EXPONENTIAL STABILITY IN MEAN SQUARE FOR A GENERAL CLASS OF DISCRETE-TIME LINEAR STOCHASTIC SYSTEMS

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ABSTRACT

The problem of the mean square exponential stability for a class of discrete-time linear stochastic systems subject to independent random perturbations and Markovian switching is investigated. The case of the linear systems whose coefficients depend both to present state and the previous state of the Markov chain is considered. Three different definitions of the concept of exponential stability in mean square are introduced and it is shown that they are not always equivalent. One definition of the concept of mean square exponential stability is done in terms of the exponential stability of the evolution defined by a sequence of linear positive operators on an ordered Hilbert space. The other two definitions are given in terms of different types of exponential behavior of the trajectories of the considered system. In our approach the Markov chain is not prefixed. The only available information about the Markov chain is the sequence of probability transition matrices and the set of its states. In this way one obtains that if the system is affected by Markovian jumping the property of exponential stability is independent of the initial distribution of the Markov chain.

The definition expressed in terms of exponential stability of the evolution generated by a sequence of linear positive operators, allows us to characterize the mean square exponential stability based on the existence of some quadratic Lyapunov functions.

The results developed in this paper may be used to derive some procedures for designing stabilizing controllers for the considered class of discrete-time linear stochastic systems in the presence of a delay in the transmission of the date.

Keywords: discrete-time linear stochastic systems, independent random perturbations, Markov chains, mean square exponential stability, Lyapunov operators, delay in transmission of the date.

1 INTRODUCTION

Stability is one of the main tasks in the analysis and synthesis of a controller in many control problems such as: linear quadratic regulator, H_2 optimal control, H_∞-control and other robust control problems (see e.g. [9, 15, 25] for the continuous time case or [1, 4, 5, 14] for the discrete time case and references therein).

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For linear stochastic systems there are various types of stability. However one of the most popular among them is "***exponential stability in mean square***" (ESMS). This is due to the existence of some efficient algorithms to check this property.

In the literature, the problem of ESMS of discrete-time linear stochastic systems was investigated separately for the linear systems subject to independent random perturbations and for linear systems affected by perturbations described by a Markov chain. For the readers' convenience we refer to [21, 22, 26] for the case of discrete time linear stochastic systems with independent random perturbations and to [2, 3, 11, 12, 13, 14, 16, 17, 18, 19, 20, 23] for the case of discrete time linear stochastic systems with Markovian jumps perturbations. The majority of the works where the problem of ESMS of discrete time linear systems with Markovian switching is investigated, deals with the so called time invariant case. That is the case of discrete time linear systems with the matrix coefficients not depending upon time and the Markov chain being a homogeneous one. There are papers [18, 20, 23] where the matrix coefficients of the system are time dependent and the Markov chain is homogeneous. There are also papers [11, 16] where the matrix coefficients of the discrete time system do not depend upon time but the Markov chain is an inhomogeneous one.

In [8] was considered the general situation of the discrete-time time-varying linear stochastic systems corrupted by independent random perturbations and by jump Markov perturbations. Four different definitions of the concept of ESMS were introduced and it was shown that they are not always equivalent. The general case of discrete-time time-varying linear systems with Markovian switching was investigated in [10].

The aim of the present paper is to extend the results of [8] to the case of discrete-time time-varying linear stochastic systems whose coefficients depend both to the present state η_t and the previous state η_{t-1} of the Markov chain. We assume that the matrix coefficients may depend upon time and the Markov chain is not necessarily a homogeneous one. Such systems arise in connection with the problem of designing of a stabilizing feedback gain in the presence of some delay in transmission of the date either on the channel from the sensors to controller or between controller and actuators. For this class of systems we introduce three different definitions of the concept of exponential stability in mean square. One of these definitions characterizes the concept of exponential stability in mean square in terms of exponential stability of the evolution generated by a suitable sequence of linear positive operators associated to the considered stochastic system. This type of exponential stability in mean square which we will called "strong exponential stability in mean square" (SESMS) is equivalent to the existence of a quadratic Lyapunov function; meaning that it is equivalent to the solvability of some systems of linear matrix equations or linear matrix inequations.

Other two types of exponential stability in mean square are stated in terms of exponential behavior of the state space trajectories of the considered stochastic systems.

We show that the three definitions of the exponential stability in mean square are not always equivalent. Also, we prove that under some additional assumptions a part of types of exponential stability in mean square become equivalent. In the case of discrete-time linear stochastic systems with periodic coefficients all these three types of exponential stability in mean square introduced in the paper become equivalent.

It must be remarked that in our approach the Markov chain is not prefixed. The only available information about the Markov chain consists of the sequence of transition probability matrices $\{P_t\}_{t\geq 0}$ and the set $\mathcal{D}$ of its states. The initial distributions of the Markov chain do not play any role in defining and characterizing the exponential stability in mean square. However, under some additional assumptions (see Theorem 3.4, Theorem 3.7) it is sufficient to have exponentially stable behavior of the state space trajectories of the considered stochastic system corresponding to a suitable Markov chain (for example the Markov chain with initial distribution $\pi_0 = (\frac{1}{N}, ..., \frac{1}{N})$) in order to have exponentially stable evolution of state space trajectories of the stochastic system corresponding to any Markov chain having the same sequence of transition probability matrix $\{P_t\}_{t\geq 0}$ and the same set of the states $\mathcal{D}$.

The outline of the paper is the follows:

Section 2 contains the description of the mathematical model of the systems under consideration together with the definitions of the three types of exponential stability in mean square.

In Section 3 we investigate the relations existing between the types of exponential stability in mean square introduced before. One shows that in the absence of some additional assumptions they are not equivalent.

Section 4 contains several Lyapunov type criteria for ESMS while in Section 5 we show how a part of the results of [8] can be recovered from the results proved in this paper.

In Section 6 we illustrate the applicability of the methodology developed in this paper to the problem of designing of the stabilizing static output feedback in the presence of some delays in the transmission of the data. Finally, in Section 7 we provide the proof of a representation theorem stated in Section 2. This proof was moved at the end of the paper due to its strong probabilistic character.

2 THE PROBLEM

2.1 Description of the systems

Let us consider discrete-time linear stochastic systems of the form:

$$x(t+1) = [A_0(t,\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} A_k(t,\eta_t,\eta_{t-1})w_k(t)]x(t) \tag{2.1}$$

$t \geq 1$, $t \in \mathbf{Z}_+$, where $x \in \mathbf{R^n}$ and $\{\eta_t\}_{t\geq 0}$ is a Markov chain defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with the state space the finite set $\mathcal{D} = \{1, 2, ..., N\}$ and the sequence of transition probability matrices $\{P_t\}_{t\geq 0}$. This means that for $t \geq 0$, P_t are stochastic matrices of size N, with the property:

$$\mathcal{P}\{\eta_{t+1} = j \mid \mathcal{G}_t\} = p_t(\eta_t, j) \tag{2.2}$$

for all $j \in \mathcal{D}$, $t \geq 0$, $t \in \mathbf{Z}_+$, where $\mathcal{G}_t = \sigma[\eta_0, \eta_1, ..., \eta_t]$ is the σ-algebra generated by the random variables $\eta_s, 0 \leq s \leq t$, $\{w(t)\}_{t\geq 0}$ is a sequence of independent random vectors, $w(t) = (w_1(t), ..., w_r(t))^*$.

It can be seen that if $\mathcal{P}\{\eta_t = i\} > 0$ then

$$\mathcal{P}\{\eta_{t+1} = j \mid \eta_t = i\} = p_t(i, j) \tag{2.3}$$

$p_t(i, j)$ being the entries of the transition probability matrix P_t. For more details concerning the properties of Markov chains and of the sequences $\{P_t\}_{t\geq 0}$ of stochastic matrices we refer to [6].

If $P_t = P$ for all $t \geq 0$ then the Markov chain is known as a *homogeneous Markov chain*.

In this paper we investigate several aspects of the issue of exponential stability in mean square of the solution $x = 0$ of the system (2.1). Our aim is to relieve some difficulties which are due to the fact that the coefficients of the system are time varying and η_t is an inhomogeneous Markov chain. A motivation to study the problem of exponential stability in mean square for the case of stochastic systems of type (2.1) is provided in Section 6 from below.

In [8] a detailed investigation of the problem of exponential stability in mean square for the following special case of (2.1)

$$x(t+1) = [A_0(t,\eta_t) + \sum_{k=1}^{r} A_k(t,\eta_t)w_k(t)]x(t) \tag{2.4}$$

$t \geq 0$, $t \in \mathbf{Z}_+$ was made. In that paper was shown that there are several ways to introduce the concept of the exponential stability in mean square in the case of systems of type (2.4). Also was proved that those different definitions of exponential stability in mean square are not always equivalent. In Section

5 from below we shall show that part of the results from [8] can be recovered as special cases of the results proved in the present paper.

Throughout this paper we assume that the following assumptions are fulfilled:

$\mathbf{H_1}$) The processes $\{\eta_t\}_{t\geq 0}$ and $\{w(t)\}_{t\geq 0}$ are independent stochastic processes.

$\mathbf{H_2}$) $E[w(t)] = 0, t \geq 0$, $E[w(t)w^T(t)] = I_r,\ t \geq 0, I_r$ being the identity matrix of size r.

Set $\pi_t = (\pi_t(1), \pi_t(2), ..., \pi_t(N))$ the distribution of the random variable η_t. That is $\pi_t(i) = \mathcal{P}\{\eta_t = i\}$. It can be verified that the sequence $\{\pi_t\}_{t\geq 0}$ solves:

$$\pi_{t+1} = \pi_t P_t, \quad t \geq 0. \tag{2.5}$$

Remark 2.1 a) From (2.5) it follows that it is possible that $\pi_t(i) = 0$ for some $t \geq 1$, $i \in \mathcal{D}$ even if $\pi_0(j) > 0$, $1 \leq j \leq N$. This is specific for the discrete-time case. It is known (see [6]) that in the continuous time case $\pi_t(i) > 0$ for all $t > 0$ and $i \in \mathcal{D}$, if $\pi_0(j) > 0$ for all $1 \leq j \leq N$.

b) The only available information concerning the system (2.1) is the set $\mathcal{D}$ and the sequences $\{P_t\}_{t\in\mathbf{Z}_+}$, $\{A_k(t,i,j)\}_{t\geq 1}, 0 \leq k \leq r, i,j \in \mathcal{D}$. The initial distributions of the Markov chain are not prefixed. Hence, throughout the paper by a Markov chain we will understand any triple $\{\{\eta_t\}_{t\geq 0},\ \{P_t\}_{t\geq 0},\ \mathcal{D}\}$ where $\mathcal{D}$ is a fixed set $\mathcal{D} = \{1, 2, ..., N\}$, $\{P_t\}_{t\geq 0}$ is a given sequence of stochastic matrices and $\{\eta_t\}_{t\geq 0}$ is an arbitrary sequence of random variables taking values in $\mathcal{D}$ and satisfying (2.2).

Remark 2.1 a) allows us to define the following subsets of the set $\mathcal{D}$:

$$\mathcal{D}_s = \{i \in \mathcal{D} | \pi_s(i) > 0\} \tag{2.6}$$

for each integer $s \geq 0$.

Set $\mathcal{A}(t) = A_0(t, \eta_t, \eta_{t-1}) + \sum_{k=1}^{r} A_k(t, \eta_t, \eta_{t-1}) w_k(t), t \geq 1$ and define $\Theta(t,s)$ as follows: $\Theta(t,s) = \mathcal{A}(t-1)\mathcal{A}(t-2)...\mathcal{A}(s)$ if $t \geq s+1$ and $\Theta(t,s) = I_n$ if $t = s, s \geq 1$.

Any solution $x(t)$ of (2.1) verifies

$$x(t) = \Theta(t,s)x(s)$$

$\Theta(t,s)$ will be called the fundamental (random) matrix solution of (2.1).

2.2 Lyapunov type operators

Let $\mathcal{S}_n \in \mathbf{R}^{n\times n}$ be linear subspace of symmetric matrices. Set $\mathcal{S}_n^N = \mathcal{S}_n \oplus \mathcal{S}_n \oplus ... \oplus \mathcal{S}_n$. One can see that $\mathcal{S}_n^N$ is a real ordered Hilbert space (see [7], Example 2.5(iii)). The usual inner product on $\mathcal{S}_n^N$ is

$$\langle X, Y \rangle = \sum_{i=1}^{N} Tr(Y(i)X(i)) \tag{2.7}$$

for all $X = (X(1), ..., X(N))$ and $Y = (Y(1), ..., Y(N))$ from $\mathcal{S}_n^N$.

Consider the sequences $\{A_k(t,i,j)\}_{t\geq 1}, A_k(t,i,j) \in \mathbf{R}^{n\times n}, 0 \leq k \leq r,\ i,j \in \mathcal{D},\ \{P_t\}_{t\geq 0}, P_t = (p_t(i,j)) \in \mathbf{R}^{N\times N}$.

Based on these sequences we construct the linear operators $\Upsilon_t : \mathcal{S}_n^N \to \mathcal{S}_n^N$, $\Upsilon_t S = (\Upsilon_t S(1), ..., \Upsilon_t S(N))$ with

$$\Upsilon_t S(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_{t-1}(j,i) A_k(t,i,j) S(j) A_k^T(t,i,j) \tag{2.8}$$

$t \geq 1, S \in \mathcal{S}_n^N$.

If $A_k(t,i,j), p_t(i,j)$ are related to the system (2.1) then the operators Υ_t are called the Lyapunov type operators associated to the system (2.1).

By direct calculation based on the definition of the adjoint operator with respect to the inner product (2.7) one obtains that $\Upsilon_t^* S = (\Upsilon_t^* S(1), ..., \Upsilon_t^* S(N))$ with

$$\Upsilon_t^* S(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p_{t-1}(i,j) A_k^T(t,j,i) S(j) A_k(t,j,i) \tag{2.9}$$

$t \geq 1, S \in \mathcal{S}_n^N$.

Let $R(t,s)$ be the linear evolution operator defined on $\mathcal{S}_n^N$ by the sequence $\{\Upsilon_t\}_{t\geq 1}$. Hence $R(t,s) = \Upsilon_{t-1}\Upsilon_{t-2}...\Upsilon_s$ if $t \geq s+1$ and $R(t,s) = \mathcal{I}_{\mathcal{S}_n^N}$ if $t = s \geq 1$.

If $X(t)$ is a solution of discrete time linear equation on $\mathcal{S}_n^N$:

$$X_{t+1} = \Upsilon_t X_t \tag{2.10}$$

then $X_t = R(t,s)X_s$ for all $t \geq s \geq 1$.

The next result provides a relationship between the operators $R^*(t,s)$ and the fundamental matrix solution $\Theta(t,s)$ of the system (2.1).

Theorem 2.2 *Under the assumption* $\mathbf{H}_1, \mathbf{H}_2$ *the following equality holds:*

$$(R^*(t,s)H)(i) = E[\Theta^T(t,s)H(\eta_{t-1})\Theta(t,s)|\eta_{s-1} = i] \tag{2.11}$$

for all $H = (H(1), ..., H(N)) \in \mathcal{S}_n^N, t \geq s \geq 1, i \in \mathcal{D}_{s-1}$.

Proof: see section 7.

2.3 Definitions of mean square exponential stability

Now we are in position to state the concept of exponential stability in mean square of the zero state equilibrium of the system (2.1).

In this subsection we introduce three different definitions of the concept of exponential stability in mean square. The first definition introduces the exponential stability in mean square for the system (2.1) in terms of the exponential stability of the evolution defined by the deterministic discrete time equation (2.10). The other two definitions are expressed in terms of exponentially stable behavior of the state space trajectories of the system (2.1).

Definition 2.1 a) We say that the zero state equilibrium of the system (2.1) is *strongly exponentially stable in mean square* (SESMS) if there exist $\beta \geq 1, q \in (0,1)$ such that

$$\|R(t,s)\| \leq \beta q^{t-s} \tag{2.12}$$

for all $t \geq s \geq 1$.

Since $\mathcal{S}_n^N$ is a finite dimensional linear space, in (2.12) one can take any norm from $\mathcal{B}(\mathcal{S}_n^N)$.

Definition 2.2 We say that the zero state equilibrium of the system (2.1) is *exponentially stable in mean square with conditioning* (ESMS-C) if there exist $\beta \geq 1, q \in (0,1)$ such that for any sequence of independent random vectors $\{w(t)\}_{t\geq 1}$ and for any Markov chain $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ which satisfy $\mathbf{H}_1, \mathbf{H}_2$ we have:

$$E[|\Theta(t,s)x|^2|\eta_{s-1} = i] \leq \beta q^{t-s}|x|^2 \tag{2.13}$$

for all $t \geq s \geq 1, x \in \mathbf{R}^n, i \in \mathcal{D}_{s-1}$.

Definition 2.3 We say that the zero state equilibrium of the system (2.1) is *exponentially stable in mean square* (ESMS) if there exist $\beta \geq 1, q \in (0,1)$ such that for any sequence of independent random vectors $\{w(t)\}_{t\geq 1}$ and for any Markov chain $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ which satisfy $\mathbf{H}_1, \mathbf{H}_2$ we have:

$$E[|\Theta(t,s)x|^2] \leq \beta q^{t-s}|x|^2 \tag{2.14}$$

for all $t \geq s \geq 1, x \in \mathbf{R}^n$.

It can be seen that the concept of strong exponential stability in mean square introduced by the Definition 2.1 does not depend upon the initial distribution of the Markov chain. It depends only on the sequences $\{A_k(t,i,j)\}_{t\geq 0}$, $\{P_t\}_{t\geq 0}$. Also it must be remarked that in the definitions of the exponential stability in mean square in terms of the state space trajectories, the sequences $\{w(t)\}_{t\geq 0}, \{\eta_t\}_{t\geq 0}$ are not prefixed. We shall see later (see Theorem 3.4 and Theorem 3.7) that under some additional assumptions the exponentially stable behavior of the trajectories of the system (2.1) for a suitable pair $(\{w(t)\}_{t\geq 0}, \{\eta_t\}_{t\geq 0})$ is enough to guarantee the exponentially stable behavior of the trajectories of the system (2.1) for all pairs $(\{w(t)\}_{t\geq 0}, \{\eta_t\}_{t\geq 0})$ which verify $\mathbf{H_1}, \mathbf{H_2}$.

3 EXPONENTIAL STABILITY IN MEAN SQUARE

In this section we establish the relations between the concepts of exponential stability in mean square introduced in Definition 2.1 -Definition 2.3. Firstly we shall show that in the general case of the time varying system 2.1 these definitions are not, in general,equivalent. Finally we show that in the case of systems 2.1 with periodic coefficients these definitions become equivalent.

3.1 The general case

On the space $\mathcal{S}_n^N$ one introduced the norm $|\cdot|_1$ by:

$$|X|_1 = max_{i\in\mathcal{D}}|X(i)| \tag{3.1}$$

where $|X(i)|$ is the spectral norm of the matrix $X(i)$. Together with the norm $|\cdot|_2$ induced by the inner product (2.7), the norm $|\cdot|_1$ will play an important role in the characterization of the strong exponential stability (SESMS).

If $T : \mathcal{S}_n^N \to \mathcal{S}_n^N$ is a linear operator then $\|T\|_k$ is the operator norm induced by $|\cdot|_k, k = 1, 2$.

We recall that (see Proposition 2.2 in [7]) if T is a linear and positive operator on $\mathcal{S}_n^N$ then

$$\|T\|_1 = |TJ|_1 \tag{3.2}$$

where $J = (I_n, ..., I_n) \in \mathcal{S}_n^N$.

Since $\|R(t,s)\|$ and $\|R^*(t,s)\|_1$ are equivalent, from (2.12) it follows that $\|R^*(t+1,t)\|_1 \leq \beta_1, t \geq 0$, hence by using (3.2) and (2.9) one obtains:

Corollary 3.1 *If the zero state equilibrium of the system 2.1 is (SESMS) then* $\{\sqrt{p_{t-1}(i,j)}A_k(t,j,i)\}_{t\geq 1}, i,j \in \mathcal{D}, 0 \leq k \leq r,$ *are bounded sequences.*

Now we prove:

Theorem 3.2 *Under the assumptions* $\mathbf{H}_1, \mathbf{H}_2$ *we have:*

(i) *If the zero state equilibrium of the system (2.1) is SESMS then it is ESMS-C.*

(ii) *If the zero state equilibrium of the system (2.1) is ESMS-C then it is ESMS.*

The proof of (i) follows immediately from (2.11). (ii) follows from the inequality

$$E[|\Theta(t,s)x|^2] \leq \sum_{i\in\mathcal{D}_{s-1}} E[|\Theta(t,s)x|^2|\eta_{s-1} = i].$$

As we can see in Example 3.5 and Example 3.6 from below the validity of the converse implications from the above theorem is not true in the absence of some additional assumptions.

Definition 3.3 We say that a stochastic matrix $P_t \in \mathbf{R}^{N\times N}$ is a *non-degenerate stochastic matrix* if for any $j \in \mathcal{D}$ there exists $i \in \mathcal{D}$ such that $p_t(i,j) > 0$.

We remark that if $P_t, t \geq 0$ are nondegenerate stochastic matrices then from (2.5) it follows that $\pi_t(i) > 0, t \geq 1, i \in \mathcal{D}$ if $\pi_0(i) > 0$ for all $i \in \mathcal{D}$.

Now we have:

Theorem 3.4 *If for all $t \geq 0$, P_t are nondegenerate stochastic matrices then the following are equivalent;*

(i) The zero state equilibrium of the system (2.1) is SESMS.

(ii) The zero state equilibrium of the system (2.1) is ESMS-C.

(iii) There exist a sequence of independent random vectors $\{w(t)\}_{t\geq 1}$ and a Markov chain $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ with $\mathcal{P}(\eta_0 = i) > 0, i \in \mathcal{D}$ satisfying $\mathbf{H}_1, \mathbf{H}_2$ such that

$$E[|\Theta(t,s)x|^2 | \eta_{s-1} = i] \leq \beta q^{t-s}|x|^2 \tag{3.3}$$

for all $t \geq s \geq 1, i \in \mathcal{D}, x \in \mathbf{R}^n$ where $\beta \geq 1, q \in (0,1)$.

Proof $(i) \to (ii)$ follows from Theorem 3.2 and $(ii) \to (iii)$ is obvious. It remains to prove implication $(iii) \to (i)$. Applying Theorem 2.2 for $H = J = (I_n, ..., I_n)$ one obtains from (3.3) that $x_0^T[R^*(t,s)J](i)x_0 \leq \beta q^{t-s}|x_0|^2$ for all $t \geq s \geq 1,\ i \in \mathcal{D},\ x_0 \in \mathbf{R}^n$.

This allows us to conclude that $|[R^*(t,s)J](i)| \leq \beta q^{t-s}$ for all $t \geq s \geq 1,\ i \in \mathcal{D}$. Based on (3.1) one gets $|R^*(t,s)J|_1 \leq \beta q^{t-s}$ for all $t \geq s$. Finally from (3.2) it follows that $\|R^*(t,s)\|_1 \leq \beta q^(t-s)$ for all $t \geq s \geq 1$. Thus the proof is complete, since $\|R(t,s)\|$ and $\|R(t,s)^*\|_1$ are equivalent.

The next two examples show that the converse implication in Theorem 3.2 are not always true.

Example 3.5 Consider the system (2.1) in the particular case $n = 1, N = 2$ described by:

$$x(t+1) = [a_0(t, \eta_t, \eta_{t-1}) + \sum_{k=1}^{r} a_k(t, \eta_t, \eta_{t-1})w_k(t)]x(t) \tag{3.4}$$

where $a_k(t,i,j) = 0, i,j \in \{1,2\}, t \geq 1, 1 \leq k \leq r, a_0(t,i,1) = 0, a_0(t,i,2) = 2^{\frac{t-1}{2}}, t \geq 1, i \in \{1,2\}$. The transition probability matrix is

$$P_t = \begin{pmatrix} 1 - \frac{1}{4^{t+1}} & \frac{1}{4^{t+1}} \\ 1 - \frac{1}{4^{t+1}} & \frac{1}{4^{t+1}} \end{pmatrix}, t \geq 0. \tag{3.5}$$

We have

$$\sqrt{p_{t-1}(2,1)}a_0(t,1,2) = (1 - \frac{1}{4^t})^{\frac{1}{2}} 2^{\frac{t-1}{2}}.$$

Hence the sequence $\{\sqrt{p_{t-1}(2,1)}a_0(t,1,2)\}_{t\geq 1}$ is unbounded. Then we deduce via Corollary 3.1 that the zero state equilibrium of the system (3.4) is not SESMS. On the other hand one sees that the transition probability matrix (3.5) is a non-degenerate stochastic matrix. Thus, via Theorem 3.4, we deduce that the zero state equilibrium of (3.4) cannot be ESMS-C.

We show now that the zero state equilibrium of (3.4) is ESMS. We write

$$E[|\Phi(t,s)x|^2] = x^2 E[a_0^2(t-1, \eta_{t-1}, \eta_{t-2})a_0^2(t-2, \eta_{t-2}, \eta_{t-3})...a_0^2(s, \eta_s, \eta_{s-1})] =$$

$$x^2 \sum_{i_{t-1}=1}^{2} \sum_{i_{t-2}=1}^{2} ... \sum_{i_{s-1}=1}^{2} a_0^2(t-1, i_{t-1}, i_{t-2})a_0^2(t-2, i_{t-2}, i_{t-3})...a_0^2(s, i_s, i_{s-1})$$

$$\mathcal{P}\{\eta_{t-1} = i_{t-1}, \eta_{t-2} = i_{t-2}, ..., \eta_{s-1} = i_{s-1}\} =$$

$$x^2 \mathcal{P}(\eta_{s-1} = 2)a_0^2(s,2,2)...a_0^2(t-2,2,2)p_{s-1}(2,2)p_s(2,2)...p_{t-3}(2,2)$$

$$[a_0^2(t-1,1,2)p_{t-2}(2,1) + a_0^2(t-1,2,2)p_{t-2}(2,2)] \leq$$

$$x^2 2^{s-1+s+...+t-3} \frac{1}{4^{s+s+1+...t-2}} 2^{t-2} = x^2 2^{\frac{(t+s-3)(t-s)}{2}} \frac{1}{2^{(t+s-2)(t-s-1)}}$$

if $t \geq s+2$. Finally one obtains that

$$E[|\Theta(t,s)x|^2] \leq x^2 \frac{1}{\sqrt{2}^{t^2-s^2-3t-s+4}} \tag{3.6}$$

if $t \geq s+2$.

For $t \geq s+2$ we have $t^2-4t-s^2+4=(t-2)^2-s^2 \geq 0$. This means that $t^2-s^2-3t-s+4 \geq t-s$. From (3.6) one obtains that

$$E[|\Phi(t,s)x|^2] \leq x^2 \frac{1}{\sqrt{2}^{t-s}} x^2 \tag{3.7}$$

for all $t \geq s+2, s \geq 1$. Further we compute

$$E[|\Theta(s+1,s)x|^2] = x^2 E[a_0^2(s,\eta_s,\eta_{s-1})] = x^2 a_0^2(s,1,2)\mathcal{P}\{\eta_{s-1}=2,\eta_s=1\}+$$

$$a_0^2(s,2,2)\mathcal{P}\{\eta_{s-1}=2,\eta_s=2\} \leq x^2 2^{s-1}\mathcal{P}\{\eta_{s-1}=2\}[p_{s-1}(2,1)+p_{s-1}(2,2)].$$

Thus we get

$$E[|\Theta(s+1,s)x|^2] \leq x^2 2^{s-1}\mathcal{P}\{\eta_{s-1}=2\}, \ \ (\forall) s \geq 1. \tag{3.8}$$

Take $s \geq 2$ and write

$$E[|\Theta(s+1,s)x|^2] \leq x^2 2^{s-1}[\mathcal{P}\{\eta_{s-2}=1\}p_{s-2}(1,2)+\mathcal{P}\{\eta_{s-2}=2\}p_{s-2}(2,2)] \leq x^2 2^s \frac{1}{4^{s-1}}$$

which leads to

$$E[|\Theta(s+1,s)x|^2] \leq x^2, \ \ \forall s \geq 2. \tag{3.9}$$

Further

$$E[|\Theta(2,1)x|^2] = E[a_0^2(1,\eta_1,\eta_0)]x^2 = x^2[\mathcal{P}\{\eta_0=2,\eta_1=1\}+\mathcal{P}\{\eta_0=2,\eta_1=2\}] = x^2\mathcal{P}\{\eta_0=2\} \ \ i.e.$$

$$E[|\Theta(2,1)x|^2] \leq x^2. \tag{3.10}$$

Finally

$$E[|\Theta(s,s)x|^2] = x^2. \tag{3.11}$$

Combining (3.7),(3.9),(3.10),(3.11) we conclude that

$$E[|\Theta(t,s)x|^2] \leq \sqrt{2}\frac{1}{\sqrt{2}^{t-s}}$$

for all $t \geq s \geq 1, x \in \mathbf{R}$ and thus one obtains that the zero state equilibrium of (3.4) is ESMS.

Example 3.6 Consider the system (2.1) in the particular case $N=2, A_0(t,i,1)=\mathbf{O} \in \mathbf{R}^{n\times n}, A_0(t,i,2)=itI_n, i \in \{1,2\}, t \geq 1$ and $A_k(t,i,j)=\mathbf{O} \in \mathbf{R}^{n\times n}, k \geq 1, i,j \in \{1,2\}, t \geq 1$. The transition probability matrix is $P=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. We have

$$\sqrt{p(2,1)}A_0(t,1,2)=tI_n$$

thus one obtains via Corollary 3.1 that the zero state equilibrium of the considered system cannot be SESMS.

On the other hand we have $\mathcal{P}\{\eta_t=2\}=0$ a.s. for all $t \geq 1$. This leads to $\eta_t=1 a.s., t \geq 1$. Hence $\Theta(t,s)=0 a.s.$ if $t \geq \max\{3,s+1\}, s \geq 1$ for any Markov chain $(\{\eta_t\}_{t\geq 0},P,\{1,2\})$. This shows that in this particular case the zero state equilibrium of the considered system is both ESMS-C, as well as ESMS.

3.2 The periodic case

In this subsection we show that under the periodicity assumption the concepts of exponential stability introduced by Definitions 2.1, 2.2, 2.3 become equivalent.

Firstly we introduce:

Definition 3.1 We say that the zero state equilibrium of the system (2.1) is *asymptotically stable in mean square* (ASMS) if for any sequence of independent random vectors $\{w(t)\}_{t\geq 1}$ and for any Markov chain $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ which satisfy $\mathbf{H}_1, \mathbf{H}_2$ we have:

$$\lim_{t\to\infty} E[|\Theta(t,1)x|^2] = 0, \forall x \in \mathbf{R}^n.$$

Now we are in position to state:

Theorem 3.7 *Assume that there exists an integer $\theta \geq 1$ such that $A_k(t+\theta,i,j) = A_k(t,i,j), 0 \leq k \leq r, i,j \in \mathcal{D}, P_{t+\theta} = P_t, t \geq 0$. Under these conditions the following are equivalent:*

(i) The zero state equilibrium of the system (2.1) is (SESMS).

(ii) The zero state equilibrium of the system (2.1) is (ESMS-C).

(iii) The zero state equilibrium of the system (2.1) is (ESMS).

(iv) The zero state equilibrium of the system (2.1) is (ASMS).

(v) There exist a sequence of independent random vectors $\{w(t)\}_{t\geq 1}$ and a Markov chain $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ with $\mathcal{P}(\eta_0 = i) > 0, i \in \mathcal{D}$ satisfying $\mathbf{H}_1, \mathbf{H}_2$ such that

$$\lim_{t\to\infty} E[|\Theta(\theta t,1)x|^2] = 0 \tag{3.12}$$

for all $x \in \mathbf{R}^n$.

(vi) $\rho[R(\theta+1,1)] < 1$ where $\rho[\cdot]$ is the spectral radius.

Proof. The implications $(i) \to (ii) \to (iii)$ follows from Theorem 3.2. The implications $(iii) \to (iv) \to (v)$ are straightforward. Now we prove the implication $(v) \to (vi)$.

From (3.12) together with the equality:

$$E[|\Theta(\theta t,1)x|^2] = \sum_{i=1}^{N} \pi_0(i) E[|\Theta(\theta t,1)x|^2 | \eta_0 = i]$$

we deduce that

$$\lim_{t\to\infty} E[\Theta(\theta t,1)x|^2 | \eta_0 = i] = 0, i \in \mathcal{D}. \tag{3.13}$$

Based on (2.11) and (3.13) one gets

$$\lim_{t\to\infty} x^T[(R^*(\theta t,1)J)(i)]x = 0. \tag{3.14}$$

If we take into account the definition of the norm $|\cdot|_1$ we may write

$$\lim_{t\to\infty} |R^*(\theta t,1)J|_1 = 0$$

or equivalently

$$\lim_{t\to\infty} \|R^*(\theta t,1)\|_1 = 0. \tag{3.15}$$

Since $\|\cdot\|_1, \|\cdot\|_2$ are equivalent norms on $\mathcal{S}_n^N$ and $\|R^*(\theta t,1)\|_2 = \|R(\theta t,1)\|_2$ one obtains from (3.15) that $\lim_{t\to\infty} \|R(\theta t,1)\|_1 = 0$. Using the fact that $R(\theta t+1,1) = \Upsilon_{\theta t} R(\theta t,1) = \Upsilon_\theta R(\theta t,1)$ we deduce that

$$\lim_{t\to\infty} \|R(\theta t+1,1)\|_1 = 0. \tag{3.16}$$

Based on periodicity of the coefficients, one shows inductively that $R(\theta t+1,1) = (R(\theta+1,1))^t$ for all $t \geq 1$. Thus (3.16) may be rewritten $\lim_{t\to\infty} \|(R(\theta+1),1)^t\|_1 = 0$.

From the definition of the spectral radius we conclude that (3.16) is equivalent to $\rho[R(\theta+1,1)] < 1$. This shows that (vi) holds. If (vi) is true then there exist $\beta \geq 1, q \in (0,1)$ such that $\|(R(\theta+1,1))^t\|_1 \leq \beta q^t$. Further one shows in a standard way that there exists $\beta_1 \geq \beta$ such that $\|R(t,s)\|_1 \leq \beta_1 q^{(t-s)}$ for all $t \geq s \geq 1$ (for more details one can see the proof of implication $(vi) \to (i)$ in Theorem 4.1 in [8]). Thus we obtained that the implication $(vi) \to (i)$ holds and the proof is complete.

Definition 3.2 We say that the system (2.1) is in the time invariant case if $A_k(t,i,j) = A_k(i,j)$ for all $t \geq 1$, $i,j \in \mathcal{D}$, $0 \leq k \leq r$ and $P_t = P$ for all $t \geq 0$.

In this case we have $\Upsilon_t = \Upsilon$, for all $t \geq 1$. One sees that the system (2.1) is in the time invariant case if and only if it is periodic with period $\theta = 1$. Hence, the equivalences from the above theorem hold in the time invariant case too. In this case, the statement (vi) becomes $\rho(\Upsilon) < 1$.

4 LYAPUNOV TYPE CRITERIA

In this section we present several conditions for exponential stability in mean square expressed in terms of solvability of some suitable systems of linear matrix equations or linear matrix inequations. The results of this section are special cases of those stated in a more general framework in [7]. That is why we present them here without proofs.

4.1 The general case

In the light of Theorem 3.2 it follows that the Lyapunov type criteria are necessary and sufficient conditions for SESMS but they are only sufficient conditions for ESMS. Direct from the above Definition 2.1 and Theorem 3.4 in [7] applied to Lyapunov type operators Υ_t we obtain:

Theorem 4.1 *Under the assumptions* $\mathbf{H_1}, \mathbf{H_2}$ *the following are equivalent:*

(i) *The zero state equilibrium of the system (2.1) is SESMS.*

(ii) *The system of backward linear equations*

$$X_t(i) = \sum_{k=o}^{r}\sum_{j=1}^{N} p_{t-1}(i,j)A_k^T(t,j,i)X_{t+1}(j)A_k(t,j,i) + I_n \tag{4.1}$$

$t \geq 1$, $i \in \mathcal{D}$ *has a bounded solution* $X_t = (X_t(1), ..., X_t(N))$ *with* $X_t(i) \geq I_n, t \geq 1$.

(iii) *There exist a bounded sequence* $\{Y_t\}_{t\geq 1} \in \mathcal{S}_n^N$ *and scalars* $\alpha > 0$, $\delta > 0$ *such that*

$$\sum_{k=0}^{r}\sum_{j=1}^{N} p_{t-1}(i,j)A_k^T(t,j,i)Y_{t+1}(j)A_k(t,j,i) - Y_t(i) \leq -\alpha I_n \tag{4.2}$$

$Y_t(i) \geq \delta I_n, t \geq 1$, $i \in \mathcal{D}$.

Remark 4.2 Even the system (4.1),((4.2)respectively) consist of an infinite number of equations (inequations respectively) the criteria derived in Theorem 4.1 may be useful to obtain sufficient conditions for ESMS in the general time varying case. This can be illustrated by the next simple example.

Example 4.3 Consider the system (2.1) in the special case $n = 1$:

$$x(t+1) = [a_0(t,\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} a_k(t,\eta_t,\eta_{t-1})w_k(t)]x(t) \tag{4.3}$$

$t \geq 1$, where $a_k(t,i,j) \in \mathbf{R}$ are such that

$$\sup_{t\geq 1} \sum_{k=0}^{r}\sum_{j=1}^{N} p_{t-1}(i,j)a_k^2(t,j,i) < 1. \tag{4.4}$$

Under this condition the zero state equilibrium of (4.3) is SESMS.

Indeed if (4.4) holds then the corresponding system (4.2) associated to (4.3) is fulfilled for $Y_t(i) = 1, t \geq 1, i \in \mathcal{D}$.

4.2 The periodic case

From Theorem 3.5 in [7] one obtains that if the coefficients of (4.1) are periodic with period θ, the unique bounded and positive solution of (4.1) is periodic with the same period θ. Also, if the system of inequalities (4.2) has a bounded and uniform positive solution then it has a periodic solution. This allows us to obtain the following specialized version of Theorem 4.1.

Theorem 4.4 *Under the assumptions of Theorem 3.7, with $\theta \geq 2$, the following are equivalent:*

(i) The zero state equilibrium of (2.1) is ESMS.

(ii) The system of linear matrix equations

$$X_t(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_{t-1}(i,j)A_k^T(t,j,i)X_{t+1}(j)A_k(t,j,i) + I_n \qquad 1 \leq t \leq \theta - 1$$

$$X_\theta(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_{\theta-1}(i,j)A_k^T(\theta,j,i)X_1(j)A_k(\theta,j,i) + I_n \tag{4.5}$$

$i \in \mathcal{D}$ has a solution $X_t = (X_t(1), ..., X_t(N))$ with $X_t(i) > 0$.

(iii) There exist positive definite matrices $Y_t(i), 1 \leq t \leq \theta, i \in \mathcal{D}$, which solve the following system of LMI's:

$$\sum_{k=0}^{r}\sum_{j=1}^{N} p_{t-1}(i,j)A_k^T(t,j,i)Y_{t+1}(j)A_k(t,j,i) - Y_t(i) < 0, \qquad 1 \leq t \leq \theta - 1$$

$$\sum_{k=0}^{r}\sum_{j=1}^{N} p_{\theta-1}(i,j)A_k^T(\theta,j,i)Y_1(j)A_k(\theta,j,i) - Y_\theta(i) < 0 \tag{4.6}$$

$i \in \mathcal{D}$.

It is easy to see that under the conditions of Theorem 3.7 the sequence Υ_t can be extended in a natural way by periodicity, to the whole set of integers $\mathbf{Z}$. In this case we may use Theorem 3.7 *(ii)* and Theorem 3.9 from [7] to obtain:

Theorem 4.5 *Under the assumptions of Theorem 3.7 with $\theta \geq 2$ the following are equivalent:*

(i) The zero state equilibrium of the system (2.1) is ESMS.

(ii) The system of linear matrix equations

$$X_{t+1}(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_{t-1}(j,i)A_k(t,i,j)X_t(j)A_k^T(t,i,j) + I_n \qquad 1 \leq t \leq \theta - 1 \tag{4.7}$$

$$X_1(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_{\theta-1}(j,i)A_k(\theta,i,j)X_\theta(j)A_k^T(\theta,i,j) + I_n$$

$i \in \mathcal{D}$ has a solution $X_t = (X_t(1), ..., X_t(N))$ such that $X_t(i) > 0, i \in \mathcal{D}$.

(*iii*) *There exist positive definite matrices* $Y_t(i), 1 \leq t \leq \theta, i \in \mathcal{D}$, *which solve the following system of LMI's:*

$$\sum_{k=0}^{r}\sum_{j=1}^{N} p_{t-1}(j,i)A_k(t,i,j)Y_t(j)A_k^T(t,i,j) - Y_{t+1}(i) < 0 \qquad 1 \leq t \leq \theta - 1 \tag{4.8}$$

$$\sum_{k-0}^{r}\sum_{j=1}^{N} p_{\theta-1}(j,i)A_k(\theta,i,j)Y_\theta(j)A_k^T(\theta,i,j) - Y_1(i) < 0$$

$i \in \mathcal{D}$.

Remark 4.6 The system of linear equations (4.5), (4.7) and the system of linear inequations (4.6), (4.8) have $\hat{n}$ scalar equations (inequations respectively), with $\hat{n}$ scalar unknowns, where $\hat{n} = \frac{n(n+1)}{2}N\theta$.

4.3 The time invariant case

Using Theorem 3.5 (*iii*), Theorem 3.7 (*iii*) and Theorem 3.9 in [7] one obtains the following Lyapunov type criteria for exponential stability in mean square for the system (2.1) in the time invariant case.

Corollary 4.7 *If the system is in the time invariant case, the following are equivalent:*

(*i*) *The zero state equilibrium of the system (2.1) is ESMS.*

(*ii*) *The system of linear equations*

$$X(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)A_k^T(j,i)X(j)A_k(j,i) + I_n \tag{4.9}$$

$i \in \mathcal{D}$, *has a solution* $X = (X(1), ..., X(N))$ *with* $X(i) > 0, i \in \mathcal{D}$.

(*iii*) *There exist positive definite matrices* $Y(i), i \in \mathcal{D}$, *which solve the following system of LMI's*:

$$\sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)A_k^T(j,i)Y(j)A_k(j,i) - Y(i) < 0 \tag{4.10}$$

$i \in \mathcal{D}$.

Corollary 4.8 *Under the conditions of Corollary 4.7 the following are equivalent:*

(*i*) *The zero state equilibrium of the system (2.1) is ESMS.*

(*ii*) *The system of linear matrix equations*

$$X(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(j,i)A_k(i,j)X(j)A_k^T(i,j) + I_n \tag{4.11}$$

$i \in \mathcal{D}$ *has a solution* $X = (X(1), ..., X(N))$ *with* $X(i) > 0, i \in \mathcal{D}$.

(*iii*) *There exist positive definite matrices* $Y(i), i \in \mathcal{D}$, *which solve the following system of LMI's*:

$$\sum_{k=0}^{r}\sum_{j=1}^{N} p(j,i)A_k(i,j)Y(j)A_k^T(i,j) - Y(i) < 0 \tag{4.12}$$

$i \in \mathcal{D}$.

Remark 4.9 The system of linear equations (4.9) and (4.11) and the system of linear inequations (4.10) and (4.12) have $\check{n}$ scalar equations (inequations respectively) with $\check{n}$ scalar unknowns, where $\check{n} = \frac{n(n+1)}{2}N$.

5 THE CASE OF THE SYSTEMS WITH COEFFICIENTS DEPENDING ONLY ON η_t

There are two ways to write (2.4) in the form of (2.1):

$$\tilde{x}(t+1) = [\tilde{A}(t, \eta_t, \eta_{t-1}) + \sum_{k=1}^{r} \tilde{A}_k(t, \eta_t, \eta_{t-1}) w_k(t)]\tilde{x}(t) \tag{5.1}$$

$t \geq 1$ or

$$\hat{x}(t+1) = [\hat{A}_0(t, \eta_t, \eta_{t-1}) + \sum_{k=1}^{r} \hat{A}_k(t, \eta_t, \eta_{t-1}) \hat{w}_k(t)]\hat{x}(t), t \geq 1 \tag{5.2}$$

where $\tilde{A}_k(t, i, j) = A_k(t, i),\ i, j \in \mathcal{D}, 0 \leq k \leq r, \tilde{x}(t) = x(t), t \geq 1$ and $\hat{A}_k(t, i, j) = A_k(t-1, j),\ i, j \in \mathcal{D}, 0 \leq k \leq r, \hat{x}(t) = x(t-1), \hat{w}_k(t) = w_k(t-1), 1 \leq k \leq r, t \geq 1$ respectively.

In this section we shall see how a part of the results concerning the exponential stability in mean square of the zero solution of the system (2.4) are recovered from the results proved in the previous sections of the present paper.

Let $\tilde{\Theta}(t, s), \hat{\Theta}(t, s)$, be the fundamental matrix solutions of the system (5.1) and the system (5.2) respectively.

$\tilde{\Upsilon}_t, \tilde{R}(t, s), \hat{\Upsilon}_t, \hat{R}(t, s)$ stand for the Lyapunov type operators and corresponding linear evolution operators associated to the system (5.1) and the system (5.2), respectively.

If $\Phi(t, s)$ is the fundamental matrix solution of the system (2.4) then the following equalities hold:

$$\tilde{\Theta}(t, s) = \Phi(t, s), t \geq s \geq 1, \tag{5.3}$$

$$\hat{\Theta}(t, s) = \Phi(t-1, s-1), t \geq s \geq 1. \tag{5.4}$$

In [8] the following Lyapunov type operators were associated to the system (2.4): $\mathcal{L}_t S = (\mathcal{L}_t S(1),\ ...,\ \mathcal{L}_t S(N))$ by

$$\mathcal{L}_t S(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p_t(j, i) A_k(t, j) S(j) A_k^T(t, j) \tag{5.5}$$

for all $1 \leq i \leq N,\ t \geq 0$ and $\Lambda_t S = (\Lambda_t S(1),\ ...,\ \Lambda_t S(N))$,

$$\Lambda_t S(i) = \sum_{k=0}^{r} A_k(t, i) \sum_{j=1}^{N} p_{t-1}(j, i) S(j) A_k^T(t, i) \tag{5.6}$$

$t \geq 1$ for all $1 \leq i \leq N,\ S = (S(1),\ ...,\ S(N)) \in \mathcal{S}_n^N$.

Let $T(t, s), S(t, s)$, be the linear evolution operators defined by the sequences $\{\mathcal{L}_t\}_{t \geq 0}$ and $\{\Lambda_t\}_{t \geq 1}$ respectively.

The following four equalities are straightforward:

$$\tilde{\Upsilon}_t H = \Lambda_t H,\ t \geq 1, \tag{5.7}$$

$$\hat{\Upsilon}_t H = \mathcal{L}_{t-1} H,\ t \geq 1, \tag{5.8}$$

for all $H = (H(1), H(2), , H(N)) \in \mathcal{S}_n^N$,

$$\tilde{R}(t, s) = S(t, s),\ t \geq s \geq 1, \tag{5.9}$$

$$\hat{R}(t,s) = T(t-1,s-1),\ t \geq s \geq 1. \tag{5.10}$$

For the readers convenience we recall the following definition:

Definition 5.1(see Definition 3.5 in [8]).

a) We say that the zero state equilibrium of the system (2.4) is **strongly exponentially stable in mean square** (SESMS) if the corresponding sequence of Lyapunov operators $\{\mathcal{L}_t\}_{t\geq 0}$ generates an exponentially stable evolution on $\mathcal{S}_n^N$.

b) We say that the zero state equilibrium of the system (2.4) is **exponentially stable in mean square with conditioning of type I** (ESMS-CI) if there exist $\beta \geq 1,\ q \in (0,1)$ such that for any sequence of independent random vectors $\{w(t)\}_{t\geq 0}$ and for any Markov chain $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ which satisfy $\mathbf{H_1}, \mathbf{H_2}$ we have $E[|\Phi(t,s)x_0|^2|\eta_s = i] \leq \beta q^{t-s}|x_0|^2$ for all $t \geq s,\ i \in \mathcal{D}_s,\ s \geq 0,\ x_0 \in \mathbf{R}^n$.

c) We say that the zero state equilibrium of the system (2.4) is **exponentially stable in mean square with conditioning of type II** (ESMS-CII) if there exist $\beta \geq 1, q \in (0,1)$ such that for any sequence of independent random vectors $\{w(t)\}_{t\geq 0}$ and for any Markov chain which satisfy $\mathbf{H_1}, \mathbf{H_2}$ we have $E[|\Phi(t,s)x_0|^2|\eta_{s-1} = i] \leq \beta q^{t-s}|x_0|^2$ for all $t \geq s,\ i \in \mathcal{D}_{s-1},\ s \geq 1,\ x_0 \in \mathbf{R}^n$.

d) We say that the zero state equilibrium of the system (2.4) is **exponentially stable in mean square** (ESMS) if there exist $\beta \geq 1,\ q \in (0,1)$ such that for any sequence of independent random vectors $\{w(t)\}_{t\geq 0}$ and for any Markov chain $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ satisfying $\mathbf{H_1}, \mathbf{H_2}$ we have $E[|\Phi(t,s)x_0|^2] \leq \beta q^{t-s}|x_0|^2$ for all $t \geq s \geq 0,\ x_0 \in \mathbf{R}^n$.

Together with the concept of strong exponential stability in mean square introduced in Definition 5.1 (a) we define a new type of SESMS which were not considered in [8].

Definition 5.2 We say that the zero state equilibrium of the system (2.4) is **strongly exponentially stable in mean square of second kind** SESMS-II if there exist $\beta \geq 1, q \in (0,1)$ such that

$$\|S(t,s)\|_1 \leq \beta q^{t-s}$$

for all $t \geq s \geq 1$.

In view of the last definition the concept of SESMS introduced in Definition 5.1 (a) will be called **strong exponential stability in mean square of the first kind** (SESMS-I).

From Theorem 2.8 in [8] one obtains:

Corollary 5.3 (i) *If the zero state equilibrium of the system (2.4) is SESMS-I then it is SESMS-II*

(ii) *If $\{A_k(t,i)\}_{t\geq 1}, 0 \leq k \leq r, i \in \mathcal{D}$ are bounded sequences, then the zero state equilibrium of the system (2.4) is SESMS-I if and only if it is SESMS-II.*

The Example 3.10 and Example 3.11 in [8], show that there exist systems of type (2.4) which are SESMS-II but they are not SESMS-I. Therefore the class of the systems of type (2.4) for which the concept of SESMS-II holds is wider than the class of systems (2.4) for which the property for SESMS-I is true.

From (5.3)-(5.4), (5.9)-(5.10) and Corollary 5.3 one obtains:

Theorem 5.4 *Under the assumptions* $\mathbf{H_1}, \mathbf{H_2}$ *we have:*

(i) *The zero state equilibrium of the system (2.4) is SESMS-I if and only if the zero state equilibrium of the system (5.2) is SESMS.*

(ii) *The zero state equilibrium of the system (2.4) is SESMS-II if and only if the zero state equilibrium of the system (5.1) is SESMS.*

(iii) *If $\{A_k(t,i)\}_{t\geq 0}, 0 \leq k \leq r, i \in \mathcal{D}$ are bounded sequences, then the zero state equilibrium of the system (2.4) is SESMS-I if and only if the zero state equilibrium of the system (5.1) is SESMS.*

(iv) *The zero state equilibrium of the system (2.4) is ESMS-CII if and only if the zero state equilibrium* *of the system (5.1) is ESMS-C.*

(v) The zero state equilibrium of the system (2.4) is ESMS-CI if and only if the zero state equilibrium of the system (5.2) is ESMS-C.

(vi) The zero state equilibrium of the system (2.4) is ESMS if and only if the zero equilibrium of (5.2) is ESMS.

Remark 5.5 a) From Theorem 3.4 and Theorem 5.4 (v) one recovers Theorem 3.8 in [8] for $\mu_k(t) = 1$.

b) From Theorem 3.4 and Theorem 5.4 $(iii), (iv)$ one obtains Theorem 3.9 in [8] for $\mu_k(t) = 1$.

c) From Theorem 4.1 and Theorem 5.4 one obtains Theorem 3.13 and Theorem 3.14 in [8] for $\mu_k(t) = 1$.

6 APPLICATIONS

In this section we illustrate the applicability of the results concerning the exponential stability in mean square for the systems of type (2.1) derived in the previous sections to the H_2 control problem of the designing of a stabilizing static output feedback in the presence of some delays in the transmission of the date.

Let us consider the discrete-time time-invariant controlled linear system described by:

$$x(t+1) = [A_0(\eta_t) + \sum_{k=1}^{r} A_k(\eta_t) w_k(t)] x(t) + B(\eta_t) u(t) \qquad (6.1)$$
$$y(t) = C(\eta_t) x(t), \quad t \geq 0,$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ is the vector of the control inputs and $y(t) \in \mathbf{R}^p$ is the vector of the measurements, $w_k(t)$ and η_t are as in the previous sections. The aim is to design a control law of the form

$$u(t) = K(\eta_t) y(t) \qquad (6.2)$$

with the property that the trajectories of the corresponding closed-loop system satisfy:

$$\lim_{t \to \infty} E[|x(t)|^2] = 0. \qquad (6.3)$$

In the case when on the channel from the sensor to the controller there exists a delay in transmission of the measurement then, the control (6.2) is replaced by

$$u(t) = K(\eta_t) y(t-1). \qquad (6.4)$$

If the delay occurs on the channel from controller to actuators then instead of (6.2) we will have:

$$u(t) = K(\eta_{t-1}) y(t-1). \qquad (6.5)$$

In this section we solve the following two problems:

$\mathbf{P_1}$: *Find a set of sufficient conditions which guarantee the existence of the matrices $K(i) \in \mathbf{R}^{m \times p}, i \in \mathcal{D}$, such that the trajectories of the closed-loop system obtained by coupling (6.4) with (6.1) will satisfy a condition of type (6.3).*

$\mathbf{P_2}$: *Find a set of conditions which guarantee the existence of the matrices $K(i) \in \mathbf{R}^{m \times p}$ with the property that the trajectories of the closed-loop system obtained by coupling the control (6.5) with the system (6.1) verify a condition of type (6.3).*

In both cases indicate some feasible procedures to compute the feedback gains $K(i), i \in \mathcal{D}$.

Coupling a control (6.4) to the system (6.1) one obtains the following closed-loop system:

$$x(t+1) = A_0(\eta_t)x(t) + B(\eta_t)K(\eta_t)C(\eta_{t-1})x(t-1) + \sum_{k=1}^{r} A_k(\eta_t)x(t)w_k(t). \tag{6.6}$$

The closed-loop system obtained combining (6.5) to (6.1) is:

$$x(t+1) = A_0(\eta_t)x(t) + B(\eta_t)K(\eta_{t-1})C(\eta_{t-1})x(t-1) + \sum_{k=1}^{r} w_k(t)A_k(\eta_t)x(t). \tag{6.7}$$

Setting $\xi(t) = (x^T(t), x^T(t-1))^T$ we obtain the following version of (6.6) and (6.7):

$$\xi(t+1) = [\tilde{A}_0(\eta_t, \eta_{t-1}) + \sum_{k=1}^{r} \tilde{A}_k(\eta_t, \eta_{t-1})w_k(t)]\xi(t), t \geq 1, \tag{6.8}$$

$$\xi(t+1) = [\hat{A}_0(\eta_t, \eta_{t-1}) + \sum_{k=1}^{r} \hat{A}_k(\eta_t, \eta_{t-1})w_k(t)]\xi(t), t \geq 1 \tag{6.9}$$

respectively, where

$$\tilde{A}_0(i,j) = \begin{pmatrix} A_0(i) & B(i)K(i)C(j) \\ I_n & 0 \end{pmatrix}, \hat{A}_0(i,j) = \begin{pmatrix} A_0(i) & B(i)K(j)C(j) \\ I_n & 0 \end{pmatrix} \tag{6.10}$$

$$\tilde{A}_k(i,j) = \hat{A}_k(i,j) = \begin{pmatrix} A_k(i) & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, the system (6.8) as well as (6.9) are time invariant version of the system (2.1).

The condition (6.3) is equivalent to

$$\lim_{t\to\infty} E[|\xi(t)|^2] = 0. \tag{6.11}$$

Based on Theorem 3.7 (for $\theta = 1$) one obtains that (6.11) is equivalent to exponential stability in mean square. Therefore to find some conditions which guarantee the existence of the feedbacks gains $K(i)$ with the desired property, we may apply the Lyapunov type criteria derived in Section 4.

The main tools in the derivation of the results in this section on the following well known lemmas.

Lemma 6.1 *(The projection lemma) [24]. Let $\mathcal{Z} = \mathcal{Z}^T \in \mathbf{R}^{n\times n}, \mathcal{U} \in \mathbf{R}^{m\times n}, \mathcal{V} \in \mathbf{R}^{p\times n}$ be given matrices, $n \geq \max\{m,p\}$. Let $\mathcal{U}^\perp, \mathcal{V}^\perp$ be full column rank matrices such that $\mathcal{U}\mathcal{U}^\perp = 0$ and $\mathcal{V}\mathcal{V}^\perp = 0$. Then the following are equivalent:*

(i) The linear matrix inequation:

$$\mathcal{Z} + \mathcal{U}^T K \mathcal{V} + \mathcal{V}^T K^T \mathcal{U} < 0$$

with the unknown matrix $K \in \mathbf{R}^{m\times p}$ is solvable.

(ii) $(\mathcal{U}^\perp)^T \mathcal{Z} \mathcal{U}^\perp < 0$,

$(\mathcal{V}^\perp)^T \mathcal{Z} \mathcal{V}^{\perp|} < 0$.

Lemma 6.2 *(Finsler's lemma)[24]. Let $\mathcal{Z} = \mathcal{Z}^T \in \mathbf{R}^{n\times n}, \mathcal{C} \in \mathbf{R}^{p\times n}, n > p$ be given. Take $\mathcal{C}^\perp$ a full column rank matrix such that $\mathcal{C}\mathcal{C}^\perp = 0$. Then the following are equivalent:*

(i) There exist a scalar μ such that $\mathcal{Z} + \mu\mathcal{C}^T\mathcal{C} < 0$.

(ii) $(\mathcal{C}^\perp)^T \mathcal{Z} \mathcal{C}^\perp < 0$.

Combining the above two lemmas one obtains

Corollary 6.3 *With the previous notations the following statements are equivalent:*

(i) The linear inequation

$$\mathcal{Z} + \mathcal{U}^T K \mathcal{V} + \mathcal{V}^T K^T \mathcal{U} < 0 \tag{6.12}$$

with the unknown $K \in \mathbf{R}^{m\times p}$ is solvable.

(ii) There exist the scalars μ_1, μ_2 such that

$$\mathcal{Z} + \mu_1 \mathcal{U}^T \mathcal{U} < 0$$

$$\mathcal{Z} + \mu_2 \mathcal{V}^T \mathcal{V} < 0.$$

In [24] a parametrization of the whole class of solutions of (6.12) is given.

Before to state the main results of this section we remark that in (6.10) we have the following decomposition:

$$\begin{aligned} \tilde{A}_0(i,j) &= \mathcal{A}_0(i) + \mathcal{B}_0(i)K(i)\mathcal{C}_0(j) \\ \hat{A}_0(i,j) &= \mathcal{A}_0(i) + \mathcal{B}_0(i)K(j)\mathcal{C}_0(j) \end{aligned} \tag{6.13}$$

where $\mathcal{A}_0(i) = \begin{pmatrix} A_0(i) & 0 \\ I_n & 0 \end{pmatrix} \in \mathbf{R}^{2n\times 2n}, \mathcal{B}_0(i) = \begin{pmatrix} B(i) \\ 0 \end{pmatrix} \in \mathbf{R}^{2n\times m}, \mathcal{C}_0(j) = (0 \quad C(j)) \in \mathbf{R}^{p\times 2n}$.

Now we prove:

Theorem 6.4 *Assume that there exist the symmetric matrices $Y(i) \in \mathbf{R}^{2n\times 2n}$ and the scalars $\mu_1(i)$ and $\mu_2(i), i \in \mathcal{D}$, satisfying the following systems of LMI's:*

$$\begin{pmatrix} \Psi_{1i}(Y) - Y(i) & \Psi_{2i}(Y) & 0 \\ \Psi_{2i}^T(Y) & \Psi_{3i}(Y) & \Psi_{4i}(Y) \\ 0 & \Psi_{4i}^T(Y) & -\mu_1(i)I_p \end{pmatrix} < 0 \tag{6.14}$$

$$\sum_{j=1}^{N} p(j,i)\mathcal{A}_0(i)Y(j)\mathcal{A}_0^T(i) + \sum_{k=1}^{r}\sum_{j=1}^{N} p(j,i)\tilde{A}_k(i,j)Y(j)\tilde{A}_k^T(i,j) - Y(i) + \mu_2(i)\mathcal{B}_0(i)\mathcal{B}_0^T(i) < 0 \tag{6.15}$$

where

$$\Psi_{1i}(Y) = \sum_{k=1}^{r}\sum_{j=1}^{N} p(j,i)\tilde{A}_k(i,j)Y(j)\tilde{A}_k^T(i,j),$$

$$\Psi_{2i}(Y) = (\sqrt{p(1,i)}\mathcal{A}_0(i)Y(1),, \sqrt{p(N,i)}\mathcal{A}_0(i)Y(N)),$$

$$\Psi_{3i}(Y) = -diag(Y(1),, Y(N)),$$

$$\Psi_{4i}^T(Y) = (\sqrt{p(1,i)}\mathcal{C}_0(1)Y(1),, \sqrt{p(N,i)}\mathcal{C}_0(N)Y(N)).$$

Under these conditions there exist stabilizing feedback gains $K(i) \in \mathbf{R}^{m\times p}, i \in \mathcal{D}$ such that the zero state equilibrium of the system (6.8) is ESMS.

Moreover the matrices $K(i)$ may be obtained as solutions of the following uncoupled LMI's:

$$\tilde{\mathcal{Z}}(i) + \tilde{\mathcal{U}}^T(i)K(i)\tilde{\mathcal{V}}(i) + \tilde{\mathcal{V}}^T(i)K^T(i)\mathcal{U}(i) < 0 \tag{6.16}$$

$i \in \mathcal{D}$, where

$$\tilde{\mathcal{Z}}(i) = \begin{pmatrix} \Psi_{1i}(\tilde{Y}) - \tilde{Y}(i) & \Psi_{2i}(\tilde{Y}) \\ \Psi_{2i}^T(\tilde{Y}) & \Psi_{3i}(\tilde{Y}) \end{pmatrix}$$

$$\tilde{\mathcal{U}}(i) = (\mathcal{B}_0^T(i), 0, , 0) \in \mathbf{R}^{m\times\tilde{n}}, \tilde{\mathcal{V}}(i) = (0, \sqrt{p(1,i)}\mathcal{C}_0(1)\tilde{Y}(1), , \sqrt{p(N,i)}\mathcal{C}_0(N)\tilde{Y}(N)) \in \mathbf{R}^{p\times\tilde{n}} \quad (6.17)$$

where $\tilde{n} = 2n(N+1), \tilde{Y} = (\tilde{Y}(1), , \tilde{Y}(N))$ *being a solution of (6.14), (6.15).*

Proof Applying Corollary 4.8 we obtain that the zero state equilibrium of the system (6.8) is ESMS if and only if there exist positive definite matrices $Y(i) \in \mathbf{R}^{2n\times 2n}, i \in \mathcal{D}$ such that

$$\sum_{k=0}^{r}\sum_{j=1}^{N} p(j,i)\tilde{A}_k(i,j)Y(j)\tilde{A}_k(i,j) - Y(i) < 0, i \in \mathcal{D}.$$

Based on the Schur complement one obtains that the last inequality is equivalent to:

$$\begin{pmatrix} \Psi_{1i}(Y) - Y(i) & \tilde{\Psi}_{2i}(Y) \\ \tilde{\Psi}_{2i}^T(Y) & \Psi_{3i}(Y) \end{pmatrix} < 0 \quad (6.18)$$

where $\tilde{\Psi}_{2i}(Y) = (\sqrt{p(1,i)}\tilde{A}_0(i,1)Y(1), , \sqrt{p(N,i)}\tilde{A}_0(i,N)Y(N))$.

It is straightforward to check based on (6.13) that (6.18) can be written in the form (6.16)-(6.17) with $\tilde{Y}(i)$ replaced by $Y(i)$. On the other hand if $(\tilde{Y}(i), \tilde{\mu}_1(i), \tilde{\mu}_2(i), i \in \mathcal{D})$ is a solution of (6.14)-(6.15), one obtains via the Schur complement technique that

$$\tilde{\mathcal{Z}}(i) + \tilde{\mu}_1^{-1}(i)\tilde{\mathcal{V}}^T(i)\tilde{\mathcal{V}}(i) < 0 \quad (6.19)$$

$$\tilde{\mathcal{Z}}(i) + \tilde{\mu}_2(i)\tilde{\mathcal{U}}^T(i)\tilde{\mathcal{U}}(i) < 0. \quad (6.20)$$

Applying Corollary 6.3 we conclude from (6.19) and (6.20) that (6.16) is solvable. Any solution of (6.16) will be a stabilizing feedback gain. Thus the proof ends.

Let us consider the case of control law (6.5).

In this case we have:

Theorem 6.5: *Assume that there exist matrices* $Y(i) \in \mathbf{R}^{2n\times 2n}$ *and the scalars* $\mu_1(i), \mu_2(i)$ *which satisfy the following system of LMI's:*

$$\sum_{j=1}^{N} p(i,j)\mathcal{A}_0^T(j)Y(j)\mathcal{A}_0(j) + \sum_{k=1}^{r}\sum_{j=1}^{N} p(i,j)\hat{A}_k^T(j,i)Y(j)\hat{A}_k(j,i) - Y(i) + \mu_1(i)\mathcal{C}_0^T(i)\mathcal{C}_0(i) < 0 \quad (6.21)$$

$$\begin{pmatrix} \Gamma_{1i}(Y) - Y(i) & \hat{\Gamma}_{2i}(Y) & 0 \\ (\hat{\Gamma}_{2i}^*)^T(Y) & \Gamma_{3i}(Y) & \Gamma_{4i}(Y) \\ 0 & \Gamma_{41}^T(Y) & -\mu_2(i)I_m \end{pmatrix} < 0 \quad (6.22)$$

where

$$\Gamma_{1i}(Y) = \sum_{k=1}^{r}\sum_{j=1}^{N} p(i,j)\hat{A}_k^T(j,i)Y(j)\hat{A}_k(j,i)$$

$$\hat{\Gamma}_{2i}(Y) = (\sqrt{p(i,1)}\mathcal{A}_0^T(1)Y(1), , \sqrt{p(i,N)}\mathcal{A}_0^T(N)Y(N))$$

$$\Gamma_{3i}(Y) = \Psi_{3i}(Y)$$

$$\Gamma_{4i}^T(Y) = (\sqrt{p(i,1)}\mathcal{B}_0^T(1)Y(1), , \sqrt{p(i,N)}\mathcal{B}_0(N)Y(N)).$$

Under these conditions there exist feedback gains $K(i) \in \mathbf{R}^{m\times p}$ *such that the zero state equilibrium of the corresponding system (6.9) is ESMS.*

Moreover, if $(\hat{Y}(i), \hat{\mu}_1(i), \hat{\mu}_2(i), i \in \mathcal{D})$ *is a solution of (6.21), (6.22) then for each* $i, K(i)$ *is obtained as solution of the following LMI*

$$\hat{\mathcal{Z}}(i) + \hat{\mathcal{U}}^T(i)K(i)\hat{\mathcal{V}}(i) + \hat{\mathcal{V}}^T(i)K^T(i)\hat{\mathcal{U}}(i) < 0 \quad (6.23)$$

where

$$\hat{\mathcal{Z}}(i) = \begin{pmatrix} \Gamma_{1i}(\hat{Y}) - \hat{Y}(i) & \hat{\Gamma}_{2i}(\hat{Y}) \\ \hat{\Gamma}_{2i}^T(\hat{Y}) & \Gamma_{3i}(\hat{Y}) \end{pmatrix}, \hat{\mathcal{U}}(i) = (\mathcal{C}_0(i), 0, , 0) \in \mathbf{R}^{p\times\tilde{n}} \tag{6.24}$$
$$\hat{\mathcal{V}}(i) = (0, \sqrt{p(i,1)}\mathcal{B}_0^T(1)\hat{Y}(1), , \sqrt{p(i,N)}\mathcal{B}_0^T(N)\hat{Y}(N)) \in \mathbf{R}^{m\times\tilde{n}}.$$

Proof Applying Corollary 4.7 one obtains that the zero state equilibrium of (6.9) is ESMS if and only if there exist positive definite matrices $Y(i) \in \mathbf{R}^{2n\times 2n}, i \in \mathcal{D}$ which verify:

$$\sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)\hat{A}_k^T(j,i)Y(j)\hat{A}_k(j,i) - Y(i) < 0, i \in \mathcal{D}.$$

This is equivalent to

$$\begin{pmatrix} \Gamma_{1i}(Y) - Y(i) & \Gamma_{2i}(Y) \\ \Gamma_{2i}^T(Y) & \Gamma_{3i}(Y) \end{pmatrix} < 0 \tag{6.25}$$

where $\Gamma_{1i}(Y), \Gamma_{3i}(Y)$ are as before and $\Gamma_{2i}(Y) = (\sqrt{p(i,1)}\hat{A}_0^T(1,i)Y(1), , \sqrt{p(i,N)}\hat{A}_0^T(N,i)Y(N))$.

Based on (6.13) one obtains that (6.25) may be written as (6.23)-(6.24) with $Y(i)$ instead of $\hat{Y}(i)$. On the other hand if $(\hat{Y}(i), \hat{\mu}_1(i), \hat{\mu}_2(i), i \in \mathcal{D})$ is a solution of (6.20), (6.21) one obtains using the Schur complement technique

$$\hat{\mathcal{Z}}(i) + \hat{\mu}_1(i)\hat{\mathcal{U}}^T(i)\hat{\mathcal{U}}(i) < 0$$
$$\hat{\mathcal{Z}}(i) + \hat{\mu}_2^{-1}(i)\hat{\mathcal{V}}^T(i)\hat{\mathcal{V}}(i) < 0.$$

Applying Corollary 6.3 we conclude that (6.23) is solvable. This allows us to compute the stabilizing gains $K(i), i \in \mathcal{D}$ and thus the proof ends.

Remark 6.6 a) It must be remarked that to obtain uncoupled LMIs (6.16) and (6.22) respectively, we used Corollary 4.8 , in the first case and Corollary 4.7 in the second case. We feel that this is a good motivation to deduce stability criteria based on Lyapunov type operators Υ_t (see Theorem 4.5 and Corollary 4.8) as well as based on its adjoint operators Υ_t^* (see Theorem 4.4 and Corollary 4.7).

b) The system (6.1) is assumed to be time invariant for the sake of simplicity. Feasible conditions could be obtained in the periodic case with $\theta \geq 2$, via Theorem 4.4 and Theorem 4.5.

7 PROOF OF THEOREM 2.2

In this section we provide a proof of the representation theorem stated in subsection 2.2. Firstly we introduce the following $\sigma-$algebras:

$$\mathcal{F}_t = \sigma[w(s), 0 \leq s \leq t]$$

$$\mathcal{H}_t = \mathcal{F}_t \vee \mathcal{G}_t, \quad t \geq 0$$

$$\tilde{\mathcal{H}}_t = \mathcal{H}_{t-1} \vee \sigma[\eta_t], \quad t \geq 1$$

$$\tilde{\mathcal{H}}_0 = \mathcal{G}_0.$$

For the readers convenience we recall the following auxiliary result (see corollary7.2 in [8]).

Lemma 7.1 *Under the assumption* $\mathbf{H_1}, \mathbf{H_2}$ *we have:*

$$E[\chi_{\{\eta_{t+1}=j\}}|\mathcal{H}_t] = p_t(\eta_t, j) \ \ a.s. \tag{7.1}$$

for all $t \in \mathbf{Z}_+, j \in \mathcal{D}$ where χ_M is the indicator function of the set.

It must be remarked that (7.1) is the extension of (2.2) to the joint process $\{(w(t), \eta_t)\}_{t\geq 0}$.

Now we are in position to prove the Theorem 2.2.

We consider the family of linear operators $\tilde{\mathcal{V}}(t,s) : \mathcal{S}_n^N \to \mathcal{S}_n^N, t \geq s \geq 1$ defined as follows: $(\tilde{\mathcal{V}}(t,s)H)(i) = E[\Theta^T(t,s)H(\eta_{t-1})\Theta(t,s)|\eta_{s-1} = i]$ if $i \in \mathcal{D}_{s-1}$ and $(\tilde{\mathcal{V}}(t,s)H)(i) = (R^*(t,s)H)(i)$ if $i \in \mathcal{D} \setminus \mathcal{D}_{s-1}$ for all $H \in \mathcal{S}_n^N$.

Firstly we write

$$\begin{aligned}\Theta^T(t+1,s)H(\eta_t)\Theta(t+1,s) &= \Theta^T(t,s)\mathcal{A}^T(t)H(\eta_t)\mathcal{A}(t)\Theta(t,s) = \\ \sum_{j=1}^{N}\Theta^T(t,s)(A_0(t,j,\eta_{t-1}) &+ \sum_{k=1}^{r}A_k(t,j,\eta_{t-1})w_k(t))^T \\ H(j)(A_0(t,j,\eta_{t-1}) &+ \sum_{l=1}^{r}A_l(t,j,\eta_{t-1})w_l(t))\Theta(t,s)\chi_{\{\eta_t=j\}}.\end{aligned} \tag{7.2}$$

Since $\Theta(t,s)$ and $\chi_{\{\eta_t=j\}}$ are $\tilde{\mathcal{H}}_t$-measurable one obtains:

$$E[\Theta^T(t,s)(A_0(t,j,\eta_{t-1}) + \sum_{k=1}^{r}A_k(t,j,\eta_{t-1})w_k(t))^T H(j)(A_0(t,j,\eta_{t-1})+$$

$$\sum_{l=1}^{r}A_l(t,j,\eta_{t-1})w_l(t))\Theta(t,s)\chi_{\{\eta_t=j\}}|\tilde{\mathcal{H}}_t] = \chi_{\{\eta_t=j\}}\Theta^T(t,s)E[(A_0(t,j,\eta_{t-1})+$$

$$+\sum_{k=1}^{r}A_k(t,j,\eta_{t-1})w_k(t))^T H(j)(A_0(t,j,\eta_{t-1}) + \sum_{l=1}^{r}A_l(t,j,\eta_{t-1})w_l(t))|\tilde{\mathcal{H}}_t]\Theta(t,s) =$$

$$= \chi_{\{\eta_t=j\}}\Theta^T(t,s)\sum_{k=0}^{r}A_k^T(t,j,\eta_{t-1})H(j)A_k(t,j,\eta_{t-1})\Theta(t,s).$$

For the last equality we take into account that η_{t-1} are $\tilde{\mathcal{H}}_t$-measurable while $w_k(t)$ are independent of the σ-algebra $\tilde{\mathcal{H}}_t$.

In this case we have

$$E[w_k(t)|\tilde{\mathcal{H}}_t] = 0$$
$$E[w_k(t)w_l(t)|\tilde{\mathcal{H}}_t] = \delta_{kl},$$

$\delta_{kl} = 1 \text{if} k = l$ and $\delta_{kl} = 0$ if $k \not= l$.

Since $\mathcal{H}_{t-1} \subset \tilde{\mathcal{H}}_t$ and $\Theta(t,s), \eta_{t-1}$ are $\mathcal{H}_{t-1}$-measurable we may use the property of the conditional expectation to obtain

$$\begin{aligned}E[\chi_{\{\eta_t=j\}}\Theta^T(t,s)(A_0(t,j,\eta_{t-1}) &+ \sum_{k=1}^{r}A_k(t,j,\eta_{t-1})w_k(t))^T H(j)(A_0(t,j,\eta_{t-1}) + \\ \sum_{l=1}^{r}A_l(t,j,\eta_{t-1})w_l(t))\Theta(t,s)|\mathcal{H}_{t-1}] &= \Theta^T(t,s)\sum_{k=0}^{r}(A_k^T(t,j,\eta_{t-1}) \cdot \\ H(j)A_k(t,j,\eta_{t-1}))\ \Theta &\ (t,s)E[\chi_{\{\eta_t=j\}}|\mathcal{H}_{t-1}].\end{aligned} \tag{7.3}$$

Based on Lemma 7.1 one gets:

$$\begin{aligned}&\Theta^T(t,s)\sum_{k=0}^{r}A_k^T(t,j,\eta_{t-1})H(j)A_k(t,j,\eta_{t-1})\Theta(t,s)E[\chi_{\{\eta_t=j\}}|\mathcal{H}_{t-1}] = \\ &\quad p_{t-1}(\eta_{t-1},j)\Theta^T(t,s)\sum_{k=0}^{r}(A_k^T(t,j,\eta_{t-1})H(j)A_k(t,j,\eta_{t-1}))\Theta(t,s).\end{aligned} \tag{7.4}$$

Combining (7.2), (7.3) and (7.4) with (2.9) we obtain:

$$\begin{aligned} E[\Theta^T(t+1,s)H(\eta_t)\Theta(t+1,s)|\mathcal{H}_{t-1}] = \\ \Theta^T(t,s)(\Upsilon_t^* H)(\eta_{t-1})\Theta(t,s) \end{aligned} \tag{7.5}$$

for all $t \geq s \geq 1$.

The inclusion $\sigma(\eta_{s-1}) \subseteq \mathcal{H}_{t-1}$ allows us to obtain from (7.5) that

$$\begin{aligned} E[\Theta^T(t+1,s)H(\eta_t)\Theta(t+1,s)|\eta_{s-1}] = \\ E[\Theta^T(t,s)(\Upsilon_t^* H)(\eta_{t-1})\Theta(t,s)(\eta_{s-1})]. \end{aligned} \tag{7.6}$$

If $i \in \mathcal{D}_{s-1}$ then (7.6) leads to:

$$\begin{aligned} E[\Theta^T(t+1,s)H(\eta_t)\Theta(t+1,s)|\eta_{s-1}=i] = \\ E[\Theta^T(t,s)(\Upsilon_t^* H)(\eta_{t-1})\Theta(t,s)|\eta_{s-1}=i]. \end{aligned} \tag{7.7}$$

Having in mind the definition of $\tilde{\mathcal{V}}(t,s)$ we see that (7.7) may be rewritten as:

$$(\tilde{\mathcal{V}}(t+1,s)H)(i) = [\tilde{\mathcal{V}}(t,s)(\Upsilon^* H)](i) \tag{7.8}$$

for all $i \in \mathcal{D}_{s-1}, H \in \mathcal{S}_n^N$.

As in the proof of Theorem 3.2 in [8] one establishes that (7.8) still holds for $i \in \mathcal{D} \setminus \mathcal{D}_{s-1}$.

Thus we may conclude that

$$\tilde{\mathcal{V}}(t+1,s) = \tilde{\mathcal{V}}(t,s)\Upsilon_t^*, \qquad t \geq s \geq 1. \tag{7.9}$$

On the other hand it is easy to see that $\tilde{\mathcal{V}}(s,s)H = H$ for all $H \in \mathcal{S}_n^N$. Hence $\tilde{\mathcal{V}}(s,s) = \mathcal{I}_{\mathcal{S}_n^N} = R^*(s,s)$.

This equality together with (7.9) shows that $\tilde{\mathcal{V}}(t,s) = R^*(t,s)$ and thus the proof ends.

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