Stochastic differential equations with jumps; Liapunov exponents, asymptotic behaviour of solutions and applications to financial mathematics

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1 Introduction

The evolution of a piecewise continuous process $\{(y(t), \lambda(t)) \mid t \geqslant 0\}$ is analyzed in connection with the stochastic rule of derivation by the use of polynomial solutions for some (possibly degenerate) elliptic equations. In our setting, the continuous process $\{y(t) \in \mathbb{R}^n \mid t \geqslant 0\}$ is generated as a solution of a stochastic differential system depending on a piecewise constant process $\{\lambda(t) \in S \subset \mathbb{R}^n \mid t \geqslant 0\}$. The set of test functions $\varphi \in \mathcal{P}_p(y; \lambda)$ is consisting in all polynomial functions with respect to the state variables $y = (y_1, \ldots, y_n)$, whose coefficients are continuous and bounded functions of $\lambda \in S$.

It agrees with the elliptic operators L_λ depending on the parameter $\lambda \in S$ which determine the "drift part" when a pondered functional $\{\exp(\gamma t)\varphi(y(t), \lambda(t)) \mid t \geqslant 0\}$ is considered and the decomposition formula

$$\exp(\gamma t)\varphi(y(t), \lambda(t)) = \psi_t(y(\cdot), \lambda(\cdot)) + \int_0^t \exp(\gamma s)\left[\gamma\varphi + L_\lambda(\varphi)\right](y(s), \lambda(s))\,\mathrm{d}s \\ + M_t(y(\cdot), \lambda(\cdot)),\ t \geqslant 0$$

is valid. Here, $L_\lambda : \mathcal{P}_p(y;\lambda) \to \mathcal{P}_p(y;\lambda)$ is an elliptic operator, $\{M_t(y(\cdot),\lambda(\cdot)) \mid t \geqslant 0\}$ is a continuous martingale and $\{\psi_t(y(\cdot),\lambda(\cdot)) \mid t \geqslant 0\}$ is a piecewise constant process.

The above decomposition formula is meaningfull when describing the asymptotic behaviour of the augmented process $\{\exp(\gamma t)\|y(t)\|^2 \mid t \geqslant 0\}$, provided the nontrivial solution $\varphi_\gamma \in \mathcal{P}_2(y;\lambda)$ of the elliptic equation

$$\gamma\varphi_\gamma(y;\lambda) + \|y\|^2 + L_\lambda(\varphi_\gamma)(y,\lambda) = 0,\ \forall y \in \mathbb{R}^n, \lambda \in S \subset \mathbb{R}^n$$

satisfies the nonsingularity condition

$$\delta(\gamma)\|y\|^2 \leqslant \varphi_\gamma(y,\lambda) \leqslant \frac{1}{C(\gamma)}\|y\|^2,$$

for some positive constants $\delta(\gamma)$, $C(\gamma)$.

The same decomposition formula is the main tool for the construction of admissible strategies in a "feedback shape", when a portofolio problem associated with an American Option is analyzed.

The paper contains four sections and the last two parts are dedicated to applications encompassing asymptotic behaviour of linear stochastic differential systems depending on a parameter $\lambda \in S \subset \mathbb{R}^n$ and the construction of admissible strategies in a "feedback shape", when the portofolio problem depends on a nontrivial solution $\varphi_\gamma \in \mathcal{P}_2(y;\lambda)$ satisfying an elliptic inequality.

The first two parts are concerned with the resolution of some elliptic equations (with singularities), without involving the basic probability field and this is achieved by using developments in series of polynomials of second degree. The solutions φ_γ of elliptic equations are significant when describing the stochastic integral equation fulfilled by the process $\{\exp(\gamma t)\varphi_\gamma(y(t,x),\lambda(t)) \mid t \geqslant 0\}$, which leads us to a corresponding asymptotic behaviour when $\{\exp(\gamma t)\|y(t,x)\|^2 \mid t \geqslant 0\}$ is considered.

2 Definitions, setting of the problems and auxiliary results

Let $W(t,\omega_1) : [0,\infty) \times \Omega_1 \to \mathbb{R}^m$ be a standard Wiener process over the complete filtered probability space $\{\Omega_1, \mathcal{F}^1 \supset \{\mathcal{F}_t^1\}_{t\geqslant 0}, \mathbb{P}_1\}$. Let also $\lambda(t,\omega_2)$:

$[0, \infty) \times \Omega_2 \to S \subset \mathbb{R}^n$ be a piecewise constant process given on a complete probability field $\{\Omega_2, \mathcal{F}^2 \supset \{\mathcal{F_\sqcup}^2\}_{t \geqslant 0}, \mathbb{P}_2\}$, such that

$$\lambda(t, \omega_2) = \lambda(t_k(\omega_2), \omega_2) = \lambda_k(\omega_2) \text{ for } t \in [t_k(\omega_2), t_{k+1}(\omega_2)),\ k = 0, 1, \dots,$$

where $0 = t_0(\omega_2) < t_1(\omega_2) < \cdots < t_k(\omega_2) < t_{k+1}(\omega_2) < \infty$ is an increasing sequence of random variables, satisfying $t_k(\omega_2) \to \infty$, a.e. ω_2 and λ_k is a random vector, for each $k \geqslant 0$.

For the sake of simplicity, we shall not mention anymore the argument $\omega_1 \in \Omega_1$ and write $W(t)$ for the standard m–dimensional Wiener process. Set $\hat{\lambda}(t) = \lambda(t, \omega_2)$, $t \geqslant 0$ and $\hat{t}_k = t_k(\omega_2)$, $k \geqslant 0$, for an arbitrarily fixed $\omega_2 \in \Omega_2$.

We are next given a finite set of vector fields $g_i(z; \lambda) \in \mathbb{R}^n$ of the following form

$$g_i(z; \lambda) = a_i(\lambda) + A_i(\lambda)z,\ i = 0, 1, \dots, m,\ \lambda \in S,\ z \in \mathbb{R}^n, \tag{1}$$

where the $(n \times n)$–matrices $A_i(\lambda)$ and the n–dimensional vectors $a_i(\lambda)$ are assumed continuous and bounded. For each $x \in \mathbb{R}^n$ define a piecewise continuous process $\{\hat{z}(t, x); t \geqslant 0\}$, as the unique solution of the following dynamical system

$$\begin{cases} d\hat{z}(t) = g_0(\hat{z}(t); \hat{\lambda}(t)) + \displaystyle\sum_{j=1}^{m} g_j(\hat{z}(t); \hat{\lambda}(t))\, \mathrm{d}W_j(t),\ t \in [\hat{t}_k, \hat{t}_{k+1}), \\ \hat{z}(\hat{t}_k) = \hat{z}_{\text{-}}(\hat{t}_k) + \delta\hat{\lambda}(\hat{t}_k),\ \text{for any } k \geqslant 1,\ \delta \in [0, 1], \\ \hat{z}(0) = x + \delta\hat{\lambda}(0), \end{cases} \tag{2}$$

where we denoted $\hat{z}_{\text{-}}(\hat{t}_k) = \lim\limits_{t \uparrow \hat{t}_k} \hat{z}(t)$.

By definition, the unique solution of the system (2) is a piecewise continuous and $\{\mathcal{F}_t^1\}$–adapted process $\{\hat{z}(t, x); t \geqslant 0\}$, such that at each deterministic instant $t = \hat{t}_k$, the jump $\hat{z}(t_k, x) - \hat{z}_{\text{-}}(t_k, x) = \delta\hat{\lambda}(t_k)$ will occur.

Definition 1. Denote by $\mathcal{P}_p(z; \lambda)$ the set consisting in the polynomial functions of p degree with respect to the variables $(z_1, z_2, \dots, z_n) = z$, whose coefficients are continuous and bounded functions of $\lambda \in S$; $\mathcal{P}_p(z) \subset \mathcal{P}_p(z; \lambda)$ consists of constant coefficients polynomials.

Consider the following piecewise continuous scalar process

$$\hat{V}_z^\gamma(t, x) = \exp(\gamma t)\varphi(\hat{z}(t, x); \hat{\lambda}(t)) + \int_0^t \exp(\gamma s) f(\hat{z}(s, x))\, \mathrm{d}s,\ t \geqslant 0, \tag{3}$$

where γ is a constant, $f \in \mathcal{P}_p(z)$ and $\varphi \in \mathcal{P}_p(z;\lambda)$. We provide a stochastic rule of differentiation involving the piecewise continuous process $\{\hat{V}_z^\gamma(t,x), t \geqslant 0\}$ which contains both a continuous component, and a piecewise constant process defined as

$$\sum_{0<\hat{t}_k\leqslant t} \exp(\gamma \hat{t}_k)\psi_k(\hat{z}(\cdot,x)) = \hat{D}_z^\gamma(t,x), \tag{4}$$

where

$$\psi_k(\hat{z}(\cdot,x)) = \varphi(\hat{z}(\hat{t}_k,x);\hat{\lambda}(\hat{t}_k)) - \varphi(\hat{z}_{\text{-}}(\hat{t}_k,x);\hat{\lambda}(\hat{t}_{k-1}))$$

and $\hat{z}_{\text{-}}(\hat{t}_k,x) = \lim_{t\uparrow\hat{t}_k} \hat{z}(t,x)$, for any $k \geqslant 1$.

Lemma 1. *Let $f \in \mathcal{P}_p(z)$, $\varphi \in \mathcal{P}_p(z;\lambda)$ define the piecewise continuous process $\{\hat{V}_z^\gamma(t,x), t \geqslant 0\}$ as in* (3)*, corresponding to the unique solution $\{\hat{z}(t,x), t \geqslant 0\}$ of the system* (2)*. Then $\hat{V}_z^\gamma(t,x)$ satisfies the following integral equation*

$$\begin{aligned}\hat{V}_z^\gamma(t,x) =& \varphi(\hat{z}(0,x);\hat{\lambda}(0)) + \int_0^t \exp(\gamma s)[\gamma\varphi + f + L(\varphi)](\hat{z}(s,x);\hat{\lambda}(s))\,\mathrm{d}s \\ &+ \sum_{j=1}^m \int_0^t \exp(\gamma s)\langle \partial_z\varphi(\hat{z}(s,x);\hat{\lambda}(s)), g_j(\hat{z}(s,x);\hat{\lambda}(s))\rangle\,\mathrm{d}W_j(s) + \hat{D}_z^\gamma(t,x),\end{aligned} \tag{5}$$

for any $t \geqslant 0$ and $x \in \mathbb{R}^n$, where the piecewise constant process $\{\hat{D}_z^\gamma(t,x), t \geqslant 0\}$ is defined in (4) *and the elliptic operator $L : \mathcal{P}_p(z;\lambda) \to \mathcal{P}_p(z;\lambda)$ is given by*

$$L(\varphi)(z;\lambda) = \langle \partial_z\varphi(z,\lambda), g_0(z;\lambda)\rangle + \frac{1}{2}\sum_{j=1}^m \langle \partial_z^2\varphi(z,\lambda)g_j(z;\lambda), g_j(z;\lambda)\rangle, \tag{6}$$

for any $z \in \mathbb{R}^n$ and $\lambda \in S$.

Proof. The stochastic rule of differentiation stated in (5) can be obtained by applying the standard rule of stochastic differentiation associated with the continuous process $\{\hat{V}_z^\gamma(t,x), t \in [\hat{t}_k, \hat{t}_{k+1})\}$, for each k. To this respect, rewrite

$$\hat{V}_z^\gamma(t,x) = \hat{V}_z^\gamma(\hat{t}_k,x) + [U_k(t,\hat{z}(t,x)) - U_k(\hat{t}_k,\hat{z}(\hat{t}_k,x))] + \int_{\hat{t}_k}^t \exp(\gamma s)f(\hat{z}(s,x))\,\mathrm{d}s \tag{7}$$

for $t \in [\hat{t}_k, \hat{t}_{k+1})$, where

$$U_k(t,z) = \exp(\gamma t)\varphi(z, \hat{\lambda}(\hat{t}_k)) \text{ and } z = \hat{z}(t,x), t \in [\hat{t}_k, \hat{t}_{k+1})$$

fulfill the usual conditions of the stochastic rule of differentiation. We get

$$\begin{aligned} U_k(t, \hat{z}(t,x)) - U_k(\hat{t}_k, \hat{z}(\hat{t}_k, x)) = \int_{\hat{t}_k}^{t} \exp(\gamma s)[\gamma\varphi + L(\varphi)](\hat{z}(s,x); \hat{\lambda}(\hat{t}_k))\,\mathrm{d}s \\ + \sum_{j=1}^{m} \int_{\hat{t}_k}^{t} \exp(\gamma s)\langle \partial_z \varphi(\hat{z}(s,x); \hat{\lambda}(\hat{t}_k)), g_j(\hat{z}(s,x); \hat{\lambda}(\hat{t}_k))\rangle \,\mathrm{d}W_j(s), \end{aligned}$$

for every $t \in [\hat{t}_k, \hat{t}_{k+1})$. It follows, using also (7)

$$\begin{aligned} \hat{V}_z^{\gamma}(t,x) = \hat{V}_z^{\gamma}(\hat{t}_k, x) + \int_{\hat{t}_k}^{t} \exp(\gamma s)[\gamma\varphi + f + L(\varphi)](\hat{z}(s,x); \hat{\lambda}(\hat{t}_k))\,\mathrm{d}s \\ + \sum_{j=1}^{m} \int_{\hat{t}_k}^{t} \exp(\gamma s)\langle \partial_z \varphi(\hat{z}(s,x); \hat{\lambda}(\hat{t}_k)), g_j(\hat{z}(s,x); \hat{\lambda}(\hat{t}_k))\rangle \,\mathrm{d}W_j(s), \end{aligned} \tag{8}$$

for $t \in [\hat{t}_k, \hat{t}_{k+1})$, where, by definition,

$$\hat{V}_z^{\gamma}(\hat{t}_k, x) = U_k(\hat{t}_k, \hat{z}(\hat{t}_k, x)) + \int_0^{\hat{t}_k} \exp(\gamma s) f(\hat{z}(s,x))\,\mathrm{d}s. \tag{9}$$

On the other hand, we notice that

$$U_k(\hat{t}_k, \hat{z}(\hat{t}_k, x)) = U_{k-1}(\hat{t}_k, \hat{z}_{\text{-}}(\hat{t}_k, x)) + \exp(\gamma \hat{t}_k)\psi_k(\hat{z}(\cdot, x)),$$

for $k \geqslant 1$. $U_{k-1}(\hat{t}_k, \hat{z}_{\text{-}}(\hat{t}_k, x))$ is computed as

$$\begin{aligned} U_{k-1}(\hat{t}_k, \hat{z}_{\text{-}}(\hat{t}_k, x)) = & U_{k-1}(\hat{t}_{k-1}, \hat{z}(\hat{t}_{k-1}, x)) + \int_{\hat{t}_{k-1}}^{\hat{t}_k} \exp(\gamma s)[\gamma\varphi + L(\varphi)](\hat{z}(s,x); \hat{\lambda}(\hat{t}_{k-1}))\,\mathrm{d}s \\ & + \sum_{j=1}^{m} \int_{\hat{t}_{k-1}}^{\hat{t}_k} \exp(\gamma s)\langle \partial_z \varphi(\hat{z}(s,x); \hat{\lambda}(\hat{t}_{k-1})), g_j(\hat{z}(s,x); \hat{\lambda}(\hat{t}_{k-1}))\rangle \,\mathrm{d}W_j(s), \end{aligned}$$

Inserting now the last two formulas into (9), we get

$$
\begin{aligned}
\hat{V}_z^\gamma(\hat{t}_k, x) =& \hat{V}_z^\gamma(\hat{t}_{k-1}, x) + \int_{\hat{t}_{k-1}}^{\hat{t}_k} \exp(\gamma s)[\gamma\varphi + f + L(\varphi)](\hat{z}(s,x); \hat{\lambda}(\hat{t}_{k-1}))\, \mathrm{d}s \\
& + \sum_{j=1}^{m} \int_{\hat{t}_{k-1}}^{\hat{t}_k} \exp(\gamma s)\langle \partial_z \varphi(\hat{z}(s,x); \hat{\lambda}(\hat{t}_{k-1})), g_j(\hat{z}(s,x); \hat{\lambda}(\hat{t}_{k-1}))\rangle \,\mathrm{d}W_j(s) \\
& + \exp(\gamma \hat{t}_k)\psi_k(\hat{z}(\cdot, x)).
\end{aligned} \tag{10}
$$

Iterating now the last formula, we rewrite the equation (8) in the same way it appears in the statement (5). $\square$

Problem A. For a fixed $f \in \mathcal{P}_p(z)$, find a nonzero constant γ and $\varphi_\gamma \in \mathcal{P}_p(z;\lambda)$, such that the piecewise continuous process $\{\hat{V}_z^\gamma(t,x), t \geqslant 0\}$ defined in (3) is a quasimartingale, i.e.

$$
\begin{aligned}
\hat{V}_z^\gamma(t, x) =& \varphi_\gamma(\hat{z}(0,x); \hat{\lambda}(0)) + \hat{D}_z^\gamma(t,x) \\
& + \sum_{j=1}^{m} \int_0^t \exp(\gamma s)\langle \partial_z \varphi_\gamma(\hat{z}(s,x); \hat{\lambda}(s)), g_j(\hat{z}(s,x); \hat{\lambda}(s))\rangle \,\mathrm{d}W_j(s), \ t \geqslant 0,
\end{aligned} \tag{11}
$$

where $\hat{D}_z^\gamma(t,x) = \sum_{0<\hat{t}_k \leqslant t} \exp(\gamma \hat{t}_k)\psi_k(\hat{z}(\cdot,x))$, for any $k \geqslant 1$. and

$$
\psi_k(\hat{z}(\cdot, x) = \varphi_\gamma(\hat{z}(\hat{t}_k, x); \hat{\lambda}(\hat{t}_k)) - \varphi_\gamma(\hat{z}_{\text{-}}(\hat{t}_k, x); \hat{\lambda}(\hat{t}_{k-1})).
$$

Problem B. For a fixed $f \in \mathcal{P}_p(z)$, find a nonzero constant γ and $\varphi_\gamma \in \mathcal{P}_p(z;\lambda)$ solution of the following elliptic equation

$$
\gamma\varphi_\gamma(z;\lambda) + f(z) + L(\varphi_\gamma)(z;\lambda) = 0, \ \text{for any } z \in \mathbb{R}^n, \lambda \in S, \tag{12}
$$

where the linear operator $L : \mathcal{P}_p(z;\lambda) \to \mathcal{P}_p(z;\lambda)$ is defined as in (6).

Remark 1. Notice that each solution (γ, φ_γ) of *Problem B* is a solution for *Problem A*. To this respect, the scalar process $\{\hat{V}_z^\gamma(t,x), t \geqslant 0\}$ defined in (3) is a quasimartingale expressed as in (11), provided that the stochastic rule of differentiation contained in the Lemma 1 is used for the given solution of *Problem B*.

Lemma 2. *Let $f \in \mathcal{P}_2(z)$ and the vector fields $g_i(z;\lambda)$, $i = 0, 1, \ldots, m$, be given as in* (1)*. Let $\gamma < 0$ be such that*

$$|\gamma| > 4\big(\|g_0\| + \frac{1}{2}\sum_{j=1}^{m} \|g_j\|^2\big), \tag{13}$$

where $\|g_j\| = \sup_{\lambda\in S}(\|a_j(\lambda)\| + \|A_j(\lambda)\|)$.

Then there exists $\varphi_\gamma \in \mathcal{P}_2(z;\lambda)$ satisfying the elliptic equation (12)*. In addition, φ_γ stands for the sum of the following convergent series*

$$\varphi_\gamma(z;\lambda) = \frac{1}{|\gamma|}\big(\sum_{k=0}^{\infty} L^k_{|\gamma|}(f)(z;\lambda)\big),\ (z,\lambda) \in \mathbb{R}^n \times S, \tag{14}$$

where $L_{|\gamma|}(f)(z;\lambda) = \dfrac{1}{|\gamma|}L(f)(z;\lambda)$.

Proof. The estimates of the polynomials $L^k_{|\gamma|}(f)(z;\lambda) \in \mathcal{P}_2(z;\lambda)$, $k \geqslant 1$, are obtained by an induction argument. Each $\varphi \in \mathcal{P}_2(z;\lambda)$ is written as

$$\varphi(z;\lambda) = C(\lambda) + \langle h_0(\lambda), z\rangle + \langle H(\lambda)z, z\rangle,$$

where $H(\lambda) = (h_1(\lambda), h_2(\lambda), \ldots, h_n(\lambda)$ and $C(\lambda) \in \mathbb{R}$, $h_i(\lambda) \in \mathbb{R}^n$, $i = 0, 1, \ldots, n$ are continuous and bounded functions with respect to $\lambda \in S$.

Denote $\|\varphi\| = \|C\| + \sum_{i=0}^{n}\|h_i\|$, where $\|C\| = \sup_{\lambda\in S}|C(\lambda)|$ and $\|h_i\| = \sup_{\lambda\in S}\|h_i(\lambda)\|$, using $\|h_i(\lambda)\| = \sum_{j=1}^{n}|h_i^j(\lambda)|$, $i = 0, 1, \ldots, n$. It is easily seen that

$$|\varphi(z;\lambda)| \leqslant |C(\lambda|(1 + \|z\|^2) + \|h_0(\lambda)\|(1 + \|z\|^2) + (\sum_{i=1}^{n}\|h_i(\lambda)\|))(1 + \|z\|^2)$$

and

$$|\varphi(z;\lambda)| \leqslant \|\varphi\|(1 + \|z\|^2),\ \forall\,(z,\lambda) \in \mathbb{R}^n \times S.$$

An elementary computation shows that

$$\begin{cases} \text{(a) } \|L_{|\gamma|}(\varphi)\| \leqslant \dfrac{4}{|\gamma|}\Big[\|g_0\|^2 + \dfrac{1}{2}\displaystyle\sum_{j=1}^{n}\|g_j\|^2\Big]\|\varphi\|, \\ \text{(b) } |L_{|\gamma|}(\varphi)(z;\lambda)| \leqslant \|\varphi\|\dfrac{4}{|\gamma|}\Big[\|g_0\|^2 + \dfrac{1}{2}\displaystyle\sum_{j=1}^{n}\|g_j\|^2\Big](1 + \|z\|^2), \end{cases} \tag{15}$$

for any $(z,\lambda) \in \mathbb{R}^n \times S$. Here we use $\|g_i\| = \sup\limits_{\lambda\in S}(\|a_i(\lambda)\|+\|A_i(\lambda)\|)$, $\|a_i(\lambda)\| = \sum\limits_{j=1}^{n} |a_i^j(\lambda)|$, $\|A_i(\lambda)\| = \sum\limits_{j=1}^{n} \|h_i^j(\lambda)\|$ if $a_i(\lambda) = \mathrm{col}\big(a_i^1(\lambda), a_i^2(\lambda), \dots, a_i^n(\lambda)\big)$ and $A_i(\lambda) = (h_i^1(\lambda), h_i^2(\lambda), \dots, h_i^n(\lambda))$.

Using an induction argument, we easily prove that

$$\big|L_{|\gamma|}^k(f)(z;\lambda)\big| \leqslant \left[\frac{4}{|\gamma|}\left(\|g_0\|^2 + \frac{1}{2}\sum_{j=1}^{n}\|g_j\|^2\right)\right]^k \|f\|(1+\|z\|^2), \tag{16}$$

for any $(z,\lambda) \in \mathbb{R}^n \times S$ and $k \geqslant 0$. To this respect, recall that $L_{|\gamma|}^{k+1} : \mathcal{P}_2(z;\lambda) \to \mathcal{P}_2(z;\lambda)$ can be written as

$$L_{|\gamma|}^{k+1}(f)(z,\lambda) = L_{|\gamma|}\big(L_{|\gamma|}^k(f)\big)(z,\lambda),$$

for $k \geqslant 0$ and assume that $L_{|\gamma|}^k(f) \in \mathcal{P}_2(z;\lambda)$ and

$$\big\|L_{|\gamma|}^k(f)\big\| \leqslant \left[\frac{4}{|\gamma|}\left(\|g_0\|^2 + \frac{1}{2}\sum_{j=1}^{n}\|g_j\|^2\right)\right]^k \|f\|,$$

for some fixed $k \geqslant 1$. Then, using the estimates (15) for $\varphi = L_{|\gamma|}^k(f)$, we get

$$\begin{aligned}\big\|L_{|\gamma|}^{k+1}(f)\big\| = \big\|L_{|\gamma|}(\varphi)\big\| &\leqslant \frac{4}{|\gamma|}\left[\|g_0\|^2 + \frac{1}{2}\sum_{j=1}^{n}\|g_j\|^2\right]\|\varphi\| \\ &\leqslant \left[\frac{4}{|\gamma|}\left(\|g_0\|^2 + \frac{1}{2}\sum_{j=1}^{n}\|g_j\|^2\right)\right]^{k+1} \|f\| = \rho^{k+1}\|f\|\end{aligned}$$

and

$$\big|L_{|\gamma|}^{k+1}(f)(z,\lambda)\big| \leqslant \rho^{k+1}\|f\|(1+\|z\|^2),$$

for any $(z,\lambda) \in \mathbb{R}^n \times S$, where $\rho = \dfrac{4}{|\gamma|}\big[\|g_0\|^2 + \dfrac{1}{2}\sum\limits_{j=1}^{n}\|g_j\|^2\big]$.

It shows that the estimate (16) is satisfied for any $k \geqslant 0$ and assuming that $\rho < 1$ (see (13)), we easily get that the series in (14) is convergent. In addition, using (16) we obtain that

$$|\varphi_\gamma(z,\lambda)| \leqslant \frac{1}{|\gamma|}\left(\sum_{k=0}^{\infty}\rho^k\right)\|f\|(1+\|z\|^2),$$

for any $(z, \lambda) \in \mathbb{R}^n \times S$, where

$$\frac{1}{|\gamma|}\sum_{k=0}^{\infty}\rho^k = \frac{1}{|\gamma|(1-\rho)} = \Big[|\gamma| - 4\big(\|g_0\|^2 + \frac{1}{2}\sum_{j=1}^{n}\|g_j\|^2\big)\Big]^{-1} > 0.$$

By definition, $\varphi_\gamma \in \mathcal{P}_2(z;\lambda)$ and using (14) we get that φ_γ satisfies the elliptic equation (12), as the following simple computation shows

$$L(\varphi_\gamma)(z;\lambda) = L_{|\gamma|}\sum_{k=0}^{\infty} L^k_{|\gamma|}(f)(z,\lambda) = \sum_{k=1}^{\infty} L^k_{|\gamma|}(f)(z,\lambda) = -\gamma\varphi_\gamma(z;\lambda) - f(z),$$

for any $(z, \lambda) \in \mathbb{R}^n \times S$. The proof is complete. $\square$

Remark 2. The functional $\varphi_\gamma \in \mathcal{P}_2(z;\lambda)$ constructed in the Lemma 2 is meaningfull for the asymptotic analysis of the piecewise continuous process $\{\hat{V}^\gamma_z(t,x), t \geqslant 0\}$, provided $\varphi_\gamma(z;\lambda)$ is a quadratic positive form satisfying

$$\delta(\gamma)\|z\|^2 \leqslant \varphi_\gamma(z;\lambda) \leqslant \frac{1}{C(\gamma)}\|z\|^2,$$

for any $\lambda \in S$. Here $\delta(\gamma)$ and $C(\gamma)$ are positive constants.

It can be accomplished by imposing the following additional conditions on the vector fields $g_i(z;\lambda)$, $i = 0, 1, \ldots, m$ given in (1)

$$\begin{cases} \text{(a) } g_i(z;\lambda) = A_i(\lambda)z,\ i = 0, 1, \ldots, m, \\ \text{(b) } A_j(\lambda)A_k(\lambda) = A_k(\lambda)A_j(\lambda),\ j, k = 0, 1, \ldots, m,\ \lambda \in S, \end{cases} \tag{17}$$

where the $A_i(\lambda)$ are continuous and bounded (symmetric) matrices defined on S.

Lemma 3. *Assume that the vector fields $g_i(z;\lambda)$ given in (1) satisfy the asumptions (17). Let $f(z) = \|z\|^2$ and the constant γ fulfilling*

$$\gamma < 0,\ |\gamma| > 4\Big[[\|g_0\| + \frac{1}{2}\sum_{j=1}^{m}\|g_j\|^2\Big], \tag{18}$$

where $\|g_j\| = \sup_{\lambda \in S}\|A_j(\lambda)\|$, $j = 0, 1, \ldots, m$.

Then there exists $\varphi_\gamma \in \mathcal{P}_2(z;\lambda)$ fulfilling the following equations

$$\begin{cases} \text{(a) } \gamma\varphi_\gamma(z;\lambda) + \|z\|^2 + L(\varphi_\gamma)(z;\lambda) = 0, \ \forall (z,\lambda) \in \mathbb{R}^n \times S, \\ \text{(b) } \varphi_\gamma(z;\lambda) = \langle R_\gamma(\lambda)z, z\rangle, \ \delta(\gamma)\|z\|^2 \leqslant \varphi_\gamma(z;\lambda) \leqslant \dfrac{1}{C(\gamma)}\|z\|^2, \end{cases} \tag{19}$$

where

$$L(\varphi_\gamma)(z;\lambda) = \langle \partial_z \varphi(z,\lambda), A_0(\lambda)z\rangle + \frac{1}{2}\sum_{j=1}^{m} \langle \partial_z^2 \varphi(z;\lambda)\ A_j(\lambda)z, A_j(\lambda)z\rangle,$$

$C(\gamma)$, $\delta(\gamma)$ are positive constants, $R_\gamma(\lambda) = [|\gamma| I_n - B(\lambda)]^{-1}$ and $B(\lambda) = 2A_0(\lambda) + \sum\limits_{j=1}^{m} A_j^2(\lambda)$.

Proof. The conditions of the Lemma 2 are satisfied and let $\varphi_\gamma \in \mathcal{P}_2(z;\lambda)$ be the solution of the elliptic equation (12), given by the series

$$\varphi_\gamma(z;\lambda) = \frac{1}{|\gamma|}\left[\sum_{k=0}^{\infty} L_{|\gamma|}^k(f)(z,\lambda)\right], \ (z,\lambda) \in \mathbb{R}^n \times S,$$

where $f(z) = \|z\|^2$ and the linear operator $L_{|\gamma|} = \dfrac{1}{|\gamma|}L$ satisfies

$$L_{|\gamma|}(\varphi)(z,\lambda) = \frac{1}{|\gamma|}\Big[\langle \partial_z \varphi(z,\lambda), A_0(\lambda)z\rangle + \frac{1}{2}\sum_{j=1}^{m}\langle \partial_z^2 \varphi(z;\lambda) A_j(\lambda)z, A_j(\lambda)z\rangle\Big].$$

By definition,

$$L_{|\gamma|}(f)(z,\lambda) = \frac{1}{|\gamma|}\left[\left\langle \big(2A_0(\lambda) + \sum_{j=1}^{m} A_j^2(\lambda)\big)z, z\right\rangle\right] = \frac{1}{|\gamma|}\langle B(\lambda)z, z\rangle.$$

It is obvious that the symmetric matrix $B(\lambda)$ is commuting with each $A_i(\lambda)$, $i = 0, 1, \ldots, m$. Therefore,

$$L_{|\gamma|}^k(f)(z,\lambda) = \left\langle \left(\frac{B(\lambda)}{|\gamma|}\right)^k z, z\right\rangle.$$

Here, the matrix $B(\lambda)$ satisfies

$$\|B(\lambda)\| \leqslant 2\|A_0(\lambda)\| + \sum_{j=1}^{m} \|A_j(\lambda)\|^2 \leqslant 2\|g_0\| + \sum_{j=1}^{m} \|g_j\|^2 = \nu,$$

for any $\lambda \in S$. Define $\|B\| = \sup_{\lambda \in S} \|B(\lambda)\|$ and using (18) we get

$$\rho = \frac{\|B\|}{|\gamma|} \leqslant \frac{\nu}{|\gamma|} < \frac{1}{2} \tag{20}$$

and the following matrix series is convergent

$$R_\gamma(\lambda) = \frac{1}{|\gamma|} \sum_{k=0}^{\infty} \left(\frac{B(\lambda)}{|\gamma|} \right)^k, \text{ for any } \lambda \in S, \tag{21}$$

if $|\gamma| > 2\nu$. As a consequence, we obtain

$$\varphi_\gamma(z;\lambda) = \frac{1}{|\gamma|} \left\langle \sum_{k=0}^{\infty} \left(\frac{B(\lambda)}{|\gamma|} \right)^k z, z \right\rangle = \langle R_\gamma(\lambda) z, z \rangle, \text{ for any } \lambda \in S,$$

provided $|\gamma| > 2\nu$. Recall that the symmetric matrix $R_\gamma(\lambda)$ fulfills

$$R_\gamma(\lambda) = \frac{1}{|\gamma|} \left[I_n - \frac{B(\lambda)}{|\gamma|} \right]^{-1}. \tag{22}$$

The conclusion (19) is a direct application of the following argument. Let $z = T(\lambda)y$ be an orthogonal mapping, i.e. $T^*(\lambda) = [T(\lambda)]^{-1}$, such that

$$T^{-1}(\lambda) B(\lambda) T(\lambda) = D(\lambda) = \operatorname{diag}(\gamma_1(\lambda), \dots, \gamma_n(\lambda))$$

and write $\varphi_\gamma(T(\lambda)y;\lambda) = \langle T^{-1}(\lambda) R_\gamma(\lambda) T(\lambda) y, y \rangle$ as follows

$$\varphi_\gamma(T(\lambda)y;\lambda) = \frac{1}{|\gamma|} \left\langle \sum_{k=0}^{\infty} \left(\frac{D(\lambda)}{|\gamma|} \right)^k y, y \right\rangle = \langle Q_\gamma(\lambda) y, y \rangle, \ y \in \mathbb{R}^n,$$

where $Q_\gamma(\lambda) = \operatorname{diag}\big[(|\gamma| - \gamma_1(\lambda))^{-1}, \dots, (|\gamma| - \gamma_n(\lambda))^{-1}\big]$. Using $B(\lambda) e_k(\lambda) = \gamma_k(\lambda) e_k(\lambda)$, $k = 1, 2, \dots, n$, where $T(\lambda) = \|e_1(\lambda), e_2(\lambda), \dots, e_n(\lambda)\|$, it is easily seen that $\|B(\lambda)\| \geqslant \sum_{k=1}^{n} |\gamma_k(\lambda)| = |\hat{\gamma}(\lambda)|$ and $|\hat{\gamma}(\lambda)| \leqslant \|B\|$, for any $\lambda \in S$.

The hypothesis $|\gamma| > 2\nu$ (see (18)) leads us to the conclusion that the matrix $Q_\gamma(\lambda)$ is positively defined, uniformly with respect to $\lambda \in S$ and

$$\delta(\gamma)\|y\|^2 \leqslant \varphi_\gamma(T(\lambda)y;\lambda) \leqslant \frac{1}{C(\gamma)}\|y\|^2,$$

with some positive constants $\delta(\gamma)$, $C(\gamma)$. Take now $y = T^{-1}(\lambda)z$ and using $\|y\|^2 = \|z\|^2$, we obtain

$$\frac{1}{C(\gamma)}\|z\|^2 \geqslant \varphi_\gamma(z;\lambda) = \langle R_\gamma z, z\rangle \geqslant \delta(\gamma)\|z\|^2,\ \forall z \in \mathbb{R}^n.$$

□

Remark 3. The stochastic rule of differentiation given in Lemma 1 and the result stated in Lemma 3 are relevant for the piecewise continuous process $\{\hat{z}(t,x); t \geqslant 0\}$, defined as the unique solution of system (2). To this respect, we use the following decomposition

$$\hat{z}(t,x) = \hat{z}_c(t,x) + \delta\hat{\lambda}(t),\ t \geqslant 0, \tag{23}$$

where $\delta \in [0,1]$ is fixed and the continuous component $\{\hat{z}_c(t,x); t \geqslant 0\}$ stands for the unique solution of the linear stochastic system

$$\begin{cases} d\hat{z}_c(t) & = A_0(\hat{\lambda}(t))\hat{z}_c(t)dt + \sum_{j=1}^m A_j(\hat{\lambda}(t))\hat{z}_c(t)dW_j(t),\ t \geqslant 0, \\ \hat{z}_c(0) & = x. \end{cases} \tag{24}$$

The stochastic rule of differentiation written for $\hat{z}_c(t,x)$ (which equals $\hat{z}(t,x)$ when $\delta = 0$) reads

$$\begin{aligned} \hat{V}^\gamma_{z_c}(t,x) =& \varphi_\gamma(x;\hat{\lambda}(0)) + \hat{D}^\gamma_{z_c}(t,x) + \int_0^t \exp(\gamma s)\ [\gamma\varphi_\gamma + f \\ &+ L(\varphi_\gamma)](\hat{z}_c(s,x);\hat{\lambda}(s))ds \\ &+ \sum_{j=1}^m \int_0^t \exp(\gamma s)\langle \partial_z\varphi_\gamma(\hat{z}_c(s,x);\hat{\lambda}(s)), A_j(s,x);\hat{\lambda}(s))\hat{z}_c(s,x)\rangle dW_j(s), \end{aligned} \tag{25}$$

where

$$\hat{D}^\gamma_{z_c}(t,x) = \sum_{0<\hat{t}_k\leqslant t} \exp(\gamma\hat{t}_k)\ \psi_k(\hat{z}_c(\cdot,x))$$

and

$$\psi_k(\hat{z}_c(\cdot,x)) = \varphi_\gamma(\hat{z}_c(\hat{t}_k,x);\hat{\lambda}(\hat{t}_k)) - \varphi_\gamma(\hat{z}_c(\hat{t}_k,x);\hat{\lambda}(\hat{t}_{k-1})), k \geqslant 1.$$

Lemma 4. *Let the hypotheses of Lemma 3 be in force and denote*

$$\hat{\mathcal{U}}_\gamma(t,x) = \exp(\gamma t)\varphi_\gamma(\hat{z}_c(t,x);\hat{\lambda}(t)), t \geqslant 0, x \in \mathbb{R}^n,$$

for $f(z) = \|z\|^2$ *and* $\varphi_\gamma(z;\lambda) = \langle R_\gamma z, z\rangle$. *Then the piecewise continuous process* $\hat{\mathcal{U}}_\gamma(t,x)$ *fulfills the following integral equation*

$$\begin{aligned}\hat{\mathcal{U}}_\gamma(t,x) =& \varphi_\gamma(\hat{z}_c(0,x);\hat{\lambda}(0)) + \hat{D}^\gamma_{z_c}(t,x) + \int_0^t \Big[-C(\gamma)\hat{\mathcal{U}}_\gamma(s,x) + \\ & + \exp(\gamma s)\beta_\gamma(\hat{z}_c(s,x);\hat{\lambda}(s))\Big] ds \\ & + \sum_{j=1}^m \int_0^t \exp(\gamma s)\langle \partial_z \varphi_\gamma(\hat{z}_c(s,x);\hat{\lambda}(s)), A_j(\hat{\lambda}(s))\hat{z}_c(s,x)\rangle dW_j(s),\end{aligned} \tag{26}$$

where

$$\begin{cases} \beta_\gamma(z;\lambda) & = C(\gamma)\varphi_\gamma(z;\lambda) - \|z\|^2 \leqslant 0, \ \forall (z,\lambda) \in \mathbb{R}^n \times S, \\ \hat{D}^\gamma_{z_c}(t,x) & = \sum_{0<\hat{t}_k \leqslant t} \exp(\gamma \hat{t}_k) \Big[\varphi_\gamma(\hat{z}_c(\hat{t}_k,x);\hat{\lambda}(\hat{t}_k)) - \varphi_\gamma(\hat{z}_c(\hat{t}_k,x);\hat{\lambda}(\hat{t}_{k-1}))\Big]. \end{cases}$$

Proof. By hypothesis, the stochastic rule of differentiation given in Lemma 1 is fulfilled for the continuous process $\{\hat{z}_c(t,x); t \geqslant 0\}$ and if we apply it for $\{\hat{V}^\gamma_{z_c}(t,x)\}$ we get the equation (25). Using the elliptic equation

$$\gamma\varphi_\gamma(z;\lambda) + \|z\|^2 + L(\varphi_\gamma)(z;\lambda) = 0, \forall\ (z,\lambda) \in \mathbb{R}^n \times S$$

and the integral equation (25), we obtain that the piecewise continuous process $\{\hat{\mathcal{U}}_\gamma(t,x);\ t \geqslant 0\}$ satisfies

$$\begin{aligned}\hat{\mathcal{U}}_\gamma(t,x) =& \varphi_\gamma(x;\hat{\lambda}(0)) + \hat{D}^\gamma_{z_c}(t,x) - \int_0^t \exp(\gamma s)\ \|\hat{z}_c(s,x)\|^2 ds \\ & + \sum_{j=1}^m \int_0^t \exp(\gamma s)\ \langle \partial_z \varphi_\gamma(\hat{z}_c(s,x);\hat{\lambda}(s)), A_j(\hat{\lambda}(s))\hat{z}_c(s,x)\rangle dW_j(s).\end{aligned} \tag{27}$$

The conclusion (19) of Lemma 3 shows that the following equation is satisfied

$$-\|z\|^2 = -C(\gamma)\varphi_\gamma(z;\lambda) + \beta_\gamma(z;\lambda),$$

where $\beta_\gamma(z;\lambda) \leqslant 0$, for any $(z,\lambda) \in \mathbb{R}^n \times S$.

Rewrite now (27) using the last formula and we thus get fulfilled the conclusion (26). □

Remark 4. Assuming that the set $S \subset \mathbb{R}^n$ is chosen such that

$$R_\gamma(\hat{\lambda}(t)) = R_\gamma(\hat{\lambda}(0)), \text{ for any } t = \hat{t}_k, k \geqslant 0 \text{ (see } \hat{\lambda}(\hat{t}_k) \in S),$$

then the piecewise constant process $\{\hat{D}^\gamma_{z_c}(t,x); t \geqslant 0\}$ is vanishing and the integral equation (26) can be written accordingly. In addition, the same equation can be reduced to the following one

$$\begin{aligned}\hat{\mathcal{U}}_\gamma(t,x) =& \varphi_\gamma(x;\hat{\lambda}(0)) - \int_0^t C(\gamma)\hat{\mathcal{U}}_\gamma(s,x)ds + 2\sum_{j=1}^m \int_0^t \hat{\mathcal{U}}_\gamma(s,x)\hat{\mu}_j(s)dW_j(s)\\ &+ \int_0^t \exp(\gamma s)\beta_\gamma(\hat{z}_c(s,x);\hat{\lambda}(s))ds,\end{aligned} \tag{28}$$

where $\mu_j(\lambda): S \to \mathbb{R}, j = 1,\ldots,m$ are given such that $A_j(\lambda) = \mu_j(\lambda)I_n$ and $\hat{\mu}_j(t) = \mu_j(\hat{\lambda}(t))$.

If it is the case, then define an exponential martingale

$$\xi(t) = \exp\left[2\sum_{j=1}^m \left(\int_0^t \hat{\mu}_j(s)dW_j(s) - \int_0^t (\hat{\mu}_j(s))^2 ds\right)\right], t \geqslant 0, \tag{29}$$

which fulfills the equation

$$\begin{cases} d\xi(t) &= 2\xi(t)\sum_{j=1}^m \hat{\mu}_j(t)dW_j(t), t \geqslant 0;\\ \xi(0) &= 1.\end{cases} \tag{30}$$

Lemma 5. *Let the hypothesis* (17), (18) *and* (19) *be in force. In addition, assume that* $A_j(\lambda) = \mu_j(\lambda)I_n$, $j = 1,2,\ldots,m$ *and the matrix* $B(\lambda) = 2A_0(\lambda)+\sum_{j=1}^m A_j^2(\lambda)$ *is constant. Denote* $\hat{h}_\gamma(t,x) = \exp(\gamma t)\|\hat{z}_c(t,x)\|^2$, $t \geqslant 0$. *Then*

$$0 \leqslant \hat{\mathcal{U}}_\gamma(t,x) \leqslant \varphi_\gamma(x;\hat{\lambda}(0)) \exp(-C(\gamma)t)\xi(t), \tag{31}$$

$$0 \leqslant \hat{h}_\gamma(t,x) \leqslant \frac{\varphi_\gamma(x;\hat{\lambda}(0))}{\delta(\gamma)} \exp(-C(\gamma)t)\xi(t), \tag{32}$$

for any $t \geqslant 0$, $x \in \mathbb{R}^n$ *where the process* $\{\xi(t); t \geqslant 0\}$ *is defined in* (29).

Proof. By hypothesis, the conclusions of the Lemma 4 are satisfied and using the Remark 4 we rewrite the integral equation (26) as it is given in (28). Set $\Phi_\gamma(t) = \exp(-C(\gamma)t)\xi(t)$, $t \geqslant 0$. Represent the solution $\hat{\mathcal{U}}_\gamma(t,x)$ of equation (26) as

$$\hat{\mathcal{U}}_\gamma(t,x) = \Phi_\gamma(t)\Big[\varphi_\gamma(x;\hat{\lambda}(0)) + \int_0^t \Phi_\gamma^{-1}(s)\exp(\gamma s)\beta_\gamma(\hat{z}_c(s,x);\hat{\lambda}(s))\,\mathrm{d}s\Big]. \quad (33)$$

Recall that $\beta_\gamma(z;\lambda) \leqslant 0$, for any $(z,\lambda) \in \mathbb{R}^n \times S$. Thus the conclusion (31) follows. Notice that $\hat{h}_\gamma(t,x) \leqslant \frac{1}{\delta(\gamma)}\hat{\mathcal{U}}_\gamma(t,x)$ and we easily get (32). □

Remark 5. A relaxation of the used assumptions consists in admitting a common eigen subspace generated by the constant vectors $\{q_1, q_2, \ldots, q_d\} \subset \mathbb{R}^n$, $d \leqslant n$. To this respect, the real symmetric $(d \times d)$–matrices $B_i(\lambda)$, $i \in \{0, 1, \ldots, m\}$, will exist such that

$$Q^* A_i(\lambda) z = B_i(\lambda) y,\ i \in \{0,1,\ldots,m\}, \lambda \in S, z \in \mathbb{R}^n,$$

where $Q = [q_1, q_2 \ldots, q_d]$, $y = \mathrm{col}(q_1^* z, q_2^*, \ldots, q_d^* z) \in \mathbb{R}^d$. The original vector fields $g_i(z,\lambda) = a_i(\lambda) + A_i(\lambda)z$ will be replaced by

$$h_i(y,\lambda) = b_i(\lambda) + B_i(\lambda)y,\ b_i(\lambda) = Q^* a_i(\lambda),$$

where $b_i(\lambda) \in \mathbb{R}^d$ and the $(d \times d)$–matrices $B_i(\lambda)$ are continuous and bounded.

For each $x \in \mathbb{R}^n$, define a piecewise continuous process $\{\hat{y}(t,x); t \geqslant 0\}$ as the unique solution of the following dynamic system

$$\begin{cases} d\hat{y}(t) = h_0(\hat{y}(t);\hat{\lambda}(t))\,\mathrm{d}t + \displaystyle\sum_{j=1}^m h_j(\hat{y}(t);\hat{\lambda}(t))\,\mathrm{d}W_j(t),\ t \in [\hat{t}_k, \hat{t}_{k+1}), \\ \hat{y}(\hat{t}_k) = \hat{y}_{\text{-}}(\hat{t}_k) + Q^*\hat{\lambda}(\hat{t}_k), \quad \text{for any } k \geqslant 1, \\ \hat{y}(0) = Q^*[x + \hat{\lambda}(0)]. \end{cases} \quad (34)$$

The analysis contained in Lemmas 1–5 has an adequate version for the piecewise continuous process $\{\hat{y}(t,x); t \geqslant 0\}$ and, in particular, the following decomposition can be used

$$\hat{y}(t,x) = \hat{y}_c(t,x) + Q^*\hat{\lambda}(t).$$

Here, the continuous argument $\hat{y}_c(t,x)$ stands for the unique solution of the linear stochastic system

$$\begin{cases} d\hat{y}_c(t) = h_0(\hat{y}_c(t);\hat{\lambda}(t))\,\mathrm{d}t + \sum_{j=1}^{m} h_j(\hat{y}_c(t);\hat{\lambda}(t))\,\mathrm{d}W_j(t),\ t \geqslant 0, \\ \hat{y}_c(0) = Q^* x. \end{cases} \tag{35}$$

Denote by $\mathbb{E}_1$ the expectation with respect to the probability measure $\mathbb{P}_1$.

Problem C. Prove that under suitable conditions imposed on the matrices $B_i(\lambda)$, we get the asymptotic behaviour of the continuous process $\{\hat{y}(t,x); t \geqslant 0\}$, described as

(a) $\exp(\gamma t)\mathbb{E}_1\|\hat{y}(t,x)\|^2 \to 0$, as $t \to \infty$, for each $x \in \mathbb{R}^n$, provided $\gamma < 0$ and $|\gamma| > \|B_0\| + \frac{1}{2}\sum_{j=1}^{m}\|B_j\|^2$, where $h_i(y;\lambda) = B_i(\lambda)y$ and $\|B_i\| = \sup_{\lambda\in S}\|B_i(\lambda)\|$.

(b) For any γ as above, $\lim_{t\uparrow\infty}\exp(\gamma t)\|\hat{y}(t,x)\|^2 = 0$, a.s. $\mathbb{P}_1$.

3 Solutions of the *Problem C* and main results

Assume that there exist $q_1, q_2, \ldots, q_d \in \mathbb{R}^n$, $d \leqslant n$ and the symmetric $d \times d$-matrices $B_i(\lambda)$, such that

$$Q^* g_i(z;\lambda) = B_i(\lambda)y,\ i \in \{0,1,\ldots,m\},\ \lambda \in S,\ z \in \mathbb{R}^n,$$

where $g_i(z;\lambda) = a_i(\lambda) + A_i(\lambda)z$, $Q = [q_1, q_2, \ldots, q_d]$ and $y = \mathrm{col}(q_1^* z, q_2^* z, \ldots, q_d^* z)$.

Associate the linear stochastic system

$$\begin{cases} \mathrm{d}\hat{y}(t) = B_0(\hat{\lambda}(t))\hat{y}(t)\,\mathrm{d}t + \sum_{j=1}^{m} B_j(\hat{\lambda}(t))\hat{y}(t)dW_j(t),\ t \geqslant 0, \\ \hat{y}(0) = Q^* x,\ x \in \mathbb{R}^n. \end{cases} \tag{36}$$

Moreover, the matrices $B_i(\lambda)$ are assumed bounded, continuous and mutually commuting, i.e.

(*a*) $B_i(\lambda)B_j(\lambda) = B_j(\lambda)B_i(\lambda)$, for any $\lambda \in S$,

(*b*) $B(\lambda) = 2B_0(\lambda) + \sum_{j=1}^{m} B_j^2(\lambda)$ is a constant matrix.

Theorem 1. *Let $\hat{y}(t)$ be the unique solution of the system* (36) *and $\gamma < 0$ such that $|\gamma| > 4\left[\|B_0\| + \frac{1}{2}\sum_{j=1}^m \|B_j\|^2\right]$. Then*

$$\lim_{t\uparrow\infty} \exp(\gamma t)\,\mathbb{E}_1\left[\|\hat{y}(t,x)\|^2\right] = 0, \text{ for any } x \in \mathbb{R}^n. \tag{37}$$

Proof. Let $\varphi_\gamma \in \mathcal{P}_2(z;\lambda)$ be the positively defined quadratic functional $\varphi_\gamma(y;\lambda) = \langle R_\gamma(\lambda)y, y\rangle$, such that the conclusions in the Lemma 3 are fulfilled. Here $z \in \mathbb{R}^n$ is replaced by $y \in \mathbb{R}^d$ and the mentioned equations are replaced by

$$\begin{cases} (a)\ \gamma\varphi_\gamma(y;\lambda) & + \|y\|^2 + L(\varphi_\gamma)(y;\lambda) = 0,\ \forall (y,\lambda) \in \mathbb{R}^n \times S, \\ (b)\ \delta(\gamma)\|y\|^2 & \leqslant \varphi_\gamma(y;\lambda) \leqslant \frac{1}{C(\gamma)}\|y\|^2, \end{cases} \tag{38}$$

where

$$L(\varphi_\gamma)(y;\lambda) = \langle \partial_y \varphi(y,\lambda), B_0(\lambda)y\rangle + \frac{1}{2}\sum_{j=1}^m \langle \partial_y^2 \varphi(y;\lambda)\ B_j(\lambda)y, B_j(\lambda)y\rangle,$$

$C(\gamma), \delta(\gamma)$ are positive constants, $R_\gamma(\lambda) = \left[|\gamma| I_d - B(\lambda)\right]^{-1}$ and $B(\lambda) = 2B_0(\lambda) + \sum_{j=1}^m B_j^2(\lambda)$.

Set $\hat{\mathcal{U}}_\gamma(t,x) = \exp(\gamma t)\varphi_\gamma(\hat{y}(t,x);\hat{\lambda}(t)), t \geqslant 0, x \in \mathbb{R}^n$. Then the piecewise continuous process $\{\hat{\mathcal{U}}_\gamma(t,x); t \geqslant 0\}$ fulfills all the necessary conditions in order to to apply Lemma 4. As a consequence, $\hat{\mathcal{U}}_\gamma(t,x)$ is a solution of the following integral equation

$$\begin{aligned} \hat{\mathcal{U}}_\gamma(t,x) =& \varphi_\gamma(\hat{y}(0,x);\hat{\lambda}(0)) + \hat{D}_y^\gamma(t,x) + \int_0^t \Big[- C(\gamma)\hat{\mathcal{U}}_\gamma(s,x) + \\ & + \exp(\gamma s)\beta_\gamma(\hat{y}(s,x);\hat{\lambda}(s))\Big] ds \\ & + \sum_{j=1}^m \int_0^t \exp(\gamma s)\langle \partial_y \varphi_\gamma(\hat{y}(s,x);\hat{\lambda}(s)), B_j(\hat{\lambda}(s))\hat{y}(s,x)\rangle dW_j(s), \end{aligned} \tag{39}$$

where

$$\begin{cases} \beta_\gamma(y;\lambda) & = C(\gamma)\varphi_\gamma(y;\lambda) - \|y\|^2 \leqslant 0,\ \forall (y,\lambda) \in \mathbb{R}^n \times S, \\ \hat{D}_y^\gamma(t,x) & = \sum_{0<\hat{t}_k\leqslant t} \exp(\gamma \hat{t}_k)\left[\varphi_\gamma(\hat{y}(\hat{t}_k,x);\hat{\lambda}(\hat{t}_k)) - \varphi_\gamma(\hat{y}(\hat{t}_k,x);\hat{\lambda}(\hat{t}_{k-1}))\right]. \end{cases}$$

Notice that the last term in (39) is a martingale on the filtered probability space $\{\Omega_1, \mathcal{F}^1, \{\mathcal{F}_t^1\}_{t\geqslant 0}, \mathbb{P}_1\}$. Taking now the expectation in the equation (39) and setting $v(t,x) = \mathbb{E}_1 \hat{\mathcal{U}}_\gamma(t,x)$, we get

$$v(t,x) = v(0,x) + D(t,x) + \int_0^t \left[-C(\gamma)\, v(s,x) + \exp(\gamma s)\, \beta(s,x)\right] ds, \quad (40)$$

where

$$D(t,x) = \sum_{0<\hat{t}_k\leqslant t} \exp(\gamma \hat{t}_k) \mathbb{E}_1 \left\{ \left\langle \left[R_\gamma(\hat{\lambda}(\hat{t}_k)) - R_\gamma(\hat{\lambda}(\hat{t}_{k-1})) \right] \hat{y}(\hat{t}_k, x), \hat{y}(\hat{t}_k, x) \right\rangle \right\}$$

and

$$\beta(s,x) = \mathbb{E}_1[\beta_\gamma(\hat{y}(s,x); \hat{\lambda}(s))] \leqslant 0, \text{ for any } s \geqslant 0,\ x \in \mathbb{R}^n.$$

Notice that the matrix $R_\gamma(\lambda)$ is expressed as

$$R_\gamma(\lambda) = \frac{1}{|\gamma|} \sum_{k=0}^{\infty} \left(\frac{B(\lambda)}{|\gamma|} \right)^k, \text{ where } B(\lambda) = 2B_0(\lambda) + \sum_{j=1}^m B_j^2(\lambda).$$

In addition, as far as $B(\lambda)$ is a constant matrix, we easily get that $R_\gamma(\lambda)$ is also constant, for any $\lambda \in S$ and $\gamma < 0$. Therefore, the piecewise constant component $\{D(t,x); t \geqslant 0\}$ is vanishing and by the application of a standard integral representation formula we obtain

$$\begin{aligned} 0 \leqslant v(t,x) &= \exp[-C(\gamma)t] \left(v(0,x) + \int_0^t \exp(C(\gamma)s) \exp(\gamma s) \beta(s,x) ds \right) \\ &\leqslant \exp(-C(\gamma)t)\ v(0,x), \text{ for any } t \geqslant 0, x \in \mathbb{R}^n. \end{aligned} \quad (41)$$

Notice that

$$\exp(\gamma t) \mathbb{E}_1 \|\hat{y}(t,x)\|^2 \leqslant \frac{1}{\delta(\gamma)} v(t,x)$$

and using (41) we get the conclusion fulfilled. □

Theorem 2. *Let the assumptions of Theorem 1 be in force. Moreover, let* $B_j(\lambda) = \mu_j(\lambda) I_d$, $j = 1, \ldots, m$, *and* $\mu_j(\lambda)$ *is continuous and bounded.*

Define the exponential martingale

$$\xi(t) = \exp\left[2 \sum_{j=1}^m \left(\int_0^t \hat{\mu}_j(s) dW_j(s) - \int_0^t (\hat{\mu}_j(s))^2 ds \right) \right], t \geqslant 0, \quad (42)$$

where $\hat{\mu}_j(t) = \mu_j(\hat{\lambda}(t))$*. Then*

$$\exp(\gamma t)\ \|\hat{y}(t,x)\|^2 \leqslant \exp[-C(\gamma)t]\ \xi(t)\ \left(\frac{\|\hat{y}(0,x)\|^2}{\delta(\gamma)}\right)$$

and $\{\hat{y}(t,x); t \geqslant 0\}$ *is the unique solution of the system* (37)*.*

Proof. By hypothesis, the integral equation (27) is fulfilled for $\hat{\mathcal{U}}_\gamma(t,x)$ defined in Theorem 1. Using the fact that the matrix $R_\gamma(\lambda)$ is constant, we get that $\hat{D}^\gamma_y(t,x) = 0$, for any $t \geqslant 0, x \in \mathbb{R}^n$. As a consequence, (27) becomes

$$\begin{aligned}\hat{\mathcal{U}}_\gamma(t,x) =& \varphi_\gamma(\hat{y}(0,x);\hat{\lambda}(0)) + \int_0^t \Big[- C(\gamma)\hat{\mathcal{U}}_\gamma(s,x) + \exp(\gamma s)\beta_\gamma(\hat{y}(s,x);\hat{\lambda}(s))\Big] ds \\ &+ 2\sum_{j=1}^m \int_0^t \hat{\mathcal{U}}_\gamma(s,x)\ \hat{\mu}_j(s) dW_j(s).\end{aligned} \tag{43}$$

A standard integral representation formula allows us to write

$$0 \leqslant \hat{\mathcal{U}}_\gamma(t) \leqslant \exp[-C(\gamma)t]\ \xi(t)\ \Big[\varphi_\gamma(\hat{y}(0,x),\hat{\lambda}(0))\Big], \forall\, t \geqslant 0. \tag{44}$$

Notice that

$$\exp(\gamma t)\ \|\hat{y}(t,x)\|^2 \leqslant \frac{1}{\delta(\gamma)}\ \hat{\mathcal{U}}_\gamma(t,x),\ t \geqslant 0,\ x \in \mathbb{R}^n$$

and the estimate (44) leads us to the conclusion. □

4 Applications

4.1 Piecewise continuous processes and elliptic equations associated with Financial Mathematics

We are given a piecewise constant process $\lambda(t,\omega_2) : [0,\infty) \times \Omega_2 \to S \subset \mathbb{R}^n$ on a complete probability field $\{\Omega_2, \mathcal{F}^2, \mathbb{P}_2\}$, such that

$$\lambda(t,\omega_2) = \lambda_k(\omega_2),\ \text{for } t \in [t_k(\omega_2), t_{k+1}(\omega_2)),\ k \geqslant 0,\ \omega_2 \in \Omega_2,$$

where $0 = t_0(\omega_2) < t_1(\omega_2) < \ldots < t_k(\omega_2) < t_{k+1}(\omega_2) < \ldots \leqslant \infty$ is an increasing sequence of random variables, satisfying $t_k(\omega_2) \to \infty$, *a.e.* ω_2 and λ_k is a random vector, for each $k \geqslant 0$.

For each $\hat{\omega}_2 \in \Omega_2$ arbitrarily fixed, denote $\hat{\lambda}(t) = \lambda(t, \hat{\omega}_2), t \geqslant 0$ and $\hat{t}_k = t_k(\hat{\omega}_2), k \geqslant 0$. By definition, $\hat{\lambda}(t) : [0, \infty) \to S$ is a deterministic piecewise constant function, satisfying $\hat{\lambda}(t) = \hat{\lambda}_k$, for $t \in [\hat{t}_k, \hat{t}_{k+1})$, where $\hat{\lambda}_k = \lambda_k(\hat{\omega}_2)$.

For every $x \in \mathbb{R}^d$, let $\{\hat{y}(t, x); t \geqslant 0\}$ be the unique solution of the following stochastic differential system

$$\begin{cases} dy(t) & = g_0(y(t); \hat{\lambda}(t))dt + \sum_{j=1}^{m} g_j(y(t); \hat{\lambda}(t))dW_j(t), \ t \geqslant 0, \ y \in \mathbb{R}^d \supseteq \mathbb{R}^n, \\ y(0) & = x, \end{cases} \tag{45}$$

where the vector fields

$$g_i(y; \lambda) = a_i(\lambda) + A_i(\lambda)y, \ i = 1, \ldots, m, \ \lambda \in S, \ y \in \mathbb{R}^d, \tag{46}$$

are assumed continuous and bounded with respect to $\lambda \in S$. The linear shape of $g_0(y; \lambda)$ is not required and a global Lipschitz condition with respect to $y \in \mathbb{R}^d$ is equally suitable. Here $W(t, \omega_1) : [0, \infty) \times \Omega_1 \to \mathbb{R}^m$ stands for a standard Wiener process over the complete filtered probability space $\{\Omega_1, \mathcal{F}^1, \{\mathcal{F}_t^1\}_{t \geqslant 0}, \mathbb{P}_1\}$ and a solution of the equation (45) is constructed using the Itô's stochastic calculus.

A portofolio problem (American Option) and its admissible strategies can be described by a value function of the following form

$$V(t, x) = \theta_0(t, x)y_0(t) + \langle \theta(t, x), \hat{y}(t, x) \rangle, \ t \geqslant 0, \ x \in \mathbb{R}^d, \tag{47}$$

where $y_0(t) = \exp(-\gamma t)$ and $\theta_0(t, x) \in \mathbb{R}, \theta(t, x) \in \mathbb{R}^d$ are some $\mathcal{F}_t^1$–adapted processes, for each fixed $x \in \mathbb{R}^d$.

Denote by $\mathcal{P}_p(y; \lambda)$ the set consisting of p degree polynomials with respect to the variables $(y_1, \ldots, y_d) = y$, whose coefficients are continuous and bounded functions of $\lambda \in S$. $\mathcal{P}_p(y) \subset \mathcal{P}_p(y; \lambda)$ will be the set of constant coefficients polynomials.

An admissible strategy $(\theta_0(t, x), \theta(t, x)) \in \mathbb{R}^{d+1}$ corresponding to the value function $V(t, x)$ depends on a fixed $\varphi \in \mathcal{P}_p(y; \lambda)$ and has to fulfil the following constraints

$$\overline{V}(t, x) = \exp(\gamma t)V(t, x) \geqslant \exp(\gamma t) \ \varphi(\hat{y}(t, x), \hat{\lambda}(t)), \ t \geqslant 0, \tag{48}$$

$$\overline{V}(t,x)=\overline{V}(\hat{t}_k,x)+\int_{\hat{t}_k}^{t}\exp(\gamma s)\langle\theta(s,x),ds\ \hat{y}(s,x)\rangle,\ t\in[\hat{t}_k,\hat{t}_{k+1}),\tag{49}$$

for each $k\geqslant 0$ and $x\in\mathbb{R}^d$, where the abbreviation

$$\langle\theta(s,x),ds\ \hat{y}(s,x)\rangle=\langle\theta(s,x),g_0(\hat{y}(s,x),\hat{\lambda}(s))\rangle ds+\sum_{j=1}^{m}\langle\theta(s,x),g_j(\hat{y}(s,x),\hat{\lambda}(s))\rangle dW_j(s)$$

is used.

In order to find such strategies, we need to emphasize those conditions which allow to get them in a "feedback shape"

$$\theta(t,x)=\partial_y\varphi(\hat{y}(t,x);\hat{\lambda}(t)),\ \forall t\geqslant 0,\ x\in\mathbb{R}^d\tag{50}$$

and

$$\theta_0(\hat{t}_k,x)=\exp(\gamma\hat{t}_k)\ \varphi(0,\hat{\lambda}_k).\tag{51}$$

Remark 6. For the sake of simplicity, in computing admissible strategies we shall include the "feedback shape" (50) and (51) in the definition of such strategies and we look for appropriate (γ,φ_γ), $\varphi_\gamma\in\mathcal{P}_2(y;\lambda)$, such that the equations (48) and (49) are fulfilled. We emphasize that this approach will lead us to an admissible $(\theta_0(t,x),\theta(t,x))\in\mathbb{R}^{d+1}$, provided

(a) $\varphi_\gamma\in\mathcal{P}_2(y;\lambda)$ is a convex function with respect to $y\in\mathbb{R}^d$;

(b) (γ,φ_γ) is a nontrivial solution of the following elliptic inequality

$$\gamma\ \varphi(y;\lambda)+\sum_{j=1}^{m}\frac{1}{2}\left\langle\partial_y^2\varphi_\gamma(y;\lambda)\ g_j(y;\lambda),g_j(y;\lambda)\right\rangle\leqslant 0,\ (y,\lambda)\in\mathbb{R}^d\times S.\tag{52}$$

The "feedback shape" (50) agrees with the constraints (48) and (49) without involving the convexity property (a) and the analysis can be reduced to the elliptic inequality (52).

On the other hand, it induces a piecewise continuous component process $\{\theta_0(t,x);t\geqslant 0\}$, which is measured as a random variable at each instant $t=\hat{t}_k, k\geqslant 1$, such that

$$\theta_0(\hat{t}_k,x)+\exp(\gamma\hat{t}_k)\ \langle\partial_y\varphi_\gamma(\hat{y}(\hat{t}_k,x);\hat{\lambda}_k),\ \hat{y}(\hat{t}_k,x)\rangle\geqslant\exp(\gamma\hat{t}_k)\ \varphi_\gamma(\hat{y}(\hat{t}_k,x);\hat{\lambda}_k).$$

It can be simplified by choosing the "feedback shape" (51) provided the statement (a) in the Remark 6 takes place.

Problem A_1. For a convex $f \in \mathcal{P}_2(y)$, find a nonzero constant γ and a convex function $\varphi_\gamma \in \mathcal{P}_2(y;\lambda)$ of $y \in \mathbb{R}^d$, such that

$$F_k^\gamma(t,x) = \exp(\gamma t)\ \varphi_\gamma(\hat{y}(t,x);\hat{\lambda}_k) + \int_{\hat{t}_k}^t \exp(\gamma s) f(\hat{y}(s,x))ds,\ t \in [\hat{t}_k, \hat{t}_{k+1}) \tag{53}$$

can be represented as

$$F_k^\gamma(t,x) = F_k^\gamma(\hat{t}_k,x) + \int_{\hat{t}_k}^t \exp(\gamma s)\langle \partial_y \varphi_\gamma(\hat{y}(s,x);\hat{\lambda}_k), ds\hat{y}(s,x)\rangle,\ k \geqslant 0, \tag{54}$$

where the abbreviation

$$\begin{aligned}\langle \theta_k(s,x), ds\ \hat{y}(s,x)\rangle &= \langle \theta_k(s,x), g_0(\hat{y}(s,x),\hat{\lambda}(s))\rangle ds \\ &\quad + \sum_{j=1}^m \langle \theta_k(s,x), g_j(\hat{y}(s,x),\hat{\lambda}(s))\rangle dW_j(s)\end{aligned}$$

is used.

Problem B_1. For a convex $f \in \mathcal{P}_2(y)$, find a nonzero constant γ and a convex function $\varphi_\gamma \in \mathcal{P}_2(y;\lambda)$ with respect to $y \in \mathbb{R}^d$, such that the following elliptic equation is satisfied

$$\gamma\ \varphi_\gamma(y;\lambda) + f(y) + \hat{L}(\varphi_\gamma(y;\lambda)) = 0,\ \text{for any } (y,\lambda) \in \mathbb{R}^d \times S, \tag{55}$$

where $\hat{L} : \mathcal{P}_2(y;\lambda) \to \mathcal{P}_2(y;\lambda)$ stands for the linear operator defined as

$$\hat{L}(\varphi(y;\lambda)) = \sum_{j=1}^m \frac{1}{2}\left\langle \partial_y^2 \varphi(y;\lambda)\ g_j(y;\lambda), g_j(y;\lambda)\right\rangle. \tag{56}$$

Remark 7. Each nontrivial solution $(f,\gamma,\varphi_\gamma)$ of the Problem B_1 solves the Problem A_1. To this respect, applying the standard rule of stochastic differentiation, for each $F_k^\gamma(t,x)$, $t \in [\hat{t}_k,\hat{t}_{k+1})$ defined in (53) we get the integral representation given in (54).

Consider the test function $\mathcal{U}(t,y) = \exp(\gamma t)\varphi_\gamma(y;\hat{\lambda}_k)$, $t \in [\hat{t}_k,\hat{t}_{k+1})$ associated with the continuous process $\{\hat{y}(t,x); t \in [\hat{t}_k,\hat{t}_{k+1})\}$ satisfying (45).

Let $(f,\gamma,\varphi_\gamma)$ be a solution of the Problem B_1. Then, for each $k\geqslant 0$ we rewrite $\mathcal{U}(t,\hat{y}(t,x))$ as

$$\mathcal{U}(t,\hat{y}(t,x)) = \exp(\gamma\hat{t}_k)\varphi_\gamma(\hat{y}(\hat{t}_k,x);\hat{\lambda}_k) + \int_{\hat{t}_k}^t \exp(\gamma s)\left[\gamma\varphi_\gamma + L(\varphi_\gamma)(\hat{y}(s,x);\hat{\lambda}_k)\right] ds$$
$$+\sum_{j=1}^m \int_{\hat{t}_k}^t \exp(\gamma s)\langle\partial_y\varphi_\gamma(\hat{y}(s,x);\hat{\lambda}_k), g_j(\hat{y}(s,x);\hat{\lambda}_k)\rangle dW_j(s), \tag{57}$$

for any $t\in[\hat{t}_k,\hat{t}_{k+1})$ and $k\geqslant 0$, where $L:\mathcal{P}_2(y;\lambda)\to\mathcal{P}_2(y;\lambda)$ is the linear operator defined as

$$L(\varphi(y;\lambda)) = \langle\partial_y\varphi(y,\lambda), g_0(y;\lambda)\rangle + \frac{1}{2}\left\langle\partial_y^2\varphi(y;\lambda)\ g_j(y;\lambda), g_j(y;\lambda)\right\rangle.$$

L may be rewritten as

$$L(\varphi(y;\lambda)) = \langle\partial_y\varphi(y,\lambda), g_0(y;\lambda)\rangle + \hat{L}(\varphi(y;\lambda)),$$

where the operator $\hat{L}$ is given in the formula (56). Then

$$F_k^\gamma(t,x) = \mathcal{U}(t,\hat{y}(t,x)) + \int_{\hat{t}_k}^t \exp(\gamma s) f(\hat{y}(s,x))ds,\ t\in[\hat{t}_k,\hat{t}_{k+1})$$

or equivalently

$$F_k^\gamma(t,x) = \exp(\gamma t)\ \varphi_\gamma(\hat{y}(\hat{t}_k,x);\hat{\lambda}_k) + \int_{\hat{t}_k}^t \exp(\gamma s)\ \langle\partial_y\varphi_\gamma(\hat{y}(s,x);\hat{\lambda}_k), ds\hat{y}(s,x)\rangle$$
$$+\int_{\hat{t}_k}^t \exp(\gamma s)[\gamma\varphi_\gamma + f + \hat{L}(\varphi_\gamma)](\hat{y}(s,x);\hat{\lambda}_k)ds. \tag{58}$$

Notice that if $(f,\gamma,\varphi_\gamma)$ is a solution of the Problem B_1, then the last integral term in (58) is vanishing and the integral equation (54) is established.

When defining admissible strategies $(\theta_0(t,x),\theta(t,x))\in\mathbb{R}^{d+1}$ such that the constraints (48), (49) and (50) are satisfied, we may use the integral representation given in the Problem A_1 without imposing a convexity property for the function $\varphi_\gamma \in \mathcal{P}_2(y;\lambda)$. In order to get a sequence of deterministic values $\{\theta_0(\hat{t}_k,x); k \geqslant 0\}$, such that

$$\begin{aligned}\overline{V}(\hat{t}_k,x) =&\theta_0(\hat{t}_k,x) + \exp(\gamma\hat{t}_k)\langle\partial_y\varphi_\gamma(\hat{y}(\hat{t}_k,x);\hat{\lambda}_k),\hat{y}(\hat{t}_k,x)\rangle \\ &\geqslant \exp(\gamma\hat{t}_k)\varphi_\gamma(\hat{y}(\hat{t}_k,x);\hat{\lambda}_k) = F_k^\gamma(\hat{t}_k,x),\end{aligned} \tag{59}$$

we need to assume that φ_γ is a convex function with respect to $y \in \mathbb{R}^d$ and that

$$\theta_0(\hat{t}_k,x) = \exp(\gamma\hat{t}_k)\varphi_\gamma(0;\hat{\lambda}_k) \tag{60}$$

satisfies (59).

Lemma 6. *Let $(f,\gamma,\varphi_\gamma)$ be a nontrivial solution of the Problem B_1 and assume $f(y) \geqslant 0$, for any $y \in \mathbb{R}^d$. Define*

$$\theta(t,x) = \partial_y\varphi_\gamma(\hat{y}(t,x);\hat{\lambda}(t)),\ t \geqslant 0,\ x \in \mathbb{R}^d \tag{61}$$

and let $\{\theta_0(t,x); t \geqslant 0\}$ be the piecewise continuous process satisfying the integral equation (49)*, where $\theta_0(\hat{t}_k,x)$ is defined in* (60)*. Then $(\theta_0(t,x),\theta(t,x)) \in \mathbb{R}^{d+1}$ is an admissible strategy (see the formulas* (48) *and* (49)*) satisfying the "feedback shape"* (50) *and* (51)*.*

Proof. By hypothesis, the conditions of the Remark 7 are fulfilled and $(f,\gamma,\varphi_\gamma)$ is a nontrivial solution of the Problem A_1. We get

$$\begin{aligned}\exp(\gamma t)\ \varphi_\gamma(\hat{y}(t,x);\hat{\lambda}_k) =&\exp(\gamma\hat{t}_k)\ \varphi_\gamma(\hat{y}(\hat{t}_k,x);\hat{\lambda}_k) \\ &+ \int_{\hat{t}_k}^t \exp(\gamma s)\ \langle\partial_y\varphi_\gamma(\hat{y}(s,x);\hat{\lambda}_k), d s\hat{y}(s,x)\rangle \\ &- \int_{\hat{t}_k}^t \exp(\gamma s) f(\hat{y}(s,x))ds,\end{aligned} \tag{62}$$

for every $t \in [\hat{t}_k,\hat{t}_{k+1})$ and $k \geqslant 0$.

On the other hand, the value function

$$\overline{V}(t,x) = \theta_0(t,x) + \exp(\gamma t)\langle\partial_y\varphi_\gamma(\hat{y}(t,x);\hat{\lambda}_k),\hat{y}(t,x)\rangle,\ t \in [\hat{t}_k,\hat{t}_{k+1})$$

satisfies

$$\overline{V}(t,x) \geqslant \exp(\gamma t)\varphi_\gamma(\hat{y}(t,x);\hat{\lambda}_k),\ t \in [\hat{t}_k,\hat{t}_{k+1}).$$

Setting $\theta_0(\hat{t}_k, x) = \exp(\gamma \hat{t}_k)\varphi(0; \hat{\lambda}_k)$, for every $k \geqslant 0$, we get the constraints (59) fulfilled, by virtue of the fact that the gradient $\partial_y \varphi_\gamma(y; \lambda)$ of a convex function satisfies

$$\langle \partial_y \varphi_\gamma(y_2; \lambda) - \partial_y \varphi_\gamma(y_1; \lambda),\ y_2 - y_1 \rangle \geqslant 0, \text{ for any } y_1, y_2 \in \mathbb{R}^d \text{ and } \lambda \in S. \tag{63}$$

Define now $\{\theta_0(t, x);\ t \in [\hat{t}_k, \hat{t}_{k+1})\}$ as the solution of the integral equation (49), where

$$\theta_0(\hat{t}_k, x) = \exp(\gamma \hat{t}_k)\varphi_\gamma(0; \hat{\lambda}_k)$$

is fixed. As a consequence, using the equation (62) we get the remaining conclusion of the Lemma. □

Remark 8. The analysis contained in the Lemma 6 shows that an admissible strategy with a "feedback shape" can be constructed using a nontrivial solution $(f, \gamma, \varphi_\gamma)$ of the Problem B_1. For a fixed $f \in \mathcal{P}_2(y)$, a solution (γ, φ_γ) of the elliptic equation (55) is constructed using the following series

$$\varphi_\gamma(y; \lambda) = \frac{1}{|\gamma|}\left[\sum_{k=0}^{\infty} \hat{L}_{|\gamma|}^k(f)(y; \lambda)\right], \text{ for } \gamma < 0, \tag{64}$$

where $\hat{L}_{|\gamma|} = \frac{1}{|\gamma|}\hat{L}$ and $\hat{L} : \mathcal{P}_2(y; \lambda) \to \mathcal{P}_2(y; \lambda)$ stands for the linear operator defined in the Problem B_1.

As far as the linear operator $\hat{L}_{|\gamma|}$ is acting on $\mathcal{P}_2(y; \lambda)$, for the sake of simplicity we shall assume that $f(y) = (\langle q, y \rangle)^2$, where $q \neq 0$ is a common eigen vector of the matrices $A_j(\lambda)$, such that $A_j^*(\lambda)q = \mu_j(\lambda)q$ and $\mu_j : S \to \mathbb{R}$ is continuous and bounded, for any $1 \leqslant j \leqslant m$.

Lemma 7. *Let $f \in \mathcal{P}_2(y)$ and $g_j(y; \lambda) = A_j(\lambda)y + a_j(\lambda)$, $j = 1, \ldots, m$, be given as above. Let $\gamma < 0$ such that $\frac{\|\mu\|}{|\gamma|} \leqslant 1$, where $\mu(\lambda) = \sum_{j=1}^m \mu_j^2(\lambda)$ and $\|\mu\| = \sup_{\lambda \in S} \mu(\lambda)$. Then the function*

$$\varphi_\gamma(y; \lambda) = \frac{1}{|\gamma|}\left[\sum_{k=0}^{\infty} \hat{L}_{|\gamma|}^k(f)(y; \lambda)\right] \tag{65}$$

$$= \frac{1}{|\gamma| - \mu(\lambda)}\left[f(y) + \frac{b(\lambda)}{|\gamma|}\langle q, y \rangle + \frac{a(\lambda)}{|\gamma|}\right], y \in \mathbb{R}^d, \lambda \in S \tag{66}$$

is a solution of the elliptic equation (55), where $b(\lambda) = 2\sum_{j=1}^m \mu_j(\lambda)\langle q, a_j(\lambda)\rangle$ and $a(\lambda) = \sum_{j=1}^m (\langle q, a_j(\lambda)\rangle)^2$.

Proof. By hypothesis, we easily see that

$$\hat{L}(f)(y;\lambda) = \sum_{j=1}^{m} [A_j(\lambda)y + a_j(\lambda)]^* \, q \, q^* \, [A_j(\lambda)y + a_j(\lambda)] \tag{67}$$
$$= \sum_{j=1}^{m} (\langle q, A_j(\lambda)y + a_j(\lambda)\rangle)^2 = \mu(\lambda)f(y) + b(\lambda)\langle q, y\rangle + a(\lambda).$$

Hence

$$\hat{L}_{|\gamma|}(f)(y;\lambda) = \frac{\mu(\lambda)}{|\gamma|} f(y) + \frac{b(\lambda)}{|\gamma|}\langle q, y\rangle + \frac{a(\lambda)}{|\gamma|}. \tag{68}$$

An induction argument leads us to

$$\hat{L}^k_{|\gamma|}(f)(y;\lambda) = \left(\frac{\mu(\lambda)}{|\gamma|}\right)^k f(y) + \left(\frac{\mu(\lambda)}{|\gamma|}\right)^{k-1} \left[\frac{b(\lambda)}{|\gamma|}\langle q, y\rangle\right] \tag{69}$$
$$+ \left(\frac{\mu(\lambda)}{|\gamma|}\right)^{k-1} \left[\frac{a(\lambda)}{|\gamma|}\right], \text{ for any } k \geqslant 1.$$

Denote $\rho_\gamma(\lambda) = \frac{\mu(\lambda)}{|\gamma|}$ and

$$S(\lambda) = \sum_{k=0}^{\infty} [\rho_\gamma(\lambda)]^k = \frac{|\gamma|}{|\gamma| - \mu(\lambda)},$$

where $\rho_\gamma(\lambda) < 1$, for any $\lambda \in S$ is used (see $\frac{\|\mu\|}{|\gamma|} \leqslant 1$). Inserting the formula (69) in (65), we obtain

$$\varphi_\gamma(y;\lambda) = \frac{1}{|\gamma|} S(\lambda) f(y) + \frac{1}{|\gamma|} S(\lambda) \frac{b(\lambda)}{|\gamma|}\langle q, y\rangle + \frac{1}{|\gamma|} S(\lambda) \frac{a(\lambda)}{|\gamma|}$$

and substituting $S(\lambda) = \frac{|\gamma|}{|\gamma| - \mu(\lambda)}$ we get the conclusion fulfilled. □

Remark 9. The solution of the Problem B_1 constructed in the Lemma 7 makes use of a special convex function $f(y) = (\langle q, y\rangle)^2$, with $q \in \mathbb{R}^d$ as a common eigen vector of the matrices $A_j(\lambda)$, $j = 1, \ldots, m$.

Assuming that there exist several eigen vectors $Q = (q_1, \ldots, q_s)$, $s \leqslant d$, such that

$$Q^* A_j(\lambda) = \mu_j(\lambda) Q^*, \ \mu_j(\lambda) \in \mathbb{R}, \ j = 1, \ldots, m, \tag{70}$$

then $f(y) = \langle Q^*y, Q^*y\rangle$ agrees with the conclusion of the Lemma 7 and the computation of the convex function $\varphi_\gamma \in \mathcal{P}_2(y)$ follows the same procedure.

In addition, for an arbitrarily fixed $y_0 \in \mathbb{R}^d$, we may consider a convex function

$$f(y) = \langle Q^*(y - y_0), Q^*(y - y_0)\rangle, \tag{71}$$

where $\hat{z}(t, x) = \hat{y}(t, x) - y_0$, $t \geqslant 0$, satisfies the following linear system

$$\begin{cases} dz(t) &= h_0(z(t); \lambda)dt + \sum_{j=1}^m h_j(z(t); \lambda)dW_j(t),\ t \geqslant 0 \\ z(0) &= x - y_0. \end{cases} \tag{72}$$

Here $h_i(z; \lambda) = A_i(\lambda)z + d_i(\lambda),\ d_i(\lambda) = a_i(\lambda) + A_i(\lambda)y_0,\ i = 0, 1, \ldots, m$ replaces the original vector fields $g_i(y; \lambda$ of the system (45) and the function $f(z) = \langle Q^*z, Q^*z\rangle$ satisfies (70).

We conclude the above given analysis by the following

Theorem 3. *Let $g_j(y; \lambda) = A_j(\lambda)y + a_j(\lambda)$ be given such that the $(d \times d)$ matrix $A_j(\lambda)$ and the vector $a_j(\lambda) \in \mathbb{R}^d$ are continuous and bounded with respect to $\lambda \in S \subset \mathbb{R}^n$, for any $j = 1, \ldots, m$ and $d \leqslant n$. Consider a continuous vector field $g_0(y; \lambda) \in \mathbb{R}^d$ which is globally Lipschitz continuous with respect to $y \in \mathbb{R}^d$, uniformly in $\lambda \in S$.*

Define a convex function $f \in \mathcal{P}_2(y)$ by

$$f(y) = \langle Q^*(y - y_0), Q^*(y - y_0)\rangle, \tag{73}$$

where $y_0 \in \mathbb{R}^d$ is arbitrarily fixed and $Q = (q_1, \ldots, q_s),\ q_i \in \mathbb{R}^d,\ s \leqslant d$ stand for some common eigen vectors satisfying

$$Q^*A_j(\lambda) = \mu_j(\lambda)Q^*,\ \mu_j(\lambda) \in \mathbb{R},\ j = 1, \ldots, m. \tag{74}$$

Let $\gamma < 0$ be such that $\frac{\|\hat{\mu}\|}{|\gamma|} < 1$, where $\mu(\lambda) = \sum_{j=1}^m \mu_j^2(\lambda)$ and $\|\hat{\mu}\| = \sup_{k \geqslant 0} \mu(\hat{\lambda}_k)$.

Then

$$\begin{aligned} \varphi_\gamma(y; \lambda) =& \frac{1}{|\gamma|}\left[\sum_{k=0}^\infty \hat{L}^k_{|\gamma|}(f)(y; \lambda)\right] = \frac{1}{|\gamma| - \mu(\lambda)} \\ &\times \left[f(y) + \langle\frac{b(\lambda)}{|\gamma|}, Q^*(y - y_0)\rangle + \frac{a(\lambda)}{|\gamma|}\right],\ y \in \mathbb{R}^d,\ \lambda \in \hat{S} \subset S, \end{aligned} \tag{75}$$

is a solution of the elliptic equation (55), where $b(\lambda) = 2\sum_{j=1}^m \mu_j(\lambda)Q^*d_j(\lambda)$, $a(\lambda) = \sum_{j=1}^m \|Q^*d_j(\lambda)\|^2$, $d_j(\lambda) = a_j(\lambda) + A_j(\lambda)y_0$, $j = 1, \ldots, m$, and $\hat{S} = \{\hat{\lambda}_k; k \geqslant 0\} \subset S$.

Proof. Using the linear mapping $z = y - y_0$, we rewrite

$$f(y) = \hat{f}(z) = \langle Q^* z, Q^* z \rangle$$

and the solution $\{\hat{y}(t,x); t \geqslant 0\}$ satisfying (45) is shifted into $\hat{z}(t,x) = \hat{y}(t,x) - y_0$, which satisfies the system (72). Here $h_j(z;\lambda) = A_j(z;\lambda)z + d_j(\lambda)$, $j = 1,\dots,m$ and $h_0(z;\lambda) = g_0(z + y_0;\lambda)$.

The procedure employed in the proof of the Lemma 7 is applicable here and the convex function $\varphi_\gamma \in \mathcal{P}_2(y;\lambda)$ given in (75) satisfies the equation (55). $\square$

Theorem 4. *Assume that the vector fields* $g_i(y;\lambda), i = 1,\dots,m$ *and* $\gamma < 0$ *satisfy the hypothesis of the Theorem 1. Let* $(f,\gamma,\varphi_\gamma)$ *be the nontrivial solution of the Problem* $B.1$*, such that the formulas* (73)*,* (74)*,* (75) *are fulfilled. Define*

$$\theta(t,x) = \partial_y \varphi_\gamma(\hat{y}(t,x); \hat{\lambda}(t)),\ t \geqslant 0,\ x \in \mathbb{R}^d \tag{76}$$

and let $\{\theta_0(t,x); t \geqslant 0\}$ *be the piecewise continuous process satisfying the integral equations* (49)*, where*

$$\theta_0(\hat{t}_k, x) = \exp(\gamma \hat{t}_k)\ \partial_y \varphi_\gamma(y_0; \hat{\lambda}_k),\ k \geqslant 0,\ x \in \mathbb{R}^d \tag{77}$$

and y_0 *comes from the formula* (73) *Then* $(\theta_0(t,x), \theta(t,x)) \in \mathbb{R}^{d+1}$ *is an admissible strategy corresponding to the value function*

$$V(t,x) = \theta_0(t,x) y_0(t) + \langle \theta(t,x), \hat{y}(t,x) - y_0 \rangle,\ t \geqslant 0,\ x \in \mathbb{R}^d.$$

Proof. By hypothesis, the nontrivial solution $(f,\gamma,\varphi_\gamma)$ of the *Problem* $B.1$ constructed in the Theorem 4 fulfills the conditions assumed in the Lemma 6. The "feedback shape" recommended by the equations (60) and (61) uses the deterministic values $\theta_0(\hat{t}_k, x) = \exp(\gamma \hat{t}_k)\varphi_\gamma(0; \hat{\lambda}_k)$, for $k \geqslant 0$, which are not correlated with the special form that we obtain here for the convex functions $f \in \mathcal{P}_2(y)$, $\varphi_\gamma \in \mathcal{P}_2(y;\lambda)$.

According to the expression of φ_γ given in the formula (75), the simplest values are obtained for $y = y_0 \in \mathbb{R}^d$, i.e.

$$\varphi_\gamma(y_0, \hat{\lambda}_k) = \frac{1}{|\gamma| - \mu(\hat{\lambda}_k)} \frac{a(\hat{\lambda}_k)}{|\gamma|},\ k \geqslant 0.$$

This is a slight changing in the definition of the "feedback shape" (see the formulas (50) and (51)) and it agrees with the linear mapping $z = y - y_0$ used in the proof of the Theorem 3, for which $z = 0$ corresponds to the special "feedback shape" given in (60) and (61).

As a consequence, $(\theta_0(t,x), \theta(t,x)) \in \mathbb{R}^{d+1}$ defined in (76) and (77) is an admissible strategy corresponding to the value function

$$V(t,x) = \theta_0(t,x)y_0(t) + \langle \theta(t,x), \hat{y}(t,x) - y_0 \rangle,\ t \geqslant 0,\ x \in \mathbb{R}^d$$

and $\hat{z}(t,x) = \hat{y}(t,x) - y_0,\ t \geqslant 0$, is the solution of the system (72). $\square$

4.2 Application II

The dynamical systems we are going to analyze are described by the following linear stochastic differential equations

$$\begin{cases} dy(t) & = B_0(\lambda(t))\, y(t)dt + \sum_{j=1}^{m} B_j(\lambda(t))\, y(t)\, dW_j(t),\ t \geqslant 0,\ y \in \mathbb{R}^d \subset \mathbb{R}^n, \\ y(0) & = y_0, \end{cases} \tag{78}$$

where $\{W(t), t \geqslant 0\}$ is a standard m–dimensional Wiener process over a complete filtered probability space $\{\Omega_1, \mathcal{F}^1, \{\mathcal{F}_t^1\}_{t\geqslant 0}, \mathbb{P}_1\}$ and the coefficients of the matrices $B_i(\lambda),\ \lambda \in S \subset R^n,\ i = 0, 1, \ldots, m$ are continuous and bounded functions.

Here $\{\lambda(t) \in S; t \geqslant 0\}$ is a piecewise constant process defined on another complete probability space $\{\Omega_2, \mathcal{F}^2, \mathbb{P}_2\}$, such that

$$\lambda(t, \omega_2) = \lambda(t_k, \omega_2),\ t \in [t_k, t_{k+1}),\ k = 0, 1, \ldots,\ \omega_2 \in \Omega_2,$$

where $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} < \infty$ is an increasing sequence with $\lim_{k\to\infty} t_k = \infty$. The piecewise constant process $\{\lambda(t);\ t \geqslant 0\}$ may represent a sequence of measurements produced on a piecewise continuous process.

The stochastic differential system (78) and its continuous solution $\{y(t; y_0);\ t \geqslant 0\}$ are viewed as perturbations acting on the projected piecewise constant process

$$\left\{\lambda_d(t) = \Pr_{\mathbb{R}^d} \lambda(t);\ t \geqslant 0\right\} = \left\{\lambda_d(t, \omega_2) = \Pr_{\mathbb{R}^d} \lambda(t, \omega_2);\ t \geqslant 0,\ \omega_2 \in \Omega_2\right\}.$$

To this respect, the piecewise continuous process $\{z(t, y_0); t \geqslant 0\}$, defined as

$$z(t, y_0) = y(t, y_0) + \lambda_d(t), \tag{79}$$

has the same asymptotic behaviour in the mean square $\mathbb{E}_1$ (with respect to the probability $\mathbb{P}_1$) as $\{\lambda_d(t);\ t \geqslant 0\}$, provided $\lim_{t\to\infty} \mathbb{E}_1\|y(t, y_0)\|^2 = 0$.

A standard rule of stochastic differentiation applied to the continuous process $\{y(t, y_0);\ t \geqslant 0\}$ leads us to the following equation

$$\mathbb{E}_1 \|y(t, y_0)\|^2 = \|y_0\|^2 + \int_0^t \mathbb{E}_1 \left\langle \left[2B_0(\lambda(s)) + \sum_{j=1}^m B_j^2(\lambda(s))\right] y(s, y_0), y(s, y_0) \right\rangle ds,$$

for any $t \geqslant 0$.

A decreasing property for the smooth deterministic function $\mathbb{E}_1\|y(t, y_0)\|^2$, $t \geqslant 0$ is valid if we impose the condition $\langle B(\lambda)y, y\rangle \leqslant 0$, for any $y \in \mathbb{R}^d$ and $\lambda \in S$, where $B(\lambda) = 2B_0(\lambda) + \sum_{j=1}^m B_j^2(\lambda)$.

Such a nice behaviour of an unknown perturbation can hardly be accepted, even if we assume

$$B_i(\lambda) = \mu_i(\lambda)A,\ i = 0, 1, \ldots, m, \tag{80}$$

where $\{\mu_i(\lambda); \lambda \in S\}$ is a continuous and bounded function and A is a symmetric matrix.

The hypothesis (80) allows to define some scalar components of the matrix $B(\lambda) = 2B_0(\lambda) + \sum_{j=1}^m B_j^2(\lambda)$ as

$$\begin{aligned} &B(\lambda) = T^{-1}D(\lambda)T,\ \text{where } D(\lambda) = diag(\rho_1(\lambda), \ldots, \rho_d(\lambda)) \\ &\rho_k(\lambda) = 2\mu_0(\lambda)a_k + \left[\textstyle\sum_{j=1}^m (\mu_j(\lambda))^2\right] a_k^2,\ k = 1, \ldots, d, \end{aligned} \tag{81}$$

and $T : \mathbb{R}^d \to \mathbb{R}^d$ is an orthogonal mapping ($T^* = T^{-1}$), such that $TAT^{-1} = diag(a_1, \ldots, a_d)$.

The main results obtained for the *Problem C* (see Theorems 1 and 2) are restricted to the case when the scalar components $\rho_k(\lambda)$, $k = 1, \ldots, d$, are constants functions with respect to $\lambda \in S$.

Here the analysis will be focussed considering that the assumption (80) is satisfied and the scalar components defined in (81) fulfil the decreasing property

$$\rho_s(\lambda(t_{k+1})) \leqslant \rho_s(\lambda(t_k)),\ \text{for any } s = 1, \ldots, d \text{ and } k \geqslant 0. \tag{82}$$

Problem C_1 Find a constant $\gamma < 0$, such that the unique solution $\{y_\gamma(t, y_0); t \geqslant 0\}$ satisfying the augmented system

$$\begin{cases} dy(t) = \left[\frac{1}{2}\gamma y(t) + B_0(\lambda(t))y(t)\right] dt + \sum_{j=1}^m B_j(\lambda(t))y(t)dW_j(t),\ t \geqslant 0 \\ y(0) = y_0 \end{cases} \tag{83}$$

fulfills $\lim_{t\to\infty} \mathbb{E}_1\|y_\gamma(t,y_0)\|^2 = 0$, for any $y_0 \in \mathbb{R}^d$.

Lemma 8. *Let the matrices $B_i(\lambda)$, $\lambda \in S$, $i = 0,1,\ldots,m$ and the piecewise constant process $\{\lambda(t); t \geqslant 0\}$ are given such that the hypothesis* (80) *and* (82) *are fulfilled. Let $\gamma < 0$ satisfying*

$$|\gamma| > 2\left(\|g_0\| + \frac{1}{2}\sum_{j=1}^{m}\|g_j\|^2\right), \text{ where } \|g_i\| = \|A\|\sup_{\lambda\in S}|\mu_i(\lambda)|, i = 0,1,\ldots,m. \tag{84}$$

Then

$$\lim_{t\to\infty} \mathbb{E}_1\|y_\gamma(t,y_0)\|^2 = 0, \text{ for any } y_0 \in \mathbb{R}^d, \tag{85}$$

where $\{y_\gamma(t,y_0); t \geqslant 0\}$ stands for the unique solution of the system (83).
In addition,

$$\mathbb{E}_1\|y_\gamma(t,y_0)\|^2 \leqslant C(y_0,\gamma)\exp(-c(\gamma)t),\ t \geqslant 0,$$

with some positive constants $C(y_0,\gamma), c(\gamma)$.

Proof. We notice that $y_\gamma(t,y_0) = \exp\left(\frac{\gamma t}{2}\right) y(t,y_0)$, $t \geqslant 0$. In addition,

$$\|y_\gamma(t,y_0)\|^2 = \exp(\gamma t)\|y(t,y_0\|^2,\ t \geqslant 0$$

and the conclusion (84) can be rewritten as

$$\lim_{t\to\infty} \exp(\gamma t)\ \mathbb{E}_1\|y(t,y_0)\|^2 = 0,$$

for any $y_0 \in \mathbb{R}^d$, where the constant $\gamma < 0$ and the matrices $B_i(\lambda)$, $i = 0,1,\ldots,m$ (see (80)) satisfy the assumptions in the Lemma 4. Denote

$$u(t,y_0) = \exp(\gamma t)\varphi_\gamma(y(t,y_0);\lambda(t)),\ t \geqslant 0,\ y_0 \in \mathbb{R}^d,$$

where $\varphi_\gamma(y;\lambda) = \langle R_\gamma(\lambda)y, y\rangle$ fulfills

$$\delta(\gamma)\|y\|^2 \leqslant \varphi_\gamma(y;\lambda) \leqslant \frac{1}{C(\gamma)}\|y\|^2, \text{ for any } \lambda \in S,\ y \in \mathbb{R}^d \tag{86}$$

and some positive constants $\delta(\gamma)$, $C(\gamma)$.

Applying the result given in the Lemma 1, we get the following integral equation

$$
\begin{aligned}
u(t, y_0) =& \varphi_\gamma(y_0; \lambda(0)) + \psi(t; y_0) \\
&+ \int_0^t [-C(\gamma) u(s, y_0) + \exp(\gamma s) \beta(y(s, y_0); \lambda(s))] ds \\
&+ \sum_{j=1}^m \int_0^t \exp(\gamma s) \left\langle \partial_y \varphi_\gamma(y(s, y_0); \lambda(s)), B_j(\lambda(s)) y(s, y_0) \right\rangle dW_j(s),
\end{aligned}
\tag{87}
$$

for any $t \geqslant 0$, where the piecewise constant process $\{\psi(t; y_0); t > 0\}$ is defined as

$$
\psi(t; y_0) = \sum_{0 < t_k \leqslant t} \exp(\gamma t_k) \left[\varphi_\gamma(y(t_k, y_0); \lambda(t_k)) - \varphi_\gamma(y(t_k, y_0); \lambda(t_{k-1})) \right] \tag{88}
$$

In addition,

$$
\beta(y; \lambda) = C(\gamma) \varphi_\gamma(y; \lambda) - \|y\|^2 \leqslant 0, \text{ for any } (y, \lambda) \in \mathbb{R}^d \times S
$$

and as a consequence $\exp(\gamma s)\ \beta(y(s, y_0); \lambda(s)) = \alpha(s; y_0)$ satisfies

$$
\alpha(s; y_0) \leqslant 0, \text{ for any } s \geqslant 0,\ y_0 \in \mathbb{R}^d.
$$

Notice that the last term in (87) is a martingale with respect to the probability $\mathbb{P}_1$. Taking the mean value $\mathbb{E}_1$ in the equation (87), we obtain an integral equation for the unknown $u_1(t, y_0) = \mathbb{E}_1 u(t, y_0)$,

$$
u_1(t, y_0) = c_0 + \psi_1(t, y_0) + \int_0^t [-C(\gamma) u_1(s, y_0) + \alpha_1(s, y_0)] ds,\ t \geqslant 0, \tag{89}
$$

where $c_0 = \varphi_\gamma(y_0; \lambda(0)) \geqslant 0, \quad \psi_1(t, y_0) = \mathbb{E}_1 \psi(t, y_0)$ and

$$
\alpha_1(s, y_0) = \mathbb{E}_1 \alpha(s, y_0) \leqslant 0, \text{ for any } s \geqslant 0,\ y_0 \in \mathbb{R}^d.
$$

We are going to prove that

$$
\psi(t, y_0) \leqslant 0\ (\psi_1(t, y_0) = \mathbb{E}_1 \psi(t, y_0) \leqslant 0) \tag{90}
$$

for any $t \geqslant 0$, $y_0 \in \mathbb{R}^d$ provided the hypothesis (82) is assumed. Even more, assuming that the conditions (82) are satisfied, we get

$$
\varphi_\gamma(y(t_k, y_0); \lambda(t_k)) - \varphi_\gamma(y(t_k, y_0); \lambda(t_{k-1})) \leqslant 0, \text{ for any } k \geqslant 1 \tag{91}
$$

and $y_0 \in \mathbb{R}^d$, where $\varphi_\gamma(y;\lambda) = \langle R_\gamma(\lambda)y, y\rangle$.

Here the matrix $R_\gamma(\lambda)$ is constructed as a value of a convergent series

$$R_\gamma(\lambda) = \frac{1}{|\gamma|}\sum_{k=0}^{\infty}\left(\frac{B(\lambda)}{|\gamma|}\right)^k = \frac{1}{|\gamma|}\sum_{k=0}^{\infty}\left[\frac{2\,\mu_0(\lambda)A + \left(\sum_{j=1}^m \mu_j^2(\lambda)\right)A^2}{|\gamma|}\right]^k,$$

where $B(\lambda) = 2B_0(\lambda) + \sum_{j=1}^m B_j^2(\lambda)$ and the hypothesis (80) is used.

To this respect, let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an orthogonal mapping ($T^* = T^{-1}$), such that

$$TAT^{-1} = \text{diag}\,(a_1,\ldots,a_d),\ a_i \in \mathbb{R} \text{ (see } A \text{ is symmetric).}$$

Using the conditions (80), we rewrite

$$B(\lambda) = T^{-1}D(\lambda)T \text{ and } R_\gamma(\lambda) = T^{-1}\left[\frac{1}{|\gamma|}\sum_{k=0}^{\infty}\left(\frac{D(\lambda)}{|\gamma|}\right)^k\right]T,$$

where $D(\lambda) =$ diag $(\rho_1(\lambda),\ldots,\rho_d(\lambda))$ and

$$\rho_s(\lambda) = 2\mu_0(\lambda)a_s + \left(\sum_{j=1}^m \mu_j^2(\lambda)\right)a_s^2,\ s = 1,\ldots,d.$$

Therefore, the matrix $R_\gamma(\lambda)$ satisfies

$$R_\gamma(\lambda) = T^{-1}M_\gamma(\lambda)T,$$

where

$$M_\gamma(\lambda) = \text{ diag } \left[(|\gamma| - \rho_1(\lambda))^{-1},\ldots,(|\gamma| - \rho_d(\lambda))^{-1}\right],$$

with $\gamma| > |\rho_s(\lambda)|$, for any $s = 1,\ldots,d$ and $\lambda \in S$.

The sequence of diagonal matrices $\{M_\gamma(\lambda(t_k))\}_{k\geqslant 0}$ is decreasing and the inequalities (91) hold for any $k \geqslant 1$. Using (91) we easily see that both the inequality (90) and

$$\exp(\gamma t_k)[\varphi_\gamma(y(t_k,y_0);\lambda(t_k)) - \varphi_\gamma(y(t_k,y_0);\lambda(t_{k-1}))] = u(t_k,y_0) - u(t_k-,y_0) \leqslant 0$$

hold for any $k \geqslant 1$, where $u(t_k-,y_0) = \lim_{t\uparrow t_k} u(t,y_0)$.

As a consequence, the piecewise smooth function $u_1(t, y_0) = \mathbb{E}_1 u(t, y_0)$ has a sequence of jumps satisfying

$$u_1(t_k, y_0) - u_1(t_k-, y_0) \leqslant 0, \text{ for any } k \geqslant 1, \ y_0 \in \mathbb{R}^d.$$

As far as $\{u_1(t; y_0); t \in (t_k, t_{k+1})\}$ fulfills

$$u_1(t, y_0) = u_1(t_k, y_0) + \int_{t_k}^{t} [-C(\gamma) u_1(s, y_0)]\, ds + \int_{t_k}^{t} \alpha_1(s, y_0) ds,$$

where $\alpha_1(s, y_0) \leqslant 0$, for any $s \geqslant 0$, $y_0 \in \mathbb{R}^d$, we easily get that

$$0 \leqslant u_1(t, y_0) \leqslant u_1(t_k, y_0) \exp[-c(\gamma)(t - t_k)],$$

for any $t \in [t_k, t_{k+1})$ and $k \geqslant 0$.

An induction argument allows to get

$$0 \leqslant u_1(t, y_0) \leqslant C_0 \exp(-C(\gamma)t), \text{for any } t \geqslant 0, \ y_0 \in \mathbb{R}^d, \tag{92}$$

where $C_0 = \varphi_\gamma(y_0; \lambda(0)) \geqslant 0$.

The inequality (92) shows that the conclusion (85) follows in a stronger sense and it holds an exponential stability in the mean square.

To this respect, using (86) and (92) we get

$$\begin{aligned} \exp(\gamma t)\mathbb{E}_1 \|y(t, y_0)\|^2 &\leqslant \frac{1}{\delta(\gamma)} \exp(\gamma t) \mathbb{E}_1 \varphi_\gamma(y(t, y_0); \lambda(t)) \\ &\leqslant \frac{1}{\delta(\gamma)} u_1(t, y_0) \leqslant \frac{C_0}{\delta(\gamma)} \exp(-C(\gamma)t), \end{aligned}$$

for any $t \geqslant 0$, $y_0 \in \mathbb{R}^d$, which shows that

$$\mathbb{E}_1 \|y_\gamma(t, y_0)\|^2 \leqslant \frac{C_0}{\delta(\gamma)} \exp(-C(\gamma)t), \ t \geqslant 0.$$

□

Remark 10. The result in the above Lemma describes the asymptotic behaviour of the continuous process $\{y_\gamma(t; y_0); t \geqslant 0\}$ using "the mean square norme" $|||\cdot|||^2 = \mathbb{E}_1 \|\cdot\|^2$, which is difficult to be measured. In a real situation we are measuring the functional $\|y_\gamma(t; y_0)\|^2$, $t \geqslant 0$, for some fixed trajectories of the continuous process $\{y_\gamma(t; y_0); t \geqslant 0\}$ and a pathwise description of the asymptotic behaviour is necessary.

To this respect, we are using the integral equation (87), by imposing adequate conditions on the matrices $B_i(\lambda)$, $i = 0, 1, \dots, m$,

$$B_0(\lambda) = \mu_0(\lambda)A,\ B_j(\lambda) = \mu_j(\lambda)I_d, j = 1, \dots, m, \tag{93}$$

where A is a symmetric matrix, I_d stands for the $(d \times d)$ unity matrix and each $\mu_i(\lambda), \lambda \in S$ is continuous and bounded.

Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an orthogonal mapping, such that

$$TAT^{-1} = \text{diag}\ (a_1, \dots, a_d),\ \text{where}\ a_i \in \mathbb{R}$$

and define the following scalar components of the matrix $B(\lambda) = 2\mu_0(\lambda)A + \left(\sum_{j=1}^{m} \mu_j^2(\lambda)\right) I_d$,

$$\rho_s(\lambda) = 2\mu_0(\lambda)a_s + \sum_{j=1}^{m} \mu_j^2(\lambda), s = 1, \dots, d,\ \lambda \in S.$$

Assume that $\rho_s(\lambda)$ satisfy a decreasing property

$$\rho_s(\lambda(t_{k+1})) \leqslant \rho_s(\lambda(t_k)),\ \text{for any}\ s = 1, \dots, d \text{ and } k \geqslant 0, \tag{94}$$

where the piecewise constant process $\{\lambda(t); t \geqslant 0\}$ is entering the system (78).

Lemma 9. *Let the matrices $B_i(\lambda)$, $\lambda \in S$, $i = 0, 1, \dots, m$ and the piecewise constant process $\{\lambda(t); t \geqslant 0\}$ be given such that the hypothesis* (93) *and* (94) *are satisfied. Let $\gamma < 0$ be such that*

$$|\gamma| > 2\left(\|g_0\| + \frac{1}{2}\sum_{j=1}^{m} \|g_j\|^2\right),$$

where $\|g_0\| = \|A\| \sup_{\lambda \in S} |\mu_0(\lambda)|$ and $\|g_j\| = \sup_{\lambda \in S} |\mu_j(\lambda)|$, $j = 1, \dots, m$.
Define the exponential martingale $\{\xi(t); t \geqslant 0\}$ as

$$\xi(t) = \exp\left\{2\left[\sum_{j=1}^{m} \int_0^t \mu_j(\lambda(s))dW_j(s) - \int_0^t [\mu_j(\lambda(s))]^2 ds\right]\right\},$$

where $\mu_j \in \mathbb{R}$ and $\lambda(t)$ are given in (93).

Then there exist some positive constants $\delta(\gamma)$ and $C(\gamma)$ such that

$$\|y_\gamma(t, y_0)\|^2 \leqslant \frac{C_0(y_0)}{\delta(\gamma)} \exp(-C(\gamma)t)\xi(t), \text{ for any } t \geqslant 0,\ y_0 \in \mathbb{R}^d, \tag{95}$$

where $C_0(y_0)$ is a continuous function taking positive values and $\{y_\gamma(t), y_0); t \geqslant 0\}$ is the unique solution of the system (83).

Proof. By hypotheses, the following integral equation is valid

$$\begin{array}{ll} u(t, y_0) & = \varphi_\gamma(y_0; \lambda(0)) + \psi(t; y_0) \\ & + \int_0^t [-C(\gamma)u(s, y_0) + \exp(\gamma s)\ \beta(y(s, y_0); \lambda(s))]\, ds \\ & + \sum_{j=1}^m \int_0^t \exp(\gamma s)\langle \partial_y \varphi_\gamma(y(s, y_0); \lambda(s)), B_j(\lambda(s))y(s, y_0)\rangle dW_j(s), \end{array} \tag{96}$$

for any $t \geqslant 0$ and $y_0 \in \mathbb{R}^d$, where

$$\varphi_\gamma(y; \lambda) = \langle R_\gamma(\lambda)y, y\rangle$$

and

$$\beta(y; \lambda) = C(\gamma)\varphi_\gamma(y; \lambda) - \|y\|^2$$

satisfy

$$\left\{ \begin{array}{ll} \delta(\gamma)\|y\|^2 & \leqslant \varphi_\gamma(y; \lambda) \leqslant \frac{1}{C(\gamma)}\|y\|^2, \\ \beta(y; \lambda) & \leqslant 0 \end{array} \right. \tag{97}$$

and $\delta(\gamma),\ C(\gamma)$ are some positive constants.

Here $u(t, y_0) = \exp(\gamma t)\ \varphi_\gamma(y(t, y_0); \lambda(t))$ and

$$\psi(t, y_0) = \sum_{0<t_k\leqslant t} \exp(\gamma t_k)[\varphi_\gamma(y(t_k, y_0); \lambda(t_k)) - \varphi_\gamma(y(t_k, y_0); \lambda(t_{k-1}))].$$

Set $c_0(y_0) = \varphi_\gamma(y_0; \lambda(0))$ and notice that the conclusion (95) is satisfied, provided the piecewise continuous process $\{y(t; y_0); t \geqslant 0\}$ fulfills

$$u(t, y_0) \leqslant C_0(y_0) \exp[-C(\gamma)t]\ \xi(t), \text{ for any } t \geqslant 0,\ y_0 \in \mathbb{R}^d. \tag{98}$$

This last inequality will be proved using the following integral representation of the solution in (96),

$$u(t, y_0) = u(t_k, y_0)\ \phi(t, t_k) + \phi(t, t_k) \int_{t_k}^t \phi^{-1}(s, t_k)\beta(s, y_0)ds, \tag{99}$$

for any $t \in [t_k, t_{k+1}),\ k \geqslant 0$, where

$$\phi(t,s) = \exp[-C(\gamma)(t-s)]\ \xi(t,s),\ t \geqslant s \geqslant 0,\ \xi(t,s) = \ \xi(t)\ \xi^{-1}(s)$$

and

$$\beta(s,y_0) = \beta(y(s,y_0);\lambda(s))\exp(\gamma s) \leqslant 0.$$

The piecewise continuous process $\{u(t;y_0); t \geqslant 0\}$ has a sequence of jumps

$$u(t_k,y_0) - u(t_{k^-},y_0) = (\exp\gamma t_k)[\varphi_\gamma(y(t_k,y_0);\lambda(t_k)) - \varphi_\gamma(y(t_k,y_0);\lambda(t_{k-1}))]$$

and in the same way as in the proof of the Lemma 8 it folows that each jump satisfies

$$u(t_k,y_0) - u(t_{k^-},y_0) \leqslant 0,\ \text{for any } k \geqslant 1,\ y_0 \in \mathbb{R}^d, \tag{100}$$

provided the hypothesis (94) is assumed.

According to the hypothesis (93), we rewrite the coefficients from the diffusion part of (96) as

$$\exp(\gamma s)\langle \partial_y \varphi_\gamma(y(s,y_0);\lambda(s)), B_j(\lambda(s))\ y(s,y_0)\rangle = 2\mu_j(\lambda(s))\ u(s,y_0)$$

and (96) becomes

$$\begin{aligned} u(t,y_0) =& u(t_k,y_0) + \int_{t_k}^{t} [-C(\gamma)u(s,y_0)]ds + \int_{t_k}^{t} \beta(s,y_0)ds \\ &+ \sum_{j=1}^{m} \int_{t_k}^{t} 2u(s,y_0)\ \mu_j(\lambda(s))dW_j(s),\ \text{for any } t \in [t_k,t_{k+1}). \end{aligned} \tag{101}$$

It is easily seen that the solution $u(t,y_0),\ t \in [t_k,t_{k+1})$ satisfying (101) can be represented as in (99). Then, by virtue of (99), (100) and an induction argument we get (98) fulfilled. □

5 Final comments

Generally, the asymptotic behaviour of a continuous solution $\{z(t,x); t \geqslant 0\}$ of a stochastic differential system driven by the linear vector fields $g_i(z;\lambda) = a_i(\lambda) + A_i(\lambda)z,\ i = 1,\ldots,m$ can be associated with the following stochastic integral equation

$$
\begin{aligned}
\exp(\gamma t)\ \varphi_\gamma(z(t,x);\hat{\lambda}(t)) &= \varphi_\gamma(x;\hat{\lambda}(0)) + D_z^\gamma(t,x) \qquad (102)\\
&+ \int_0^t \exp(\gamma s)\, [\gamma\varphi_\gamma + L(\varphi_\gamma)]\, (z(s,x);\hat{\lambda}(s))ds\\
&+ \sum_{j=1}^m \int_0^t \exp(\gamma s) \left\langle \partial_z\varphi_\gamma(z(s,x);\hat{\lambda}(s)), g_j(z(s,x);\hat{\lambda}(s)) \right\rangle dW_j(s),\ t \geqslant 0,
\end{aligned}
$$

where we applied the stochastic rule of differentiation given in the Lemma 1 and used the same notations as in the previous sections. The process $\{z(t,x); t \geqslant 0\}$ stands for the unique solution of the linear system

$$
\begin{cases} dz(t) &= g_0(z;\hat{\lambda}(t))dt + \sum_{j=1}^m g_j(z;\hat{\lambda}(t))dW_j(t),\ t \geqslant 0\\ z(0) &= x. \end{cases} \qquad (103)
$$

The piecewise constant process $\{D_z^\gamma(t,x); t \geqslant 0\}$ is defined as

$$
D_z^\gamma(t,x) = \sum_{0<\hat{t}_k \leqslant t} \exp(\gamma \hat{t}_k)\ [\varphi(z(\hat{t}_k,x);\hat{\lambda}(\hat{t}_k)) - \varphi(z(\hat{t}_k,x);\hat{\lambda}(\hat{t}_{k-1}))] \qquad (104)
$$

and the elliptic operator $L : \mathcal{P}_2(z;\lambda) \to \mathcal{P}_2(z;\lambda)$ is given by

$$
L(\varphi)(z;\lambda) = \langle \partial_z\varphi(z,\lambda), g_0(z;\lambda)\rangle + \frac{1}{2}\sum_{j=1}^m \langle \partial_z^2\varphi(z,\lambda)\ g_j(z;\lambda), g_j(z;\lambda)\rangle. \qquad (105)
$$

Notice that the equations (102)–(105) are obtained from (2), (4), (5) and (6) by choosing $\delta = 0$ in (2) and replacing $\mathcal{P}_p(z;\lambda)$ by $\mathcal{P}_2(z;\lambda)$.

The aim of this comment is to point out that the drift part $[\gamma\varphi_\gamma + L(\varphi_\gamma)]$ in equation (102) may be replaced by

$$
[\gamma\varphi_\gamma + L(\varphi_\gamma)](z,\lambda) = (c\gamma)\varphi_\gamma(z,\lambda) - f(z), \qquad (106)
$$

where $c \in (0,1)$ is arbitrarily fixed and $f \in \mathcal{P}_2(z;\lambda)$ satisfies $f(z) \geqslant 0$, for any $z \in \mathbb{R}^n$, provided $\gamma_c = (1-C)\gamma < 0$ and $\varphi_\gamma \in \mathcal{P}_2(z;\lambda)$ is found such that

$$
\gamma_c\varphi_\gamma(z;\lambda) + f(z) + L(\varphi_\gamma)(z,\lambda) = 0,\ \forall (z,\lambda) \in \mathbb{R}^n \times S.
$$

Denote $\mathcal{U}_\gamma(t,z) = \exp(\gamma t)\varphi_\gamma(z;\hat{\lambda}(t))$ and using (106) we rewrite the integral equation (102) as

$$\mathcal{U}_\gamma(t,z(t,x)) = \mathcal{U}_\gamma(0,x) + D_z^\gamma(t,x) + \int_0^t (c\gamma)\mathcal{U}_\gamma(s,z(s,x))ds \tag{107}$$
$$-\int_0^t \exp(\gamma s) f(z(s,x))ds + \sum_{j=1}^m \int_0^t \left\langle \partial_z \mathcal{U}_\gamma(z(s,x)), g_j(z(s,x);\hat{\lambda}(s)) \right\rangle dW_j(s),$$

for any $t \geqslant 0,\ x \in \mathbb{R}^n$.

The integral equation (107) can be considered as a basis for any asymptotic evaluation of the process $\{\mathcal{U}_\gamma(t,z(t,x)); t \geqslant 0\}$.

On the other hand, considering a stochastic dynamic system driven by the linear vector fields

$$g_i(y;\lambda) = A_i(\lambda)y + a_i(\lambda),\ i \in \{1,2,\ldots,m\}, y \in \mathbb{R}^n \tag{108}$$

and depending on a parameter $\lambda \in S \subset \mathbb{R}^d$, we need to point out those conditions insuring the weak convergence of the probabilities measures $\{P_\gamma(t,x;\cdot) : t \geqslant 0\}$ generated by a piecewise continuous solution $\{y_\gamma(t,x) : t \geqslant 0\}$ satisfying a corresponding augmented system. In this respect, a continous and bounded mapping $y_0(\lambda) : S \to \mathbb{R}^n$ anihilating the action of the given vector fields

$$g_i(y_0;\lambda) = 0,\ i \in \{0,1,\ldots,m\}, \lambda \in S \tag{109}$$

plays a crucial role. For a fixed piecewise constant function $\hat{\lambda}(t) : [0,\infty) \to S$, denote $\hat{y}_0(t) = y_0(\hat{\lambda}(t)),\ t \geqslant 0$ the corresponding piecewise constant trajectory containing only stationary points of the vector fields $g_i(y;\hat{\lambda}(t)),\ i \in \{0,1,\ldots,m\},\ t \geqslant 0$. Let $\{y_\gamma(t,x) : t \geqslant 0\}$ be the piecewise continous process satisfying the following augmented system

$$\begin{cases} d_t y = d_t\hat{y}_0(t) + \dfrac{\gamma}{2}(y - \hat{y}_0(t)) + g_0(y;\hat{\lambda}(t))dt + \displaystyle\sum_{j=1}^m g_j(y;\hat{\lambda}(t))dW_j(t),\ t \geqslant 0 \\ y(0) = x \in \mathbb{R}^n,\ y \in \mathbb{R}^n \end{cases} \tag{110}$$

where $\gamma < 0$ is a constant.

By definition, the solution $\{y_\gamma(t,x) : t \geqslant 0\}$ can be decomposed into a continuous component $\{z_\gamma(t,x) : t \geqslant 0\}$ and $\{\hat{y}_0(t) : t \geqslant 0\}$

$$y_\gamma(t,x) = z_\gamma(t,x) + \hat{y}_0(t),\ t \geqslant 0, x \in \mathbb{R}^n \tag{111}$$

such that $z_\gamma(t,x) = y_\gamma(t,x) - \hat{y}_0(t),\ t \geqslant 0$ is the unique solution of the following linear homogenous system

$$\begin{cases} d_t z = A_0(\hat{\lambda}(t))zdt + \sum_{j=1}^{m} A_j(\hat{\lambda}(t))dW_j(t) + \frac{\gamma}{2}zdt,\ t \geqslant 0, z \in \mathbb{R}^n, \\ z(0) = x \end{cases} \tag{112}$$

Using (111) and (112) we see easily that $y_\gamma(t,x) = z_\gamma(t,x) + \hat{y}_0(t)$ is the solution of the system (113) provided $\{z_\gamma(t,x) : t \geqslant 0\}$ fulfils (112).

In addition, the results given in the Lemmas 8 and 9 can be applied to the linear system (112) and as far as the weak limits of the set $\{P_\gamma(t,x;\cdot\}_{t\uparrow\infty}$ are concerned we use Lemma 8 which says that for $\gamma < 0$ and $|\gamma|$ sufficiently large we get

$$\lim_{t\to\infty} E_1\|z_\gamma(t,x)\|^2 = 0 \tag{113}$$

exponentially. Denote $M \subset \mathcal{P}(\mathbb{R}^n)$ the set of the probabilities, measures concentrated on the closure of the set $G = \{y_0(\lambda) : \lambda \in S\}$,

$$M = \{P \in \mathcal{P}(\bar{G})\}. \tag{114}$$

Then using (113) and [1] we get that any weak limit

$$\lim_{t_n\to\infty} P_\gamma(t,x;\cdot) = \hat{P}_\gamma(\cdot),\ \text{for some sequence } t_n \uparrow \infty, \tag{115}$$

belongs to the fixed set M and, in addition, the set

$$M_\gamma = \{\hat{P}_\gamma \in \mathcal{P}(\mathbb{R}^n) : \hat{P}_\gamma \text{ is a weak limit point satisfying (8)}\} \tag{116}$$

coincides with $\hat{M}$, i.e. $M_\gamma = \hat{M}$, where $\hat{M} = \{P \in \mathcal{P}(\hat{G})\}$ and $\hat{G} \in \mathbb{R}^n$ is the closure of the set $\{\hat{y}_0(t) : t \in [0,\infty)\} = G_0$.

Finally, we notice that the "weak asymptotic behaviour" of some trajectories $\{y_\gamma(t,x,\hat{\omega}) : t \geqslant 0, \hat{\omega} \in \Omega,$ fixed$\}$ can be described using the decomposition (4) and Lemma 9.

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