The linear quadratic optimization problem for a class of discrete-time stochastic linear systems

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Abstract

In this paper the problem of the optimization of a quadratic cost functional along the trajectories of a discrete-time affine stochastic system affected by jumping Markov perturbations and independent random perturbations is investigated. Both the case of finite time horizon as well as the infinite time horizon are considered. The optimal control is constructed using a suitable solution for a system of discrete-time Riccati type equations. A set of sufficient conditions assuring the existence of the desired solutions of the discrete-time Riccati equations is provided. A tracking problem is also solved.

Keywords: linear quadratic problems, tracking problems, discrete time stochastic systems, Markov chains, independent random perturbations.

1 Introduction

The state space approach for the problem of minimization of a quadratic cost functional along the trajectories of a linear (affine) controlled system has a long history. Such an optimization problem is usually known as linear quadratic optimization problem (LQOP). Starting with the pioneer work of Kalman [32] the solution of such an optimization problem is closely related to the existence of some suitable solutions of a matrix differential (difference) equations of Riccati type.

In the continuous time stochastic framework different aspects of the LQOP were investigated in [2, 16, 30, 43, 44] for the case of controlled systems described by Ito differential equations and [9, 29, 35, 37, 42, 44] in the case of controlled systems described by differential equations with Markov jumping. In [45] the linear quadratic optimization problem for stochastic systems and cost functionals with indefinite sign was considered and solved.

In [18, 22] it was shown that the solution of the linear quadratic optimization problem strongly depends upon the class of admissible controls. It was proved that for a given cost functional and for different sets of admissible controls the solution of the optimization problem is constructed either using the maximal solution or the minimal solution of the corresponding Riccati differential equation.

In the discrete-time stochastic framework the linear quadratic optimization problem was separately investigated for systems with independent random perturbations and systems with Markov perturbations, respectively. Thus, for the case of discrete-time stochastic systems with independent random

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perturbations we refer to [40, 38, 46], while for discrete-time systems with Markovian switching we mention [1, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14, 12, 27, 28, 29, 31, 35, 36, 37, 39, 42].

In the present paper we consider an optimization problem asking the minimization of a quadratic cost functional along the trajectories of a discrete-time affine stochastic system subject to Markovian switching and independent random perturbations. Both the case of finite time horizon and the case of infinite time horizon are studied. Sufficient conditions ensuring the existence of the solutions for a corresponding system of a discrete-time Riccati type equations involved in the construction of the optimal control are given. As an application we give the solution of a tracking problem.

Lately there is an increasing interest in the investigation of different control problems related to discrete-time linear stochastic systems corrupted by independent random perturbations and Markovian switching. For the readers convenience we refer to a recent paper [15] where several optimization problems having the cost functional given by the terminal value of the expectation or of the variance of an output are studied.

The outline of our paper is as follows: Section 2 contains the description of the mathematical model under consideration as well as the setting of the optimization problem under investigation. The main results are in section 3. Thus, after some auxiliary results given in subsection 3.1 the solution of LQOP in the finite time horizon is given in subsection 3.2. Subsection 3.3 contains a detailed investigation of the problem of the existence of the stabilizing solution for the system of discrete-time Riccati type equations associated to the considered optimization problem. Finally, in subsection 3.4 we provide the solution of LQOP in the infinite time horizon case. Section 4 contains the solution of a tracking problem for a given reference signal. In section 5 we collect the proofs of several results used in the previous sections but also interesting in themselves.

2 Problem formulation

Let us consider the discrete-time controlled stochastic linear system described by:

$$\begin{aligned} x(t+1) = A_0(t,\eta_t)x(t) + B_0(t,\eta_t)u(t) + f_0(t,\eta_t) + \\ + \sum_{k=1}^{r}[A_k(t,\eta_t)x(t) + B_k(t,\eta_t)u(t) + f_k(t,\eta_t)]w_k(t), \quad t \geq 0, t \in \mathbf{Z}, \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbf{R}^n$, is the state vector and $u(t) \in \mathbf{R}^m$ is the vector of control inputs, $\{\eta_t\}_{t\geq 0}$ is a Markov chain defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with the state space the finite set $\mathcal{D} = \{1, 2, ..., N\}$ and the sequence of transition probability matrices $\{P_t\}_{t\geq 0}$. This means that for $t \geq 0$, P_t are stochastic matrices of size N, with the property:

$$\mathcal{P}\{\eta_{t+1} = j \mid \mathcal{G}_t\} = p_t(\eta_t, j) \tag{2.2}$$

for all $j \in \mathcal{D}$, $t \geq 0$, $t \in \mathbf{Z}_+$, where $\mathcal{G}_t = \sigma[\eta_0\ \eta_1,\ \dots, \eta_t]$ is the σ-algebra generated by the random variables $\eta_s, 0 \leq s \leq t$, $\{w(t)\}_{t\geq 0}$ is a sequence of independent random vectors $w(t) = (w_1(t), ..., w_r(t))^T$.

It can be seen that if $\mathcal{P}\{\eta_t = i\} > 0$ then

$$\mathcal{P}\{\eta_{t+1} = j \mid \eta_t = i\} = p_t(i,j) \tag{2.3}$$

$p_t(i,j)$ being the entries of the transition probability matrix P_t. For more details regarding the properties of Markov chains and sequences $\{P_t\}_{t\geq 0}$ of stochastic matrices we refer to [17].

If $P_t = P$ for all $t \geq 0$ then the Markov chain is known as a *homogeneous Markov chain*.

For each $t \in \mathbf{Z}_+$, $A_k(t,i) \in \mathbf{R}^{n\times n}$, $B_k(t,i) \in \mathbf{R}^{n\times m}$ and $f_k(t,i) \in \mathbf{R}^n$ are known.

Different properties of the sequences $\{A_k(t,i)\}_{t\geq 0}$, $\{B_k(t,i)\}_{t\geq 0}$, $\{f_k(t,i)\}_{t\geq 0}$, $0 \leq k \leq r$, will be emphasized later.

Together with the σ-algebras $\mathcal{G}_t$ we introduce the following new σ-algebras $\mathcal{F}_t = \sigma[w(s), 0 \le s \le t]$, $\mathcal{H}_t = \mathcal{G}_t \vee \mathcal{F}_t$, $\tilde{\mathcal{H}}_t = \mathcal{H}_{t-1} \vee \sigma[\eta_t]$ if $t \ge 1$ and $\tilde{\mathcal{H}}_t = \mathcal{G}_0$ if $t = 0$.

We recall that if $\mathcal{F}$ and $\mathcal{G}$ are two σ-algebras then $\mathcal{F} \vee \mathcal{G}$ stands for the smallest σ-algebra containing $\mathcal{F}$ and $\mathcal{G}$. Throughout the paper the following assumptions regarding the processes $\{\eta_t\}_{t\ge 0}$and $\{w(t)\}_{t\ge 0}$ are made:

$\mathbf{H_1}$) The processes $\{\eta_t\}_{t\ge 0}$ and $\{w(t)\}_{t\ge 0}$ are independent stochastic processes.

$\mathbf{H_2}$) $E[w(t)] = 0$, $t \ge 0$, $E[w(t)w^T(t)] = I_r,\ t \ge 0, I_r$ being the identity matrix of size r.

$\mathbf{H_3}$) For each $t \ge 0,\ P_t$ is a nondegenerate stochastic matrix. We recall that a stochastic matrix P_t is a nondegenerate stochastic matrix if for every $j \in \mathcal{D}$, there exists $i \in \mathcal{D}$ such that $p_t(i,j) > 0$.

In terms of σ-algebras the assumption $\mathbf{H_1}$ asserts that for each $t \in \mathbf{Z}_+$, $\mathcal{F}_t$ is independent of $\mathcal{G}_t$.

The following two classes of admissible controls will be involved in the paper.

a) If $0 \le t_0 < t_f \in \mathbf{Z}$, $\mathcal{U}_{t_0,t_f}$ consists of the stochastic processes $u = \{u(t), t_0 \le t \le t_f\}$, where $u(t)$ is a m-dimensional random vector with finite second moments and $\tilde{\mathcal{H}}_t$- measurable.

b) If $t_f = \infty$ and $x_0 \in \mathbf{R}^n$, $\mathcal{U}_{t_0,\infty}(x_0)$ consists of all stochastic processes $u = u(t), t_0 \le t < \infty$ where for each t, $u(t)$ is a m-dimensional random vector which is $\tilde{\mathcal{H}}_t$-measurable having the following two additional properties:

$$\begin{aligned} &\alpha) \quad E[|u(t)|^2] < \infty, \quad t \ge t_0 \\ &\beta) \quad \sup_{t \ge t_0} E[|x_u(t,t_0,x_0)|^2] < \infty \end{aligned} \tag{2.4}$$

$x_u(\cdot, t_0, x_0)$ being the solution of (2.1) determined by the control u and starting from x_0 at $t = t_0$.

It must be remarked that in the case $t_f < +\infty$ the initial value x_0 does not play any role in the definition of the admissible controls $\mathcal{U}_{t_0 t_f}$. On the other hand in the infinite time horizon case ($t_f = +\infty$) it is expected that the set of admissible controls be dependent upon the initial state x_0. This could happened due to condition (2.4).

That is why the dependence with respect to the initial state x_0 is emphasized writing $\mathcal{U}_{t_0,\infty}(x_0)$.

We associate the following two cost functionals to the system (2.1): $J(t_0, t_f, x_0, \cdot) : \mathcal{U}_{t_0,t_f} \to \mathbf{R}$ and $J(t_0, \infty, x_0, \cdot) : \mathcal{U}_{t_0,\infty}(x_0) \to \bar{\mathbf{R}}$ by

$$J(t_0, t_f, x_0, u) = E[x^T(t_f)K_f(\eta_{t_f})x(t_f) + \sum_{t=t_0}^{t_f-1} |y(t)|^2] \tag{2.5}$$

$$J(t_0, \infty, x_0, u) = \overline{\lim_{T\to\infty}} \frac{1}{T - t_0} \sum_{t=t_0}^{T} E[|y(t)|^2] \tag{2.6}$$

where

$$y(t) = C(t, \eta_t)x_u(t, t_0, x_0) + D(t, \eta_t)u(t) \tag{2.7}$$

and $x_u(t, t_0, x_0)$ is as before.

Now we are in position to formulate the two optimization problems which are solved in this paper.

OP 1. *Given $0 \le t_0 < t_f \in \mathbf{Z}$ and $x_0 \in \mathbf{R}^n$, find an admissible control $\tilde{u} \in \mathcal{U}_{t_0,t_f}$ which satisfies $J(t_0, t_f, x_0, \tilde{u}) \le J(t_0, t_f, x_0, u)$ for all $u \in \mathcal{U}_{t_0,t_f}$.*

OP 2. *Given $t_0 \ge 0, x_0 \in \mathbf{R}^n$ find a control $\tilde{u} \in \mathcal{U}_{t_0,\infty}(x_0)$ such that $J(t_0, \infty, x_0, \tilde{u}) < \infty$ and $J(t_0, \infty, x_0, \tilde{u}) \le J(t_0, \infty, x_0, u)$ for all $u \in \mathcal{U}t_0, \infty(x_0)$.*

In the case of the cost functional (2.6) it is not known that there exists $u \in \mathcal{U}_{t_0,\infty}(x_0)$ such that

$$J(t_0, \infty, x_0, u) < +\infty. \tag{2.8}$$

That is why it is natural to introduce the following definition:

Definition 2.1

We say that the optimization problem **OP 2** is well posed if for every $x_0 \in \mathbf{R}^n$ and $t_0 \in \mathbf{Z}_+$ there exists $u \in \mathcal{U}_{t_0,\infty}(x_0)$ such that (2.8) is fulfilled.

In the construction of the optimal control $\tilde{u}$ in the above optimization problems a crucial role is played by the solutions of the following system of discrete time stochastic generalized Riccati equations (DTSGRE):

$$X(t,i) = \sum_{k=0}^{r} A_k^T(t,i)\Pi_i(t, X(t+1))A_k(t,i) + C^T(t,i)C(t,i) -$$
$$-[\sum_{k=0}^{r} A_k^T(t,i)\Pi_i(t, X(t+1))B_k(t,i) + C^T(t,i)D(t,i)][D^T(t,i)D(t,i) +$$
$$\sum_{k=0}^{r} B_k^T(t,i)\Pi_i(t, X(t+1))B_k(t,i)]^{-1}[\sum_{k=0}^{r} B_k^T(t,i)\Pi_i(t, X(t+1))A_k(t,i) + D^T(t,i)C(t,i)] \tag{2.9}$$

where

$$\Pi_i(t,Y) = \sum_{j=1}^{N} p_t(i,j)Y(j) \tag{2.10}$$

for all $Y = (Y(1), ..., Y(N))$.

For the problem **OP 1** we need the solution of (2.9) with the terminal condition $X(t_f, i) = K_f(i)$, while in the case of problem **OP 2** a global solution of (2.9), called stabilizing solution, is involved.

Later we will provide a set of conditions which guarantee the existence of the above mentioned solutions of (2.9). Finally we mention that (2.9) contains different types of discrete-time Riccati equations involved in the solution of the linear quadratic optimization problems, as particular cases, and H_2-control problems both in deterministic framework and stochastic framework [1, 4, 27, 28, 29, 30, 38, 39, 43, 44, 46].

3 The solution of the optimization problems

3.1 Some preliminaries:

Let $V : Z_+ \times \mathbf{R}^n \times \mathcal{D} \to \mathbf{R}$ be a quadratic function in x of the form $V(t,x,i) = x^T X(t,i)x + 2x^T \kappa(t,i)$ where $X(t,i) = X^T(t,i) \in \mathbf{R}^{n\times n}$ and $\kappa(t,i) \in \mathbf{R}^n$ are given. For the readers convenience we recall the following result (see [21]):

Lemma 3.1 Under the assumption $\mathbf{H_1}$ and $\mathbf{H_2}$ we have

$$E[\chi_{\{\eta_{t+1}=j\}}|\mathcal{H}_t] = p_t(\eta_t, j) \tag{3.1}$$

a.s. for all $t \in \mathbf{Z}_+$ and $j \in \mathcal{D}$.

We remark that (3.1) is the extension of (2.2) to the joint process $\{w(t), \eta_t\}_{t\geq 0}$.

Now we prove:

Lemma 3.2 *Let $0 \leq t_0 < t_f$ and $u \in \mathcal{U}_{t_0,t_f}$. If $x(t)$, $t_0 \leq t \leq t_f$ is a trajectory of the system (2.1) determined by the input u with $x(t_0)$ $\tilde{\mathcal{H}}_{t_0}$-measurable and $E|x(t_0)|^2 < \infty$, then we have:*

$$E[V(t+1, x(t+1), \eta_{t+1})|\eta_{t_0}] = E[W(t, x(t), u(t), \eta_t)|\eta_{t_0}] \tag{3.2}$$

for all $t_0 \le t \le t_f - 1$, *where* $W(t,x,u,i) = \sum_{k=0}^{r}\{x^T A_k^T(t,i)\Pi_i(t,X(t+1))A_k(t,i)x + u^T B_k^T(t,i)\Pi_i(t,X(t+1))B_k(t,i)u + f_k^T(t,i)\Pi_i(t,X(t+1))f_k(t,i) + 2x^T A_k^T(t,i)\Pi_i(t,X(t+1))B_k(t,i)u + 2x^T A_k^T(t,i)\Pi(t,X(t+1))f_k(t,i) + 2u^T B_k^T(t,i)\Pi(t,X(t+1))f_k(t,i)\} + 2[A_0(t,i)x + B_0(t,i)u + f_0(t,i)]^T \Pi_i(t,\kappa(t+1))$ *with* $\Pi_i(t,X(t+1))$ *and* $\Pi_i(t,\kappa(t+1))$ *are defined as in (2.10) with* $X(t+1,i)$ *and* $\kappa(t+1,i)$ *instead of* $Y(i)$.

Proof. Taking into-account that $x(t+1)$ is $\mathcal{H}_t$- measurable we have via Lemma 3.1: $E[V(t+1,x(t+1),j)\chi_{\{\eta_{t+1}=j\}}|\mathcal{H}_t] = V(t+1,x(t+1),j)E[\chi_{\{\eta_{t+1}=j\}}|\mathcal{H}_t] = V(t+1,x(t+1),j)p_t(\eta_t,j)$ a.s.

Since $\tilde{\mathcal{H}}_t \subseteq \mathcal{H}_t$ and η_t is $\tilde{\mathcal{H}}_t$-measurable, we get

$$E[V(t+1,x(t+1),j)\chi_{\{\eta_{t+1}=j\}}|\tilde{\mathcal{H}}_t] = p_t(\eta_t,j)E[V(t+1,x(t+1),j)|\tilde{\mathcal{H}}_t] a.s. \tag{3.3}$$

On the other hand from the assumption $\mathbf{H_1}$ and $\mathbf{H_2}$ one obtains that: $E[w_k(t)|\tilde{\mathcal{H}}_t] = E[w_k(t)] = 0$ and $E[w_k(t)w_l(t)|\tilde{\mathcal{H}}_t] = E[w_k(t)w_l(t)] = \delta_{kl}$ where $\delta_{kl} = 1$ if $k = l$ and $\delta_{kl} = 0$ if $k \neq l$. Thus from (2.1), (3.3) and the equality

$$V(t+1,x(t+1),\eta_{t+1}) = \sum_{j=1}^{N} V(t+1,x(t+1),j)\chi_{\{\eta_{t+1}=j\}} \tag{3.4}$$

we obtain:

$$E[V(t+1,x(t+1),\eta_{t+1})|\tilde{\mathcal{H}}_t] = W(t,x(t),u(t),\eta_t). \tag{3.5}$$

Since $\sigma[\eta_{t_0}] \subset \tilde{\mathcal{H}}_t$ the conclusion follows taking the conditional expectation with respect to η_{t_0} in (3.5). Thus the proof is complete.

From the above lemma we obtain:

Corollary 3.3 Under the assumptions of Lemma 3.2 we have the following equality:

$$E[V(t_f,x(t_f),\eta_{t_f})|\eta_{t_0}] + \sum_{t=t_0}^{t_f-1} E[|C(t,\eta_t)x(t) + D(t,\eta_t)u(t)|^2|\eta_{t_0}] =$$

$$= E[V(t_0,x(t_0),\eta_{t_0})|\eta_{t_0}] + \sum_{t=t_0}^{t_f-1} E[(x^T(t),1,1,u^T(t)]\tilde{W}(t,\eta_t)(x^T(t),1,1,u^T(t))^T|\eta_{t_0}] \tag{3.6}$$

where

$$\tilde{W}(t,i) = \begin{pmatrix} W_{11}(t,i) & W_{12}(t,i) & W_{13}(t,i) & \mathcal{G}_i(t,X(t+1)) \\ W_{12}^T(t,i) & W_{22}(t,i) & W_{23}(t,i) & W_{24}(t,i) \\ W_{13}^T(t,i) & W_{23}^T(t,i) & 0 & W_{34}(t,i) \\ \mathcal{G}_i^T(t,X(t+1)) & W_{24}^T(t,i) & W_{34}^T(t,i) & \mathcal{R}_i(t,X(t+1)) \end{pmatrix} \tag{3.7}$$

with

$W_{11}(t,i) = \sum_{k=0}^{r} A_k^T(t,i)\Pi_i(t,X(t+1))A_k(t,i) - X(t,i) + C^T(t,i)C(t,i)$

$W_{12}(t,i) = \sum_{k=0}^{r} A_k^T(t,i)\Pi_i(t,X(t+1)f_k(t,i)$

$W_{13}(t,i) = A_0^T(t,i)\Pi_i(t,\kappa(t+1)) - \kappa(t,i)$

$\mathcal{G}_i(t,X(t+1)) = C^T(t,i)D(t,i) + \sum_{k=0}^{r} A_k^T(t,i)\Pi_i(t,X(t+1))B_k(t,i)$

$W_{22}(t,i) = \sum_{k=0}^{r} f_k^T(t,i)\Pi_i(t,X(t+1))f_k(t,i)$

$W_{23}(t,i) = f_0^T(t,i)\Pi_i(t,\kappa(t+1))$

$W_{24}(t,i) = \sum_{k=0}^{r} f_k^T(t,i)\Pi_i(t,X(t+1))B_k(t,i)$

$W_{34}(t,i) = \Pi_i^T(t,\kappa(t+1))B_0(t,i)$

$\mathcal{R}_i(t,X(t+1)) = D^T(t,i)D(t,i) + \sum_{k=0}^r B_k^T(t,i)\Pi_i(t,X(t+1))B_k(t,i)$.

If $X(t) = (X(t,1),...,X(t,N))$, $t_0 \le t \le t_f$ are such that $\mathcal{R}_i(t,X(t,i))$ are invertible then we define:

$$F^X(t,i) = -\mathcal{R}_i^{-1}(t,X(t+1))\mathcal{G}_i^T(t,X(t+1)), \quad i \in \mathcal{D}. \tag{3.8}$$

For each sequence $\{X(t)\}_{t\ge t_0}$, $X(t) = (X(t,1),...,X(t,N))$ such that $\mathcal{R}_i(t,X(t+1))$ is invertible, we associate the following backward affine equation:

$$\kappa(t,i) = (A_0(t,i) + B_0(t,i)F^X(t,i))^T\Pi_i(t,\kappa(t+1)) + g(t,i) \tag{3.9}$$

where $g(t,i) = \sum_{k=0}^r (A_k(t,i) + B_k(t,i)F^X(t,i))^T\Pi_i(t,X(t+1))f_k(t,i)$,$i \in \mathcal{D}$,$t_0 \le t \le t_f - 1$.

Now we recall the following straightforward fact:

Lemma 3.4 Let $\mathcal{F}(\xi,u) = (\xi^T u^T)\begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_2 \\ \mathcal{M}_2^T & \mathcal{M}_3 \end{pmatrix}\begin{pmatrix} \xi \\ u \end{pmatrix}$ be a quadratic form where $det\mathcal{M}_3 \ne 0$.

Then

$$\mathcal{F}(\xi,u) = \xi^T(\mathcal{M}_1 - \mathcal{M}_2\mathcal{M}_3^{-1}\mathcal{M}_2^T)\xi + (u + \mathcal{M}_3^{-1}\mathcal{M}_2^T\xi)^T\mathcal{M}_3(u + \mathcal{M}_3^{-1}\mathcal{M}_2^T\xi). \tag{3.10}$$

Combining Corollary 3.3 and Lemma 3.4 one obtains:

Proposition 3.5 Let $X(t) = (X(t,1),...,X(t,N))$ be a solution of the system of Riccati type equations (2.9). Let $\kappa(t) = (\kappa(t,1),...,\kappa(t,N))$, $\kappa(t,i) \in \mathbf{R}^n$ be a solution of the corresponding backward affine equation (3.9). We have:

$$E[V(t_f,x(t_f),\eta_{t_f})|\eta_{t_0}] + \sum_{t=t_0}^{t_f-1} E[|y(t)|^2|\eta_{t_0}] = E[V(t_0,x(t_0),\eta_{t_0})|\eta_{t_0}] + \sum_{t=t_0}^{t_f-1} E[\mu(t,\eta_t)|\eta_{t_0}]$$

$$+\sum_{t=t_0}^{t_f-1} E[(u(t) - F^X(t,\eta_t)x(t) - \psi(t,\eta_t))^T\mathcal{R}_{\eta_t}(t,X(t+1))(u(t) - F^X(t,\eta_t)x(t) - \psi(t,\eta_t))|\eta_{t_0}]$$

for all $u \in \mathcal{U}_{t_0,t_f}$, with

$$\begin{aligned}\mu(t,i) = \sum_{k=0}^r f_k^T(t,i)\Pi_i(t,X(t+1))f_k(t,i) - (\Pi_i^T(t,\kappa(t+1))B_0(t,i) + \\ \sum_{k=0}^r f_k^T(t,i)\Pi_i(t,X(t+1))B_k(t,i))\mathcal{R}_i^{-1}(t,X(t+1))(B_0^T(t,i)\Pi_i(t,\kappa(t+1)) + \\ \sum_{k=0}^r B_k^T(t,i)\Pi_i(t,X(t+1))f_k(t,i)) + 2f_0^T(t,i)\Pi_i(t,\kappa(t,i))\end{aligned} \tag{3.11}$$

$$\psi(t,i) = -\mathcal{R}_i^{-1}(t,X(t+1))[B_0^T(t,i)\Pi_i(t,\kappa(t+1)) + \sum_{k=0}^r B_k^T(t,i)\Pi_i(t,X(t+1))f_k(t,i)] \tag{3.12}$$

and $F^X(t,i)$ is as in (3.8), $x(t)$ being the solution with the properties as in Lemma 3.2 .

The proof follows immediately applying Lemma 3.4 with $\mathcal{M}_3 = \mathcal{R}_{\eta_t}(t,X(t+1))$. It must be remarked that if $X(t)$ is a solution of (2.9) then it is tacitly assumed that $\mathcal{R}_i(t,X(t+1))$ is invertible.

3.2 Solution of the problem OP1

With regards to the optimization problem **OP1**, we prove:

Theorem 3.6 *Assume that in the cost functional (2.5) we have:*

a) $K_f(i) \geq 0, \;\; i \in \mathcal{D}$.

b) $D^T(t,i)D(t,i) > 0, \; t_0 \leq t \leq t_f - 1, \; i \in \mathcal{D}$.

Let $\hat{X}(t) = (\hat{X}(t,1), ..., \hat{X}(t,N))$ *be the solution of the system (2.9) which verifies the terminal condition* $\hat{X}(t_f, i) = K_f(i), \; i \in \mathcal{D}$.

Let $\hat{\kappa}(t) = (\hat{\kappa}(t,1), ..., \hat{\kappa}(t,N))$ *be the solution of the corresponding backward affine equation (3.9), with the terminal condition* $\hat{\kappa}(t_f, i) = 0, \;\; i \in \mathcal{D}$. *Under these conditions the optimal control in the optimization problem* **OP1** *is given by*

$$\hat{u}(t) = \hat{F}(t, \eta_t)\hat{x}(t) + \hat{\psi}(t, \eta_t) \tag{3.13}$$

where $\hat{F}(t,i) = F^{\hat{X}}(t,i)$ *and* $\hat{\psi}(t,i)$ *is as in (3.12) with* $(\hat{X}(t,i), \hat{\kappa}(t,i))$ *instead of* $(X(t,i), \kappa(t,i))$ *and* $\hat{x}(t)$ *is a solution of the problem with given initial values:*

$$\begin{aligned} x(t+1) &= [A_0(t,\eta_t) + B_0(t,\eta_t)\hat{F}(t,\eta_t)]x(t) + \hat{f}_0(t,\eta_t) + \\ &+ \sum_{k=1}^{r} w_k(t)[(A_k(t,\eta_t) + B_k(t,\eta_t)\hat{F}(t,\eta_t))x(t) + \hat{f}_k(t,\eta_t)] \end{aligned} \tag{3.14}$$

$x(t_0) = x_0$ *and* $\hat{f}_k(t,i) = B_k(t,i)\hat{\psi}(t,i) + f_k(t,i), 0 \leq k \leq r$.

The optimal value is

$$J(t_0, t_f, x_0, \hat{u}) = \sum_{l=1}^{N} \pi_{t_0}(l)\{x_0^T \hat{X}(t_0, l)x_0 + 2x_0^T \hat{\kappa}(t_0, l) + \sum_{t=t_0}^{t_f-1}\sum_{j=1}^{N} p^t(l,j)\hat{\mu}(t,j)\} \tag{3.15}$$

where $\pi_{t_0}(l) = \mathcal{P}\{\eta_{t_0} = l\}$ *is the distribution of the Markov chain and* $p^t(l,j)$ *are the elements of the matrix* $P^t = P_{t_0}P_{t_0+1}...P_{t-1}$, $\hat{\mu}(t,i)$ *being as in (3.11).*

Proof. First we show that under assumptions a) and b) the solution $\hat{X}(t)$ is well defined for $t_0 \leq t \leq t_f - 1$ and $\hat{X}(t,i) \geq 0$.

To this end we remark that for every $i \in \mathcal{D}$ the right hand side of (2.9) is the Schur complement of the 2×2 block of the matrix

$$\Delta_i(t, X(t+1)) = \begin{pmatrix} \Delta_{1i}(t, X(t+1)) & \mathcal{G}_i(t, X(t+1)) \\ \mathcal{G}_i^T(t, X(t+1)) & \mathcal{R}(t, X(t+1)) \end{pmatrix}$$

where

$$\Delta_{1i}(t, X(t+1)) = \sum_{k=0}^{r} A_k^T(t,i)\Pi_i(t, X(t+1))A_k(t,i) + C^T(t,i)C(t,i)$$

$\mathcal{G}_i(t, X(t+1)), \mathcal{R}_i(t, X(t+1))$ are as before.

It is easy to check that

$$\begin{aligned} \Delta_i(t, X(t+1)) = \sum_{k=0}^{r} &\begin{pmatrix} A_k^T(t,i) \\ B_k^T(t,i) \end{pmatrix} \Pi_i(t, X(t+1)) \begin{pmatrix} A_k(t,i) & B_k(t,i) \end{pmatrix} \\ &+ \begin{pmatrix} C^T(t,i) \\ D^T(t,i) \end{pmatrix} \begin{pmatrix} C(t,i) & D(t,i). \end{pmatrix} \end{aligned} \tag{3.16}$$

From (3.16) it follows that $\Delta_i(t, X(t+1)) \geq 0$ if $X(t+1, j) \geq 0$ for all $j \in \mathcal{D}$.

From the assumptions a) and b) in the statement one obtains that $\mathcal{R}_i(t_f - 1, \hat{X}(t_f)) > 0$.

Based on the Schur complement technique one obtains that

$$\hat{X}(t_f - 1) = \Delta_{1i}(t_f - 1, \hat{X}(t_f)) - \mathcal{G}_i(t_f - 1, \hat{X}(t_f))\mathcal{R}_i^{-1}(t_f - 1, \hat{X}(t_f))\mathcal{G}_i^T(t_f - 1, \hat{X}(t_f)) \geq 0.$$

Further, by induction, one proves that

$$\hat{X}(t,i) = \Delta_{1i}(t, \hat{X}(t+1)) - \mathcal{G}_i(t, \hat{X}(t+1))\mathcal{R}_i^{-1}(t, \hat{X}(t+1))\mathcal{G}_i^T(t, \hat{X}(t+1)) \geq 0.$$

Since $\mathcal{R}_i(t, \hat{X}(t+1)) > 0$ $\hat{X}(t,i)$ is well defined and $\hat{X}(t,i) \geq 0$ for all $t_0 \leq t \leq t_f$.

On the other hand $\hat{\kappa}(t,i)$ is well defined as solution of (3.9). Applying Proposition 3.5 for the pair $(\hat{X}(t,i), \hat{\kappa}(t,i))$ and taking the expectation one gets:

$$J(t_0, t_f, x_0, u) = E[x_0^T \hat{X}(t_0, \eta_{t_0})x_0 + 2x_0^T \hat{\kappa}(t_0, \eta_{t_0})] + \sum_{t=t_0}^{t_f-1} E[\hat{\mu}(t, \eta_t)] + \sum_{t=t_0}^{t_f-1} E[(u(t) - \hat{u}(t))^T \mathcal{R}_{\eta_t}(t, \hat{X}(t+1))(u(t) - \hat{u}(t))]. \tag{3.17}$$

where $\hat{u}(t)$ is given by (3.13).

The fact that $\hat{u} \in \mathcal{U}_{t_0,t_f}$ follows from its formula. From (3.17) we deduce that

$$J(t_0, t_f, x_0, u) \geq J(t_0, t_f, x_0, \hat{u}) = E[V(t_0, x_0, \eta_{t_0})] + \sum_{t=t_0}^{t_f-1} E[\hat{\mu}(t, \eta_t)].$$

The fact that $\mathcal{R}_{\eta_t}(t, \hat{X}(t+1)) > 0$ if $\hat{X}(t,i) \geq 0$ was used. (3.17) also shows that $\hat{u}$ is the unique optimal control. To obtain the optimal value of the cost functional, we write:

$$\sum_{t=t_0}^{t_f-1} E[\hat{\mu}(t, \eta_t)] = \sum_{t=t_0}^{t_f-1} \sum_{j=1}^{N} \pi_t(j)\hat{\mu}(t,j) \tag{3.18}$$

where $\pi_t(j) = \mathcal{P}\{\eta_t = j\}$.

Set $\pi_t = (\pi_t(1), ..., \pi_t(N))$. It is known that $\pi_{t+1} = \pi_t P_t$, for all $t \geq 0$.

Hence $\pi_t = \pi_{t_0} P_{t_0} P_{t_0+1} ... P_{t-1}$, or $\pi_t = \pi_{t_0} P^t$, P^t being as in the statement. This leads to

$$\pi_t(j) = \sum_{l=1}^{N} \pi_{t_0}(l)p^t(l,j). \tag{3.19}$$

Substituting (3.19) in (3.18) one obtains (3.15) and thus the proof ends.

3.3 Special global solutions of DTSGRE

In this subsection we investigate the problem of the existence of some special global solutions of (DTSGRE) (2.9) as: the minimal solution and the stabilizing solution.

With regards to the matrix coefficients of the system (2.9) we make the following assumption:

$\mathbf{H_4}$): (i) $\{A_k(t,i)\}_{t\geq 0}$, $\{B_k(t,i)\}_{t\geq 0}$, $0 \leq k \leq r$, $\{C(t,i)\}_{t\geq 0}$, $\{D(t,i)\}_{t\geq 0}$ are bounded sequences.

(ii) a) there exists $\delta_0 > 0$ not depending upon t such that $R(t,i) := D^T(t,i)D(t,i) \geq \delta_0 I_m$, $\forall (t,i) \in Z_+ \times \mathcal{D}$.

b) $C^T(t,i)D(t,i) = 0, \forall (t,i) \in Z_+ \times \mathcal{D}$.

One can see that if the assumption (ii) a) is fulfilled then (ii) b) takes place without any loss of generality. This may be obtained if in the system (2.1) as well as in the output (2.7) one makes the following change of the control parameters: $u(t) = -(D^T(t,\eta_t)D(t,\eta_t))^{-1}D^T(t,\eta_t)C(t,\eta_t)x_t + v(t)$

where $v(t)$ is the vector of new control parameters. For details regarding the continuous time case, one can see [18] or [22] chapter 5. Under the assumption $\mathbf{H_4}(ii)$ the feedback gain (3.8) becomes:

$$F^X(t,i) = -(R(t,i) + \sum_{k=0}^{r} B_k^T(t,i)\Pi_i(t,X(t+1))B_k(t,i))^{-1}(\sum_{k=0}^{r} B_k^T(t,i)\Pi_i(t,X(t+1))A_k(t,i)) \quad (3.20)$$

The following results will be repeatedly used in the next developments:

Lemma 3.7 *Let $\{X(t)\}_{t_0\le t\le t_1}$ be a solution of (DTSGRE) (2.9) and $W(t) = (W(t,1),...,W(t,N)), W(t,i) \in \mathbf{R}^{m\times n}$ be given. Then $\{X(t)\}_{t_0\le t\le t_1}$ verifies the following modified system of discrete time equations:*

$$X(t,i) = \sum_{k=0}^{r}(A_k(t,i) + B_k(t,i)W(t,i))^T\Pi_i(t,X(t+1))(A_k(t,i) + B_k(t,i)W(t,i)) + C^T(t,i)C(t,i)$$

$$+W^T(t,i)R(t,i)W(t,i) - (W(t,i) - F^X(t,i))^T\mathcal{R}_i(t,X(t+1))(W(t,i) - F^X(t,i)).$$

Proof follows by direct calculations. It is omitted for shortness.

We mention that any time we refer to a solution of (DTSGRE) (2.9) we assume tacitly that $\mathcal{R}_i(t,X(t+1))$ is invertible.

Based on the previous Lemma one easily obtains the following comparison result:

Proposition 3.8 *Let $X_l(t) = (X_l(t,1), X_l(t,2), ..., X_l(t,N)), l \in \{1,2\}, t_0 \le t \le t_1$ be two solutions of (DTSGRE) (2.9) with the properties:*

a) $\mathcal{R}_i(t,X_1(t+1)) > 0,\ \ t_0 \le t \le t_1 - 1$;

b) $X_2(t_1,i) \ge X_1(t_1,i),\ \ \forall i \in \mathcal{D}$.

Under these conditions we have $X_2(t,i) \ge X_1(t,i)$ for all $t_0 \le t \le t_1, i \in \mathcal{D}$.

Proof. Let $F_l(t) = (F_l(t,1), F_l(t,2), ..., F_l(t,N))$ be defined by $F_l(t,i) = F^{X_l}(t,i), l = 1,2, i \in \mathcal{D}, t_0 \le t \le t_1$. Applying successively the previous Lemma to the system (2.9) verified by $X_1(t)$ and $X_2(t)$ respectively and for $W(t) = F_2(t)$, we obtain:

$$X_2(t,i) - X_1(t,i) = \sum_{k=0}^{r}(A_k(t,i) + B_k(t,i)F_2(t,i))^T\Pi_i((t,X_2(t+1) - X_1(t+1))(A_k(t,i)$$

$$+B_k(t,i)F_2(t,i)) + M(t,i) \quad (3.21)$$

where $M(t,i) = (F_2(t,i) - F_1(t,i))^T\mathcal{R}_i(t,X_1(t+1))(F_2(t,i) - F_1(t,i))$.

Based on assumption a) one deduces that $M(t,i) \ge 0$ for all $t_0 \le t \le t_1 - 1, i \in \mathcal{D}$. The conclusion follows now inductively from (3.21) and thus the proof ends.

Consider the following system associated to (2.1) for $f_k(t,i) = 0$:

$$x(t+1) = (A_0(t,\eta_t)x(t) + B_0(t,\eta_t)u(t) + \sum_{k=1}^{r} w_k(t)(A_k(t,\eta_t)x(t) + B_k(t,\eta_t)u(t)),\ t \ge 0. \quad (3.22)$$

Definition 3.1. We say that the system (3.22) is stochastic stabilizable if there exist bounded sequences $\{F(t,i)\}_{t\ge 0},\ i \in \mathcal{D}$, such that the trajectories of the closed-loop system

$$x(t+1) = [A_0(t,\eta_t) + B_0(t,\eta_t)F(t,\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(t,\eta_t) + B_k(t,\eta_t)F(t,\eta_t))]x(t) \quad (3.23)$$

satisfy

$$E[|x(t,t_0,x_0)|^2|\eta_{t_0} = i] \le \beta q^{t-t_0}|x_0|^2 \quad (3.24)$$

for all $t \geq t_0,\ i \in \mathcal{D}$ with $\pi_{t_0}(i) > 0$, where $\beta \geq 1,\ q \in (0,1)$ are independent upon t, t_0, x_0.

The sequences $\{F(t,i)\}_{t\geq 0}$, involved in the above definition will be called stabilizing feedback gains. If the coefficients of the system (3.22) are periodic, with period $\theta \geq 1$, the definition of the stochastic stabilizability will be restricted to the class of θ-periodic stabilizing feedback gains.

According to the terminology introduced in [21] the definition of stochastic stabilizability could be restated as follows: the system (3.22) is stochastic stabilizable if there exist bounded sequences $\{F(t,i)\}_{t\geq 0}$ such that the zero state equilibrium of the system (3.23) is exponentially stable in mean square with conditioning of type I (ESMS-CI).

Let us consider the following linear system obtained from (2.1) and the output (2.7), for $f_k(t,i) = 0$ and $u(t) = 0$:

$$x(t+1) = (A_0(t,\eta_t) + \sum_{k=1}^{r} w_k(t) A_k(t,\eta_t) x(t) \tag{3.25}$$
$$y(t) = C(t,\eta_t)x(t).$$

Definition 3.2. We say that the system (3.25) is stochastic detectable if there exist bounded sequences $\{K_k(t,i)\}_{t\geq 0},\ i \in \mathcal{D}, 0 \leq k \leq r$ such that the zero state equilibrium of the system

$$x(t+1) = (A_0(t,\eta_t) + K_0(t,\eta_t)C(t,\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(t,\eta_t) + K_k(t,\eta_t)C(t,\eta_t)))x(t)$$

is (ESMS-CI).

Different aspects regarding the concepts of stochastic stability, stochastic stabilizability and stochastic detectability can be found in [13, 19, 20, 21, 24, 25, 26, 34, 38, 39, 41].

Now we prove:

Theorem 3.9 *Assume: a) The assumptions* $\mathbf{H_1} - \mathbf{H_4}$ *are fulfilled.*

b) The system (3.22) is stochastic stabilizable.

Under these assumptions (DTSGRE) (2.9) has a bounded solution $\tilde{X}(t) = (\tilde{X}(t,1), \tilde{X}(t,2), ..., \tilde{X}(t,N))$ *such that* $\tilde{X}(t,i) \geq 0, (t,i) \in Z_+ \times \mathcal{D}$. *The solution* $\tilde{X}(t)$ *is minimal in the class of global bounded and positive semidefinite solutions of (2.9). Moreover if there exists an integer* $\theta \geq 1$ *such that* $A_k(t+\theta,i) = A_k(t,i),\ B_k(t+\theta,i) = B_k(t,i),\ 0 \leq k \leq r,\ C(t+\theta,i) = C(t,i),\ D(t+\theta,i) = D(t,i)$, $t \in \mathbf{Z}_+, P_{t+\theta} = P_t,\ i \in \mathcal{D}$, *then* $\tilde{X}(t+\theta,i) = \tilde{X}(t,i)$, *for all* $t \in \mathbf{Z}_+,\ i \in \mathcal{D}$.

Proof. For each $\tau \geq 1, \tau \in \mathbf{Z}_+$ let $X_\tau(t) = (X_\tau(t,1), ..., X_\tau(t,N))$ be the solution of (2.9) with the terminal condition $X_\tau(\tau,i) = 0, i \in \mathcal{D}$. Proceeding as in the proof of Theorem 3.6 one obtains that $X_\tau(\cdot)$ are well defined and $X_\tau(t,i) \geq 0$ for $0 \leq t \leq \tau$. If $1 \leq \tau_1 < \tau_2$ we have $X_{\tau_2}(\tau_1,i) \geq 0 = X_{\tau_1}(\tau_1,i), i \in \mathcal{D}$. Applying Proposition 3.8 for $X_l(t) = X_{\tau_l}(t), l = 1,2$, we conclude that

$$X_{\tau_1}(t,i) \leq X_{\tau_2}(t,i) \tag{3.26}$$

for all $0 \leq t \leq \tau_1, i \in \mathcal{D}$. On the other hand, the assumption b) in the statement together with Theorem 3.8 [21] guarantee the existence of the bounded sequences $\{F_0(t,i)\}_{t\geq 0}, i \in \mathcal{D}$, such that the following system of coupled Lyapunov type equations

$$X_0(t,i) = \sum_{k=0}^{r} [A_k(t,i) + B_k(t,i)F_0(t,i)]^T \Pi_i(t, X_0(t+1))[A_k(t,i) + B_k(t,i)F_0(t,i)]$$
$$+C^T(t,i)C(t,i) + F_0^T(t,i)R(t,i)F_0(t,i), i \in \mathcal{D} \tag{3.27}$$

has a bounded solution $X_0(t) = (X_0(t,1), ..., X_0(t,N)), X_0(t,i) \geq 0, i \in \mathcal{D}, t \in \mathbf{Z}_+$.

Applying Lemma 3.7 with $W(t,i) = F_0(t,i)$ to (2.9) verified by $X_\tau(t)$ one obtains that $X_0(t) - X_\tau(t), 0 \le t \le \tau$, solves the following equation

$$X_0(t,i) - X_\tau(t,i) = \sum_{k=0}^{r}(A_k(t,i) + B_k(t,i)F_0(t,i))^T\Pi_i(t, X_0(t+1) - X_\tau(t+1))(A_k(t,i) \\ +B_k(t,i)F_0(t,i)) + \tilde{M}(t,i) \tag{3.28}$$

where $\tilde{M}(t,i) = (F_0(t,i) - F_\tau(t,i))^T\mathcal{R}_i(t, X_\tau(t+1))(F_0(t,i) - F_\tau(t,i)), F_\tau(t,i) = F^{X_\tau}(t,i)$. It can be seen that $\tilde{M}(t,i) \ge 0, 0 \le t \le \tau - 1, i \in \mathcal{D}$. Since $X_0(\tau,i) - X_\tau(\tau,i) = X_0(\tau,i) \ge 0$ one obtains inductively from (3.28) that $X_0(t,i) - X_\tau(t,i) \ge 0, \forall 0 \le t \le \tau, i \in \mathcal{D}$. This allows us to conclude that

$$0 \le X_\tau(t,i) \le X_0(t,i) \le cI_n \tag{3.29}$$

for all $0 \le t \le \tau, \tau \ge 1, i \in \mathcal{D}$ where $c > 0$ is independent of τ and t. From (3.26) and (3.29) one obtains that the sequences $\{X_\tau(t,i)\}_{\tau \ge 1}, i \in \mathcal{D}$ are convergent. Let $\tilde{X}(t,i) = lim_{t\to\infty}X_\tau(t,i)$. It follows that $0 \le \tilde{X}(t,i) \le cI_n$. Moreover $\tilde{X}(t) = (\tilde{X}(t,1), \tilde{X}(t,2), ..., \tilde{X}(t,N))$ is a global solution of (2.9). If $\hat{X}(t) = (\hat{X}(t,1), \hat{X}(t,2), ..., \hat{X}(t,N))$ is another bounded solution of (DTSGRE) (2.9) with $\hat{X}(t,i) \ge 0$ for all $t, i \in \mathbf{Z}_+ \times \mathcal{D}$ then $\hat{X}(\tau,i) \ge 0 = X_\tau(\tau,i), \forall\ \tau \ge 1$. Applying Proposition 3.8 we deduce that $\hat{X}(t,i) \ge X_\tau(t,i)$ for all $0 \le t \le \tau, i \in \mathcal{D}$. Taking the limit for $\tau \to \infty$ we deduce that $\hat{X}(t,i) \ge \tilde{X}(t,i)$ for arbitrary $(t,i) \in \mathbf{Z}_+ \times \mathcal{D}$ and thus we obtain that $\tilde{X}(t)$ is the minimal solution.

If the coefficients of DTSGRE (2.9) are periodic sequences with period $\theta \ge 1$ we define $\check{X}_\tau(t) = (\check{X}_\tau(t,1), \check{X}_\tau(t,2), ..., \check{X}_\tau(t,N))$ by $\check{X}_\tau(t,i) = X_{\tau+\theta}(t+\theta, i)$, $0 \le t \le \tau, i \in \mathcal{D}$. It is easy to check that $\check{X}_\tau(t)$ is also a solution of DTSGRE (2.9) and $\check{X}_\tau(\tau,i) = 0 = X_\tau(\tau,i)$. Therefore $\check{X}_\tau(t,i) = X_\tau(t,i)$ for all $0 \le t \le \tau, i \in \mathcal{D}$. This leads to $\lim_{\tau\to\infty}\check{X}_\tau(t,i) = \lim_{\tau\to\infty}X_\tau(t,i) = \tilde{X}(t,i)$. On the other hand $\lim_{\tau\to\infty}\check{X}_\tau(t,i) = \lim_{\tau\to\infty}X_{\tau+\theta}(t+\theta,i) = \tilde{X}(t+\theta,i)$. This allows us to conclude that $\tilde{X}(t,i) = \tilde{X}(t+\theta,i)$ and thus the proof ends.

Definition 3.3. We say that $\tilde{X}(t) = (\tilde{X}(t,1), ...\tilde{X}(t,N))$, $t \in \mathbf{Z}_+$ is a stabilizing solution of the system (2.9) if the zero state equilibrium of the closed-loop system

$$x(t+1) = [A_0(t,\eta_t) + B_0(t,\eta_t)\tilde{F}(t,\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(t,\eta_t) + B_k(t,\eta_t)\tilde{F}(t,\eta_t))]x(t) \tag{3.30}$$

is (ESMS-CI), where

$$\tilde{F}(t,i) = -(R(t,i) + \sum_{k=1}^{r} B_k^T(t,i)\Pi_i(t, \tilde{X}(t+1))B_k(t,i))^{-1}(\sum_{k=0}^{r} B_k^T(t,i)\Pi_i(t, \tilde{X}(t+1))A_k(t,i)). \tag{3.31}$$

With respect to the stabilizing solution of (2.9) we first prove:

Theorem 3.10 (uniqueness): *Under the assumption* $\mathbf{H_1} - \mathbf{H_4}$ *the (DTSGRE) (2.9) has at most one bounded and stabilizing solution* $\tilde{X}(t)$ *satisfying the additional condition* $\mathcal{R}_i(t, \tilde{X}(t+1)) > 0, t \in \mathbf{Z}_+, i \in \mathcal{D}$.

Proof. Let us assume that (2.9) has at least two bounded and stabilizing solutions $X_l(t) = (X_l(t,1), ..., X_l(t,N)), l = 1,2$, which verify the additional condition $\mathcal{R}_i(t, X_l(t+1)) > 0$. Set $F_l(t,i) = F^{X_l}(t,i), l = 1,2$. Applying Lemma 3.7 with $W(t,i) = F_2(t,i)$ to the equation (2.9) verified by $X_2(t,i)$ and $X_1(t,i)$ respectively, one obtains that the sequence $\{X_2(t) - X_1(t)\}_{t\ge0}$ is a bounded solution of the linear equation on $\mathcal{S}_n^N$:

$$Z(t,i) = \sum_{k=0}^{r}(A_k(t,i) + B_k(t,i)F_2(t,i))^T\Pi_i(t, Z(t+1))(A_k(t,i) + B_k(t,i)F_2(t,i)) + M_2(t,i) \tag{3.32}$$

where $M_2(t,i) = (F_2(t,i) - F_1(t,i))^T\mathcal{R}_i(t, X_1(t+1))(F_2(t,i) - F_1(t,i))$. From the assumption in the statement it follows that $M_2(t,i) \ge 0, (t,i) \in \mathbf{Z}_+ \times \mathcal{D}$. Since $X_2(t)$ is a stabilizing solution of (2.9) one obtains via Theorem 3.5 in [20] that the equation (3.32) has a unique bounded solution and that solution is positive semindefinite. This allows us to conclude that $X_2(t,i) - X_1(t,i) \geq 0$. Applying again Lemma 3.7 with $W(t,i) = F_1(t,i)$ we obtain in the same way that $X_1(t,i) - X_2(t,i) \geq 0$. Hence $X_1(t,i) = X_2(t,i)$ and thus the proof is complete.

Lemma 3.11 *Assume:*

a) The assumptions $\mathbf{H_1} - \mathbf{H_4}$ *are fulfilled.*

b) The system (3.25) is stochastic detectable.

Under these assumptions any bounded solution $\tilde{X}(t) = (\tilde{X}(t,1), ..., \tilde{X}(t,N))$ *of DTSGRE (2.9) with* $\tilde{X}(t,i) \geq 0$ *for all* $t \geq 0, i \in \mathcal{D}$ *is a stabilizing solution.*

Proof. Let $\tilde{X}(t) = (\tilde{X}(t,1), \tilde{X}(t,2), ..., \tilde{X}(t,N))$ be a bounded and positive semi-definite solution of (2.9). Set $\tilde{F}(t,i) = F^{\tilde{X}}(t,i)$, $(t,i) \in Z_+ \times calD$. Applying Lemma 3.7 with $W(t,i) = \tilde{F}(t,i)$ one obtains that $\tilde{X}(t)$ solves:

$$\begin{aligned}\tilde{X}(t,i) = \sum_{k=0}^{r}(A_k(t,i) + B_k(t,i)\tilde{F}(t,i))^T \Pi_i(t, \tilde{X}(t+1))(A_k(t,i)\\ +B_k(t,i)\tilde{F}(t,i)) + \tilde{C}^T(t,i)\tilde{C}(t,i)\end{aligned} \tag{3.33}$$

where $\tilde{C}(t,i) = \begin{pmatrix} C(t,i) \\ R^{\frac{1}{2}}(t,i)\tilde{F}(t,i) \end{pmatrix}$.

Now we show that under assumption b) the system (3.33) is stochastic detectable. To this end we take $\tilde{K}_k(t,i) \in \mathcal{R}^{n\times(p+m)}$, $\tilde{K}_k(t,i) = (K_k(t,i) \quad -B_k(t,i)R^{\frac{-1}{2}}(t,i))$ where $K_k(t,i)$ are provided by the assumption b).

One obtains that the corresponding closed-loop system $x(t+1) = [A_0(t,\eta_t) + \tilde{K}_0(t,\eta_t)\tilde{C}(t,\eta_t) +$ $+ \sum_{k=1}^{r} w_k(t)(A_k(t,\eta_t) + \tilde{K}_k(t,\eta_t)\tilde{C}(t,\eta_t))]x(t)$ coincides with

$$x(t+1) = (A_0(t,\eta_t) + K_0(t,\eta_t)C(t,\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(t,\eta_t) + K_k(t,\eta_t)C(t,\eta_t)))x(t)$$

which is ESMS-CI.

Thus we conclude that the system (3.33) is stochastic detectable. Applying a slightly modified version of the Theorem 4.8 in [20] to equation (3.33) we conclude that the zero state equilibrium of the system (3.30) is ESMS-CI and thus the proof ends.

At the end of this subsection we prove:

Theorem 3.12. Assume: a) The hypotesis $\mathbf{H_1} - \mathbf{H_4}$ are fulfilled. b) The system (3.22) is stochastic stabilizable. c) The system (3.25) is stochastic detectable.

Under these conditions DTSGRE (2.9) has a bounded and stabilizing solution $\tilde{X}(t) = (\tilde{X}(t,1), \tilde{X}(t,2), ..., \tilde{X}(t,N)), \tilde{X}(t,i) \geq 0$ for all $(t,i) \in Z_+ \times \mathcal{D}$.

Moreover if the coefficients of the DTSGRE (2.9) are periodic sequences with period $\theta \geq 1$ then $\tilde{X}(t)$ is a periodic solution with the same period θ.

Proof. It follows immediately form Theorem 3.9, Lemma 3.11 and Theorem 3.10.

3.4 The solution of OP2

In this subsection we solve the problem $OP2$ under the following additional assumption:

$\mathbf{H_5}$ $\{f_k(t,i)\}_{t\geq 0}, 0 \leq k \leq r, i \in \mathcal{D}$ are bounded sequences.

Let us consider the system of backward affine equations

$$\kappa(t,i) = (A_0(t,i) + B_0(t,i)\tilde{F}(t,i))^T \Pi_i(t, \kappa(t+1)) + \tilde{g}(t,i), \tag{3.34}$$

$i \in \mathcal{D}$, where

$$\tilde{g}(t,i) = \sum_{k=0}^{r}(A_k(t,i) + B_k(t,i)\tilde{F}(t,i))^T \Pi_i(t, \tilde{X}(t+1)) f_k(t,i)$$

$\tilde{F}(t,i)$ being the stabilizing feedback gain determined by the stabilizing solution of (2.9).

Lemma 3.13 Under the assumptions of Theorem 3.12 together with $\mathbf{H_5}$, the system of backward affine equations (3.34) has a unique bounded solution on $\mathbf{Z}_+$, $\tilde{\kappa}(t) = (\tilde{\kappa}(t,1), ..., \tilde{\kappa}(t,N))$.

Proof-see subsection 5.2.

Based on the stabilizing solution $\tilde{X}(t)$ of (2.9) and the unique bounded solution $\tilde{\kappa}(t)$ of (3.34) we construct the following control law:

$$\tilde{u}(t) = \tilde{F}(t,\eta_t)\tilde{x}(t) + \tilde{\psi}(t,\eta_t) \tag{3.35}$$

where $\tilde{F}(t,i)$ is defined as in (3.31),

$$\tilde{\psi}(t,i) = -\mathcal{R}_i^{-1}(t,\tilde{X}(t+1))[B_0^T(t,i)\Pi_i(t,\tilde{\kappa}(t+1)) + \sum_{k=0}^{r} B_k^T(t,i)\Pi_i(t,\tilde{X}(t+1))f_k(t,i)] \tag{3.36}$$

and $\tilde{x}(t)$ is the solution of the closed-loop system

$$\begin{aligned} \tilde{x}(t+1) = [A_0(t,\eta_t) + B_0(t,\eta_t)\tilde{F}(t,\eta_t)]\tilde{x}(t) + \tilde{f}_0(t,\eta_t) + \\ \sum_{k=1}^{r} w_k(t)[(A_k(t,\eta_t) + B_k(t,\eta_t)\tilde{F}(t,\eta_t))\tilde{x}(t) + \tilde{f}_k(t,\eta_t)] \\ \tilde{x}(t_0) = x_0 \end{aligned} \tag{3.37}$$

where $\tilde{f}_k(t,i) = B_k(t,i)\tilde{\psi}(t,i) + f_k(t,i), 0 \leq k \leq r, (t,i) \in \mathbf{Z}_+ \times \mathcal{D}$.

We have:

Lemma 3.14 *Under the assumptions of Lemma 3.13 the following hold:*

(i) For each $x_0 \in \mathbf{R}^n$, $\tilde{u} \in \mathcal{U}_{t_0\infty}(x_0)$;

(ii) $J(t_0, \infty, x_0, \tilde{u}) < +\infty$.

Proof-see subsection 5.3.

For each $(t_0, x_0) \in \mathbf{Z}_+ \times \mathbf{R}^n$ we introduce the sets

$$\tilde{\mathcal{U}}_{t_0\infty}(x_0) = \{u \in \mathcal{U}_{t_0\infty}(x_0) | J(t_0, \infty, x_0, u) < +\infty\}.$$

Under the conditions of the above lemma it follows that $\tilde{u} \in \tilde{\mathcal{U}}_{t_0\infty}(x_0)$.

Moreover based on Corollary 5.4 (i) below one obtains that if the linear system (3.22) is stochastic stabilizable and if the assumptions $\mathbf{H_1} - \mathbf{H_5}$ are fulfilled then each $(t_0, x_0) \in \mathbf{Z}_+ \times \mathbf{R}^n$ the set $\tilde{\mathcal{U}}_{t_0\infty}(x_0)$ contains the controls of the form

$$u(t) = F(t,\eta_t)\hat{x}(t) + h(t)$$

for arbitrary stabilizing feedback gain $\{F(t,i)\}_{t\geq 0}, i \in \mathcal{D}$ and for arbitrary stochastic process $\{h(t)\}_{t\geq 0}$ with the properties:

a) for each $t \in \mathbf{Z}_+, h(t)$ is $\tilde{\mathcal{H}}_t$-stabilizable;

b) $\sup_{t\geq 0} E[|h(t)|^2] < \infty$;

$\hat{x}(t)$ is the solution of

$$x(t+1) = [A_0(t,\eta_t) + B_0(t,\eta_t)F(t,\eta_t)]x(t) + \check{f}_0(t) + \sum_{k=1}^{r} w_k(t)[(A_k(t,\eta_t) + B_k(t,\eta_t)F(t,\eta_t))x(t) + \check{f}_k(t)]$$

where

$$\check{f}_k(t) = f_k(t, \eta_t) + B_k(t, \eta_t)h(t).$$

Thus we obtain:

Corollary 3.15 If the assumptions in Theorem 3.12 and the assumption $\mathbf{H_5}$ are fulfilled, the problem **OP2** is well possed.

The main result of this section is:

Theorem 3.16 *Assume that:*

a) The hypotheses $\mathbf{H_1} - \mathbf{H_5}$ *are fulfilled.*

b) The system (3.22) is stochastic stabilizable.

c) The system (3.25) is stochastic detectable.

Under these conditions the optimal control problem **OP2** *is given by (3.35)-(3.37).*

The optimal value of the cost functional is given by

$$J(t_0, \infty, x_0, \tilde{u}) = \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{t=0}^{T} \sum_{l=1}^{N} \sum_{j=1}^{N} \Pi_0(l) p^t(l, j) \tilde{\mu}(t, j) \tag{3.38}$$

where $\tilde{\mu}(t, j)$ *is defined as in (3.11) based on* $\tilde{X}(t)$ *and* $p^t(l, j)$ *are as in Theorem 3.6.*

Proof. It follows immediately by combining Proposition 3.5, Lemma 3.13 and Lemma 3.14.

Remark 3.1. If the condition (2.4) from the definition of the set of admissible controls $\mathcal{U}_{t_0\infty}(x_0)$ is replaced by

$$\lim_{t\to\infty} E[|x_u(t, t_0, x_0)|^2] = 0 \tag{3.39}$$

one obtains a new class of admissible controls $\hat{\mathcal{U}}_{t_0\infty}(x_0)$.

It is obvious that $\hat{\mathcal{U}}_{t_0\infty}(x_0) \subset \mathcal{U}_{t_0\infty}(x_0)$.

Thus we may consider a new optimization problem asking the minimization of the cost functional (2.6) over the set of admissible controls $\hat{\mathcal{U}}_{t_0\infty}(x_0)$. To be sure that condition (3.39) is satisfied, the assumption $\mathbf{H_5}$ must be replaced with a stronger one:

$\mathbf{H_6}$: $\lim_{t\to\infty} f_k(t, i) = 0, i \in \mathcal{D}, 0 \leq k \leq r$.

One proves that under the assumptions $\mathbf{H_1} - \mathbf{H_4}, \mathbf{H_6}$ the unique bounded solution of (3.34) satisfies $\lim_{t\to\infty} \tilde{\kappa}(t) = 0$.

Furthermore if $\tilde{\psi}(t, i)$ is defined by (3.36) we have $\lim_{t\to\infty} \tilde{\Psi}(t, i) = 0, i \in \mathcal{D}$. Applying Corollary 5.4 (ii) below, one obtains that the solution of (3.37) will satisfy $\lim_{t\to\infty} E[|\tilde{x}(t)|^2] = 0$. This shows that the control $\tilde{u}(t)$ defined by (3.35)-(3.37) belongs to the new class of admissible controls $\hat{\mathcal{U}}_{t_0\infty}(x_0)$.

Reasoning as in the proof of Theorem 3.16 one obtains that if the assumption $\mathbf{H_5}$ is replaced by $\mathbf{H_6}$ the control $\tilde{u}(t)$ defined by (3.35)-(3.37) achieves the optimal value of the cost functional (2.6) with respect to both classes of admissible controls $\mathcal{U}_{t_0\infty}(x_0)$ as well as $\hat{\mathcal{U}}_{t_0\infty}(x_0)$.

4 Applications to tracking problems

Consider the discrete time controlled system described by

$$x(t+1) = A_0(t, \eta_t)x(t) + B_0(t, \eta_t)u(t) + \sum_{k=1}^{r} [A_k(t, \eta_t)x(t) + B_k(t, \eta_t)u(t)]w_k(t), \tag{4.1}$$

$t \geq t_0$, $x(t_0) = x_0$.

Let $\{r(t)\}_{t\geq 0}$, $r(t) \in \mathbf{R}^n$ be a given signal called **reference signal**. The control problem we want to solve is to find a control $\tilde{u}(t)$ which minimizes the deviation $x(t) - r(t)$.

For a more rigorous setting of this problem let us introduce the following cost functionals:

$$\begin{aligned} &J(t_0, t_f, x_0, u) = E\{(x(t_f) - r(t_f))^T \kappa_f(\eta_t)(x(t_f) - r(t_f)) + \\ &\sum_{t=t_0}^{t_f - 1} [(x(t) - r(t))^T M(t, \eta_t)(x(t) - r(t)) + u^T(t) R(t, \eta_t) u(t)]\} \end{aligned} \tag{4.2}$$

in the case of finite time horizon and

$$J(t_0, \infty, x_0, u) = \overline{\lim}_{T\to\infty} \frac{1}{T - t_0} \sum_{t=t_0}^{T} E[(x(t) - r(t))^T M(t, \eta_t)(x(t) - r(t)) + u^T(t) R(t, \eta_t) u(t)] \tag{4.3}$$

in the case of infinite time horizon, where $M(t,i) = M^T(t,i) \geq 0$, $K_f(i) = K_f^T(i) \geq 0$, $R(t,i) = R^T(t,i) > 0$ and $x(t) = x_u(t, t_0, x_0)$.

The tracking problems considered in this section ask for finding a control law $u_{opt} \in \mathcal{U}_{t_0,t_f}$ ($u_{opt} \in \mathcal{U}_{t_0,\infty}(x_0)$ respectively) in order to minimize the cost (4.2) (the cost (4.3) respectively). If we set $\xi(t) = x(t) - r(t)$ then we obtain $\xi(t+1) = A_0(t,\eta_t)\xi(t) + B_0(t,\eta_t)u(t) + f_0(t,\eta_t) + \sum_{k=1}^{r}[A_k(t,\eta_t)\xi(t) + B_k(t,\eta_t)u(t) + f_k(t,\eta_t)]w_k(t)$ and the cost functionals

$$J(t_0, t_f, x_0, u) = E[\xi^T(t_f) K_f(\eta_t) \xi(t_f) +$$

$$\sum_{t=t_0}^{t_f - 1} (\xi^T(t) M(t, \eta_t) \xi(t) + u^T(t) R(t, \eta_t) u(t))]$$

and

$$J(t_0, infty, x_0, u) = \overline{\lim}_{T\to\infty} \frac{1}{T - t_0} \sum_{t=t_0}^{T} E[\xi^T(t) M(t, \eta_t) \xi(t) + u(t)^T R(t, \eta_t) u(t)]$$

where

$$\begin{aligned} f_0(t,i) &= A_0(t,i) r(t) - r(t+1) \\ f_k(t,i) &= A_k(t,i) r(t), \quad 1 \leq k \leq r, t \geq 0, \end{aligned} \tag{4.4}$$

Let us consider the following system of Riccati type equations:

$$\begin{aligned} X(t,i) = \sum_{k=0}^{r} A_k^T(t,i) \Pi_i(t, X(t+1)) A_k(t,i) + -[\sum_{k=0}^{r} A_k^T(t,i) \Pi_i(t, X(t+1)) B_k(t,i)][R(t,i) \\ + \sum_{k=0}^{r} B_k^T(t,i) \Pi_i(t, X(t+1)) B_k(t,i)]^{-1} [\sum_{k=0}^{r} B_k^T(t,i) \Pi_i(t, X(t+1)) A_k(t,i)] + M(t,i) \end{aligned} \tag{4.5}$$

It is easy to see that (4.5) is the special case of (2.9) with $C(t,i) = \begin{pmatrix} M^{\frac{1}{2}}(t,i) \\ 0 \end{pmatrix}$, $D(t,i) = \begin{pmatrix} 0 \\ (R(t,i))^{\frac{1}{2}} \end{pmatrix}$.

The solution of the tracking problems are derived directly from Theorem 3.6 and Theorem 3.11.

Corollary 4.1 Under the considered assumptions, the optimal control of the tracking problem described by the system (4.1) and the cost (4.2) is given by

$$\hat{u}_{opt}(t) = \hat{F}(t, \eta_t)(\hat{x}(t) - r(t)) + \hat{\psi}(t, \eta_t) \tag{4.6}$$

where $\hat{F}(t,i) = F^{\hat{X}}(t,i)$ is constructed as in (3.8) based on the solution $\hat{x}(t,i)$ of the system (4.5) with the terminal condition $\hat{X}(t_f,i) = K_f(i)$, $i \in \mathcal{D}$, $\hat{\psi}(t,i)$ is constructed as in (3.12) based on $\hat{X}(t,i)$ and $\hat{\kappa}(t)$ with $(\hat{\kappa}(t) = (\hat{\kappa}(t,1),...,\hat{\kappa}(t,N))$ is the solution of the system of backward affine equations

$$\kappa(t,i) = (A_0(t,i) + B_0(t,i)\hat{F}(t,i))^T \Pi_i(t,\hat{\kappa}(t+1)) + \hat{g}(t,i)\hat{\kappa}(t_f,i) = 0, \tag{4.7}$$

$i \in \mathcal{D}$ where $\hat{g}(t,i) = \sum_{k=0}^{r}(A_k(t,i) + B_k(t,i)\hat{F}(t,i))^T \Pi_i(t,\hat{X}(t+1)f_k(t,i)$, $f_k(t,i)$ given by (4.4), $\hat{x}(t)$ is the solution of the closed loop system:

$$\hat{x}(t+1) = [A_0(t,\eta_t) + B_0(t,\eta_t)\hat{F}(t,\eta_t) + \sum_{k=1}^{r}(A_k(t,\eta_t) + B_k(t,\eta_t)\hat{F}(t,\eta_t))w + k(t)]\hat{x}(t),$$

$t \geq t_0$, $\hat{x}(t_0) = x_0$.

The optimal cost is given by $J(t_0,t_f,x_0,\hat{u}_{opt}) = \sum_{l=1}^{N}[\pi_0(l)[(x_0 - r(t_0))^T \hat{X}(t_0,l)(x_0 - r(t_0)) +$ $+ 2(x_0 - r(t_0))^T \kappa(t_0,l) + \sum_{t=0}^{t_f-1}\sum_{j=1}^{N} p^t(l,j)\hat{\mu}(t,j)]$ where $\hat{\mu}(t,j)$ and $p^t(l,j)$ are as in Theorem 3.6.

Consider the linear system:

$$\begin{aligned} x(t+1) &= (A_0(t,\eta_t) + \sum_{k=1}^{r} A_k(t,\eta_t)w_k(t))x(t) \\ y(t) &= M^{\frac{1}{2}}(t,\eta_t)x(t). \end{aligned} \tag{4.8}$$

We have:

Corollary 4.2 Assume:

a) The hypotheses $\mathbf{H_1}, \mathbf{H_2}, \mathbf{H_3(i)}, \mathbf{H_4}$ are fulfilled.

b) The system (4.1) is stochastic stabilizable.

c) The system (4.8) is stochastic detectable.

d) $R(t,i) \geq \delta I_n > 0$ and the sequence $\{r(t)\}_{t\geq 0}$ is bounded.

Under these conditions the optimal control of the tracking problem described by system (4.1) and cost (4.3) is:

$$\tilde{u}_{opt}(t) = \tilde{F}(t,\eta_t)(\tilde{x}(t) - r(t)) + \tilde{\psi}(t,\eta_t) \tag{4.9}$$

where $\tilde{F}(t,i) = F^{\tilde{X}}(t,i)$ is constructed as in (3.8) based on the stabilizing solution $\tilde{X}(t) = ((\tilde{X}(t,1),...,\tilde{X}(t,N))$ of the system (4.5), $\tilde{\psi}(t,i)$ is given by

$$\begin{aligned} \tilde{\psi}(t,i) = -(R(t,i) + \sum_{k=0}^{r} B_k(t,i)\Pi_i(t,\tilde{X}(t+1))B_k(t,i))^{-}1[B_0^T(t,i)\Pi_i(t,\tilde{\kappa}(t+1)) \\ + \sum_{k=0}^{r} B_k^T(t,i)\Pi_i(t,\tilde{X}(t+1))f_k(t,i)] \end{aligned}$$

where $\tilde{\kappa}(t) = (\tilde{\kappa}(t,1),...,\kappa(t,N))$ is the unique bounded solution of the system of backward affine equations

$$\kappa(t,i) = (A_0(t,i) + B_0(t,i)\tilde{F}(t,i))^T \Pi_i(t,\kappa(t+1)) + \tilde{g}(t,i)$$

with

$$\tilde{g}(t,i) = \sum_{k=0}^{r}(A_k(t,i) + B_k(t,i)\tilde{F}(t,i))^T \Pi_i(t,\tilde{X}(t+1))f_k(t,i)$$

$f_k(t,i)$ be given by (4.4); $\tilde{x}(t)$ is the solution of the closed-loop system

$$\begin{aligned} \tilde{x}(t+1) = [A_0(t,\eta_t) + B_0(t,\eta_t)\tilde{F}(t,\eta_t) + \\ \sum_{k=1}^{r}(A_k(t,\eta_t) + B_k(t,\eta_t)\tilde{F}(t,\eta_t))w_k(t)]\tilde{x}(t) \end{aligned}$$

$t \geq 0$, $x(t_0) = x_0$.

5 Proofs

In this section we provide the proofs of Lemma 3.13 and Lemma 3.14 above. To prove these lemmata we need several auxiliary results.

5.1 Some auxiliary results

Let $\mathbf{R}^{n\cdot N} = \mathbf{R}^n \oplus \mathbf{R}^n \oplus ... \oplus \mathbf{R}^n$(N times). If $x \in \mathbf{R}^{n\cdot N}$ then $x = (x(1), ..., x(N))$ with $x(i) \in \mathbf{R}^n$, $x(i) = (x_1(i), x_2(i), ..., x_n(i))^T$. $\mathbf{R}^{n\cdot N}$ is a Hilbert space with the inner product

$$\langle x, y\rangle = \sum_{i=1}^{N} x^T(i)y(i) \tag{5.1}$$

for all $x, y \in \mathbf{R}^{n\cdot N}$. Together with the norm $|\cdot|_2$ induced on $\mathbf{R}^{n\cdot M}$ by the inner product (5.1) we consider also the norm

$$|x|_1 = max_{i\in\mathcal{D}}(x^T(i)x(i))^{\frac{1}{2}}. \tag{5.2}$$

If $L : \mathbf{R}^{n\cdot N} \to \mathbf{R}^{n\cdot N}$ is a linear operator then $\|L\|_k$ is the operator norm induced by $|\cdot|_k$, $k \in \{1, 2\}$. Based on the sequences $\{A_0(t, i)\}_{t\geq 0}$, $\{P_t\}_{t\geq 0}$ we construct the linear operators $\mathcal{A}_t : \mathbf{R}^{n\cdot N} \to \mathbf{R}^{n\cdot N}$ by $\mathcal{A}_t x = ((\mathcal{A}_t x)(1), ..., (\mathcal{A}_t x)(N))$ with

$$(\mathcal{A}_t x)(i) = \sum_{j=1}^{N} p_t(j, i)A_0(t, j)x(j). \tag{5.3}$$

It is easy to see that the adjoint operator of $\mathcal{A}_t$ with respect to the inner product (5.1) is given by $\mathcal{A}_t^* x = ((\mathcal{A}_t^* x)(1), ..., (\mathcal{A}_t^* x)(N))$ with

$$(\mathcal{A}_t^* x)(i) = A_0^T(t, i)\Pi_i(t, x) \tag{5.4}$$

where $\Pi_i(t, x)$ is defined as in (2.10) with $x(i)$ instead of $Y(i)$. In the sequel $\Xi(t, s)$ stands for the linear evolution operator on $R^{n\cdot N}$ defined by $\mathcal{A}_t$; that is $\Xi(t, s) = \begin{cases} \mathcal{A}_{t-1}...\mathcal{A}_s & \text{if} \quad t > s \geq 0 \\ I_{\mathbf{R}^{n\cdot N}} & \text{if} \quad t = s \end{cases}$ where $I_{\mathbf{R}^{n\cdot N}}$ the identity operator on $\mathbf{R}^{n\cdot N}$.

Consider the linear system derived from (2.1)

$$x(t+1) = [A(t, \eta_t) + \sum_{k=1}^{r} w_k(t)A_k(t, \eta_t)]x(t), \quad t \geq 0. \tag{5.5}$$

We denote $\mathbf{A}(t) = A_0(t, \eta_t) + \sum_{k=1}^{r} w_k(t)A_k(t, \eta_t)$ and define

$$\Phi(t, s) = \begin{cases} \mathbf{A}(t-1)\mathbf{A}(t-2)...\mathbf{A}(s) & \text{if} \quad t > s \geq 0 \\ I_n & \text{if} \quad t = s \end{cases} \tag{5.6}$$

We have

$$\Phi(t+1, s) = \mathbf{A}(t)\Phi(t, s) \tag{5.7}$$

$\Phi(t, s)$ is the fundamental matrix solution of (5.5). For each $s \geq 0$ we define the following subset $\mathcal{D}_s = \{i \in \mathcal{D}|\mathcal{P}\{\eta_s = i\} > 0\}$. It must be remarked that under the assumption $\mathbf{H_4}$ we have $\mathcal{D}_s = \mathcal{D}$ for all $s \geq 1$ if $\mathcal{D}_0 = \mathcal{D}$.

Lemma 5.1 Under the assumptions $\mathbf{H_1}, \mathbf{H_2}$ we have

$$(\Xi^*(t, s)x)(i) = E[\Phi^T(t, s)x(\eta_t)|\eta_s = i] \tag{5.8}$$

for all $i \in \mathcal{D}_s$, $t \geq s > 0$, $x = (x(1), ..., x(N)) \in \mathbf{R}^{n \cdot N}$.

Proof. We define the linear operators $\mathcal{U}(t,s) : \mathbf{R}^{n \cdot N} \to \mathbf{R}^{n \cdot N}$, $t \geq s \geq 0$ by

$$(\mathcal{U}(t,s)x)(i) = \begin{cases} E[\Phi^T(t,s)x(\eta_t)|\eta_s = i & \text{if} \quad i \in \mathcal{D}_s \\ (\Xi^*(t,s)x)(i) & \text{if} \quad i \in \mathcal{D} - \mathcal{D}_s \end{cases} \tag{5.9}$$

Taking successively the conditional expectation with respect to $\mathcal{H}_t$, $\tilde{\mathcal{H}}_t$ and $\sigma(\eta_s)$ and taking into account Lemma 3.1 one obtains via (5.7) that

$$E[\Phi^T(t+1,s)x(\eta_{t+1})|\eta_s] = \sum_{j=1}^{N} E[\Phi^T(t,s)A_0^T(t,\eta_t)x(j)p_t(\eta_t,j)|\eta_s]. \tag{5.10}$$

We also used the fact that $E[w_k(t)|\tilde{\mathcal{H}}_t] = E[w_k(t)] = 0$, $1 \leq k \leq r$. If $i \in \mathcal{D}_s$ (5.10) leads to :

$$E[\Phi(t+1,s)x(\eta_{t+1})|\eta_s = i] = \sum_{j=1}^{N} E[\Phi^T(t,s)A_0^T(t,\eta_t)x(j)p_t(\eta_t,j)|\eta_s = i].$$

Using the definition of $\mathcal{A}_t$ one gets:

$$E[\Phi^T(t+1,s)x(\eta_{t+1})|\eta_s = i] = E[\Phi^T(t,s)(\mathcal{A}_t^* x)(\eta_t)|\eta_s = i]. \tag{5.11}$$

Based on (5.8), the equality (5.11) may be written:

$$(\mathcal{U}(t+1,s)x)(i) = (\mathcal{U}(t,s)\mathcal{A}_t^* x)(i) \tag{5.12}$$

$i \in \mathcal{D}_s$. By direct calculation one obtains that (5.12) still holds for $i \in \mathcal{D}$. Therefore (5.12) leads to $\mathcal{U}(t+1,s) = \mathcal{U}(t,s)\mathcal{A}_t^*$. This shows that the sequence $\{\mathcal{U}(t,s)\}_{t \geq s}$ solves the same equation as $\Xi^*(t,s)$. Also we have $\mathcal{U}(s,s)x = x = \Xi(s,s)x$ for all $x \in \mathbf{R}^{n \cdot N}$. This allows us to conclude that $\mathcal{U}(t,s) = \Xi^*(t,s)$ for all $t \geq s \geq 0$ and thus the proof ends.

From the representation formula (5.8) one obtains:

Corollary 5.2 If the zero solution of (5.5) is exponentially stable in mean square with conditioning of type I (ESMS-CI) then the zero solution of the discrete time linear equation on $R^{n \cdot N}$:

$$x_{t+1} = \mathcal{A}_t x_t$$

is exponentially stable.

Lemma 5.3 *Consider the discrete-time affine equation:*

$$x(t+1) = A_0(t,\eta_t)x(t) + g_0(t) + \sum_{k=1}^{r} w_k(t)(A_k(t,\eta_t)x(t) + g_k(t)). \tag{5.13}$$

Assume: a) $\mathbf{H_1} - \mathbf{H_3}$ *hold and the zero solution of (5.5) is strongly exponentially stable in mean square (SESMS).*

b) $\{g_k(t)\}_{t \geq 0}$ *are stochastic processes with the properties:*

α) for each $t \in Z_+$, $g_k(t)$ *is* $\tilde{\mathcal{H}}_t$*- measurable.*

β) $E[|g_k(t)|^2] < \infty$, $0 \leq k \leq r$, $t \in Z_+$.

Under these conditions the trajectories of the system (5.13) satisfy:

$$E[|x(t,t_0,x_0)|^2|\eta_{t_0}] \leq c_1 q^{t-t_0}|x_0|^2 + c_2 \sum_{s=t_0}^{t-1} q^{t-s-1} \sum_{k=0}^{r} E[|g_k(s)|^2|\eta_{t_0}]$$

for all $t \geq t_0 \geq 0$, $x_0 \in \mathbf{R}^n$, *where* $c_1 > 0$, $c_2 > 0$, $q \in (0,1)$ *are independent of* t, t_0, x_0.

Proof. Based on Theorem 3.13 in [21] it follows that if the assumption a) holds the backward affine equation:

$$X(t,i) = \sum_{k=0}^{r} A_k^T(t,i)\Pi_i(t, X(t+1))A_k(t,i) + I_n$$

has a bounded solution $X(t) = (X(t,1), ..., X(t,N))$ that is

$$I_n \leq X(t,i) \leq cI_n \tag{5.14}$$

$t \geq 0$, $i \in \mathcal{D}$, $c > 0$ independent of t.

Applying Lemma 3.2 to the function $V(t,x,i) = x^T X(t,i)x$ one obtains

$$\begin{gathered} E[V(t+1, x(t+1), \eta_{t+1})|\eta_{t_0}] - E[V(t, x(t), \eta_t)|\eta_{t_0}] = -E[|x(t)|^2|\eta_{t_0}] + \\ 2\sum_{k=0}^{r} E[x^T(t)A_k^T(t,\eta_t)X(t,\eta_t)g_k(t,\eta_t)|\eta_{t_0}] + \sum_{k=0}^{r} E[g_k^T(t,\eta_t)X(t,\eta_t)g_k(t,\eta_t)|\eta_{t_0}] \end{gathered} \tag{5.15}$$

for all $t \geq t_0 \geq 0$, and arbitrary solution $x(t)$ of the system (5.13) with the initial value $x(t_0) = x_0 \in \mathbf{R}^n$.

Denoting $\varphi(t,t_0)$ the right hand side of (5.15) we may write:

$$\varphi(t,t_0) = -\frac{1}{2}E[|x(t)|^2|\eta_{t_0}] - \sum_{k=0}^{r} E[|\frac{1}{\sqrt{r+1}}x(t) - 2\sqrt{r+1}A_k^T(t,\eta_t)X(t,\eta_t)g_k(t)|^2|\eta_{t_0}]$$

$$+\sum_{k=0}^{r} E[g_k^T(t)M_k(t,\eta_t)g_k(t)|\eta_{t_0}]$$

where $M_k(t,i) = X(t,i) + 4(r+1)X(t,i)A_k(t,i)A_k^T(t,i)X(t,i)$. Hence

$$\varphi(t,t_0) \leq -\frac{1}{2}E[|x(t)|^2|\eta_{t_0}] + \sum_{k=0}^{r} E[g_k(t)M_k(t,\eta_t)g_k(t)|\eta_{t_0}]. \tag{5.16}$$

Setting $V(t) = E[V(t,x(t),\eta_t)|\eta_{t_0}]$ we deduce from (5.15) and (5.16) that

$$V(t+1) - V(t) \leq -\frac{1}{2}E[|x(t)|^2|\eta_{t_0}] + \gamma\sum_{k=0}^{r} E[|g_k(t)|^2|\eta_{t_0}] \tag{5.17}$$

where $\gamma = sup_{t\geq 0}|M_k(t,i)|, 0 \leq k \leq r, i \in \mathcal{D}$.

The fact that $\gamma < +\infty$ follows from (5.14) and the assumption a) which implies (see[21]) the boundness of the sequences $\{A_k(t,i)\}_{t\geq 0}, 0 \leq k \leq r, i \in \mathcal{D}$. Invoking again (5.14) we have further $V(t+1) \leq qV(t) + \tilde{g}(t)$ with $q = (1 - \frac{1}{2c}) \in (0,1)$ and $\tilde{g}(t) = \gamma\sum_{k=0}^{r} E[|g(t)|^2|\eta_{t_0}]$. Let $\tilde{V}(t)$ be the solution of the problem $\tilde{V}(t+1) = q\tilde{V}(t) + \tilde{g}(t), t \geq t_0, \tilde{V}(t_0) = V(t_0)$.

We have $V(t+1) - \tilde{V}(t+1) \leq q(V(t) - \tilde{V}(t)), t \geq t_0$. This allows us to obtain inductively that $V(t) - \tilde{V}(t) \leq 0, t \geq t_0$. On the other hand, from (5.18) one obtains that $\tilde{V}(t) = q^{t-t_0}V(t_0) + \sum_{s=t_0}^{t-1} q^{t-s-1}\tilde{g}(s), t \geq t_0 + 1$.

Using again (5.14) one gets

$$V(t) \leq cq^{t-t_0}|x_0|^2 + \gamma\sum_{k=0}^{r}\sum_{s=t_0}^{t-1} q^{t-s-1}E[|g_k(s)^2|\eta_{t_0}]. \tag{5.18}$$

Conclusion follows combining (5.19) and the first inequality from (5.14) and the proof is complete.

Corollary 5.4 Assume that:

a) $\mathbf{H_1} - \mathbf{H_4}$ are fulfilled.

b) The system (5.5) is ESMS-CI.

Under these conditions the following holds:

(i) If

$$sup_{t\geq 0}E[|g_k(t)|^2] < \infty, 0 \leq k \leq r \tag{5.19}$$

then the trajectories of the system (5.13) satisfy: $sup_{t\geq t_0}E[|x(t,t_0,x_0)|^2] < \infty$ for all $t \geq t_0 \geq 0$, $x_0 \in \mathbf{R}^n$.

(ii) If $lim_{t\to\infty}E[|g_k(t)|^2] = 0$, $0 \leq k \leq r$ then $lim_{t\to\infty}E[|x(t,t_0,x_0)|^2] = 0, \forall t_0 \geq 0, x \in \mathbf{R}^n$.

Proof. (i) Taking the expectation in the inequality proved in Lemma 5.3 we obtain:

$$E[|x(t,t_0,x_0)|^2] \leq c_1 q^{t-t_0}|x_0|^2 + c_2 \sum_{k=0}^{r}\sum_{s=t_0}^{t-1} q^{t-s-1}E[|g_k(s)|^2]. \tag{5.20}$$

From (5.20) and (5.21) one gets $E[|x(t,t_0,x_0)|^2] \leq c_1|x_0|^2 + c_2\frac{\gamma_1}{1-q}$ with $\gamma_1 = \sum_{k=0}^{r} sup_{t\geq 0}E[|g_k(s)|^2]$. Thus (i) is proved.

From (5.21) it follows that we have to prove

$$lim_{t\to\infty}\sum_{t=t_0}^{t-1} q^{t-s-1}E[|q_k(s)|^2] = 0 \tag{5.21}$$

$0 \leq k \leq r$. To this end we shall use Stolz-Cesaro criteria for the convergence of a sequence of real numbers.

Denoting $\xi_k(t) = \sum_{s=t_0}^{t-1} q^{t-s-1}E[|g_k(s)|^2], t \geq t_0 \geq 0$ one sees that $\xi_k(t) = \frac{\tilde{\xi}_k(t)}{q^{-t}}$ with $\tilde{\xi}_k(t) = \sum_{s=t_0}^{t-1} q^{-s-1}E[|g_k(s)|^2]$. We have $\frac{\tilde{\xi}_k(t+1)-\tilde{\xi}_k(t)}{q^{-t-1}-q^{-t}} = \frac{q^{-1}}{q^{-1}-1}E[|g_k(t)|^2]$. Hence $lim_{t\to\infty}\frac{\tilde{\xi}(t+1)-\tilde{\xi}(t)}{q^{-t-1}-q^{-t}} = 0$. This implies that $lim_{t\to\infty}\xi_k(t) = 0$ and thus the proof is complete.

5.2 Proof of Lemma 3.13

Let $\tilde{X}(t) = (\tilde{X}(t,1),...,\tilde{X}(t,N))$ be the stabilizing solution of (2.9), and $\tilde{F}(t) = (\tilde{F}(t,1),...,\tilde{F}(t,N))$ be the corresponding stabilizing feedback gain. Let $\tilde{\mathcal{A}}_t : \mathbf{R}^{n\cdot N} \to \mathbf{R}^{n\cdot N}$ defined by

$$(\tilde{\mathcal{A}}_t x)(i) = \sum_{j=1}^{N} p_t(j,i)(A_0(t,j) + B_0(t,j)\tilde{F}(t,j))x(j)$$

for all $x = (x(1),...,x(N)) \in \mathbf{R}^{n\cdot N}$. It is easy to see that the backward affine equation (3.34) may be written as

$$\kappa(t) = \tilde{\mathcal{A}}_t^*\kappa(t+1) + \tilde{g}(t) \tag{5.22}$$

with $\tilde{g}(t) = (\tilde{g}(t,1),...,\tilde{g}(t,N))$. Under the considered assumptions it follows that $|\tilde{g}(t)|_1 \leq \mu$, where $\mu > 0$ is independent of t.

From Corollary 5.2 it follows that the sequence $\{\tilde{\mathcal{A}}_t\}_{t\geq 0}$ defines an exponentially stable evolution. The conclusion of Lemma 3.8 follows now applying Theorem 3.5 in [20] to the equation (5.19).

5.3 The proof of Lemma 3.14

It is easy to see that if $f_k(t,i) = 0$, $0 \leq k \leq r$, $i \in \mathcal{D}$, $t \in \mathbf{Z}_+$ then $\tilde{\kappa}(t,i) = 0$; this leads to $\tilde{\psi}(t,i) = 0$, $t \geq 0$, $i \in \mathcal{D}$. In this case the control (3.35) reduces to $\tilde{u}(t) = \tilde{F}(t,\eta_t)\tilde{x}(t)$, $\tilde{x}(t)$ being the solution of (3.30). Therefore conditions (2.4) is fulfilled. If $f_k(t,i) \neq 0$ then, based on Corollary 5.4 (i) applied to the system (3.37) one obtains that the sequence $\{\tilde{x}(t)\}_{t\geq 0}$ is bounded in mean square.

From (3.35) one obtains that $\tilde{u}(t)$ is bounded in mean square too. These guarantee the fact that $\tilde{u}$ is an admissible control and that $J(t_0,\infty,x_0,\tilde{u}) < +\infty$ and thus the proof is complete.

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