EQUATIONS OF MOTION GENERATED BY LINEAR HAMILTONIANS ASSOCIATED TO THE JACOBI GROUP

STEFAN BERCEANU

Abstract. Using the coherent states attached to the complex Jacobi group $G^J_n = H_n \rtimes \mathrm{Sp}(n,\mathbb{R})$, based on the manifold $\mathcal{D}^J_n = \mathbb{C}\times\mathcal{D}_n$, we study some of the properties of coherent states based on the manifold $\mathcal{X}^J_n = \mathbb{C}^n\times\mathcal{H}_n$, where $\mathcal{D}_n$ ($\mathcal{H}_n$) is the Siegel ball (respectively the generalized Siegel upper half plane). Starting with the resolution of unity on $\mathcal{D}^J_n$ proved for Perelomov's coherent states attached to the Jacobi group G^J_n, we obtain the resolution of unity on $\mathcal{X}^J_n$ and the Kähler two-form ω'_n on the manifold $\mathcal{X}^J_n$. This ω'_n is a "n"-dimensional generalization of Kähler-Berndt's two-form ω'_1 on $\mathcal{X}_1$. The motion associated to a Hamiltonian linear in the generators of the Jacobi group G^J_n is described by a Matrix Riccati equation on $\mathcal{D}^J_n$.

Contents

1991 *Mathematics Subject Classification.* 81R30,32WXX,12E10,33C47,32Q15,81V80.
Key words and phrases. Coherent states, representations of coherent state Lie algebras, Jacobi group, first order holomorphic differential operators with polynomial coefficients.

1. Introduction

The coherent states offer a useful connection between classical and quantum mechanics. On the other side, Perelomov's [36] group-theoretic generalization of coherent states can be used as a tool in the study of the geometry of manifolds on which the coherent states are based [9]. It is well known that the symplectic methods have a large field of applications in Physics, in particular in classical and quantum mechanics, but also in Gaussian and Linear Optics [23, 22].

In this paper we continue the investigation of the so called Jacobi group started in [10, 11] using Perelomov's coherent states. The Jacobi group – the semidirect product of the Heisenberg-Weyl group and the symplectic group – is an important object in connection with Quantum Mechanics, Geometric Quantization, Optics [23, 22, 44, 4, 43, 32, 33, 41].

In [10] we have constructed generalized coherent states (CS) attached to the Jacobi group, $G^J_1 = H_1 \rtimes \mathrm{SU}(1,1)$, based on the homogeneous Kähler manifold $\mathcal{D}^J_1 = H_1/\mathbb{R} \times \mathrm{SU}(1,1)/\mathrm{U}(1) = \mathbb{C}^1 \times \mathcal{D}_1$. Here $\mathcal{D}_1$ denotes the unit disk $\mathcal{D}_1 = \{w \in \mathbb{C} | |w| < 1\}$, and H_n is the $(2n+1)$-dimensional real Heisenberg-Weyl group with Lie algebra $\mathfrak{h}_n$. Using this construction, we have obtained a holomorphic discrete series representation of the Jacobi algebra $\mathfrak{g}^J_1 = \mathfrak{h}_1 \rtimes \mathfrak{su}(1,1)$ by holomorphic first-order differential operators with polynomial coefficients on $\mathcal{D}^J_1$. In fact, this construction is nothing more that an explicate realization of a well known holomorphic representation [37, 34] of the so called coherent state-type groups [30, 34]. In [10] we have also emphasized that, when expressed in appropriate coordinates on the manifold $\mathcal{X}^J_1$, which, as set, is $\mathcal{X}^J_1 = \mathbb{C} \times \mathcal{H}_1$, where $\mathcal{H}_1$ is the Siegel upper half plane $\mathcal{H}_1 = \{v \in \mathbb{C} | \Im(v) > 0\}$, the Kähler two-form ω_1 derived from the Kähler potential obtained from the scalar product of Perelomov's coherent state vectors based on $\mathcal{D}^J_1$, is identical with the one considered by Kähler-Berndt [14, 16, 24, 25, 26], here denoted ω'_1. In the present paper we also give more details about this identification.

In [11] we have considered coherent states attached to the Jacobi group $G^J_n = H_n \rtimes \mathrm{Sp}(n, \mathbb{R})$, based on the manifold $\mathcal{D}^J_n = \mathbb{C}^n \times \mathcal{D}_n$, where $\mathcal{D}_n$ is the Siegel ball. In the present paper we give the Kähler two-form ω'_n on the manifold $\mathcal{X}^J_n = \mathbb{C}^n \times \mathcal{H}_n$, where $\mathcal{H}_n$ is the Siegel upper half plane obtained by the Cayley transform of the Siegel ball $\mathcal{D}_n$. This ω'_n is a "n"-dimensional generalization of Kähler-Berndt's two-form ω'_1 on $\mathcal{X}^J_1$ to the corresponding one on $\mathcal{X}^J_n$.

Let us recall several facts which were only emphasized in [10, 11]. Firstly, let me mention that the Jacobi group is in fact a realization of the squeezed states in Quantum Optics [47, 42, 18], a subject largely studied starting in the sixties, which has large applications in detection of gravitational waves, spectroscopy with two and three-level atoms in squeezed fields, quantum communications, Einstein-Podolsky-Rosen correlations, entanglement, quantum cryptography, teleportation, ... [19].

Let us also remained that the squeezed states are a particular class of "minimum uncertainty states" (MUS) — states which saturates the Heisenberg uncertainty relation. The "Gaussian pure states" ("Gaussons") [40] are more general MUSs; MUSs can be considered as CSs indexed by points of manifold $\mathcal{X}^J_n$, cf. §10.1 in [1]. The geometry of the semidirect product in §10.2 in [1] is based on the technique developed in [23], using a definition of coherent states larger that used by Perelomov [36].

The connection of our construction of coherent states based on $\mathcal{D}^J_n$ and the Gaussons is a subtle one. We have shown in [10] that the clue of this connection in the case $n = 1$ is offered by the Kähler-Berndt's construction.

Let us point out that many of the mathematical formulas which appear in the context of the Jacobi group have a direct physical interpretation. We just mention that the linear fractional transformation is nothing else than the "ABCD" law for laser beams [28, 29, 2] for a complex beam parameter; see also general results about the Gaussian Optics - the ray transfer matrix, the eikonal approximation e.g. in [31, 3, 40].

Finally, let me recall that the denomination of " Jacobi group" was firstly introduced by mathematicians in [20]. The same group is known to physicists under other names, as the Schrödinger group [35], see more references and a discussion of this remark in the second reference [11]. Also the name of "Weyl-symplectic" group is used for the same direct product of the Heisenberg-Weyl group and the symplectic group [45, 46].

The paper is laid out as follows. For self-contentedness, §2 recalls the basic facts established in [10] about the algebra $\mathfrak{g}^J_1$ and its holomorphic differential representation. §3 is devoted to comparison of our approach in [10] with that of Kähler-Berndt. We have included in Remark 5 the differential action of the generators of the Jacobi algebra $\mathfrak{g}^J_1$ expressed in the Kähler-Berndt variables on $\mathcal{X}^J_1$. §4 recalls some facts established in [11] about holomorphic representation of the Jacobi algebra $\mathfrak{g}^J_n$. In §4.2 is presented the Kähler two-form ω on $\mathcal{X}^J_n$, a generalization of Kähler-Berndt construction on $\mathcal{X}^J_1$. The last section §4.3 presents the equations of motion on $\mathcal{D}^J_n$ generated by linear Hamiltonians in the generators of the group G^J_n.

2. A holomorphic representation of the Jacobi algebra $\mathfrak{g}^J_1$

2.1. **The algebra.** The Heisenberg-Weyl group is the group with the 3-dimensional real Lie algebra

$$\mathfrak{h}_1 \equiv \mathfrak{g}_{HW} =< \mathrm{i}s1 + xa^+ - \bar{x}a >_{s\in\mathbb{R},x\in\mathbb{C}}, \tag{2.1}$$

where a^+ (a) are the boson creation (respectively, annihilation) operators which verify the CCR (2.4a).

Let us also consider the Lie algebra of the group $\mathrm{SU}(1,1)$:

$$\mathfrak{su}(1,1) =< 2\mathrm{i}\theta K_0 + yK_+ - \bar{y}K_- >_{\theta\in\mathbb{R},y\in\mathbb{C}}, \tag{2.2}$$

where the generators $K_{0,+,-}$ verify the standard commutation relations (2.4b).

The Jacobi algebra is defined as the the semi-direct sum

$$\mathfrak{g}^J_1 := \mathfrak{h}_1 \rtimes \mathfrak{su}(1,1), \tag{2.3}$$

where $\mathfrak{h}_1$ is an ideal in $\mathfrak{g}_1^J$, i.e. $[\mathfrak{h}_1, \mathfrak{g}_1^J] = \mathfrak{h}_1$, determined by the commutation relations:

$$[a, a^+] = 1, \tag{2.4a}$$
$$[K_0, K_\pm] = \pm K_\pm \ , \ [K_-, K_+] = 2K_0, \tag{2.4b}$$
$$[a, K_+] = a^+ \ , \ \left[K_-, a^+\right] = a, \tag{2.4c}$$
$$\left[K_+, a^+\right] = [K_-, a] = 0, \tag{2.4d}$$
$$\left[K_0, \ a^+\right] = \frac{1}{2}a^+, [K_0, a] = -\frac{1}{2}a. \tag{2.4e}$$

2.2. **The differential action.** We suppose that we know the derived representation $d\pi$ of the Lie algebra $\mathfrak{g}_1^J$ (2.3) of the Jacobi group G_1^J. We associate to the generators a, a^+ of the HW-group and to the generators $K_{0,+,-}$ of the group $\mathrm{SU}(1,1)$ the operators a, a^+, respectively $\boldsymbol{K}_{0,+,-}$, where $(a^+)^+ = a$, $\boldsymbol{K}_0^+ = \boldsymbol{K}_0, \boldsymbol{K}_\pm^+ = \boldsymbol{K}_\mp$, and we impose to the cyclic vector e_0 to verify simultaneously the conditions

$$ae_0 = 0, \tag{2.5a}$$
$$\boldsymbol{K}_- e_0 = 0, \tag{2.5b}$$
$$\boldsymbol{K}_0 e_0 = ke_0; \ k > 0, 2k = 2, 3, ... \tag{2.5c}$$

We have considered in (2.5c) the positive discrete series representations D_k^+ of $\mathrm{SU}(1,1)$ [5].

Perelomov's coherent state vectors associated to the group G_1^J with Lie algebra the Jacobi algebra (2.3), based on the manifold M:

$$M := H_1/\mathbb{R} \times \mathrm{SU}(1,1)/\mathrm{U}(1), \tag{2.6a}$$
$$M = \mathcal{D}_1^J := \mathbb{C} \times \mathcal{D}_1, \tag{2.6b}$$

are defined as

$$e_{z,w} := e^{za^+ + w\boldsymbol{K}_+} e_0, \ z, w \in \mathbb{C}, \ |w| < 1. \tag{2.7}$$

The general scheme associates to elements of the Lie algebra $\mathfrak{g}$ differential operators: $X \in \mathfrak{g} \rightarrow \mathbb{X} \in \mathfrak{D}_1$.

Lemma 1. *The differential action of the generators* (2.4a)-(2.4e) *of the Jacobi algebra* (2.3) *is given by the formulas:*

$$\boldsymbol{a} = \frac{\partial}{\partial z}; \ \boldsymbol{a}^+ = z + w\frac{\partial}{\partial z}; \tag{2.8a}$$
$$\mathbb{K}_- = \frac{\partial}{\partial w}; \ \mathbb{K}_0 = k + \frac{1}{2}z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}; \tag{2.8b}$$
$$\mathbb{K}_+ = \frac{1}{2}z^2 + 2kw + zw\frac{\partial}{\partial z} + w^2\frac{\partial}{\partial w}, \tag{2.8c}$$

where $z \in \mathbb{C}$, $|w| < 1$.

2.3. **The reproducing kernel.**

Lemma 2. *Let $K = K(\bar{z}, \bar{w}, z, w)$, where $z \in \mathbb{C},\ w \in \mathbb{C},\ |w| < 1$,*

$$K := (e_0, e^{\bar{z}a+\bar{w}\boldsymbol{K}_-} e^{za^+ + w\boldsymbol{K}_+} e_0). \tag{2.9}$$

Then the reproducing kernel is

$$K = (1 - w\bar{w})^{-2k} \exp \frac{2z\bar{z} + z^2\bar{w} + \bar{z}^2 w}{2(1 - w\bar{w})}. \tag{2.10}$$

More generally, the kernel $K : \mathcal{D}^J_1 \times \bar{\mathcal{D}}^J_1 \to \mathbb{C}$ is:

$$K(z, w; \bar{z}', \bar{w}') := (e_{\bar{z},\bar{w}}, e_{\bar{z}',\bar{w}'}) = (1 - w\bar{w}')^{-2k} \exp \frac{2\bar{z}'z + z^2\bar{w}' + \bar{z}'^2 w}{2(1 - w\bar{w}')}. \tag{2.11}$$

2.4. **Formulas for the Heisenberg-Weyl group H_1 and $\mathbf{SU}(1,1)$.** Let us recall some relations for the displacement operator:

$$D(\alpha) := \exp(\alpha a^+ - \bar{\alpha} a) = \exp(-\frac{1}{2}|\alpha|^2) \exp(\alpha a^+) \exp(-\bar{\alpha} a), \tag{2.12}$$

$$D(\alpha_2)D(\alpha_1) = e^{i\theta_h(\alpha_2,\alpha_1)} D(\alpha_2 + \alpha_1),\ \theta_h(\alpha_2, \alpha_1) := \Im(\alpha_2\bar{\alpha}_1). \tag{2.13}$$

Let us denote by S, *the unitary squeezed operator*, the D^k_+ representation of the group $\mathrm{SU}(1,1)$ and let us introduce the notation $\underline{S}(z) = S(w)$, where w and z, $w \in \mathbb{C},\ |w| < 1$, $z \in \mathbb{C}$, are related by (2.14c), (2.14d). We have the relations:

$$\begin{aligned}
&(2.14\mathrm{a}) & \underline{S}(z) &:= \exp(z\boldsymbol{K}_+ - \bar{z}\boldsymbol{K}_-),\ z \in \mathbb{C};\\
&(2.14\mathrm{b}) & S(w) &= \exp(w\boldsymbol{K}_+) \exp(\eta\boldsymbol{K}_0) \exp(-\bar{w}\boldsymbol{K}_-);\\
&(2.14\mathrm{c}) & w &= w(z) = \frac{z}{|z|} \tanh(|z|)\ , w \in \mathbb{C}, |w| < 1;\\
&(2.14\mathrm{d}) & z &= z(w) = \frac{w}{|w|} \operatorname{arctanh}(|w|) = \frac{w}{2|w|} \log \frac{1 + |w|}{1 - |w|};\\
&(2.14\mathrm{e}) & \eta &= \log(1 - w\bar{w}) = -2 \log(\cosh(|z|)).
\end{aligned}$$

Let us consider an element $g \in \mathrm{SU}(1,1)$,

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix},\ \text{where } |a|^2 - |b|^2 = 1. \tag{2.15}$$

Lemma 3. *The (squeezed coherent state) vector*

$$\Psi_{\alpha,w} := D(\alpha)S(w)e_0;$$

and (Perelomov's coherent state) vector

$$e_{z,w'} := \exp(za^+ + w'\boldsymbol{K}_+)e_0$$

are related by the relation

$$\Psi_{\alpha,w} = (1 - w\bar{w})^k \exp(-\frac{\bar{\alpha}}{2} z)e_{z,w}, \tag{2.16}$$

where $z = \alpha - w\bar{\alpha}$.

2.5. **The representation.** From the following proposition we can see the holomorphic action of the group Jacobi

$$G^J_1 := H_1 \rtimes \mathrm{SU}(1,1), \tag{2.17}$$

on the manifold $\mathcal{D}^J_1$ (2.6b):

Proposition 1. *Let us consider the action $S(g)D(\alpha)e_{z,w}$, where $g \in \mathrm{SU}(1,1)$ has the form* (2.15)*, $D(\alpha)$ is given by* (2.12)*, and Perelomov's coherent state vector is defined in* (2.7)*. Then we have the formula* (2.18) *and the relations* (2.19), (2.20)-(2.22) *below:*

$$S(g)D(\alpha)e_{z,w} = \lambda e_{z_1,w_1},\ \lambda = \lambda(g,\alpha;z,w), \tag{2.18}$$

$$z_1 = \frac{\alpha - \bar{\alpha}w + z}{\bar{b}w + \bar{a}};\ w_1 = g\cdot w = \frac{aw+b}{\bar{b}w+\bar{a}}, \tag{2.19}$$

$$\lambda = (\bar{a}+\bar{b}w)^{-2k}\exp(\frac{z}{2}\bar{\alpha}_0 - \frac{z_1}{2}\bar{\alpha}_2)\exp \mathrm{i}\theta_h(\alpha,\alpha_0), \tag{2.20}$$

$$\alpha_0 = \frac{z+\bar{z}w}{1-w\bar{w}}, \tag{2.21}$$

$$\alpha_2 = (\alpha+\alpha_0)a + (\bar{\alpha}+\bar{\alpha}_0)b. \tag{2.22}$$

Corollary 1. *The action of the 6-dimensional Jacobi group* (2.17) *on the 4-dimensional manifold* (2.6b)*, where $\mathcal{D}_1 = \mathrm{SU}(1,1)/\mathrm{U}(1)$, is given by equations* (2.18), (2.19)*. The composition law in G^J_1 is*

$$(g_1,\alpha_1,t_1)\circ(g_2,\alpha_2,t_2) = (g_1\circ g_2, g_2^{-1}\cdot\tilde{\alpha}_1+\alpha_2, t_1+t_2+\Im(g_2^{-1}\cdot\alpha_1\bar{\alpha}_2)), \tag{2.23}$$

where $g\cdot\tilde{\alpha} := \alpha_g$ is given by

$$\alpha_g = a\,\alpha + b\,\bar{\alpha}. \tag{2.24}$$

If g has the form given by (2.15)*, then $g^{-1}\cdot\tilde{\alpha} = \alpha_{g^{-1}} = \bar{a}\alpha - b\bar{\alpha}$.*

Remark 1. *Combining the expressions* (2.19)-(2.22)*, the factor λ in* (2.18) *can be written down as*

$$\lambda = (\bar{a}+\bar{b}w)^{-2k}\exp(-\lambda_1), \tag{2.25}$$

where

$$\lambda_1 = \frac{\bar{b}z^2 + (\bar{a}\bar{\alpha}+\bar{b}\alpha)(2z+z_0)}{2(\bar{a}+\bar{b}w)},\ z_0 = \alpha - \bar{\alpha}w, \tag{2.26}$$

or

$$\lambda_1 = \frac{\bar{b}(z+z_0)^2}{2(\bar{a}+\bar{b}w)} + \bar{\alpha}(z+\frac{z_0}{2}). \tag{2.27}$$

Note that the expression (2.25)-(2.27) *is identical with the expression given in* Theorem 1.4 *in* [20] *of the Jacobi forms, under the the identification of c,d,τ,z,μ,λ in* [20] *with, respectively, $\bar{b},\bar{a},w,z,\alpha,-\bar{\alpha}$ in our notation. Note also that the composition law* (2.23) *of the Jacobi group G^J_1 and the action of the Jacobi group on the base manifold* (2.6b) *is similar with that in the paper* [15]. See also §3 and the Corollary 3.4.4 in [16]. ■

Note that the second relation in (2.19) giving the fractional linear action of the group $\mathrm{SU}(1,1)$ on the homogeneous manifold $\mathcal{D}_1 = \mathrm{SU}(1,1)/\mathrm{U}(1)$ is the famous "ABCD"-law in Optics [28, 29, 2].

2.6. **The symmetric Fock space.** The scalar product of functions from the space $\mathcal{F}_K$ corresponding to the kernel defined by (2.11) on the manifold (2.6b) is:

(2.28)
$$(\phi,\psi) = \Lambda_1 \int_{z\in\mathbb{C};|w|<1} \bar{f}_\phi(z,w) f_\psi(z,w)(1-w\bar{w})^{2k} \exp -\frac{|z|^2}{1-w\bar{w}} \exp -\frac{z^2\bar{w}+\bar{z}^2 w}{2(1-w\bar{w})} d\nu_1,$$

where the value of the G^J_1-invariant measure $d\nu_1$

$$d\nu_1 = \frac{d\Re w d\Im w}{(1-w\bar{w})^3} d\Re z d\Im z \tag{2.29}$$

is given in (2.36) and

$$\Lambda_1 = \frac{4k-3}{2\pi^2}. \tag{2.30}$$

We consider now the variables: $z = x + \mathrm{i}y$; $w = u + \mathrm{i}v$. With the change of variables

$$X = \sqrt{\frac{1+u}{1-u^2-v^2}}(x + \frac{v}{1+u}y), \tag{2.31a}$$

$$Y = \frac{y}{\sqrt{1+u}}, \tag{2.31b}$$

we have

$$dxdy = \sqrt{1-u^2-v^2}dXdY,$$
$$d\nu_1 = \frac{dudv}{(1-u^2-v^2)^{5/2}},$$

and (2.28) becomes:

$$(\phi,\psi) = \Lambda_1 \int_{1-u^2-v^2>0} \bar{f}_\phi f_\psi \exp[-(X^2+Y^2)] dXdY (1-u^2-v^2)^{2k-\frac{5}{2}} dudv. \tag{2.32}$$

2.7. **The geometry of the manifold $\mathbb{C}\times\mathcal{D}_1$.** We calculate the Kähler potential as the logarithm of the reproducing kernel (2.11), $f := \log K$, i.e.

$$f = \frac{2z\bar{z} + z^2\bar{w} + \bar{z}^2 w}{2(1-w\bar{w})} - 2k\log(1-w\bar{w}). \tag{2.33}$$

The Kähler two-form ω_1 is given by the formula:

$$-\mathrm{i}\,\omega_1 = f_{z\bar{z}} dz\wedge d\bar{z} + f_{z\bar{w}} dz\wedge d\bar{w} - f_{\bar{z}w} d\bar{z}\wedge dw + f_{w\bar{w}} dw\wedge d\bar{w}. \tag{2.34}$$

We can write down the two-form ω_1 (2.34) as

$$-\mathrm{i}\,\omega_1 = \frac{2k}{(1-w\bar{w})^2} dw\wedge d\bar{w} + \frac{A\wedge\bar{A}}{1-w\bar{w}},\ A = dz + \bar{\alpha}_0 dw,\ \alpha_0 = \frac{z+\bar{z}w}{1-w\bar{w}}. \tag{2.35}$$

For the volume form we find:

$$\omega_1\wedge\omega_1 = 4k(1-w\bar{w})^{-3} 4\Re z \Im z \Re w \Im w. \tag{2.36}$$

It can be checked up that indeed, *the measure* $d\nu_1$ *and the fundamental two-form* ω_1 *are group-invariant at the action* (2.19) *of the Jacobi group* G^J_1 (2.17).

3. Kähler-Berndt's approach

3.1. **An outline.** Rolf Berndt -alone or in collaboration - has studied the real Jacobi group $G^J(\mathbb{R})$ in several references, from which I mention [14, 15, 16, 17]. The Jacobi group appears (see explanation in [27]) in the context of the so called *Poincaré group* or *The New Poincaré group* - the double cover of the de Sitter group $\mathrm{SO}_0(4,1)$ - investigated by Erich Kähler as the 10-dimensional group G^K which invariates a hyperbolic metric [24, 25, 26]. Kähler and Berndt have investigated the Jacobi group $G^J_0(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ acting on the manifold $\mathcal{X}^J_1 := \mathcal{H}_1 \times \mathbb{C}$, where $\mathcal{H}_1$ is the upper half plane $\mathcal{H}_1 := \{v \in \mathbb{C} | \Im(v) > 0\}$.

For self-contentedness, in Remarks 2 and 3 below, we briefly proof two results from [16], which we need in order two express the two-form ω_1 in the coordinates used by Kähler and Berndt. The main ingredient in the proof of Remark 2 below is the Iwasawa decomposition. Let us also mention that Iwasawa decomposition was largely used in applications in Optics, see e.g. [39, 41].

Remark 2. *The action of* $G^J_0(\mathbb{R})$ *on* $\mathcal{X}^J_1$ *is given by* $(g,(v,z)) \rightarrow (v_1, z_1)$, $g = (M, l)$, *where*

$$v_1 = \frac{av+b}{cv+d}, z_1 = \frac{z + l_1 v + l_2}{cv+d};\ M = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \mathrm{SL}_2(\mathbb{R}), (\mathrm{l}_1, \mathrm{l}_2) \in \mathbb{R}^2. \tag{3.1}$$

Proof. Let us use the notation of [16]. We denote $G^J(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R}) \ltimes H(\mathbb{R})$, where here $H(\mathbb{R})$ denotes the real HW group with the composition law:

$$(\lambda, \mu, \kappa)(\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \left| \begin{array}{c} X \\ X' \end{array} \right|), \left| \begin{array}{c} X \\ X' \end{array} \right| = \det \left(\begin{array}{c} X \\ X' \end{array} \right). \tag{3.2}$$

If $g = (M, X, \kappa) \in G^J(\mathbb{R})$, where $M \in \mathrm{SL}_2(\mathbb{R})$, $X = (\lambda, \mu)$, $(X, \kappa) \in \mathbb{R}^3$, then the composition law in the real Jacobi group is

$$gg' = (MM', XM' + X', \kappa + \kappa' + \left| \begin{array}{c} XM' \\ X' \end{array} \right|). \tag{3.3}$$

The action of $G^J(\mathbb{R})$ over the $H(\mathbb{R})$ is

$$M(X, \kappa)M^{-1} = (XM^{-1}, \kappa). \tag{3.4}$$

Let us consider the Iwasawa decomposition for a matrix $M \in \mathrm{SL}_2(\mathbb{R})$:

$$M = \left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array}\right) \left(\begin{array}{cc} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{array}\right) \left(\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array}\right),\ y > 0. \tag{3.5}$$

If

$$M = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right), \tag{3.6}$$

then we find for x, y, θ in (3.5)

$$x = \frac{ac+bd}{d^2+c^2};\ y = \frac{1}{d^2+c^2};\ \sin\theta = -\frac{c}{\sqrt{c^2+d^2}};\ \cos\theta = \frac{d}{\sqrt{c^2+d^2}}. \tag{3.7}$$

For $g = (M, X, \kappa) \in G^J(\mathbb{R})$, the EZ-coordinates (Eichler-Zagier, cf. the definition at p. 12 and p. 51 in [16]) are $(x, y, \theta, \lambda, \mu, \kappa)$. Let $\tau = x + \mathrm{i}y \in \mathcal{H}_1$, $z = \xi + \mathrm{i}\eta = p\tau + q$, where

$$(p, q) = XM^{-1} = (\lambda d - \mu c, -\lambda b + \mu a) \tag{3.8}$$

If we attach a "$_*$" to the results of elements of the composition rule (3.3), we have

$$x_* = \frac{AC+BD}{D^2+C^2}; y_* = \frac{1}{D^2+c^2}, \tag{3.9}$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}. \tag{3.10}$$

We find out:

$$D^2 + C^2 = c^2(a'^2 + b'^2) + d^2(c'^2 + d'^2) + 2cd(a'c' + b'd'),$$

i.e.

$$D^2 + C^2 = c^2(a'^2 + b'^2) + \frac{d^2}{y'} + 2cd\frac{x'}{y'}.$$

Similarly,

$$AC + BD = ac(a'^2 + b'^2) + (ad + bc)\frac{x'}{y'} + \frac{bd}{y'}.$$

We find for $\tau_* = x_* + \mathrm{i}y_*$

$$\tau_* = \frac{ac(a'^2+b'^2)y' + (ad+bc)x' + \mathrm{i}y' + bd}{c^2(a'^2+b'^2)y' + 2cdx' + d^2}. \tag{3.11}$$

Let us verify the first relation (3.1), in the present notation

$$\tau_* = \frac{a\tau' + b}{c\tau' + d}, \tag{3.12}$$

where

$$\tau' = x' + \mathrm{i}y' = \frac{a'c' + b'd' + \mathrm{i}}{d'^2 + c'^2}. \tag{3.13}$$

Combining (3.12), (3.13), we find out

$$\tau_* = \frac{(ax'+b)(cx'+d) + acy'^2 + \mathrm{i}y'}{(cx'+d)^2 + c^2y'^2}, \tag{3.14}$$

and we have to verify the identify (3.11) and (3.14).

In order to prove the second equation (3.1), we calculate firstly

$$(P_*, Q_*) = (LD - MC, -LB + MA),$$

where

$$(L, M) = (\lambda' + \lambda a' + \mu c', \mu' + \lambda b' + \mu d'),$$

and we find

$$P_* = \lambda'(cb'+dd') - \mu'(ca'+dc') + \lambda d - \mu c; \tag{3.15a}$$

$$Q_* = -\lambda'(ab'+bd') + \mu'(ad'+bc') - \lambda b + \mu a. \tag{3.15b}$$

Then we obtain

$$z_* := P_*\tau_* + Q_* = \frac{(P_*a + Q_*c)\tau' + P_*b + Q_*a}{c\tau' + d}.$$

The nominator E of the last expression of $\tau*$ should be identified with $E = p'\tau' + q' + \lambda\tau' + \mu$, i.e. it remains to verify that

$$\begin{aligned} P_*a + Q_*b &= p' + \lambda; \\ P_*b + Q_*d &= q' + \mu'. \end{aligned}$$

In conclusion, using the multiplication law (3.3), the Iwasawa decomposition (3.5) and the equations (3.7), (3.8), we have obtained the action of $G^J(\mathbb{R})$ on the base $\mathcal{X}^J_1$

$$g(\tau, z) = (\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}), \tag{3.16}$$

and Remark 2 is proved. ■

Let us now recall that

$$C^{-1}\mathrm{SL}_2(\mathbb{R})C = \mathrm{SU}(1,1), \text{ where } C = \left(\begin{array}{cc} \mathrm{i} & \mathrm{i} \\ -1 & 1 \end{array}\right). \tag{3.17}$$

If $M \in \mathrm{SL}_2(\mathbb{R})$ is the matrix (3.6), then, under the transformation (3.17)

$$M_* = C^{-1}MC = \left(\begin{array}{cc} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{array}\right), \ \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1, \tag{3.18}$$

where

$$2\alpha = a + d + \mathrm{i}(b - c); \ 2\beta = a - d - \mathrm{i}(b + c). \tag{3.19}$$

Now we pass to the complex group $G^J_{\mathbb{C}} = C^{-1}G^J(\mathbb{R})C$. We recall that the Jacobi group $G^J_{\mathbb{C}}$ is a *group of Harish-Chandra type*, (cf. e.g. p. 514 in [34]; see the definition in Ch. III §5 in [37] and Ch. XII.1 in [34]). Moreover, it is well known that *the Jacobi algebra* (2.3) *is a CS-Lie algebra* (cf. e.g. Theorem 5.2 in [30]). The correspondence between our notation and that of Berndt-Schmidt at p. 12 in [16] is as follows: $a^+, a, K_+, K_-, 1, K_0$ corresponds, respectively to: $Y_+, Y_-, X_+, -X_-, -Z_0, \frac{1}{2}Z$. We see that under the transformation (3.17), $g = (M, X, \kappa) \in \mathrm{SL}_2(\mathbb{R}) \ltimes H(\mathbb{R})$ is twisted to $g_* = (M_*, X_*, \kappa)$, where M_* is given by (3.18), while, due to action (3.4), $X_* = XC = (\mathrm{i}\lambda - \mu, \mathrm{i}\lambda + \mu)$.

Also the map (3.17) induces a transformation of the bounded domain $\mathcal{D}_1$ into the upper half plane $\mathcal{H}_1$ and

$$\tau \in \mathcal{H}_1 \mapsto \tau_* = C^{-1}(\tau) = \frac{\tau - \mathrm{i}}{\tau + \mathrm{i}} \in \mathcal{D}_1. \tag{3.20}$$

The action $C^{-1}G^J_0(\mathbb{R})C$ descends on the basis to the biholomorphic map: $\check{C}^{-1} : \mathcal{X}^J_1 := \mathcal{H}_1 \times \mathbb{C} \to \mathcal{D}^J_1 := \mathcal{D}_1 \times \mathbb{C}$: $(\tau, z) \mapsto (\tau_*, z_*)$. Here τ_* is given by (3.20), while $z_* = p_*\tau_* + q_*$. So, $(p, q) = (\lambda, \mu)M^{-1}$, and $(p_*, q_*) = (\lambda_*, \mu_*){M_*}^{-1}$. But $M_* = C^{-1}MC$, and

$(p_*, q_*) = (p, q)C = (-q + \mathrm{i}p, q + \mathrm{i}p)$, and we get $z_* = \frac{2\mathrm{i}z}{\tau+\mathrm{i}}$. Note that at p. 53 in [16] the factor 2i in this formula is missing.

In a different notation, we have shown that

Remark 3. *The action $C^{-1}G_0^J(\mathbb{R})C$, descends on the basis to the biholomorphic map: $\check{C}^{-1} : \mathcal{X}_1^J := \mathcal{H}_1 \times \mathbb{C} \to \mathcal{D}_1^J := \mathcal{D}_1 \times \mathbb{C}$:*

$$w = \frac{v - \mathrm{i}}{v + \mathrm{i}};\ z = \frac{2\mathrm{i}u}{v + \mathrm{i}}, w \in \mathcal{D}_1,\ v \in \mathcal{H}_1, z \in \mathbb{C}. \tag{3.21}$$

■

The $G_0^J(\mathbb{R})$-invariant closed two-form considered by Kähler-Berndt is:

$$\omega_1' = \alpha\frac{dv \wedge d\bar{v}}{(v - \bar{v})^2} + \beta\frac{1}{v - \bar{v}}B \wedge \bar{B},\ B = du - \frac{u - \bar{u}}{v - \bar{v}}dv, v, u \in \mathbb{C},\ \Im(v) > 0, \tag{3.22}$$

cf.§36 in [26]; see also §3.2 in [14], where the first term is misprinted as $\alpha\frac{dv\wedge d\bar{v}}{v-\bar{v}}$.

Under the mapping (3.21), the two-form ω_1 (2.35) reads

$$-\mathrm{i}\,\omega_1' = -\frac{2k}{(\bar{v} - v)^2}dv \wedge d\bar{v} + \frac{2}{\mathrm{i}(\bar{v} - v)}B \wedge \bar{B}, \tag{3.23}$$

i.e. (3.22). In fact, we have proved that

Remark 4. *When expressed in the coordinates $(v, u) \in \mathcal{X}_1^J$ which are related to the coordinates $(w, z) \in \mathcal{D}_1^J$ by the map* (3.21) *given by* Remark 3*, the Kähler two-form* (2.35) *is identical with the one* (3.23) *considered by Kähler-Berndt.*

If we use the EZ coordinates adapted to our notation

$$v = x + \mathrm{i}y;\ u = pv + q,\ x, p, q, y \in \mathbb{R}, y > 0, \tag{3.24}$$

the $G_0^J(\mathbb{R})$-invariant Kähler metric on $\mathcal{X}^J$ corresponding to the Kähler-Berndt's Kähler two-form ω (3.23) reads

$$ds^2 = \frac{k}{2y^2}(dx^2 + dy^2) + \frac{1}{y}[(x^2 + y^2)dp^2 + dq^2 + 2xdpdq], \tag{3.25}$$

i.e. the equation at p. 62 in [16] or the equation given at p. 30 in [14].

Equation (3.25) can be written in a form to show the positive-definiteness of the metric

$$ds^2 = \frac{k}{2y^2}(dx^2 + dy^2) + \frac{x^2 + y^2}{y}(dp + \frac{x}{x^2 + y^2}dq)^2, \tag{3.26}$$

The Kähler two-form (3.22) of Kähler-Berndt corresponds (cf. equation (9) in Ch. 36 of [24]) to the Kähler potential

$$f' = -\frac{\lambda}{2}\log\frac{v - \bar{v}}{2i} - i\pi\mu\frac{(u - \bar{u})^2}{v - \bar{v}}, u \in \mathbb{C},\ v \in \mathcal{H}_1. \tag{3.27}$$

3.2. New results.

Remark 5. *When expressed in the coordinates* $(v,u)\in\mathcal{X}^J_1=\mathcal{H}_1\times\mathbb{C}$, *related with the coordinates* $(w,z)\in\mathcal{D}^J_1=\mathcal{D}_1\times\mathbb{C}$ *by* (3.21), *the differential action of the generators* (2.4a)-(2.4e) *of the Jacobi algebra* (2.3), *given by* Lemma 1, *becomes*

$$\boldsymbol{a}=\frac{v+\mathrm{i}}{2\mathrm{i}}\frac{\partial}{\partial v};\ \boldsymbol{a}^+=\frac{2\mathrm{i}u}{v+\mathrm{i}}+\frac{v-\mathrm{i}}{2\mathrm{i}}\frac{\partial}{\partial u}; \tag{3.28a}$$

$$\mathbb{K}_-=\frac{(v+\mathrm{i})^2}{2\mathrm{i}}\frac{\partial}{\partial v}+\frac{v+\mathrm{i}}{2\mathrm{i}}u\frac{\partial}{\partial u};\ \mathbb{K}_0=k+\frac{uv}{2\mathrm{i}}\frac{\partial}{\partial u}+\frac{v^2+1}{2\mathrm{i}}\frac{\partial}{\partial v}; \tag{3.28b}$$

$$\mathbb{K}_+=-\frac{2u^2}{(v+\mathrm{i})^2}+\frac{2k(v-\mathrm{i})}{v+\mathrm{i}}+\frac{u(v-\mathrm{i})}{2\mathrm{i}}\frac{\partial}{\partial u}+\frac{(v-\mathrm{i})^2}{2\mathrm{i}}\frac{\partial}{\partial v}. \tag{3.28c}$$

We recall the expression (2.28) of the scalar product, where $d\nu_1$ is given by (2.29), which gives the resolution of unit. We introduce the change of variables given by (3.21) and we get the scalar product

$$(\phi,\psi)=\Lambda_1\int_{\mathcal{X}^J_1}\bar{f}_\phi(v,u)f_\psi(v,u)K^{-1}_1(v,u)d\nu'_1. \tag{3.29}$$

Here $K_1(v,u)$ is the value of the reproducing kernel (2.10) in the new variable (3.21), while $d\nu_1$ represents the $G^J_0(\mathbb{R})$-invariant measure on $\mathcal{X}^J_1$.

The reproducing kernel $K=(e_{\bar{z},\bar{w}},e_{\bar{z},\bar{w}})$ in the new variables (3.21) is

$$K_1=\left[\frac{|v+i|^2}{2i(\bar{v}-v)}\right]^{2k}\exp F, \tag{3.30}$$

where

$$F=2|v+i|^{-2}\left[|u|^2-\frac{(u\bar{v}-\bar{u}v)^2+(\bar{u}-u)^2}{2\mathrm{i}(\bar{v}-v)}\right], \tag{3.31}$$

or

$$F=2|v+i|^{-2}\left[|u|^2+\frac{\Im(u\bar{v})^2+(\Im u)^2}{\Im v}\right]. \tag{3.32}$$

The Kähler potential is $f_1=\log K_1$.

It is obtained

$$\frac{\partial^2F}{\partial u\partial\bar{u}}=\frac{2}{i(\bar{v}-v)}, \tag{3.33}$$

$$\frac{\partial^2F}{\partial u\partial\bar{v}}=\frac{2(u-\bar{u})}{i(\bar{v}-v)^2}, \tag{3.34}$$

$$\frac{\partial^2F}{\partial v\partial\bar{v}}=\frac{2(u-\bar{u})^2}{i(\bar{v}-v)^3}. \tag{3.35}$$

So, it can be verified that *formula* (2.34) *in the new variables* u,v *gives indeed the Kähler-Berndt two-form* ω_1 (3.23). However, note that *the Kähler-Berndt potential* (3.27) *is different of our* f_1.

We introduce the variables

$$v = x+iy, x,y\in\mathbb{R}, y>0, \tag{3.36}$$

$$u = m+in, m,n\in\mathbb{R}. \tag{3.37}$$

We have to calculate

$$\Delta = \det \begin{Vmatrix} A & B \\ C & D \end{Vmatrix} \tag{3.38}$$

where

$$\det A = \det \begin{Vmatrix} \frac{\partial \Re w}{\partial x} & \frac{\partial \Re w}{\partial y} \\ \frac{\Im w}{\partial x} & \frac{\partial \Im w}{\partial y} \end{Vmatrix} = \det \begin{Vmatrix} \frac{4x(y+1)}{E^2} & 2\frac{-x^2+(y+1)^2}{E^2} \\ -2\frac{-x^2+(y+1)^2}{E^2} & \frac{4x(y+1)}{E^2} \end{Vmatrix} = \frac{4}{E^2},$$

$$\det B = \det \frac{(\Re w, \Im w)}{(m,n)} = 0,$$

$$\det D = \det \begin{Vmatrix} \frac{\partial \Re z}{\partial m} & \frac{\partial \Re z}{\partial n} \\ \frac{\Im z}{\partial m} & \frac{\partial \Im z}{\partial n} \end{Vmatrix} = \det \begin{Vmatrix} 2\frac{y+1}{E} & -2\frac{x}{E} \\ 2\frac{x}{E} & 2\frac{y+1}{E} \end{Vmatrix} = \frac{4}{E},$$

where

$$E = x^2+(y+1)^2.$$

Also

$$1-w\bar{w} = \frac{4y}{E}.$$

We find:

$$d\nu_1' = \frac{1}{4y^3}dxdydmdn. \tag{3.39}$$

It can be verified that *the measure $d\nu_1'$ is invariant under the action* (3.1) *of the real Jacobi group $G_0^J(\mathbb{R})$*. In fact, we have obtained the "resolution of unity" on the manifold $\mathfrak{X}_1^J$:

Remark 6. *Let us consider the Jacobi group $G_0^J(\mathbb{R})$ with the composition rule* (3.2) *and the action* (3.1) *on the manifold $\mathfrak{X}_1^J$. The Kähler two-form ω_1' is given by* (3.23)*, where B si given in* (3.22)*. The symmetric Fock space $\mathfrak{F}_{K_1}$ attached to the reproducing kernel* (3.30)-(3.31)*, $K_1 : \mathfrak{X}_1^J \times \mathfrak{X}_1^J \rightarrow \mathbb{C}$, is endowed with the scalar product* (3.29)*, where the normalization constant Λ_1 is given in* (2.30)*, and the $G_0^J(\mathbb{R})$-invariant measure $d\nu_1'$ is given by* (3.39).

4. The Jacobi group G_n^J

4.1. **The symmetric Fock space.** The Heisenberg-Weyl group H_n is the nilpotent group with the $(2n+1)$-dimensional real Lie algebra isomorphic to the algebra

$$\mathfrak{h}_n =< \mathrm{i}s1 + \sum_{i=1}^{n}(x_i a_i^+ - \bar{x}_i a_i) >_{s\in\mathbb{R}, x_i\in\mathbb{C}}, \tag{4.1}$$

where a_i^+ (a_i) are the boson creation (respectively, annihilation) operators which verify the CCR

$$[a_i, a_j^+] = \delta_{ij};\ [a_i, a_j] = [a_i^+, a_j^+] = 0. \tag{4.2}$$

The vacuum verifies the relations:

$$a_i e_o = 0, i = 1, \cdots, n. \tag{4.3}$$

The displacement operator

$$D(\alpha) := \exp(\alpha a^+ - \bar{\alpha} a) = \exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha a^+)\exp(-\bar{\alpha} a), \tag{4.4}$$

verifies the composition rule:

$$D(\alpha_2)D(\alpha_1) = \mathrm{e}^{\mathrm{i}\theta_h(\alpha_2,\alpha_1)}D(\alpha_2+\alpha_1),\ \theta_h(\alpha_2,\alpha_1) := \Im(\alpha_2\bar{\alpha_1}). \tag{4.5}$$

Here we have used the notation $\alpha\beta = \sum_i \alpha_i\beta_i$, where $\alpha = (\alpha_i)$. The composition law of the HW group H_n is:

$$(\alpha_2, t_2)\circ(\alpha_1, t_1) = (\alpha_2+\alpha_1, t_2+t_1+\Im(\alpha_2\bar{\alpha_1})). \tag{4.6}$$

If we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, $(p, q) \mapsto \alpha$:

$$\alpha = p + \mathrm{i}q,\ p, q \in \mathbb{R}^n, \tag{4.7}$$

then

$$\Im(\alpha_2\bar{\alpha_1}) = (p_1^t, q_1^t)J\begin{pmatrix} p_2 \\ q_2 \end{pmatrix}, \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Jacobi algebra is the the semi-direct sum

$$\mathfrak{g}_n^J := \mathfrak{h}_n \rtimes \mathfrak{sp}(n, \mathbb{R}), \tag{4.8}$$

where $\mathfrak{h}_n$ is an ideal in $\mathfrak{g}$, i.e. $[\mathfrak{h}_n, \mathfrak{g}] = \mathfrak{h}_n$, determined by the commutation relations:

$$\begin{aligned}
[a_k^+, K_{ij}^+] &= [a_k, K_{ij}^-] = 0, && \text{(4.9a)}\\
[a_i, K_{kj}^+] &= \frac{1}{2}\delta_{ik}a_j^+ + \frac{1}{2}\delta_{ij}a_k^+, && \text{(4.9b)}\\
[K_{kj}^-, a_i^+] &= \frac{1}{2}\delta_{ik}a_j + \frac{1}{2}\delta_{ij}a_k, && \text{(4.9c)}\\
[K_{ij}^0, a_k^+] &= \frac{1}{2}\delta_{jk}a_i^+, && \text{(4.9d)}\\
[a_k, K_{ij}^0] &= \frac{1}{2}\delta_{ik}a_j. && \text{(4.9e)}
\end{aligned}$$

The generators $K^{0,+,-}$ of $\mathfrak{sp}(n, \mathbb{R})$ verify the commutation relations

$$\begin{aligned}
[K_{ij}^-, K_{kl}^-] &= [K_{ij}^+, K_{kl}^+] = 0, && \text{(4.10a)}\\
2[K_{ij}^-, K_{kl}^+] &= K_{kj}^0\delta_{li} + K_{lj}^0\delta_{ki} + K_{ki}^0\delta_{lj} + K_{li}^0\delta_{kj}, && \text{(4.10b)}\\
2[K_{ij}^-, K_{kl}^0] &= K_{il}^-\delta_{kj} + K_{jl}^-\delta_{ki}, && \text{(4.10c)}\\
2[K_{ij}^+, K_{kl}^0] &= -K_{ik}^+\delta_{jl} - K_{jk}^+\delta_{li}, && \text{(4.10d)}\\
2[K_{ji}^0, K_{kl}^0] &= K_{jl}^0\delta_{ki} - K_{ki}^0\delta_{lj}. && \text{(4.10e)}
\end{aligned}$$

Now we briefly fix the definition concerning the symplectic group. For $g \in \mathrm{GL}(2n, \mathbb{R})$, we have

$$g \in \mathrm{Sp}(n, \mathbb{R}) \ \leftrightarrow\ g^t J g = J. \tag{4.11}$$

If in (4.11) $g \in \mathrm{GL}(2n,\mathbb{C})$, then $g \in \mathrm{Sp}(n,\mathbb{C})$. We remind also that $g \in \mathrm{U}(n,n)$ iff $gKg^* = K$, where $K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Under the identification (4.7) of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, we have the correspondence

$$A \in M(2n,\mathbb{R}) \rightarrow A_{\mathbb{C}} \in M(2n,\mathbb{R})_{\mathbb{C}},\ A_{\mathbb{C}} = C^{-1}AC,\ C = \begin{pmatrix} \mathrm{i}1 & \mathrm{i}1 \\ -1 & 1 \end{pmatrix}, \tag{4.12}$$

where

$$M(2n,\mathbb{R})_{\mathbb{C}} = \left\{ \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}, P,Q \in M(n,\mathbb{C}) \right\}.$$

We extract from [38], [6], [21]

Remark 7. *To every* $g \in \mathrm{Sp}(n,\mathbb{R})$ *as in* (4.11), $g \mapsto g_c \in \mathrm{Sp(n,\mathbb{R})_{\mathbb{C}}} \equiv \mathrm{Sp(n,\mathbb{C})} \cap \mathrm{U(n,n)}$, *or denoted just* g

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \tag{4.13}$$

where

$$aa^* - bb^* = 1;\ ab^t = ba^t, \tag{4.14a}$$

$$a^*a - b^t\bar{b} = 1;\ a^t\bar{b} = b^*a. \tag{4.14b}$$

If $g \in \mathrm{Sp}(n,\mathbb{R})$ is given by (4.13), then

$$g^{-1} = \begin{pmatrix} a^* & -b^t \\ -b^* & a^t \end{pmatrix}. \tag{4.15}$$

Perelomov's coherent state vectors associated to the group G^J_n with Lie algebra the Jacobi algebra (4.8), based on the complex N-dimensional, $N = \frac{n(n+3)}{2}$, manifold M:

$$M := H_n/\mathbb{R} \times \mathrm{Sp}(n,\mathbb{R})/\mathrm{U}(n), \tag{4.16a}$$

$$M = \mathcal{D}^J_n := \mathbb{C}^n \times \mathcal{D}_n, \tag{4.16b}$$

are defined as

$$e_{z,W} = \exp(\boldsymbol{X})e_0,\ \boldsymbol{X} := \sum_i z_i a^+_i + \sum_{ij} w_{ij}\boldsymbol{K}^+_{ij},\ z \in \mathbb{C}^n; W \in \mathcal{D}_n. \tag{4.17}$$

The vector e_0 verify (4.18) and (4.19)

$$a_i e_o = 0,\ i = 1,\cdots,n. \tag{4.18}$$

$$\boldsymbol{K}^+_{ij}e_0 \neq 0, \tag{4.19a}$$

$$\boldsymbol{K}^-_{ij}e_0 = 0, \tag{4.19b}$$

$$\boldsymbol{K}^0_{ij}e_0 = \frac{k}{4}\delta_{ij}e_0. \tag{4.19c}$$

The scalar product of functions in the symmetric Fock space is [11]

$$(\phi,\psi) = \Lambda_n \int_{z\in\mathbb{C}^n;1-W\bar{W}>0} \bar{f}_\phi(z,W)f_\psi(z,W)QK^{-1}dzdW. \tag{4.20}$$

Here the density of the volume form is

$$Q=\det(1-W\bar{W})^{-(n+2)},$$

the reproducing kernel K is

$$(e_{z,W},e_{z,W})=\det(M)^{\frac{k}{2}}\exp\frac{1}{2}[2<z,Mz>+<W\bar{z},Mz>+<z,MW\bar{z}>], \tag{4.21}$$

and

$$M=(1-W\bar{W})^{-1},$$

$$dz=\prod_{i=1}^{n}\Re z_i\Im z_i;\ dW=\prod_{1\le i\le j\le n}\Re w_{ij}\Im w_{ij}, \tag{4.22}$$

$$\Lambda_n=\frac{k-3}{2\pi^{\frac{n(n+3)}{2}}}\prod_{i=1}^{n-1}\frac{(\frac{k-3}{2}-n+i)\Gamma(k+i-2)}{\Gamma[k+2(i-n-1)]}.$$

Comparatively with the case of the case of the symplectic group, a shift of p to $p-1/2$ in the normalization constant $\Lambda_n=\pi^{-n}J^{-1}(p)$ is obtained [11].

4.2. **The Kähler two form on $\mathcal{X}^J_n=\mathcal{H}_n\times\mathbb{R}^{2n}$.** On the manifold $\mathcal{D}^J_n$, we have the Kähler two-form [11]:

$$-\mathrm{i}\,\omega_n=\frac{k}{2}\mathrm{Tr}(C\wedge\bar{C})+\mathrm{Tr}(A^t\bar{M}\wedge\bar{A}), \tag{4.23}$$

where

$$A=dz+dW\bar{x},$$
$$C=\ MdW,\ M=(1-W\bar{W})^{-1}$$
$$x=(1-W\bar{W})^{-1}(z+W\bar{z}),W\in\mathcal{D}_n,z\in\mathbb{C}^n.$$

Now we consider the real Jacobi group $G^J_n(\mathbb{R})=\mathrm{Sp}(n,\mathbb{R})\ltimes H_n(\mathbb{R})$, where $H_n(\mathbb{R})$ is the real Heisenberg-Weyl group of real dimension $(2n+1)$. Let $g=(M,X,k),g'=(M',X',k')\in G^J_n(\mathbb{R})$, where $X=(\lambda,\mu)\in\mathbb{R}^{2n}$ and $(X,k)\in H_n(\mathbb{R})$. Then the composition law in $G^J_n(\mathbb{R})$ is

$$gg'=(MM',XM'+X',k+k'+XM'JX'^t). \tag{4.24}$$

We shall also consider the restricted real Jacobi group $G^J_n(\mathbb{R})_0$, consisting only of elements of the form above, but $g=(M,X)$.

We consider also the manifold

$$\mathcal{X}^J_n:=\mathcal{H}_n\times\mathbb{R}^{2n},$$

where $\mathcal{H}_n$ is Siegel upper half-plane

$$\mathcal{H}_n:=\{Z\in M(n,\mathbb{C})|Z=U+\mathrm{i}V,U,V\in M(n,\mathbb{R}),V>0,U^t=U;V^t=V\}$$

Let us consider an element $g=(M,l)$ in $G^J_n(\mathbb{R})_0$, i.e.

$$M=\left(\begin{array}{cc}A&B\\C&D\end{array}\right)\in\mathrm{Sp}(n,\mathbb{R}),\ l=(l_1,l_2)\in\mathbb{R}^{2n}, \tag{4.25}$$

and $v \in \mathcal{H}_n,\ z \in \mathbb{C}^n \equiv \mathbb{R}^{2n}$. Then the action of the group $G^J_n(\mathbb{R})_0$ on the base manifold $\mathcal{X}^J_n$ is given by the formula $(M, l) \times (v, z) \rightarrow (v_1, z_1) \in \mathcal{X}^J_n$, where

$$v_1 = (Av + B)(Cv + D)^{-1}; \tag{4.26a}$$

$$z_1 = (z + vl_1^t + l_2^t)(Cv + D)^{-1}. \tag{4.26b}$$

Now we consider the transformation

$$w = (v - \mathrm{i})(v + \mathrm{i})^{-1}; \tag{4.27a}$$

$$z = 2\mathrm{i}(v + \mathrm{i})^{-1}u \tag{4.27b}$$

of $\mathcal{X}^J_n \rightarrow \mathcal{D}^J_n$. Equation (4.27a) is nothing else than the linear fractional transformation (4.26a) corresponding the a matrix $M = C^{-1}$ where C is given by (4.12) – the Cayley transform of the Siegel half-plane $\mathcal{H}_n$ into the Siegel unit ball $\mathcal{D}_n$. Under the same transformation, $C^{-1}\mathrm{Sp}(n, \mathbb{R})C \rightarrow \mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}$, and the linear fractional transformation (4.26a) on $\mathcal{H}_n$ determined by a matrix (4.25) becomes linear fractional transformation on $\mathcal{D}_n$ with the matrix $C^{-1}MC$ [38].

Under the transformation (4.27), the two-form (2.35) on $\mathcal{D}^J_n$ becomes on $\mathcal{X}^J_n$

(4.28)

$$-\mathrm{i}\,\omega'_n = \frac{k}{2}\mathrm{Tr}(p^t \wedge \bar{p}) + \frac{2}{\mathrm{i}}\mathrm{Tr}(B^t D \wedge \bar{B}), D = (\bar{v} - v)^{-1}, p = Ddv, B = du - dvD(\bar{u} - u),$$

(4.28) is a "n"-dimensional generalization of Berndt-Kähler two-form (3.23).

We want now to determine a "resolution of unity" on $\mathcal{X}^J_n$. We apply the change of coordinates (4.27) in (4.20). We get

$$(\phi, \psi) = \Lambda \int_{\mathcal{X}^J_n} \bar{f}_\phi(v, u) f_\psi(v, u) Q_1 K_1^{-1} dv du, \tag{4.29}$$

where dv and du are written down with the convention (4.22), i.e.

$$dv = \prod_{1 \le i \le j \le n} d\Re v_{ij} d\Im v_{ij}, du = \prod_{i=1}^n d\Re u_i d\Im u_i. \tag{4.30}$$

K_1 is the reproducing kernel (4.21) in the new variables:

$$K_1(u, v; \bar{u}, \bar{v}) = \det\ {}^{k/2}\frac{|v + \mathrm{i}|^2}{A} \exp F, \tag{4.31}$$

$$\frac{1}{2}F = 2\bar{u}^t A^{-1} u - [u^t (v + \mathrm{i})^{-1}(\bar{v} + \mathrm{i})A^{-1}u + cc],\ A = 2\mathrm{i}(\bar{v} - v). \tag{4.32}$$

The expression (4.32) can be put into a form which generalizes the expression (3.31)-(3.32) in the case $n = 1$:

$$F = 2\left[\bar{u}^t L^{-1} u + (\Im u)^t \Im(L^{-1}(\Im v)^{-1} u)\right] + \Re\left[(\bar{u}^t L^{-1}\bar{v}v - u^t L^{-1}\bar{v}^2)(\Im v)^{-1} u\right], \tag{4.33}$$

where L is hermitian and symmetric matrix $L = (\bar{v} - \mathrm{i})(v + \mathrm{i})$.

Q_1, the $G^J_n(\mathbb{R})_0$-invariant measure on $\mathcal{X}^J_n$, is calculated as $Q_1 = Q\frac{\partial(z,w)}{\partial(v,u)}$, applying the Lemma at p. 398 in Berezin's paper [12]. It is obtained

$$Q_1 = 2^{-n(n+3)}[\det(2i(\bar{v} - v))]^{-(n+2)}. \tag{4.34}$$

We have also the

Remark 8. *Let us consider the real Jacobi group $G^J_n(\mathbb{R})_0$ acting on the base manifold $\mathcal{X}^J_n$ by the formula* (4.26). *Then the symmetric Fock space attached to the reproducing kernel K_1 obtained from the kernel K* (4.21) *by the substitution* (4.27) *is given by square-integrable functions with respect to the scalar product* (4.29), *where group-invariant measure Q_1 is given by* (4.34) *and the integration variables are defined in* (4.30). *The manifold $\mathcal{X}^J_n$ is endowed with the Kähler form* (4.28), *which generalize the corresponding one* (3.23) *for $n = 1$ used by Kähler-Berndt.*

4.3. **Equations of motion.** Passing on from the dynamical system problem in the Hilbert space $\mathcal{H}$ to the corresponding one on M is called sometimes *dequantization*, and the system on M is a classical one [7, 8]. Following Berezin [13], the motion on the classical phase space can be described by the local equations of motion. We consider an algebraic Hamiltonian linear in the generators of the group of symmetry

$$\boldsymbol{H} = \sum_{\lambda\in\Delta} \epsilon_\lambda \boldsymbol{X}_\lambda. \tag{4.35}$$

The classical motion and the quantum evolutions generated by the Hamiltonian are given by the same equations of motion on $M = G/H$ [7, 8]:

$$\mathrm{i}\dot{z}_\alpha = \sum_\lambda \epsilon_\lambda Q_{\lambda,\alpha}, \tag{4.36}$$

where the differential action corresponding to the operator $\boldsymbol{X}_\lambda$ in (4.35) can be expressed in a local system of coordinates as

$$\mathbb{X}_\lambda = P_\lambda + \sum_\beta Q_{\lambda,\beta}\partial_\beta.$$

Above λ denotes a root, while β denotes a positive root.

The Hamiltonian

$$H = \epsilon_i a_i + \bar{\epsilon}_i a^+_i + \epsilon^0_{ij}K^0_{ij} + \epsilon^-_{ij}K^-_{ij} + \epsilon^+_{ij}K^+_{ij}$$

implies the Matrix Riccati equation on $\mathcal{D}^J_n$

$$\left\{\begin{array}{rcl} \mathrm{i}\dot{z} & = & \epsilon + W\bar{\epsilon} + \epsilon^+ zW + \frac{1}{2}z\epsilon^0, \\ \mathrm{i}\dot{W} & = & W\epsilon^+W + W\epsilon^0 + \epsilon^-. \end{array}\right.$$

References

[1] S. T. Ali, J.-P. Antoine and J.-P. Gazeau, Coherent states, wavelets, and their generalizations, Springer-Verlag, New York (2000)

[2] J. Arnaud and H. Kogelnik, *Gaussian light beams with general astigmatism*, Appl. Optics **5**(1969) 1687-1695

[3] J. J. Arnaud, Beam and fiber optics, Academic Press (1976)

[4] H. Bacry and M. Cadilhac, *Metaplectic group and Fourier optics*, Phys. Rev. A **23** (1981) 2533-2536

[5] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. of Math. **48** (1947) 568-640

[6] V. Bargmann, Group representations on Hilbert spaces of analytic functions, in *Analytic methods in Mathematical Physics*, R.P Gilbert and R.G. Newton (Eds), Gordon and Breach, Science publishers, New York, London, Paris, 1970, pp. 27–63.

[7] S. Berceanu and A. Gheorghe, *On equations of motion on Hermitian symmetric spaces*, J. Math. Phys. **33** (1992) 998-1007

[8] S. Berceanu and L. Boutet de Monvel, *Linear dynamical systems, coherent state manifolds, flows and matrix Riccati equation*, J. Math. Phys. **34** (1993) 2353-2371

[9] S. Berceanu, *Coherent states: old Geometric methods in new quantum clothes*, Rom. Journ. Phys. **47** (2002) 353-358, deposed in arXiv hep-th/9408008; S. Berceanu, *Coherent states and global differential geometry*, in "Quantization, Coherent states and complex structures", Plenum, New York, 1995, pp. 131-140, Edited by J. P. Antoine, S. Twareque Ali, W. Lisiecki, I. M. Mladenov, A. Odzijewicz, Proceedings of the XIII Workshop of Geometric methods in Physics, Białowieża, July 9-15, Poland 1994; S. Berceanu *Geometry via coherent states*, in "GROUP21: Physical Applications and Mathematical Aspects of Geometry, Groups and Algebras", V. 1, Editors: H.-D. Doebner, W Scherer, P. Nattermann, Word Scientific, Singapore, (1997), pp. 228-232, Proceedings of the XXI International Colloquium on Group Theoretical Methods in Physics, 15-20 July 1996, Goslar, Germany, deposed in arXiv: dg-ga/9708001

[10] S. Berceanu, *A holomorphic representation of the Jacobi algebra*, Rev. Math. Phys. **18** (2006) 163-199, deposed in arXiv: math.DG/0408219

[11] S. Berceanu, *A holomorphic representation of the semidirect sum of symplectic and Heisenberg Lie algebras*, J. Geom. Symmetry Phys. 5 (2006) 5-13; S. Berceanu, *A holomorphic representation of Jacobi algebra in several dimensions*, 6th Operator Algebras International Conference: Operator Algebras and Mathematical Physics-3, Bucharest (Romania), August 10 - 17, 2005 to appear in the Proceedings; deposed in arXiv: math.DG/060404381

[12] F. A. Berezin, *Quantization in complex symmetric spaces*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975) 363-402, 472.

[13] F. A. Berezin, *Models of Gross-Neveu type are quantization of a Classical Mechanics with a nonlinear phase space*, Commun. Math. Phys. **63** (1978) 131-153

[14] R. Berndt, Sur l'arithmtique du corps des fonctions elliptiques de niveau N, in *Seminar on number theory, Paris 1982–83*, Progr. Math., 51, Birkhäuser Boston, Boston, MA (1984) 21-32

[15] R. Berndt, S. Böcherer, *Jacobi forms and discrete series representations of the Jacobi group*, Math. Z. **204** (1990) 13-44

[16] R. Berndt and R. Schmidt, Elements of the representation theory of the Jacobi group, Progress in Mathematics, 163, Birkhäuser Verlag, Basel (1998)

[17] R. Berndt, *Coadjoint orbits and representations of the Jacobi group*, IHES/M/03/37 (2003)

[18] V. V. Dodonov, *'Nonclassical' states in quantum optics: a 'squeezed' review of the first 75 years*, J. Opt. B.: Quantum semiclassical Opt. **4**, (2002) R1-R33

[19] P. D. Drummond and Z. Ficek, Editors, Quantum squeezing, Springer, Berlin (2004)

[20] M. Eichler and D. Zagier, The theory of Jacobi forms, Progress in Mathematics, 55, Birkhäuser, Boston, MA (1985)

[21] G. B. Folland, Harmonic analysis in phase space, Princeton University Press, Princeton, New Jersey, 1989.

[22] V. Guillemin and S. Sternberg, Geometric Asymptotics, American Mathematical Society, Providence, R. I. (1977)

[23] V. Guillemin and S. Sternberg, Symplectic Technics in Physics, Cambridge University Press, Cambridge (1984)

[24] E. Kähler, *Die Poincaré-Gruppe*, Rend. Sem. Mat. Fis. Milano 53 (1983) 359-390

[25] E. Kähler, *The Poincaré group*, in *Clifford algebras and their applications in mathematical physics* (Canterbury, 1985), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 183, Reidel, Dordrecht, (1986) 265-272

[26] E. Kähler, *Raum-Zeit-Individuum*, Rend. Accad. Naz. Sci. XL Mem. Mat. **5** 16 (1992) 115-177

[27] Erich Kähler: Mathematische Werke; Mathematical Works, Edited by R. Berndt, O. Riemenschneider, Walter de Gruyter, Berlin-New York (2003)

[28] M. Kogelnik, *Imagining of optical mode-Resonators with internal lenses*, Bell Sys. Tech. J. **44** (1965) 455-494

[29] H. Kogelnik and T. Li, *Laser beems and resonators*, Proc. IEEE **54** (1966) 1312-1328
[30] W. Lisiecki, *Coherent state representations. A survey*, Rep. Math. Phys. **35** (1995) 327-358
[31] R. K. Luneburg, Mathematical Theory of Optics, Univ. California Press, Berkeley and Los Angeles (1966)
[32] M. Nazarathy and J. Shamir, *Fourier optics described by operator algebra*, J.Opt.Soc.Am., **70** (1980) 150-158
[33] M. Nazarathy and J. Shamira, *First-order optics - a canonical operator representation: lossless systems*, J. Opt. Soc. Am. **72** (1982)
[34] K.-H. Neeb, Holomorphy and Convexity in Lie Theory, de Gruyter Expositions in Mathematics 28, Walter de Gruyter, Berlin-New York (2000)
[35] U. Niederer, *The maximal kinematical invariance group of the free Schrödinger equation*, Helv. Phys. Acta. **45** (1972/73) 802-810
[36] A. M. Perelomov, Generalized Coherent States and their Applications, Springer, Berlin (1986)
[37] I. Satake, Algebraic structures of symmetric domains, Publ. Math. Soc. Japan **14**, Princeton Univ. Press (1980)
[38] C. L. Siegel, Symplectic geometry, Academic Press, New York (1964)
[39] R. Simon and N. Mukunda, *Iwasawa decomposition for* $SU(1,1)$ *and the Goy effect for squeezed states*, Opt. Commun, **95**, (1993) 39-45
[40] R. Simon, E.C.G. Sudarshan and N. Mukunda, *Gaussian pure states in quantum mechanics and the symplectic group*, Phys. Rev. A. **37** (1988) 3028-3038
[41] R. Simon and Kurt Bernardo Wolf, *Structure of the set of paraxial optical systems*, J. Opt. Soc. Am. A **17** (2000) 342-355
[42] S. Sivakumar, *Studies on nonlinear coherent states*, J. Opt. B: Quantum Semiclass. Opt. **2** (2000) R61-R75
[43] D. Stoler, *Operator methods in physical optics*, J. Opt. Soc. Am. A **71** (1981) 334-341
[44] K. B. Wolf, *The Heisenberg-Weyl ring in quantum mechanics*, in *Group theory and its applications*, Vol. 3, Ed. E M Loebl, Academic Press, New York (1975) 189-247
[45] K. B. Wolf, Integral transforms in science and engineering, Plenum Publ. Corp. New York (1979)
[46] K. B. Wolf, Geometric Optics on Phase Space, Springer (2004)
[47] W.-M. Zhang, D. H. Feng and R. Gilmore, *Coherent states: theory and some applications*, Rev. Mod. Phys. **62** (1990) 867-927

October 10, 2007

(Stefan Berceanu) National Institute for Physics and Nuclear Engineering, Department of Theoretical Physics, PO BOX MG-6, Bucharest-Magurele, Romania
E-mail address: `Berceanu@theor1.theory.nipne.ro`