Theorem 3.3, Theorem 3.4 and (67)-(69) applied to the closed-loop system (15) we have

$$||G_{cl}||_{2,\mu}^2 = \sum_{i_1,i_2=1}^{N} \varepsilon_\mu(i_2)p(i_2,i_1)Tr[B_{vcl}^T(i_2)\mathcal{X}_{cl}(i_1)B_{vcl}(i_2)] \tag{70}$$

where $\mathcal{X}_{cl} = (\mathcal{X}_{cl}(1), \mathcal{X}_{cl}(2), ..., \mathcal{X}_{cl}(N)) \in \mathcal{S}_{n+n_c}^{N+}$ is the unique solution of the linear equation:

$$\mathcal{X}_{cl}(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)A_{kcl}^T(i)\mathcal{X}_{cl}(j)A_{kcl}(i) + C_{cl}^T(i)C_{cl}(i), i \in \mathcal{D} \tag{71}$$

with $\varepsilon_\mu(i_2) = \begin{cases} q^{\pi_0}(i_2), for \mu \in \{1,3\}; \\ q^{\mathcal{D}_0}(i_2), for \mu = 2; \\ \tilde{q}(i_2), for \mu = 4. \end{cases}$ Consider the system of nonlinear algebraic equations which extends to this framework the well known discrete-time algebraic Riccati equations:

$$X(i) = \sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X)A_k(i) + C_z^T(i)C_z(i) - (\sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X)B_k(i) + C_z^T(i)D_z(i))$$

$$(D_z^T(i)D_z(i) + \sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X)B_k(i))^{-1}(\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X)A_k(i) + D_z^T(i)C_z(i)), i \in \mathcal{D} \tag{72}$$

where

$$\mathcal{E}_i(X) = \sum_{j=1}^{N} p(i,j)X(j) \tag{73}$$

for all $X = (X(1), X(2), ..., X(N)) \in \mathcal{S}_n^N$.

We shall refer to such systems as discrete-time systems of generalized Riccati equations (DTSGRE). We recall that a solution $X_s = (X_s(1), X_s(2), ..., X_s(N))$ of DTSGRE (72) is called stabilizing solution if the zero state equilibrium of the closed-loop system:

$$x_s(t+1) = [A_0(\eta_t) + B_0(\eta_t)F_s(\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + B_k(\eta_t)F_s(\eta_t))]x_s(t), t \geq 0 \tag{74}$$

is ESMS, where

$$F_s(i) = -(D_z^T(i)D_z(i) + \sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_s)B_k(i))^{-1}(\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_s)A_k(i) + D_z^T(i)C_z(i)), i \in \mathcal{D}. \tag{75}$$

A set of sufficient conditions for the existence of a stabilizing solution of DTSGRE (72) were provided in [20] and they are expressed in terms of stochastic stabilizability and stochastic detectability.

In [21] a set of necessary and sufficient conditions for the existence of stabilizing solution of (72) which satisfy

$$D_z^T(i)D_z(i) + \sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_s)B_k(i) > 0, i \in \mathcal{D} \tag{76}$$

are given.

For each controller G_c of type (14) we introduce the following performances

$$J_\mu(G_c) = \sum_{i_1,i_2=1}^{N} \varepsilon_\mu(i_2)p(i_2,i_1)Tr[B_{vcl}^T(i_2)\mathcal{X}_{cl}(i_1)B_{vcl}(i_2)], \mu \in \{1,2,3,4\}. \tag{77}$$

To be sure that (77) is well defined we need only the fact that the controller G_c is stabilizing.

Further under some additional assumptions which are as in Theorem 3.3 and Theorem 3.4, respectively, $J_\mu(G_c)$ will be just the H_2-norm $\|\cdot\|_{2\mu}$ of the corresponding closed-loop system.

In the process of the designing of a H_2-optimal controller we try to minimize $J_\mu(G_c)$ for some $\mu \in \{1,2,3,4\}$.

Now we are in position to prove:

Theorem 4.1. *Assume that (72) has a stabilizing solution* $X_s = (X_s(1), X_s(2), ..., X_s(N))$ *which satisfies (76). Then*

$$\min_{G_c \in \mathcal{K}_s(G)} J_\mu(G_c) = \sum_{i_1,i_2=1}^{N} \varepsilon_\mu(i_2) p(i_2,i_1) Tr[B_v^T(i_2) X_s(i_1) B_v(i_2)], \mu \in \{1,2,3,4\}$$

. The optimal value is achieved for the zero order controller

$$\tilde{G} : u_s(t) = F_s(\eta_t) x_s(t) \tag{78}$$

where $F_s(i), i \in \mathcal{D}$ *are as in (75) and* $x_s(t)$ *is the solution of (74).*

Proof. Let us remark that in the case of the zero order controller (78) the corresponding closed-loop system is:

$$x_{cl}(t+1) = [A_0(\eta_t) + B_0(\eta_t)F_s(\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + B_k(\eta_t)F_s(\eta_t))]x(t) + B_v(\eta_t)v(t) \tag{79}$$

$$z_{cl}(t) = (C_z(\eta_t) + D_z(\eta_t)F_s(\eta_t))x_{cl}(t), t \geq 0.$$

On the other hand, by direct calculation one obtains that DTSGRE (72) verified by X_s can be rewritten as:

$$X_s(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)(A_k(i) + B_k(i)F_s(i))^T X_s(j)(A_k(i) + B_k(i)F_s(i)) +$$
$$(C_z(i) + D_z(i)F_s(i))^T(C_z(i) + D_z(i)F_s(i)), i \in \mathcal{D}. \tag{80}$$

One sees that the linear equation (71) corresponding to the closed-loop system (79) is just (80). Therefore the value of the corresponding performance is

$$J_\mu(\tilde{G}_c) = \sum_{i_1,i_2=1}^{N} \varepsilon_\mu(i_2) p(i_2,i_1) Tr[B_v^T(i_2) X_s(i_1) B_v(i_2)]. \tag{81}$$

Let G_c be an arbitrary admissible controller of type (14). Let $X_{cl}(i) = \begin{pmatrix} X_{11}(i) & X_{12}(i) \\ X_{12}^T(i) & X_{22}(i) \end{pmatrix}$ be a partition of the solution of (71) according to the partition of the coefficients of the closed-loop system.

Using (16) we obtain the following partition of (71):

$$\begin{aligned}X_{11}(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)[(A_k(i)+B_k(i)F_c(i))^T X_{11}(j)(A_k(i)+B_k(i)F_c(i)) + \\ B_{ck}^T(i)X_{12}^T(j)(A_k(i)+B_k(i)F_c(i)) + (A_k(i)+B_k(i)F_c(i))^T X_{12}(j)B_{ck}(i) + \\ B_{ck}^T(i)X_{22}(j)B_{ck}(i)] + [C_z(i)+D_z(i)F_c(i)]^T[C_z(i)+D_z(i)F_c(i)] \quad (82)\\ X_{12}(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)[(A_k(i)+B_k(i)F_c(i))^T X_{11}(j)B_k(i)C_c(i) + \\ B_{ck}^T(i)X_{12}^T(j)B_k(i)C_c(i) + (A_k(i)+B_k(i)F_c(i))^T X_{12}(j)A_{ck}(i) + \\ B_{ck}^T(i)X_{22}(j)A_{ck}(i)] + (C_z(i)+D_z(i)F_c(i))^T D_z(i)C_c(i)\\ X_{22}(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)[C_c^T(i)B_k^T(i)X_{11}(j)B_k(i)C_c(i) + A_{ck}^T(i)X_{12}^T(j)B_k(i)C_c(i) \\ +C_c^T(i)B_k^T(i)X_{12}(j)A_{ck}(i) + A_{ck}^T(i)X_{22}(j)A_{ck}(i)] + C_c^T(i)D_z^T(i)D_z(i)C_c(i).\end{aligned}$$

On the other hand the DTSGRE (72) verified by the stabilizing solution X_s can be rewritten as:

$$\begin{aligned}X_s(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)[A_k(i)+B_k(i)F_c(i)]^T X_s(j)[A_k(i)+B_k(i)F_c(i)] \\ +[C_z(i)+D_z(i)F_c(i)]^T[C_z(i)+D_z(i)F_c(i)] - [F_s(i)-F_c(i)]^T\Delta(i)[F_s(i)-F_c(i)]\end{aligned} \quad (83)$$

where

$$\Delta(i) = D_z^T(i)D_z(i) + \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)B_k^T(i)X_s(j)B_k(i) > 0. \quad (84)$$

Set $\hat{\mathcal{X}}_{cl}(i) = \mathcal{X}_{cl}(i) - \begin{pmatrix} X_s(i) & 0 \\ 0 & 0 \end{pmatrix}, i \in \mathcal{D}$. Subtracting (83) from (82) and taking into account (75) and (84) one obtains that $\hat{\mathcal{X}}_{cl} = (\hat{\mathcal{X}}_{cl}(1), \hat{\mathcal{X}}_{cl}(2), ..., \hat{\mathcal{X}}_{cl}(N))$ is the solution of the following equation:

$$\hat{\mathcal{X}}_{cl}(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)A_{kcl}^T(i)\hat{\mathcal{X}}_{cl}(j)A_{kcl}(i) + \Psi^T(i)\Delta(i)\Psi(i), i \in \mathcal{D} \quad (85)$$

where $\Psi(i) = \begin{pmatrix} F_s(i) - F_c(i) & -C_c(i) \end{pmatrix}$. Since G_c is a stabilizing controller and $\Delta(i) > 0$, it follows that the unique solution of (85) satisfies

$$\hat{\mathcal{X}}_{cl}(i) \geq 0, i \in \mathcal{D}. \quad (86)$$

The value of the performance $J_\mu(G_c)$ from (77) can be rewritten as:

$$\begin{aligned}J_\mu(G_c) = \sum_{i_1,i_2=1}^{N} \varepsilon_\mu(i_2)p(i_2,i_1)Tr[B_v(i_2)X_s(i_1)B_v(i_2)] + \\ \sum_{i_1,i_2=1}^{N} \varepsilon_\mu(i_2)p(i_2,i_1)Tr[B_{vcl}^T(i_2)\hat{\mathcal{X}}_{cl}(i_1)B_{vcl}(i_2)].\end{aligned} \quad (87)$$

Based on (81), (86), (87) one obtains that $J_\mu(G_c) \geq J_\mu(\tilde{G}_c)$ and thus the proof is complete.

Remark 4.1. The result proved in the above theorem shows that in the case of full access to the measurements of the states, the best performance with respect to all four H_2-performance criteria, is provided by the same zero order controller. In fact it is the same state feedback which provides the optimal control in the linear quadratic optimization problem.

In the second part of this section we briefly show how can be solved H_2-optimal control problems for the systems with coefficients depending upon η_t, η_{t-1}.

Let us consider the controlled system:

$$(\mathbf{G}): \begin{cases} x(t+1) = (A_0(\eta_t,\eta_{t-1}) + \sum\limits_{k=1}^{r} w_k(t)A_k(\eta_t,\eta_{t-1}))x(t) + (B_0(\eta_t,\eta_{t-1}) + \\ \qquad \sum\limits_{k=1}^{r} w_k(t)B_k(\eta_t,\eta_{t-1}))u(t) + B_v(\eta_t,\eta_{t-1})v(t) \\ y(t) = x(t) \\ z(t) = C_z(\eta_t,\eta_{t-1})x(t) + D_z(\eta_t,\eta_{t-1})u(t), t \geq 1 \end{cases} \tag{88}$$

The class of admissible controllers consist of the family of dynamic compensators of the form:

$$(\mathbf{G}_c) \begin{cases} x_c(t+1) = (A_{c0}(\eta_t,\eta_{t-1}) + \sum\limits_{k=1}^{r} w_k(t)A_{ck}(\eta_t,\eta_{t-1}))x_c(t) + (B_{c0}(\eta_t,\eta_{t-1}) + \\ \qquad \sum\limits_{k=1}^{r} w_k(t)B_{ck}(\eta_t,\eta_{t-1}))u_c(t) \\ y_c(t) = C_c(\eta_{t-1})x_c(t) + F_c(\eta_{t-1})u_c(t) \\ x_c \in \mathbf{R}^{n_c}, u_c \in \mathbf{R}^n, y_c \in \mathbf{R}^m \end{cases} \tag{89}$$

The fact that the output of the admissible controller depends only upon η_{t-1} is a constraint impose by our technique of the proof of the main result.

It remains an open problem the extension of the family of the admissible controllers to the case of those with the output depending both upon η_t and η_{t-1}.

Coupling (89) with (88), taking $u_c(t) = y(t), u(t) = y_c(t)$ one obtains the following closed-loop system:

$$(\mathbf{G}_{cl}): \begin{cases} x_{cl}(t+1) = (A_{0cl}(\eta_t,\eta_{t-1}) + \sum\limits_{k=1}^{r} w_k(t)A_{kcl}(\eta_t,\eta_{t-1}))x_{cl}(t) + B_{vcl}(\eta_t,\eta_{t-1})v(t) \\ z_{cl}(t) = C_{cl}(\eta_t,\eta_{t-1})x_{cl}(t), t \geq 1 \end{cases} \tag{90}$$

where $x_{cl}(t) = \begin{pmatrix} x^T(t) & x_c^T(t) \end{pmatrix}^T \in \mathbf{R}^{n+n_c}$, $A_{kcl}(i,j) = \begin{pmatrix} A_k(i,j) + B_k(i,j)F_c(j) & B_k(i,j)C_c(j) \\ B_{ck}(i,j) & A_{ck}(i,j) \end{pmatrix}$, $0 \leq k \leq r$, $B_{vcl}(i,j) = \begin{pmatrix} B_v(i,j) \\ 0 \end{pmatrix}$, $C_{cl}(i,j) = \begin{pmatrix} C_z(i,j) + D_z(i,j)F_c(j) & D_z(i,j)C_c(j) \end{pmatrix}$, $i,j \in \mathcal{D}$.

Let us remark if we consider system (13) with a controller of type (14) and a delay occurs on the channel between controllers and actuators (i.e. $y_c(t-1)$ is used instead of $y_c(t)$), then the closed-loop system is of the form (90). Hence it is natural to consider a H_2-control problem for system with coefficients depending upon η_t, η_{t-1}. A such problem is specific to the discrete-time framework. It has not an analogous in the continuous time case.

As in the first part of this section we denote $||\mathbf{G}_{cl}||_{2\mu}, \mu \in \{1,2,3,4\}$ the four type of H_2-norms defined for the closed-loop system by (21), (22), (23) and (64). Based on Theorem 3.1, Theorem 3.2 and equality (64) one deduces that

$$||\mathbf{G}_{cl}||_{2\mu} = \sum_{i_1,i_2=1}^{N} \varepsilon_\mu(i_2)p(i_2,i_1)Tr[B_{vcl}^T(i_1,i_2)X_{cl}(i_1)B_{vcl}(i_1,i_2), \mu \in \{1,2,3,4\} \tag{91}$$

where $\varepsilon_\mu(i_2)$ are defined as before and $X_{cl} = (X_{cl}(1), X_{cl}(2), ..., X_{cl}(N)) \in \mathcal{S}_{n+n_c}^{N+}$ is the unique solution of:

$$X_{cl}(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)A_{kcl}^T(j,i)X_{cl}(j)A_{kcl}(j,i) + \sum_{j=1}^{N} p(i,j)C_{cl}^T(j,i)C_{cl}(j,i), i \in \mathcal{D}. \tag{92}$$

As before we introduce the performances of an admissible controller (89) by

$$J_\mu(\mathbf{G}_c) = \sum_{i_1,i_2=1}^{N} \varepsilon_\mu(i_2)p(i_2,i_1)Tr[B_{vcl}^T(i_1,i_2)X_{cl}(i_1)B_{vcl}(i_1,i_2)]. \tag{93}$$

It must be remarked that to be sure that (93) is well defined we need to know that the assumptions $\mathbf{H}_1) - \mathbf{H}_3)$ are fulfilled and the zero state equilibrium of the linear closed-loop system:

$$x_{cl}(t+1) = [A_{0cl}(\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} w_k(t)A_{kcl}(\eta_t,\eta_{t-1})]x_{cl}(t)$$

is ESMS. As we proceed in the first part of this section we will minimize $H_\mu(\mathbf{G}_{cl})$ in order to obtain the solution of H_2-optimization problem for systems of type (88).

Let us consider the following discrete time system of generalized Riccati equations DTSGRE associated to (88):

$$\begin{aligned} X(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)A_k^T(j,i)X(j)A_k(j,i) + \sum_{j=1}^{N} p(i,j)C_z^T(j,i)C_z(j,i) \\ -[\sum_{j=1}^{N} p(i,j)(C_z^T(j,i)D_z(j,i) + \sum_{k=0}^{r} A_k^T(j,i)X(j)B_k(j,i))][\sum_{j=1}^{N} p(i,j)(D_z^T(j,i)D_z(j,i) \\ +\sum_{k=0}^{r} B_k^T(j,i)X(j)B_k(j,i))]^{-1}[\sum_{j=1}^{N} p(i,j)(D_z^T(j,i)C_z(j,i) + \sum_{k=0}^{r} B_k^T(j,i)X(j)A_k(j,i))]. \end{aligned} \tag{94}$$

A solution $X_s = (X_s(1), X_s(2), ..., X_s(N))$ of DTSGRE (94) is called stabilizing solution if the zero state equilibrium of the corresponding closed-loop system

$$\begin{aligned} x_s(t+1) = [A_0(\eta_t,\eta_{t-1}) + B_0(\eta_t,\eta_{t-1})F_s(\eta_{t-1}) + \\ \sum_{k=1}^{r} w_k(t)(A_k(\eta_t,\eta_{t-1}) + B_k(\eta_t,\eta_{t-1})F_s(\eta_{t-1}))]x_s(t) \end{aligned} \tag{95}$$

is ESMS where

$$\begin{aligned} F_s(i) = -[\sum_{j=1}^{N} p(i,j)(D_z^T(j,i)D_z(j,i) + \sum_{k=0}^{r} B_k^T(j,i)X(j)B_k(j,i))]^{-1} \\ \cdot[\sum_{j=1}^{N} p(i,j)(D_z^T(j,i)C_z(j,i) + \sum_{k=0}^{r} B_k^T(j,i)X(j)A_k(j,i))]. \end{aligned} \tag{96}$$

A set of conditions which are equivalent with the existence of a stabilizing solution of DTSGRE (94) with the additional property:

$$\sum_{j=1}^{N} p(i,j)(D_z^T(j,i)D_z(j,i)+\sum_{k=1}^{r} B_k^T(j,i)X_s(j)B_k(j,i)>0, i\in\mathcal{D} \tag{97}$$

can be found in [21]. Those conditions are expressed in terms of solvability of some suitable systems of LMI (see Appendix A6 from below).

Remark 4.2 One can sees that if system (88) is in the special case of (42)-(43) then DTSGRE (94) reduces to (72) and the corresponding stabilizing feedback gain (96) reduces to (75).

The next result provides the solution of the H_2-optimal control problems associated to the systems (88).

Theorem 4.2. *Assume that DTSGRE (94) has a stabilizing solution $X_s=(X_s(1),X_s(2),...,X_s(N))$ which satisfy the condition (97). Then*

$$\min_{\mathbf{G}_c\in\mathcal{K}_s(\mathbf{G})} J_\mu(\mathbf{G}_c)=\sum_{i_1,i_2=1}^{N}\varepsilon_\mu(i_2)p(i_2,i_1)Tr(B_v^T(i_1,i_2)X_s(i_1)B_v(i_1,i_2)).$$

The optimal value is achieved by the zero order controller

$$\tilde{\mathbf{G}}_c: u_s(t)=F_s(\eta_{t-1})x_s(t)$$

where $F_s(i), i\in\mathcal{D}$ are constructed in (96) and $x_s(t)$ is the solution of the closed loop system (95).

The proof is similar to the one of Theorem 4.1. It is omitted for shortness.

Remark 4.3. Due to the important role played by the stabilizing solutions of DTSGREs (72) and (94), respectively, in construction of the optimal controller in the H_2-control problems it follows that it is important to have efficient numerical procedures for computation of the stabilizing solutions. In Theorem 4.2 in [21] an iterative procedure based on Newton-Kantorovich algorithm was proposed to prove existence of the maximal solution and consequently of the stabilizing solution. That iterative procedure could be used in order to compute the stabilizing solution of (72) and (94), respectively. However, a procedure based on Newton-Kantorovich method consists in solving linear systems of high dimension at each iteration.

Therefore, it is useful to obtain numerical procedures based on solutions of some Stein equations as it happens in Kleiman procedure known in the deterministic framework. A such procedure will be provided in a future paper.

5 Appendix

For each $(t,s)\in\mathbf{Z}_+\times\mathbf{Z}_+$, we denote

$$\check{\mathcal{H}}_{t,s}=\sigma[\eta_\mu,\check{w}(\nu);0\le\mu\le t,0\le\nu\le s]$$

where either $\check{w}(\nu)=w(\nu)$ or $\check{w}(\nu)=(w(\nu),v(\nu)),\nu\ge 0$.

In the special case $t=s$ we write $\check{\mathcal{H}}_t$ instead of $\check{\mathcal{H}}_{tt}$. It is obvious that $\check{\mathcal{H}}_t=\mathcal{H}_t$ if $\check{w}(\nu)=w(\nu)$ and $\check{\mathcal{H}}_t=\hat{\mathcal{H}}_t$ if $\check{w}(\nu)=(w(\nu),v(\nu)),\nu\ge 0$. the next result can be proved following step by step the proof of Lemma 7.1 in [16].

Lemma A1. *Under the assumptions* $\mathbf{H_1}) - \mathbf{H_3})$ *if* $\Psi : \Omega \to \mathbf{R}$ *is an integrable random variable which is measurable with respect to* $\sigma[\eta_\mu, \breve{w}(\nu); \mu \geq t, \nu \geq s+1]$ *then*

$$E[\Psi|\breve{\mathcal{H}}_{ts}] = E[\Psi|\eta_t], \quad a.s.$$

From the previous lemma one obtains directly:

Corollary A1. *Under the assumptions* $\mathbf{H}_1) - \mathbf{H}_3)$ *the following equality holds:*

$$E[\chi_{\{\eta_{t+1}=j\}}|\breve{\mathcal{H}}_t] = E[\chi_{\{\eta_{t+1}=j\}}|\eta_t] = p(\eta_t, j) \quad a.s.$$

for all $j \in \mathcal{D}, t \geq 0$, *where* $\breve{\mathcal{H}}_t = \mathcal{H}_t$ *or* $\breve{\mathcal{H}}_t = \hat{\mathcal{H}}_t$.

It must be remarked that equality in the previous Corollary extends (2) to the joint process $\{\eta_t, w(t)\}_{t\geq 0}$ *or* $\{\eta_t, w(t), v(t)\}_{t\geq 0}$.

A1. Proof of Lemma 3.1.

First we write

$$x^T(t+1)H(\eta_t)x(t+1) = x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)A_0(\eta_t,\eta_{t-1})x(t) +$$
$$\sum_{k,l=1}^{r} w_k(t)w_l(t)x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)A_l(\eta_t,\eta_{t-1})x(t) + v^T(t)B_v^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})v(t)$$
$$+2\sum_{k=1}^{r} w_k(t)x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)A_k(\eta_t,\eta_{t-1})x(t) + 2x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})v(t)$$
$$+2\sum_{k=1}^{r} w_k(t)x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})v(t). \tag{98}$$

If we take into account that $x(t)$ is $\hat{\mathcal{H}}_{t-1}$-measurable, $\hat{\mathcal{H}}_{t-1} \subset \tilde{\mathcal{H}}_t$ and $w_k(t), v(t)$ are independent of $\tilde{\mathcal{H}}_t$ one obtains

$$E[x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)A_0(\eta_t,\eta_{t-1})x(t)|\tilde{\mathcal{H}}_t] =$$
$$x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)A_0(\eta_t,\eta_{t-1})x(t) \tag{99}$$
$$E[\sum_{k,l=1}^{r} w_k(t)w_l(t)x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)A_l(\eta_t,\eta_{t-1})x(t)|\tilde{\mathcal{H}}_t] =$$
$$\sum_{k,l=1}^{r} x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)A_l(\eta_t,\eta_{t-1})x(t)E[w_k(t)w_l(t)|\tilde{\mathcal{H}}_t] =$$
$$\sum_{k,l=1}^{r} x^T(t)A_k^T(\eta_t\eta_{t-1})H(\eta_t)A_l(\eta_t,\eta_{t-1})x(t)E[w_k(t)w_l(t)].$$

Based on $\mathbf{H}_1$) one obtains:

$$E[\sum_{k,l=1}^{r} w_k(t)w_l(t)x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)A_l(\eta_t,\eta_{t-1})x(t)|\tilde{\mathcal{H}}_t] = \sum_{k=1}^{r} x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)A_k(\eta_t,\eta_{t-1})x(t) \tag{100}$$

$$\begin{aligned} E[\sum_{k=1}^{r} w_k(t)x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)A(\eta_t,\eta_{t-1})x(t)|\tilde{\mathcal{H}}_t] = \\ \sum_{k=1}^{r} x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)A_k(\eta_t,\eta_{t-1})x(t)E[w_k(t)|\tilde{\mathcal{H}}_t] = \\ \sum_{k=1}^{r} x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)A_k(\eta_t,\eta_{t-1})x(t)E[w_k(t)] \end{aligned}$$

Invoking again the assumption $\mathbf{H}_1$) we conclude:

$$E[\sum_{k=1}^{r} w_k(t)x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)A_k(\eta_t,\eta_{t-1})x(t)|\tilde{\mathcal{H}}_t] = 0 \tag{101}$$

$$\begin{aligned} E[\sum_{k=1}^{r} w_k(t)x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})v(t)|\tilde{\mathcal{H}}_t] = \\ \sum_{k=1}^{r} x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})E[w_k(t)v(t)|\tilde{\mathcal{H}}_t] = \\ \sum_{k=1}^{r} x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})E[w_k(t)v(t)]. \end{aligned}$$

Based on the assumptions $\mathbf{H}_1) - \mathbf{H}_3)$ we deduce:

$$E[\sum_{k=1}^{r} w_k(t)x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})v(t)|\tilde{\mathcal{H}}_t] = 0. \tag{102}$$

Similarly

$$E[x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})v(t)|\tilde{\mathcal{H}}_t] = 0. \tag{103}$$

Invoking again $\mathbf{H}_3$) we write:

$$\begin{aligned} E[v^T(t)B_v^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})v(t)|\tilde{\mathcal{H}}_t] = \\ E[Tr(B_v^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})v(t)v^T(t))|\tilde{\mathcal{H}}_t] = \\ Tr[B_v^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})E[v(t)v^T(t)|\tilde{\mathcal{H}}_t]] = \\ Tr[B_v^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})E[v(t)v^T(t)]] = \\ Tr[B_v^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})] \end{aligned} \tag{104}$$

Combining (98)-(104) one obtains

$$E[x^T(t+1)H(\eta_t)x(t+1)|\tilde{\mathcal{H}}_t] = \sum_{k=0}^{r} x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)A_k(\eta_t,\eta_{t-1})x(t) + Tr[B_v^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})] \tag{105}$$

Further taking the conditional expectation with respect to $\hat{\mathcal{H}}_{t-1}$ in (105) one obtains:

$$E[x^T(t+1)H(\eta_t)x(t+1)|\hat{\mathcal{H}}_{t-1}] =$$

$$\sum_{k=0}^{r}\sum_{j=1}^{N} A_k^T(j,\eta_{t-1})H(j)A_k(j,\eta_{t-1})x(t)E[\chi_{\{\eta_t=j\}}|\hat{\mathcal{H}}_{t-1}] +$$

$$Tr[\sum_{j=1}^{N} H(j)B_v(j,\eta_{t-1})B_v^T(j,\eta_{t-1})E[\chi_{\{\eta_t=j\}}|\hat{\mathcal{H}}_{t-1}]]. \tag{106}$$

Applying Corollary A1 one obtains

$$E[\chi_{\{\eta_t=j\}}|\hat{\mathcal{H}}_{t-1}] = E[\chi_{\{\eta_t=j\}}|\eta_{t-1}] = p(\eta_{t-1},j) \quad a.s. \tag{107}$$

Combining (106)-(107) and taking the conditional expectation with respect to $\sigma[\eta_{s-1}] \subset \hat{\mathcal{H}}_{t-1}$ we obtain the equality in the statement and thus the proof is complete.

A2. Proof of Lemma 3.3.

Under the considered assumptions the linear equation (33)-(34) has a unique solution $\tilde{X} = (\tilde{X}(1), \tilde{X}(2), ..., \tilde{X}(N)) \in \mathcal{S}_n^{N+}$ (see also Proposition 3.2). Applying Lemma 3.1 for $H(i) = \tilde{X}(i)$ one obtains for $i \in \mathcal{D}_0$

$$E[x^T(t+1)\tilde{X}(\eta_t)x(t+1)|\eta_0 = i] = E[x^T(t)(\Upsilon^*\tilde{X})(\eta_{t-1})x(t)|\eta_0 = i)]$$

$$+\sum_{j=1}^{N} E[Tr[\tilde{X}(j)B_v(j,\eta_{t-1})B_v^T(j,\eta_{t-1})]p(\eta_{t-1},j)|\eta_0 = i]$$

$\forall x(t) = x(t,1,x_0)$ solution of (20) with the initial value x_0 at $t = 1$.

Based on (33) we deduce

$$E[x^T(t+1)\tilde{X}(\eta_t)x(t+1)|\eta_0 = i] - E[x^T(t)\tilde{X}(\eta_{t-1})x(t)|\eta_0 = i] =$$

$$-E[x^T(t)\tilde{C}(\eta_{t-1})x(t)|\eta_0 = i] + \sum_{j=1}^{N} E[Tr[\tilde{X}(j)B_v(j,\eta_{t-1})B_v^T(j,\eta_{t-1})]p(\eta_{t-1},j)|\eta_0 = i]$$

where $\tilde{C}(i)$ is defined in (34).

Further we have

$$E[x^T(t+1)\tilde{X}(\eta_t)x(t+1)|\eta_0 = i] - E[x^T(t)\tilde{X}(\eta_{t-1})x(t)|\eta_0 = i] =$$

$$-E[x^T(t)\tilde{C}(\eta_{t-1})x(t)|\eta_0 = i] + \sum_{j,i_2=1}^{N} Tr[\tilde{X}(j)B_v(j,i_2)B_v^T(j,i_2)]p(i_2,j)p^{t-1}(i,i_2) \tag{108}$$

where $p^{t-1}(i,i_2)$ is an element of P^{t-1}. On the other hand,

$$E[|C(\eta_t,\eta_{t-1})x(t)|^2|\hat{\mathcal{H}}_{t-1}] = \sum_{j=1}^{N} E[|C(j,\eta_{t-1})x(t)\chi_{\{\eta_t=j\}}|^2|\hat{\mathcal{H}}_{t-1}]$$

$$= \sum_{j=1}^{N} |C(j,\eta_{t-1})x(t)|^2 E[\chi_{\{\eta_t=j\}}|\hat{\mathcal{H}}_{t-1}]. \tag{109}$$

Using again Corollary A1 one deduces that

$$E[\chi_{\{\eta_t=j\}}|\hat{\mathcal{H}}_{t-1}] = p(\eta_{t-1}, j). \tag{110}$$

Combining (109)-(110) together with (34) we may write

$$E[|C(\eta_t, \eta_{t-1})x(t)|^2|\hat{\mathcal{H}}_{t-1}] = x^T(t)\tilde{C}(\eta_{t-1})x(t).$$

Taking the conditional expectation with respect to the event $\{\eta_0 = i\}$ in the last equality and replacing the obtained result in (108) we have

$$E[|C(\eta_t, \eta_{t-1})x(t)|^2|\eta_0 = i] = \sum_{j,i_2=1}^{N} Tr[\tilde{X}(j)B_v(j, i_2)B_v^T(j, i_2)]p(i_2, j)p^{t-1}(i, i_2)+$$

$$E[x^T(t)\tilde{X}(\eta_{t-1})x(t)|\eta_0 = i] - E[x^T(t+1)\tilde{X}(\eta_t)x(t+1)|\eta_0 = i].$$

Thus

$$\begin{aligned} &\frac{1}{l}\sum_{t=1}^{l} E[|C(\eta_t, \eta_{t-1})x(t)|^2|\eta_0 = i] = \\ &\sum_{i_1,i_2=1}^{N} [Tr[\tilde{X}(i_1)B_v(i_1, i_2)B_v^T(i_1, i_2)]p(i_2, i_1)\frac{1}{l}\sum_{t=1}^{l} p^{t-1}(i, i_2)] \\ &+\frac{1}{l}[x_0^T\tilde{X}(i)x_0 - E[x^T(l+1)\tilde{X}(\eta_l)x(l+1)|\eta_0 = i]] \end{aligned} \tag{111}$$

Based on Remark 3.1 we obtain $E[x^T(l+1)\tilde{X}(\eta_l)x(l+1)|\eta_0 = i] \leq \hat{\gamma}\frac{1}{\pi_0(i)}(1 + |x_0|^2)$. Therefore

$$\lim_{l\to\infty} \frac{1}{l}[x_0^T\tilde{X}(i)x_0 - E[x^T(l+1)\tilde{X}(\eta_l)x(l+1)|\eta_0 = i] = 0. \tag{112}$$

On the other hand from Proposition 3.3 we obtain

$$\lim_{l\to\infty} \frac{1}{l}\sum_{t=1}^{l} p^{t-1}(i, i_2) = q(i, i_2). \tag{113}$$

Taking the limit for $l \to \infty$ in (111) and taking into account (112)-(113) one obtains the equality in the statement and thus the proof is complete.

A3. Some representation formulae

The next result is a special version of the Theorem 2.2 in [19].

Proposition A1. *Under the assumptions* $\mathbf{H}_1) - \mathbf{H}_2)$ *the following equality holds:*

$$((\Upsilon^*)^{t-s}H)(i) = E[\Theta^T(t, s)H(\eta_{t-1})\Theta(t, s)|\eta_{s-1} = i]$$

for all $H = (H(1), H(2), ..., H(N)) \in \mathcal{S}_n^N$, $t \geq s \geq 1, i \in \mathcal{D}_{s-1}$.

The equality from the above proposition together with (24) and the definition of the adjoint operator allows us to obtain:

Corollary A2. *Assume that:*

a)$\mathbf{H}_1) - \mathbf{H}_2)$ *are fulfilled.*
b) The transition probability matrix P is a non-degenerate stochastic matrix.
c) $\pi_0(i) > 0, i \in \mathcal{D}$.
Then the following representation formula hold:

$$(\Upsilon^{t-s}S)(j) = \sum_{i=1}^{N} E[\Theta(t,s)S(i)\Theta^T(t,s)\chi_{\{\eta_{t-1}=j\}}|\eta_{s-1} = i] \tag{114}$$

for all $j \in \mathcal{D}$, *and* $S = (S(1), S(2), ..., S(N)) \in \mathcal{S}_n^N$.

A4. Proof of Lemma 3.4.

Based on the assumptions $\mathbf{H}_1) - \mathbf{H}_3)$ we may write

$$\begin{aligned} &E[\Theta(t,l+1)B_v(\eta_l,\eta_{l-1})B_v^T(\eta_l,\eta_{l-1})\Theta^T(t,l+1)\chi_{\{\eta_{t-1}=j\}}|\mathcal{H}_l] = \\ &\sum_{i_1,i_2=1}^{N} E[\chi_{\{\eta_l=i_1\}}\chi_{\{\eta_{l-1}=i_2\}}\Theta(t,l+1)B_v(i_1,i_2)B_v^T(i_1,i_2)\Theta^T(t,l+1)\chi_{\{\eta_{t-1}=j\}}|\mathcal{H}_l] \\ &= \sum_{i_1,i_2=1}^{N} \chi_{\{\eta_l=i_1\}}\chi_{\{\eta_{l-1}=i_2\}}E[\Theta(t,l+1)B_v(i_1,i_2)B_v^T(i_1,i_2)\Theta^T(t,l+1)\chi_{\{\eta_{t-1}=j\}}|\mathcal{H}_l] \end{aligned} \tag{115}$$

for all $s \le l \le t-1$.

Since $\Theta(t,l+1), \chi_{\{\eta_{t-1}=j\}}$ are measurable with respect to $\sigma[\eta_{s_1}, w(s_2); s_1 \ge l, s_2 \ge l+1]$, we obtain from Lemma A1 that

$$\begin{aligned} &E[\Theta(t,l+1)B_v(i_1,i_2)B_v^T(i_1,i_2)\Theta^T(t,l+1)\chi_{\{\eta_{t-1}=j\}}|\mathcal{H}_l] = \\ &E[\Theta(t,l+1)B_v(i_1,i_2)B_v^T(i_1,i_2)\Theta^T(t,l+1)\chi_{\{\eta_{t-1}=j\}}|\eta_l]. \end{aligned} \tag{116}$$

Using (116)and (115) and taking into account the second equality in Lemma 3.2 we have

$$\begin{aligned} &E[x_0(t,s)x_0^T(t,s)\chi_{\{\eta_{t-1}=j\}}] = \\ &\sum_{l=s}^{t-1}\sum_{i_1,i_2=1}^{N} \mathcal{P}\{\eta_{l-1}=i_2, \eta_l=i_1\}E[\Theta(t,l+1)B_v(i_1,i_2)B_v^T(i_1,i_2)\Theta^T(t,l+1)\chi_{\{\eta_{t-1}=j\}}|\eta_l=i_1] \\ &= \sum_{l=s}^{t-1}\sum_{i_1,i_2=1}^{N} \mathcal{P}\{\eta_{l-1}=i_2\}p(i_2,i_1)E[\Theta(t,l+1)B_v(i_1,i_2)B_v^T(i_1,i_2)\Theta^T(t,l+1)\chi_{\{\eta_{t-1}=j\}}|\eta_l=i_1]. \end{aligned} \tag{117}$$

From (3) one obtains

$$\mathcal{P}\{\eta_{l-1} = i_2\} = \sum_{i=1}^{N} \pi_0(i)p^{l-1}(i,i_2) \tag{118}$$

where $p^{l-1}(i,i_2), 1 \le i, i_2 \le N$ are the elements of P^{l-1}. Using (118) in (117), one gets:

$$E[x_0(t,s)x_0^T(t,s)\chi_{\{\eta_{t-1}=j\}}] = \sum_{l=s}^{t-1}\sum_{i_1,i_2=1}^{N} E[\Theta(t,l+1)H_l(i_1)\Theta^T(t,l+1)\chi_{\{\eta_{t-1}=j\}}|\eta_l=i_1] \tag{119}$$

where $H_l(i_1)$ is defined in the statement.

Now conclusion follows transforming the right hand side of (119) using the representation formula (114) for $S = H_l$. Thus the proof is complete.

A5. The proof of Lemma 3.5.

If the assumption $\mathbf{H}_4$) is fulfilled it follows that $\lim_{l\to\infty} H_l(i_1) = B^{\pi_0}(i_1), \forall i_1 \in \mathcal{D}$.

Using the equality proved in Lemma 3.4 we may write successively

$$\begin{aligned}
E[x_0(t,s)x_0^T(t,s)\chi_{\{\eta_{t-1}=j\}}] = \sum_{l=s}^{t-1}[(\Upsilon^{t-l-1}H_l)(j) = \\
\sum_{l=s}^{t-1}[(\Upsilon^{t-l-1}B^{\pi_0})(j) + \sum_{l=s}^{t-1}(\Upsilon^{t-l-1}(H_l - B^{\pi_0}))(j) = \\
\sum_{l=0}^{t-s-1}(\Upsilon^l B^{\pi_0})(j) + \sum_{l=s}^{t-1}(\Upsilon^{t-l-1}(H_l - B^{\pi_0}))(j).
\end{aligned} \tag{120}$$

From the assumption b) in the statement we deduce firstly that

$$\lim_{t\to\infty}\sum_{l=0}^{t-s-1}(\Upsilon^l B^{\pi_0})(j) = \sum_{l=0}^{\infty}(\Upsilon^l B^{\pi_0})(j) = Y^{\pi_0}(j). \tag{121}$$

Also from assumption b) we deduce that there exists $\beta \geq 1, q \in (0,1)$ such that

$$\|\Upsilon^l\|_1 \leq \beta q^l, \forall l \geq 0 \tag{122}$$

where $\|\cdot\|_1$ is the norm induced by $|\cdot|_1$.

If $|M|$ is the spectral norm of a symmetric matrix then based on definition of $|\cdot|_1$ in section 3.1 we deduce

$$|\sum_{l=s}^{t-1}[\Upsilon^{t-l-1}(H_l - B^{\pi_0})](j)| \leq |\sum_{l=s}^{t-l-1}\Upsilon^{t-l-1}(H_l - B^{\pi_0})|_1 \leq \sum_{l=s}^{t-1}\|\Upsilon^{t-l-1}\|_1|H_l - B^{\pi_0}|_1.$$

Further (122) allows us to write

$$|\sum_{l=s}^{t-1}[\Upsilon^{t-l-1}(H_l - B^{\pi_0})](j)| \leq \sum_{l=s}^{t-1}\beta q^{t-l-1}|H_l - B^{\pi_0}|_1. \tag{123}$$

Since $\lim_{l\to\infty}|H_l - B^{\pi_0}|_1 = 0$ and $q \in (0,1)$ one obtains from (123) that

$$\lim_{t\to\infty}\sum_{l=s}^{t-1}[\Upsilon^{t-l-1}(H_l - B^{\pi_0})](j) = 0. \tag{124}$$

Taking the limit for $t \to \infty$ in (120) and using (121), (124) one obtains

$$\lim_{t\to\infty} E[x_0(t,s)x_0^T(t,s)\chi_{\{\eta_{t-1}=j\}}] = Y^{\pi_0}(j) \tag{125}$$

$\forall j \in \mathcal{D}, s \geq 1$.

Further the representation formula (29) together with the assumption b) in the statement allows us to write

$$E[|x(t,s,x_0) - x_0(t,s)|^2] \leq \beta q^{t-s}|x_0|^2$$

$f\forall t \geq s \geq 1, x_0 \in \mathbf{R}^n$, where $\beta \geq 1, q \in (0,1)$.

Hence

$$\lim_{t\to\infty} E[x(t,s,x_0)x^T(t,s,x_0)\chi_{\{\eta_{t-1}=j\}}] = \lim_{t\to\infty} E[x_0(t,s)x_0^T(t,s)\chi_{\{\eta_{t-1}=j\}}] \quad (126)$$

for all $t \geq s \geq 1, x_0 \in \mathbf{R}^n$. The equality in the statement follows now from (125), (126) and thus the proof ends.

A6. Stabilizing solution of DTSGRE (94)

In this subsection we briefly show how we can use the result proved in [21] to obtain a set of conditions which guarantee the existence of a stabilizing solution of DTSGRE (94). We remark that in the special case of (43) the conclusions derived for (94) provide conditions which are equivalent to the existence of the stabilizing solution of DTSGRE (72).

Consider the system

$$x(t+1) = [A_0(\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} w_k(t)A_k(\eta_t,\eta_{t-1})]x(t) + [B_0(\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} w_k(t)B_k(\eta_t,\eta_{t-1})]u(t) \quad (127)$$

obtained from (88) by taking $B_v(i,j) = 0$.

Definition A1. We say that the system (127) is stochastic stabilizable if there exist $F = (F(1), F(2), ..., F(N)), F(i) \in \mathbf{R}^{m\times n}, i \in \mathcal{D}$ such that the zero state equilibrium of the closed loop system:

$$x(t+1) = [A_0(\eta_t,\eta_{t-1}) + B_0(\eta_t,\eta_{t-1})F(\eta_{t-1}) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t,\eta_{t-1}) + B_k(\eta_t,\eta_{t-1})F(\eta_{t-1}))]x(t) \quad (128)$$

$t \geq 1$ is ESMS.

Let $\Upsilon_F : \mathcal{S}_n^N \to \mathcal{S}_n^N$ be the Lyapunov type operator associated to (128). Using (26) we have $\Upsilon_F H = (\Upsilon_F H(1), \Upsilon_F H(2), ..., \Upsilon_F H(N))$,

$$\Upsilon_F H(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(j,i)[A_k(i,j) + B_k(i,j)F(j)]H(j)[A_k(i,j) + B_k(i,j)F(j)]^T \quad (129)$$

for all $H \in \mathcal{S}_n^N, i \in \mathcal{D}$.

Using Corollary 4.8 in [19] and some Schur complement techniques one obtains the following criteria for stochastic stabilizablity:

Lemma A2. *Under the assumptions* $\mathbf{H_1}) - \mathbf{H_2})$ *the following are equivalent:*

(i) The system (127) is stochastic stabilizable.

(ii) There exist $F = (F(1), F(2), ..., F(N)), F(i) \in \mathbf{R}^{m\times n}, i \in \mathcal{D}, X = (X(1), X(2), ..., X(N)) \in \mathcal{S}_n^N, X(i) > 0, i \in \mathcal{D}$ *which solve:*

$$\Upsilon_F X(i) - X(i) < 0, i \in \mathcal{D}. \quad (130)$$

(iii) There exist $X = (X(1), X(2), ..., X(N)) \in \mathcal{S}_n^N, \Gamma = (\Gamma(1), \Gamma(2), ..., \Gamma(N)), \Gamma(i) \in \mathbf{R}^{m\times n}, i \in \mathcal{D}$ *which solve the following system of LMIs:*

$$\begin{pmatrix} -X(i) & \mathcal{M}_0(i) & \mathcal{M}_1(i) & ... & \mathcal{M}_r(i) \\ \mathcal{M}_0^T(i) & -\mathbf{X} & 0 & ... & 0 \\ \mathcal{M}_1^T(i) & 0 & -\mathbf{X} & ... & 0 \\ ... & ... & ... & ... & ... \\ \mathcal{M}_r^T(i) & 0 & 0 & ... & -\mathbf{X} \end{pmatrix} < 0 \tag{131}$$

$i \in \mathcal{D}$ *where* $\mathcal{M}_k(i) = (\sqrt{p(1,i)}(A_k(i,1)X(1)+B_k(i,1)\Gamma(1)) \quad \sqrt{p(2,i)}(A_k(i,2)X(2)+B_k(i,2)\Gamma(2)) \quad ... \quad \sqrt{p(N,i)}(A_k(i,N)X(N)+B_k(i,N)\Gamma(N))), \quad 0 \le k \le r, \quad \mathbf{X} = diag(X(1), X(2), ...X(N)) \in \mathbf{R}^{n\times N}$.

Moreover if (X, Γ) *is a solution of (131) then* $F(i) = \Gamma(i)X^{-1}(i), i \in \mathcal{D}$ *provide a stabilizing feedback gain for (127).*

The adjoint operator of Υ_F with respect to the inner product (24) is given by: $\Upsilon_F^* H = ((\Upsilon_F^* H)(1), (\Upsilon_F^* H)(2), ..., (\Upsilon_F^* H)(N))$,

$$(\Upsilon_F^* H)(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)(A_k(j,i) + B_k(j,i)F(i))^T H(j)(A_k(j,i) + B_k(j,i)F(i)). \tag{132}$$

One sees that

$$(\Upsilon_F^* H)(i) = \begin{pmatrix} I_n \\ F(i) \end{pmatrix}^T \begin{pmatrix} (\Pi_1 H)(i) & (\Pi_2 H)(i) \\ ((\Pi_2 H)(i))^T & (\Pi_3 H)(i) \end{pmatrix} \begin{pmatrix} I_n \\ F(i) \end{pmatrix} \tag{133}$$

where

$$(\Pi_1 H)(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)A_k^T(j,i)H(j)A_k(j.i)$$

$$(\Pi_2 H)(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)A_k^T(j,i)H(j)B_k(j,i)$$

$$(\Pi_3 H)(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)B_k^T(j,i)H(j)B_k(j.i)$$

$i \in \mathcal{D}, H \in \mathcal{S}_n^N$. Setting $\Pi_k H = ((\Pi_k H)(1), (\Pi_k H)(2), ..., (\Pi_k H)(N))$ we may define the operator $\Pi : \mathcal{S}_n^N \to \mathcal{S}_{n+m}^N$ by

$$\Pi H = \begin{pmatrix} \Pi_1 H & \Pi_2 H \\ (\Pi_2 H)^T & \Pi_3 H \end{pmatrix} \tag{134}$$

here we use the convention of notation $B^T = (B^T(1), B^T(2), ..., B^T(N))$ if $B = (B(1), B(2), ..., B(N))$ with $B(i) \in \mathbf{R}^{n\times m}, i \in \mathcal{D}$. Using the above operators the equation (94) can be rewritten in a compact form as:

$$X = \Pi_1 X + M - (\Pi_2 X + L)(R + \Pi_3 X)^{-1}(\Pi_2 X + L)^T \tag{135}$$

where $M = (M(1), M(2), ..., M(N))$,

$$M(i) = \sum_{j=1}^{N} p(i,j)C_z^T(j,i)C_z(j,i),$$

$L = (L(1), L(2), ..., L(N))$,

$$L(i) = \sum_{j=1}^{N} p(i,j) C_z^T(j,i) D_z(j,i),$$

$R = (R(1), R(2), ..., R(N))$,

$$R(i) = \sum_{j=1}^{N} p(i,j) D_z^T(j,i) D_z(j,i), i \in \mathcal{D}.$$

Hence (135) is the time invariant version of the nonlinear equation investigated in [21]. Also the equalities (133), (134) show that the system (127) is stochastic stabilizable iff the linear positive operator Π is stabilizable in the sense of Definition 2.3 in [21].

With the above notations we may introduce the so called dissipation operator associated to (135): $\mathbf{D} : \mathcal{S}_n^N \to \mathcal{S}_{n+m}^N$ by

$$(\mathbf{D}X)(i) = \begin{pmatrix} (\Pi_1 X)(i) + M(i) - X(i) & (\Pi_2 X)(i) + L(i) \\ ((\Pi_2 X)(i) + L(i))^T & (\Pi_3 X)(i) + R(i) \end{pmatrix} \tag{136}$$

$i \in \mathcal{D}, X \in \mathcal{S}_n^N$.

The following is the time invariant version of Theorem 5.4 in [21].

Theorem A1. *Under the assumptions* $\mathbf{H_1}) - \mathbf{H_2})$ *the following are equivalent:*

(i) The system (127) is stochastic stabilizable and there exist $\hat{X} = (\hat{X}(1), \hat{X}(2), ..., \hat{X}(N)) \in \mathcal{S}_n^N$ *such that*

$$(\mathbf{D}\hat{X})(i) < 0, \qquad i \in \mathcal{D}. \tag{137}$$

(ii) The DTSGRE (94) has a stabilizing solution X_s *which satisfies (96).*

We remark that a set of conditions equivalent with the existence of a stabilizing solution of (94) which verify condition (96) consist of the solvability of the systems of LMIs (131) and (137).

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