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DISCRETE-TIME RICCATI TYPE EQUATIONS AND THE TRACKING PROBLEM

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Abstract. *This paper is an addendum of* [6]. *Sufficient conditions for the existence of the minimal solution and the stabilizing solution of a class of discrete-time coupled Riccati type equations are given. As an application we provide the solution to a tracking problem in the case of discrete-time stochastic linear systems affected by independent random perturbations and Markovian switching.*

Keywords: Riccati type equations, Tracking problem, Discrete-time stochastic systems, Markov chains, Independent random perturbations

1. **Problem Formulation.** Let us consider the discrete-time controlled stochastic linear system described by:

$$\begin{aligned} x(t+1) &= A_0(t,\eta_t)x(t) + B_0(t,\eta_t)u(t) + \\ &+ \sum_{k=1}^{r}[A_k(t,\eta_t)x(t) + B_k(t,\eta_t)u(t)]w_k(t), \quad t \geq 0, t \in \mathbf{Z}, \end{aligned} \tag{1.1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector and $u(t) \in \mathbf{R}^m$ is the vector of control inputs, $\{\eta_t\}_{t\geq 0}$ is a Markov chain defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with the state space the finite set $\mathcal{D} = \{1, 2, ..., N\}$ and the sequence of transition probability matrices $\{P_t\}_{t\geq 0}$. This means that for $t \geq 0$, $P_t = [p_t(i,j)]$ are stochastic matrices of size N, with the property:

$$\mathcal{P}\{\eta_{t+1} = j \mid \mathcal{G}_t\} = p_t(\eta_t, j) \tag{1.2}$$

for all $j \in \mathcal{D}$, $t \geq 0$, $t \in \mathbf{Z}_+$, where $\mathcal{G}_t = \sigma[\eta_0, \ \eta_1, \ \ldots \ , \eta_t]$, $\{w(t)\}_{t\geq 0}$ is a sequence of independent random vectors $w(t) = (w_1(t), ..., w_r(t))^T$. For each $t \in \mathbf{Z}_+$, $A_k(t,i) \in \mathbf{R}^{n\times n}$, $B_k(t,i) \in \mathbf{R}^{n\times m}$.

We recall that if $\mathcal{F}$ and $\mathcal{G}$ are two σ-algebras then $\mathcal{F} \vee \mathcal{G}$ stands for the smallest σ-algebra containing $\mathcal{F}$ and $\mathcal{G}$. Throughout the paper the following assumptions regarding the processes $\{\eta_t\}_{t\geq 0}$ and $\{w(t)\}_{t\geq 0}$ are made:

$\mathbf{H_1}$) The processes $\{\eta_t\}_{t\geq 0}$ and $\{w(t)\}_{t\geq 0}$ are independent stochastic processes.

$\mathbf{H_2}$) $E[w(t)] = 0$, $E[w(t)w^T(t)] = I_r$, $t \geq 0, I_r$ being the identity matrix of size r.

$\mathbf{H_3}$) For each $t \geq 0$, P_t is a nondegenerate stochastic matrix. We recall that a stochastic matrix P_t is a nondegenerate stochastic matrix if for every $j \in \mathcal{D}$ there exists $i \in \mathcal{D}$ such that $p_t(i,j) > 0$.

The following class of admissible controls will be involved in the paper $\mathcal{U}_{t_0,\infty}(x_0), t_0 \geq 0, x_0 \in \mathbf{R}^n$ consists of all stochastic processes $u = \{u(t)\}_{t_0\leq t<\infty}$ where for each t, $u(t)$ is a m-dimensional random vector which is $\widetilde{\mathcal{H}}_t$-measurable having the following two additional properties:

$$\alpha) \quad E[|u(t)|^2] < \infty, \quad t \geq t_0 \tag{1.3}$$

$$\beta) \quad \sup_{t \geq t_0} E[|x_u(t, t_0, x_0)|^2] < \infty \tag{1.4}$$

$x_u(\cdot, t_0, x_0)$ being the solution of (1.1) and $\tilde{\mathcal{H}}_t = \sigma[\eta_s, w(s'), 0 \leq s \leq t, 0 \leq s' \leq t-1]$ for $t \geq 1$ and $\tilde{\mathcal{H}}_0 = \sigma[\eta_0]$.

Let $\{r(t)\}_{t \geq 0}$, $r(t) \in \mathbf{R}^n$ be a given signal called **reference signal**. The control problem we want to solve is to find a control $\tilde{u}(t)$ which minimizes the deviation $x(t) - r(t)$.
For a more rigorous setting of this problem let us introduce the following cost functional:

$$J(t_0, \infty, x_0, u) = \overline{\lim_{T \to \infty}} \frac{1}{T - t_0} \sum_{t=t_0}^{T} E[(x(t) - r(t))^T C^T(t, \eta_t) C(t, \eta_t)(x(t) - r(t)) + u^T(t) R(t, \eta_t) u(t)] \tag{1.5}$$

where $C(t,i) \in \mathbf{R}^{p \times n}$, $R(t,i) = R^T(t,i) \in \mathbf{R}^{m \times m}$ are given matrices and $x(t) = x_u(t, t_0, x_0)$. The tracking problem considered in this section ask for finding a control law $u_{opt} \in \mathcal{U}_{t_0,\infty}(x_0)$ in order to minimize the cost (1.5).

In the construction of the optimal control $\tilde{u}$ in the above optimization problems a crucial role is played by the solutions of the following system of discrete-time stochastic generalized Riccati equations DTSGRE:

$$\begin{aligned} X(t,i) = \sum_{k=0}^{r} A_k^T(t,i) \Pi_i(t, X(t+1)) A_k(t,i) + C^T(t,i) C(t,i) \\ - [\sum_{k=0}^{r} A_k^T(t,i) \Pi_i(t, X(t+1)) B_k(t,i)][R(t,i) \\ + \sum_{k=0}^{r} B_k^T(t,i) \Pi_i(t, X(t+1)) B_k(t,i)]^{-1} [\sum_{k=0}^{r} B_k^T(t,i) \Pi_i(t, X(t+1)) A_k(t,i)] \end{aligned} \tag{1.6}$$

where

$$\Pi_i(t, Y) = \sum_{j=1}^{N} p_t(i,j) Y(j), \quad (\forall) \ Y = (Y(1), ..., Y(N)). \tag{1.7}$$

For the tracking problem associated to (1.5) a global solution of (1.6), called stabilizing solution, is involved.

In the next section we will provide a set of sufficient conditions which guarantee the existence of the stabilizing bounded solution of (1.6). Finally we mention that (1.6) contains different types of discrete-time Riccati equations involved in the solution of the linear quadratic optimization problems, as particular cases, and H_2-control problems both in deterministic framework and stochastic framework [1, 7, 8, 9, 12, 13, 14].

2. **Special Global Solutions of DTSGRE.** In this section we investigate the problem of the existence of some special global solutions of DTSGRE (1.6) such as the minimal solution and the stabilizing solution.

Regarding the matrix coefficients of the system (1.6) we make the following assumption:

$\mathbf{H_4}$): (i) $\{A_k(t,i)\}_{t \geq 0}, \{B_k(t,i)\}_{t \geq 0}, 0 \leq k \leq r$, $\{C(t,i)\}_{t \geq 0}, \{D(t,i)\}_{t \geq 0}$ are bounded sequences.
(ii) There exists $\delta_0 > 0$ not depending upon t such that $R(t,i) \geq \delta_0 I_m$, $\forall (t,i) \in Z_+ \times \mathcal{D}$.

For a solution $X(t) = (X(t,1), X(t,2) ..., X(t,N))$ of (1.6) we introduce the notation:

$$\mathcal{R}_i(t, X(t+1)) = R(t,i) + \sum_{k=0}^{r} B_k^T(t,i)\Pi_i(X(t+1))B_k(t,i), \tag{2.1}$$

$$F^X(t,i) = -\mathcal{R}_i^{-1}(t, X(t+1))[\sum_{k=0}^{r} B_k^T(t,i)\Pi_i(X(t+1))A_k(t,i)]. \tag{2.2}$$

It should be remarked that any time we refer to a solution of DTSGRE (1.6) then it is tacitly assumed that $\mathcal{R}_i(t, X(t+1))$ is invertible.

The following result will be repeatedly used in the next developments:

Lemma 2.1. *Let $\{X(t)\}_{t_0 \le t \le t_1}$ be a solution of DTSGRE (1.6) and $W(t) = (W(t,1), \dots, W(t,N)), W(t,i) \in \mathbf{R}^{m \times n}$ be given. Then $\{X(t)\}_{t_0 \le t \le t_1}$ verifies the following modified system of discrete-time equations:*

$$X(t,i) = \sum_{k=0}^{r}(A_k(t,i) + B_k(t,i)W(t,i))^T\Pi_i(t, X(t+1))(A_k(t,i) + B_k(t,i)W(t,i))$$

$$+C^T(t,i)C(t,i) + W^T(t,i)R(t,i)W(t,i) - (W(t,i) - F^X(t,i))^T\mathcal{R}_i(t, X(t+1))(W(t,i) - F^X(t,i)).$$

Proof: Follows by direct calculations, tt is omitted for shortness.

Based on the previous Lemma one easily obtains the following comparison result:

Proposition 2.2. *Let $X_l(t) = (X_l(t,1), X_l(t,2), ..., X_l(t,N)), l \in \{1,2\}, t_0 \le t \le t_1$ be two solutions of DTSGRE (1.6) with the properties:*

a) $\mathcal{R}_i(t, X_1(t+1)) > 0, \quad t_0 \le t \le t_1 - 1$;

b) $X_2(t_1, i) \ge X_1(t_1, i), \quad \forall i \in \mathcal{D}$.

Under these conditions we have $X_2(t,i) \ge X_1(t,i)$ for all $t_0 \le t \le t_1, i \in \mathcal{D}$.

Proof: Let $F_l(t) = (F_l(t,1), F_l(t,2), ..., F_l(t,N))$ be defined by $F_l(t,i) = F^{X_l}(t,i), l = 1,2, i \in \mathcal{D}, t_0 \le t \le t_1$. Applying successively the previous Lemma to the system (1.6) verified by $X_1(t)$ and $X_2(t)$ respectively and for $W(t) = F_2(t)$, we obtain:

$$X_2(t,i) - X_1(t,i) = \sum_{k=0}^{r}(A_k(t,i) + B_k(t,i)F_2(t,i))^T\Pi_i((t, X_2(t+1) - X_1(t+1))(A_k(t,i)$$

$$+B_k(t,i)F_2(t,i)) + M(t,i) \tag{2.3}$$

where $M(t,i) = (F_2(t,i) - F_1(t,i))^T\mathcal{R}_i(t, X_1(t+1))(F_2(t,i) - F_1(t,i))$.

Based on assumption a) one deduces that $M(t,i) \ge 0$ for all $t_0 \le t \le t_1 - 1, i \in \mathcal{D}$. The conclusion follows now inductively from (2.3) and thus the proof ends.

Definition 2.1. *We say that the system (1.1) is stochastic stabilizable if there exist bounded sequences $\{F(t,i)\}_{t \ge 0}$, $i \in \mathcal{D}$, such that the trajectories of the closed-loop system*

$$x(t+1) = [A_0(t,\eta_t) + B_0(t,\eta_t)F(t,\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(t,\eta_t) + B_k(t,\eta_t)F(t,\eta_t))]x(t) \tag{2.4}$$

satisfy

$$E[|x(t,t_0,x_0)|^2 | \eta_{t_0} = i] \le \beta q^{t-t_0}|x_0|^2 \tag{2.5}$$

for all $t \ge t_0$, $i \in \mathcal{D}$ with $\pi_{t_0}(i) > 0$, where $\beta \ge 1$, $q \in (0,1)$ do not depend upon t, t_0, x_0 and $\pi_t(i) = \mathcal{P}\{\eta_t = i\}$. The sequences $\{F(t,i)\}_{t \ge 0}$, involved in the above definition will be called stabilizing feedback gains. If the coefficients of the system (1.1) are periodic, with period $\theta \ge 1$, the definition of the stochastic stabilizability will be restricted to the class of θ-periodic stabilizing feedback gains. According to the terminology introduced in [5], the definition of stochastic stabilizability could be restated as follows: the system (1.1) is stochastic stabilizable if there exist bounded sequences $\{F(t,i)\}_{t \ge 0}$ such that the zero state equilibrium of the system (2.4) is exponentially stable in mean square with conditioning of type I (ESMS-CI).

Let us consider the following linear system:

$$\begin{aligned} x(t+1) &= (A_0(t,\eta_t) + \sum_{k=1}^{r} w_k(t)A_k(t,\eta_t))x(t) \qquad (2.6)\\ y(t) &= C(t,\eta_t)x(t). \end{aligned}$$

Definition 2.2. *We say that the system (2.6) is stochastic detectable if there exist bounded sequences $\{K_k(t,i)\}_{t\geq 0}$, $i \in \mathcal{D}, 0 \leq k \leq r$ such that the zero state equilibrium of the system*

$$x(t+1) = (A_0(t,\eta_t) + K_0(t,\eta_t)C(t,\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(t,\eta_t) + K_k(t,\eta_t)C(t,\eta_t)))x(t)$$

is ESMS-CI.

Different aspects regarding the concepts of stochastic stability, stochastic stabilizability and stochastic detectability can be found in [2, 3, 4, 5, 12, 13].

Now we prove:

Theorem 2.3. *Assume that: a) the assumptions $\mathbf{H_1}) - \mathbf{H_4}$) are fulfilled; b) the system (1.1) is stochastic stabilizable.*

Under these assumptions DTSGRE (1.6) has a bounded solution $\tilde{X}(t) = (\tilde{X}(t,1), \tilde{X}(t,2), \ldots, \tilde{X}(t,N))$ such that $\tilde{X}(t,i) \geq 0, (t,i) \in Z_+ \times \mathcal{D}$. The solution $\tilde{X}(t)$ is minimal in the class of global bounded and positive semidefinite solutions of (1.6). Moreover if there exists an integer $\theta \geq 1$ such that $A_k(t+\theta,i) = A_k(t,i)$, $B_k(t+\theta,i) = B_k(t,i)$, $0 \leq k \leq r$, $C(t+\theta,i) = C(t,i)$, $D(t+\theta,i) = D(t,i)$, $t \in \mathbf{Z}_+$, $P_{t+\theta} = P_t$, $i \in \mathcal{D}$, then $\tilde{X}(t+\theta,i) = \tilde{X}(t,i)$, for all $t \in \mathbf{Z}_+$, $i \in \mathcal{D}$.

Proof: For each $\tau \geq 1, \tau \in \mathbf{Z}_+$ let $X_\tau(t) = (X_\tau(t,1), ..., X_\tau(t,N))$ be the solution of (1.6) with the terminal condition $X_\tau(\tau,i) = 0, i \in \mathcal{D}$. Proceeding as in the proof of Theorem 3.4 in [6] one obtains that $X_\tau(\cdot)$ are well defined and $X_\tau(t,i) \geq 0$ for $0 \leq t \leq \tau$. If $1 \leq \tau_1 < \tau_2$ we have $X_{\tau_2}(\tau_1,i) \geq 0 = X_{\tau_1}(\tau_1,i), i \in \mathcal{D}$. Applying Proposition 2.2 for $X_l(t) = X_{\tau_l}(t), l = 1,2$, we conclude that

$$X_{\tau_1}(t,i) \leq X_{\tau_2}(t,i) \qquad (2.7)$$

for all $0 \leq t \leq \tau_1, i \in \mathcal{D}$. On the other hand, the assumption b) in the statement together with Theorem 3.8 [5] guarantee the existence of the bounded sequences $\{F_0(t,i)\}_{t\geq 0}, i \in \mathcal{D}$, such that the following system of coupled Lyapunov type equations

$$\begin{aligned} X_0(t,i) = \sum_{k=0}^{r} [A_k(t,i) + B_k(t,i)F_0(t,i)]^T \Pi_i(t, X_0(t+1))[A_k(t,i) + B_k(t,i)F_0(t,i)] \\ +C^T(t,i)C(t,i) + F_0^T(t,i)R(t,i)F_0(t,i), i \in \mathcal{D} \qquad (2.8) \end{aligned}$$

has a bounded solution $X_0(t) = (X_0(t,1), ..., X_0(t,N)), X_0(t,i) \geq 0, i \in \mathcal{D}, t \in \mathbf{Z}_+$.

Applying Lemma 2.1 with $W(t,i) = F_0(t,i)$ to (1.6) verified by $X_\tau(t)$ one obtains that $X_0(t) - X_\tau(t), 0 \leq t \leq \tau$, solves the following equation

$$\begin{aligned} X_0(t,i) - X_\tau(t,i) = \sum_{k=0}^{r} (A_k(t,i) + B_k(t,i)F_0(t,i))^T \Pi_i(t, X_0(t+1) - X_\tau(t+1))(A_k(t,i) \\ +B_k(t,i)F_0(t,i)) + \tilde{M}(t,i) \qquad (2.9) \end{aligned}$$

where $\tilde{M}(t,i) = (F_0(t,i) - F_\tau(t,i))^T \mathcal{R}_i(t, X_\tau(t+1))(F_0(t,i) - F_\tau(t,i)), F_\tau(t,i) = F^{X_\tau}(t,i)$. It can be seen that $\tilde{M}(t,i) \geq 0, 0 \leq t \leq \tau - 1, i \in \mathcal{D}$. Since $X_0(\tau,i) - X_\tau(\tau,i) = X_0(\tau,i) \geq$

0 one obtains inductively from (2.9) that $X_0(t,i) - X_\tau(t,i) \geq 0, \forall 0 \leq t \leq \tau, i \in \mathcal{D}$. This allows us to conclude that

$$0 \leq X_\tau(t,i) \leq X_0(t,i) \leq cI_n \tag{2.10}$$

for all $0 \leq t \leq \tau, \tau \geq 1, i \in \mathcal{D}$ where $c > 0$ is independent of τ and t. From (2.7) and (2.10) one obtains that the sequences $\{X_\tau(t,i)\}_{\tau \geq 1}, i \in \mathcal{D}$ are convergent. Let $\tilde{X}(t,i) = lim_{t\to\infty} X_\tau(t,i)$. It follows that $0 \leq \tilde{X}(t,i) \leq cI_n$. Moreover $\tilde{X}(t) = (\tilde{X}(t,1), \tilde{X}(t,2), ..., \tilde{X}(t,N))$ is a global solution of (1.6). If $\hat{X}(t) = (\hat{X}(t,1), \hat{X}(t,2), ..., \hat{X}(t,N))$ is another bounded solution of DTSGRE (1.6) with $\hat{X}(t,i) \geq 0$ for all $t, i \in \mathbf{Z}_+ \times \mathcal{D}$ then $\hat{X}(\tau,i) \geq 0 X_\tau(\tau,i), \forall\ \tau \geq 1$. Applying Proposition 2.2 we deduce that $\hat{X}(t,i) \geq X_\tau(t,i)$ for all $0 \leq t \leq \tau, i \in \mathcal{D}$. Taking the limit for $\tau \to \infty$ we deduce that $\hat{X}(t,i) \geq \tilde{X}(t,i)$ for arbitrary $(t,i) \in \mathbf{Z}_+ \times \mathcal{D}$ and thus we obtain that $\tilde{X}(t)$ is the minimal solution. If the coefficients of DTSGRE (1.6) are periodic sequences with period $\theta \geq 1$ we define $\check{X}_\tau(t) = (\check{X}_\tau(t,1), \check{X}_\tau(t,2), ..., \check{X}_\tau(t,N))$ by $\check{X}_\tau(t,i) = X_{\tau+\theta}(t+\theta,i)$, $0 \leq t \leq \tau, i \in \mathcal{D}$. It is easy to check that $\check{X}_\tau(t)$ is also a solution of DTSGRE (1.6) and $\check{X}_\tau(\tau,i) = 0 = X_\tau(\tau,i)$. Therefore $\check{X}_\tau(t,i) = X_\tau(t,i)$ for all $0 \leq t \leq \tau, i \in \mathcal{D}$. This leads to $\lim_{\tau\to\infty} \check{X}_\tau(t,i) = \lim_{\tau\to\infty} X_\tau(t,i) = \tilde{X}(t,i)$. On the other hand $\lim_{\tau\to\infty} \check{X}_\tau(t,i) = \lim_{\tau\to\infty} X_{\tau+\theta}(t+\theta,i) = \tilde{X}(t+\theta,i)$. This allows us to conclude that $\tilde{X}(t,i) = \tilde{X}(t+\theta,i)$ and thus the proof ends.

Definition 2.3. *We say that* $\tilde{X}(t) = (\tilde{X}(t,1), ...\tilde{X}(t,N))$, $t \in \mathbf{Z}_+$ *is a stabilizing solution of the system (1.6) if the zero state equilibrium of the closed-loop system*

$$x(t+1) = [A_0(t,\eta_t) + B_0(t,\eta_t)\tilde{F}(t,\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(t,\eta_t) + B_k(t,\eta_t)\tilde{F}(t,\eta_t))]x(t) \tag{2.11}$$

is ESMS-CI, where $\tilde{F}(t,i) = F^{\tilde{X}}(t,i)$.

Concerning the stabilizing solution of (1.6) we prove:

Theorem 2.4. (uniqueness): *Under the assumptions* $\mathbf{H_1}) - \mathbf{H_4})$ *the DTSGRE (1.6) has at most one bounded and stabilizing solution* $\tilde{X}(t)$ *satisfying the additional condition* $\mathcal{R}_i(t, \tilde{X}(t+1)) > 0, t \in \mathbf{Z}_+, i \in \mathcal{D}$.

Proof: Let us assume that (1.6) has at least two bounded and stabilizing solutions $X_l(t) = (X_l(t,1), ..., X_l(t,N)), l = 1,2$, which verify the additional condition $\mathcal{R}_i(t, X_l(t+1)) > 0$. Set $F_l(t,i) = F^{X_l}(t,i), l = 1,2$. Applying Lemma 2.1 with $W(t,i) = F_2(t,i)$ to the equation (1.6) verified by $X_2(t,i)$ and $X_1(t,i)$ respectively, one obtains that the sequence $\{X_2(t) - X_1(t)\}_{t\geq 0}$ is a bounded solution of the linear equation on $\mathcal{S}_n^N$:

$$Z(t,i) = \sum_{k=0}^{r} (A_k(t,i) + B_k(t,i)F_2(t,i))^T \Pi_i(t, Z(t+1))(A_k(t,i) + B_k(t,i)F_2(t,i)) + M_2(t,i) \tag{2.12}$$

where $M_2(t,i) = (F_2(t,i) - F_1(t,i))^T \mathcal{R}_i(t, X_1(t+1))(F_2(t,i) - F_1(t,i))$. From the assumption in the statement it follows that $M_2(t,i) \geq 0, (t,i) \in \mathbf{Z}_+ \times \mathcal{D}$. Since $X_2(t)$ is a stabilizing solution of (1.6) one obtains via Theorem 3.5 in [4] that the equation (2.12) has a unique bounded solution and that solution is positive semidefinite. This allows us to conclude that $X_2(t,i) - X_1(t,i) \geq 0$. Applying again Lemma 2.1 with $W(t,i) = F_1(t,i)$ we obtain in the same way that $X_1(t,i) - X_2(t,i) \geq 0$. Hence $X_1(t,i) = X_2(t,i)$ and thus the proof is complete.

Lemma 2.5. *Assume that: a) the assumptions* $\mathbf{H_1}) - \mathbf{H_4})$ *are fulfilled; b) the system (2.6) is stochastic detectable.*

Under these assumptions any bounded solution $\tilde{X}(t) = (\tilde{X}(t,1),...,\tilde{X}(t,N))$ *of DTSGRE (1.6) with* $\tilde{X}(t,i) \geq 0$ *for all* $t \geq 0, i \in \mathcal{D}$ *is a stabilizing solution.*

Proof: Let $\tilde{X}(t) = (\tilde{X}(t,1), \tilde{X}(t,2),...,\tilde{X}(t,N))$ be a bounded and positive semidefinite solution of (1.6). Set $\tilde{F}(t,i) = F^{\tilde{X}}(t,i)$, $(t,i) \in Z_+ \times \mathcal{D}$. Applying Lemma 2.1 with $W(t,i) = \tilde{F}(t,i)$ one obtains that $\tilde{X}(t)$ solves the equation:

$$\tilde{X}(t,i) = \sum_{k=0}^{r}(A_k(t,i) + B_k(t,i)\tilde{F}(t,i))^T \Pi_i(t, \tilde{X}(t+1))(A_k(t,i) + B_k(t,i)\tilde{F}(t,i)) + \tilde{C}^T(t,i)\tilde{C}(t,i) \tag{2.13}$$

where $\tilde{C}(t,i) = \begin{pmatrix} C(t,i) \\ R^{\frac{1}{2}}(t,i)\tilde{F}(t,i) \end{pmatrix}$.

Now we show that under assumption b) the system (2.11) with the output $\tilde{y}(t) = \tilde{C}(t,\eta_t)x(t)$ is stochastic detectable. To this end we take $\tilde{K}_k(t,i) \in \mathbf{R}^{n\times(p+m)}$, $\tilde{K}_k(t,i) = (K_k(t,i) - B_k(t,i)R^{\frac{-1}{2}}(t,i))$ where $K_k(t,i)$ are provided by the assumption b).

One obtains that the corresponding closed-loop system

$$x(t+1) = [A_0(t,\eta_t) + B_0(t,\eta_t)\tilde{F}(t,\eta_t) + \tilde{K}_0(t,\eta_t)\tilde{C}(t,\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(t,\eta_t) + B_k(t,\eta_t)\tilde{F}(t,\eta_t) + \tilde{K}_k(t,\eta_t)\tilde{C}(t,\eta_t))]x(t)$$

coincides with

$$x(t+1) = (A_0(t,\eta_t) + K_0(t,\eta_t)C(t,\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(t,\eta_t) + K_k(t,\eta_t)C(t,\eta_t)))x(t)$$

which is ESMS-CI.

Thus we conclude that the system (2.11)with the output $\tilde{y}(t) = \tilde{C}(t,\eta_t)x(t)$ is stochastic detectable. Applying a slightly modified version of the Theorem 4.8 in [4] to equation (2.13) we conclude that the zero state equilibrium of the system (2.11) is ESMS-CI and thus the proof ends.

At the end of this section we prove:

Theorem 2.6. *Assume that: a) the hypotheses* $\mathbf{H_1}) - \mathbf{H_4})$ *are fulfilled; b) the system (1.1) is stochastic stabilizable; c) the system (2.6) is stochastic detectable.*

Under these conditions the DTSGRE (1.6) has a bounded and stabilizing solution $\tilde{X}(t) = (\tilde{X}(t,1), \tilde{X}(t,2),...,\tilde{X}(t,N)), \tilde{X}(t,i) \geq 0$ *for all* $(t,i) \in Z_+ \times \mathcal{D}$*. Moreover if the coefficients of the DTSGRE (1.6) are periodic sequences with period* $\theta \geq 1$ *then* $\tilde{X}(t)$ *is a periodic solution with the same period* θ*.*

Proof: It follows immediately form Theorem 2.3, Lemma 2.5 and Theorem 2.4.

3. The Solution of the Tracking Problem.

If we set $\xi(t) = x(t) - r(t)$ then we obtain

$$\xi(t+1) = A_0(t,\eta_t)\xi(t) + B_0(t,\eta_t)u(t) + f_0(t,\eta_t) + \sum_{k=1}^{r}[A_k(t,\eta_t)\xi(t) + B_k(t,\eta_t)u(t) + f_k(t,\eta_t)]w_k(t)$$

and the cost functional

$$J(t_0,\infty,x_0,u) = \overline{\lim}_{T\to\infty}\frac{1}{T-t_0}\sum_{t=t_0}^{T} E[\xi^T(t)C^T(t,\eta_t)C(t,\eta_t)\xi(t) + u(t)^T R(t,\eta_t)u(t)]$$

where

$$f_0(t,i) = A_0(t,i)r(t) - r(t+1), \qquad f_k(t,i) = A_k(t,i)r(t), \quad 1 \leq k \leq r, t \geq 0. \tag{3.1}$$

The solution of the tracking problem is derived directly from Theorem 3.7 in [6].

Theorem 3.1. *Assume that: a) the hypotheses* $\mathbf{H_1}) - \mathbf{H_4})$ *are fulfilled; b) the system (1.1) is stochastic stabilizable; c) the system (2.6) is stochastic detectable; d) the sequence* $\{r(t)\}_{t\geq 0}$ *is bounded.*

Under these conditions the optimal control of the tracking problem described by the system (1.1) and the cost (1.5) is:

$$\tilde{u}_{opt}(t) = \tilde{F}(t,\eta_t)(\tilde{x}(t) - r(t)) + \tilde{\psi}(t,\eta_t) \tag{3.2}$$

where $\tilde{F}(t,i) = F^{\tilde{X}}(t,i)$ *is constructed as in (2.2) based on the stabilizing solution* $\tilde{X}(t) = (\tilde{X}(t,1),...,\tilde{X}(t,N))$ *of the system (1.6),* $\tilde{\psi}(t,i)$ *is given by*

$$\tilde{\psi}(t,i) = -(R(t,i) + \sum_{k=0}^{r} B_k(t,i)\Pi_i(t,\tilde{X}(t+1))B_k(t,i))^{-1}[B_0^T(t,i)\Pi_i(t,\tilde{\kappa}(t+1))$$

$$+\sum_{k=0}^{r} B_k^T(t,i)\Pi_i(t,\tilde{X}(t+1))f_k(t,i)]$$

where $\tilde{\kappa}(t) = (\tilde{\kappa}(t,1),...,\tilde{\kappa}(t,N))$ *is the unique bounded solution of the system of backward affine equations*

$$\kappa(t,i) = (A_0(t,i) + B_0(t,i)\tilde{F}(t,i))^T\Pi_i(t,\kappa(t+1)) + \tilde{g}(t,i)$$

with $\tilde{g}(t,i) = \sum_{k=0}^{r}(A_k(t,i) + B_k(t,i)\tilde{F}(t,i))^T\Pi_i(t,\tilde{X}(t+1))f_k(t,i)$, $f_k(t,i)$ *being given by (3.1);* $\tilde{x}(t)$ *is the solution of the closed-loop system*

$$\tilde{x}(t+1) = [A_0(t,\eta_t) + B_0(t,\eta_t)\tilde{F}(t,\eta_t)+$$

$$\sum_{k=1}^{r}(A_k(t,\eta_t) + B_k(t,\eta_t)\tilde{F}(t,\eta_t))w_k(t)]\tilde{x}(t), \quad t \geq 0, x(t_0) = x_0.$$

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