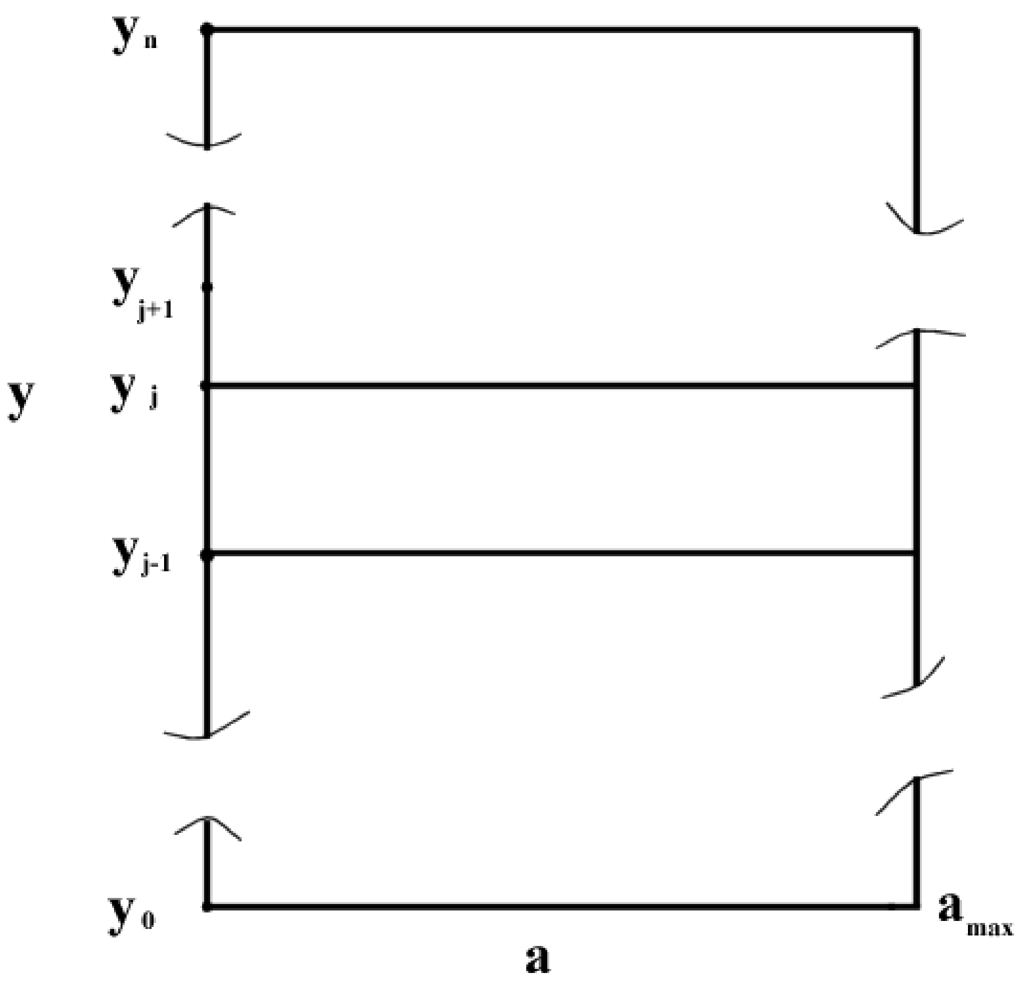
multi-layer environment: specific motivation to the model



the model: age + diffusion

Diffusion + Gurtin-MacCamy in each layer ($j = 1, \ldots, n$)

- $p_j(a, y, t); \quad y \in (y_{j-1}, y_j) \quad a \in (0, a_\dagger)$

$$\frac{\partial p_j}{\partial t} + \frac{\partial p_j}{\partial a} + \mu_0(a)p_j + \mu_j(a, S_j(t))p_j - K_j(a)\frac{\partial^2 p_j}{\partial y^2} = f_j$$

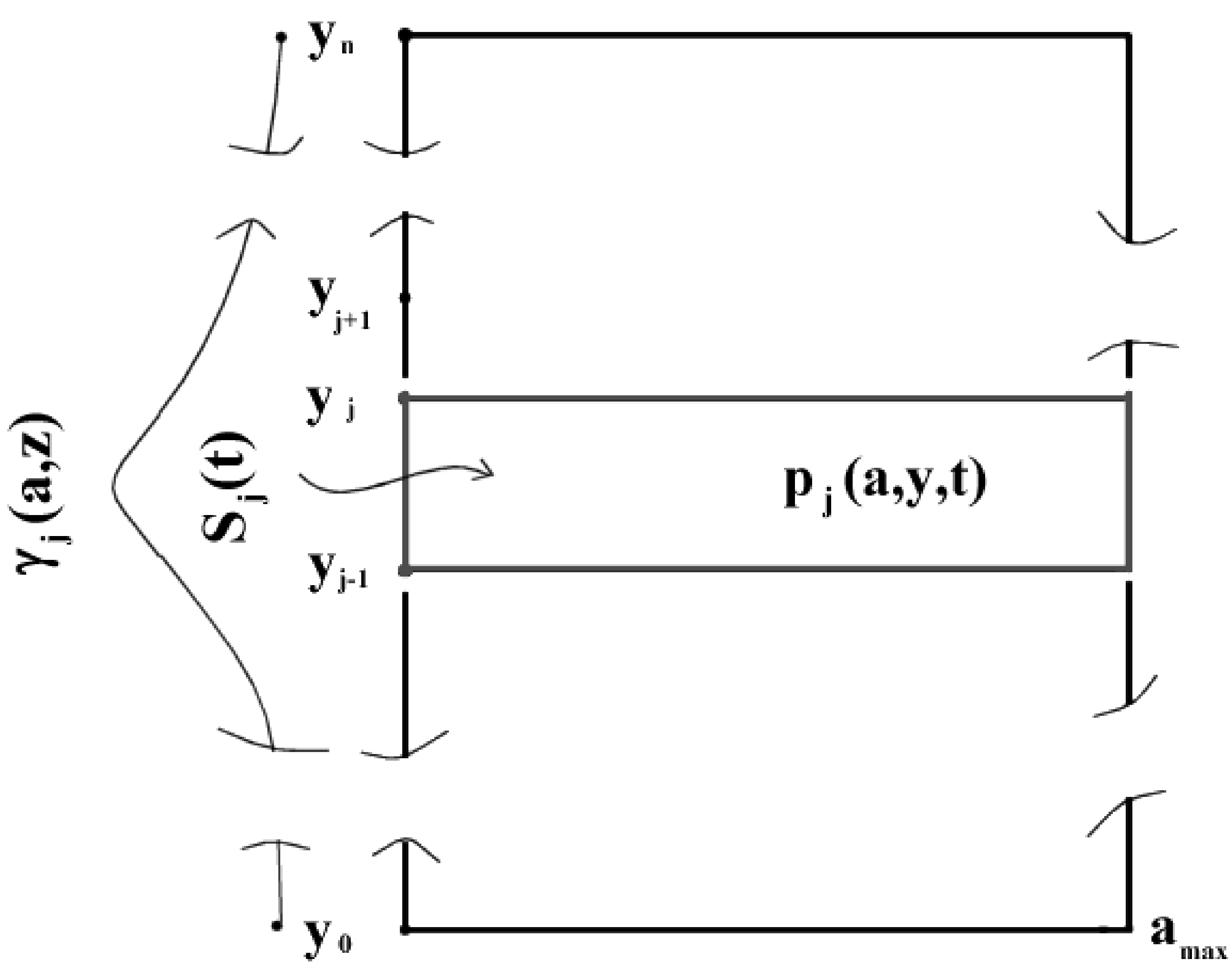
$$p_j(t, 0, y) = \int_0^{a_\dagger} \beta_j(a, S_j(t))p_j(t, a, y)da$$

$$p_j(0, a, y) = p_{j0}(a, y),$$

where

$$S_j(t) = \sum_{k=1}^{n} \int_0^{a^+} \int_{y_{k-1}}^{y_k} \gamma_j(a, z)p_k(t, a, z)dzda,$$

the model: age + diffusion

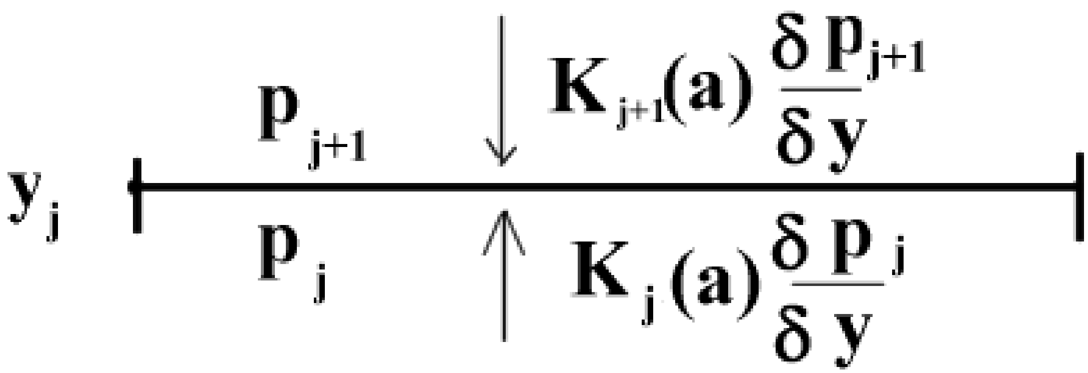


the model: age + diffusion

Continuity conditions at the interface between two layers

density $$p_j(a, y_j^-, t) = p_{j+1}(a, y_j^+, t) \quad (j = 1, \ldots, n)$$

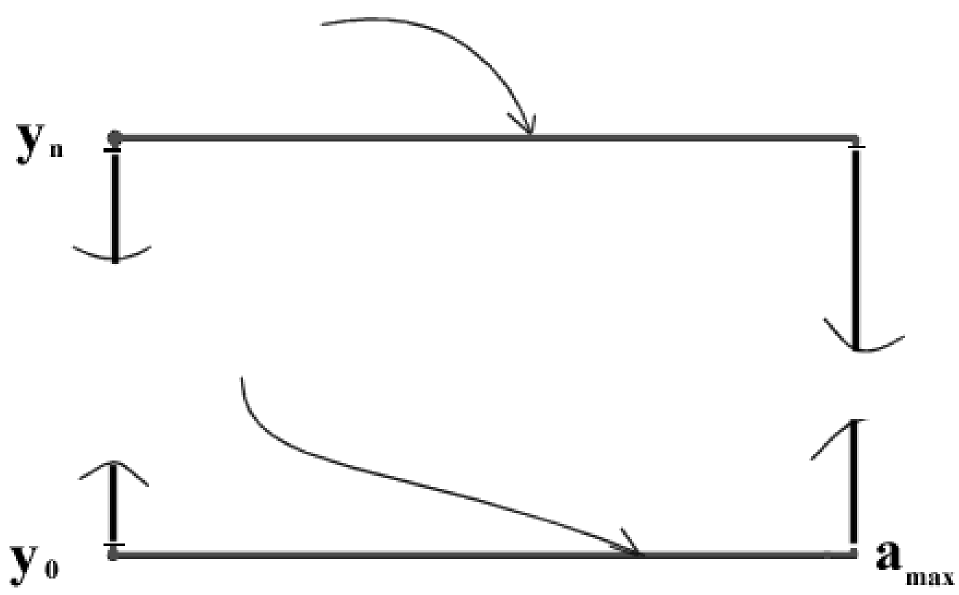
flux $$K_j(a)\frac{\partial p_j(a, y_j^-, t)}{\partial y} = K_{j+1}(a)\frac{\partial p_{j+1}(a, y_j^+, t)}{\partial y} \quad (j = 1, \ldots, n)$$



the model: age + diffusion

Null flux conditions at the boundary of the multi-layer (at $y = y_0$ and $y = y_n$)

$$K_1(a)\frac{\partial p_1(a, y_0, t)}{\partial y} = 0 \quad \textit{and} \quad K_n(a)\frac{\partial p_n(a, y_n, t)}{\partial y} = 0$$



the model: assumptions

$$\frac{\partial p_j}{\partial t} + \frac{\partial p_j}{\partial a} + \mu_0(a)p_j + \mu_j(a, S_j(t))p_j - K_j(a)\frac{\partial^2 p_j}{\partial y^2} = f_j$$

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the model: assumptions

$\mu_0(a)$

$$\mu_0(\cdot) \in L^1_{loc}([0, a_\dagger)),$$

$$\mu_0(a) \geq 0 \text{ a. e. in } [0, a_\dagger], \qquad \int_0^\infty \mu_0(a)\, da = +\infty.$$

In particular these conditions guarantee that the survival probability

$$\Pi_0(a) = e^{-\int_0^a \mu_0(\sigma) d\sigma}$$

vanishes at the maximum age $a_\dagger$.

the model: assumptions

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$\beta_j(a, S_j(t))$

$$|\beta_j(a,x) - \beta_j(a,\bar{x})| \leq L_\beta(R)\,|x - \bar{x}|\,, \qquad \text{for } \ |x| \leq R \text{ and } \ |\bar{x}| \leq R$$

$$0 \leq \beta_j(a,x) \leq \beta_+$$

$\mu_j(a, S_j(t))$

$$|\mu_j(a,x) - \mu_j(a,\bar{x})| \leq L_\mu(R)\,|x - \bar{x}|\,, \qquad \text{for } \ |x| \leq R \text{ and } \ |\bar{x}| \leq R$$

$$0 \leq \mu_j(a,x) \ \text{ with } \ \mu_j(a,0) = 0$$

the model: assumptions

$$\frac{\partial p_j}{\partial t} + \frac{\partial p_j}{\partial a} + \mu_0(a)p_j + \mu_j(a, S_j(t))p_j - K_j(a)\frac{\partial^2 p_j}{\partial y^2} = f$$

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the model: assumptions

$\gamma_j(a,z)$

$$0 \le \gamma_j(a,z) \le \gamma_+$$

$K_j(a)$

$$0 < K_0 \le K_j(a)$$

Functional framework

Slight technical simplification to get rid of $\mu_0(a)$

Change variable from $p_j(a,y,t)$ to $p_j(a,y,t) = \dfrac{p_j(a,y,t)}{e^{-\int_0^a \mu_0(\sigma)d\sigma}}$

$$\frac{\partial p_j}{\partial t} + \frac{\partial p_j}{\partial a} + \mu_0(a)p_j + \mu_j(a, S_j(t))p_j - K_j(a)\frac{\partial^2 p_j}{\partial y^2} = f_j$$
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$$p_j(0,a,y) = p_{j0}(a,y),$$

Functional framework

Put together all the layers in one single domain glueing the functions from the different layers

$$p(t,a,y) = \begin{cases} p_1(t,a,y), \; y \in (y_0, y_1), \\ \dots \\ p_n(t,a,y), \; y \in (y_{n-1}, y_n), \end{cases}$$

$$(t,a,y) \in (0,T) \times (0,a_\dagger) \times (y_0, y_n)$$

Functional framework

$$S(t,y) = \begin{cases} S_1(t,y), \ y \in (y_0, y_1), \\ \dots \\ S_n(t,y), \ y \in (y_{n-1}, y_n), \end{cases}$$

$$\beta(a,y,x) = \begin{cases} \beta_1(a,x), \quad y \in (y_0, y_1), \\ \dots \\ \beta_n(a,x), \quad y \in (y_{n-1}, y_n), \end{cases}$$

$$\mu(a,y,x) = \begin{cases} \mu_1(a,x), \quad y \in (y_0, y_1), \\ \dots \\ \mu_n(a,x), \quad y \in (y_{n-1}, y_n), \end{cases}$$

$$\dots \quad \dots \quad \dots$$

note that with the definitions above

$$S(t,y) = \int_0^{a_\dagger} \int_{y_0}^{y_n} \gamma(a,y,z)p(t,a,z)dzda.$$

and

$$\beta(a,y,S(t,y)) = \beta j(a, S_j(t)) \quad \textit{for} \quad y \in (y_{j-1}, y_j)$$

$$etc \ldots\ldots\ldots$$

Functional framework

First we consider $H = L^2(y_0, y_n)$, $V = H^1(y_0, y_n)$, *its dual* V'
and define the operator

$$A_0 : D(A_0) \subset L^2(0, a_\dagger; V) \longrightarrow L^2(0, a_\dagger; V'),$$

$$D(A_0) = \{u \in L^2(0, a_\dagger; V); u_a \in L^2(0, a_\dagger; V'),$$
$$u(0, y) = \int_0^{a_\dagger} \beta(a, y, S(y))u(a, y)da\}$$

by

$$< A_0 u, \psi >= \int_\Omega (u_a \psi + \mu(a, y, S(y))u\psi + K(a, y)u_y \psi_y)dady,$$

$$\forall \psi \in L^2(0, a_\dagger; V)$$

Functional framework

Then we embed the problem in the space

$$H_\Omega = L^2((0, a_\dagger) \times (y_0, y_n))$$

defining

$$A : D(A) \subset H_\Omega \to H_\Omega$$

on the domain

$$D(A) = \{u \in D(A_0), A_0 u \in H_\Omega\}$$

by setting

$$Au = A_0 u, \quad \forall u \in D(A)$$

Functional framework

$$\begin{cases} \dfrac{d}{dt}p(t) + Ap(t) = f \\ \\ p(0) = p_0 \end{cases}$$

an abstract Cauchy problem in H_Ω

two non-linearities

$$D(A_0) = \{u \in L^2(0, a_\dagger; V); u_a \in L^2(0, a_\dagger; V'),$$
$$u(0, y) = \int_0^{a_\dagger} \beta(a, y, S(y))u(a, y)da\}$$

$$< A_0 u, \psi >= \int_\Omega (u_a \psi + \mu(a, y, S(y))u\psi + K(a, y)u_y \psi_y)dady,$$

$$\forall \psi \in L^2(0, a_\dagger; V)$$

the functions

$$u \mapsto E(u) \equiv \mu(a, y, S(y))u(a, y)$$

$$u \mapsto F(u) \equiv \beta(a, y, S(y))u(a, y)$$

are locally Lipschitz continuous from H_Ω to H_Ω

$$\|E(u) - E(\bar{u})\|_{H_\Omega} \leq M(R) \|u - \bar{u}\|_{H_\Omega}$$

$$\|F(u) - F(\bar{u})\|_{H_\Omega} \leq B(R) \|u - \bar{u}\|_{H_\Omega}$$

Analysis of the Cauchy problem

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Lipshitz continuity allows A ito be quasi accretive

$$((\lambda I + A)u - (\lambda I + A)\bar{u}, u - \bar{u})_{H_\Omega} =$$

$$= \lambda \|u - \bar{u}\|^2_{H_\Omega} + \int_\Omega \{(u_a - \bar{u}_a)(u - \bar{u}) + (E(u) - E(\bar{u}))(u - \bar{u})\} dyda$$

$$+ \int_\Omega K(a, y)(u_y - \bar{u}_y)^2 dyda \geq$$

$$\geq \lambda \|u - \bar{u}\|^2_{H_\Omega} - \int_{y_0}^{y_n} \left[\int_0^{a_\dagger} (F(u) - F(\bar{u}))da\right]^2 dy +$$

$$- \left|\int_\Omega (E(u) - E(\bar{u}))(u - \bar{u})dyda\right| + K_0 \|u_y - \bar{u}_y\|^2_{H_\Omega} \geq$$

$$\geq (\lambda - \omega) \|u - \bar{u}\|^2_{H_\Omega}$$

Moreover A is m-accretive

$$\text{Range}\,(\lambda I + A) = H_\Omega$$

We need to solve the problem

$$\begin{aligned} & u_a + \lambda u + E(u) + K(a,y)u_{yy} = f \\ & u(0,y) = \int_0^{a_\dagger} F(u)(a,y)da \\ & u_y(a,y_0) = u_y(a,y_n) = 0 \end{aligned}$$

for any $f \in H_\Omega$.

Analysis of the Cauchy problem

Moreover A is m-accretive

$$\textit{Range}\,(\lambda I + A) = H_\Omega$$

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By the solution $u = \mathcal{P}(w)$ we get a mapping $\mathcal{P} : H_\Omega \to H_\Omega$ which is a contraction. The fixed point

$$u^* = \mathcal{P}(u^*)$$

solves the problem.

Analysis of the Cauchy problem

$$u^*_a + \lambda u^* + E(u^*) + K(a,y)u^*_{yy} = f$$
$$u^*(0,y) = \int_0^{a_\dagger} F(u^*)(a,y)da$$
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Existence, uniqueness, estimates and all that

Under the assumption of Lipschitz continuity the abstract Cauchy problem has a unique solution (non-negative if p_0 and f are non-negative) which satisfies the estimates

$$\|p(t) - \bar{p}(t)\|^2_{H_\Omega} \leq \left(\|p_0 - \bar{p}_0\|^2_{H_\Omega} + \int_0^t \|f(\tau) - \bar{f}(\tau)\|^2_{H_\Omega} \, d\tau \right) e^{\alpha_0 t},$$

$$\|p(t)\|^2_{H_\Omega} \leq \frac{1}{K_0} \left(\|p_0\|^2_{H_\Omega} + \int_0^T \|f(\tau)\|^2_{H_\Omega} \, d\tau \right) e^{\beta_0 t}$$

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$$\beta_0 = \beta_+^2 a_\dagger + 2$$

Morever, if

$$0 \leq p_0 \leq p_M \text{ a.e. } (a, y) \in \Omega \text{ and } 0 \leq f \leq f_M \text{ a.e. } (t, a, y) \in (0, T) \times \Omega$$

the solution p satisfies

$$0 \leq p(t) \leq p_M + f_M t, \text{ a.e. in } \Omega, \text{ for any } t \in [0, T]. \tag{1}$$

In order to treat the locally Lipschitz case we consider the truncated functions (arbitrary N)

$$E_N(u) = \begin{cases} E(u) & \text{for} \quad \|u\|_{H_\Omega} \leq N \\ E\left(\dfrac{Nu}{\|u\|_{H_\Omega}}\right) & \text{for} \quad \|u\|_{H_\Omega} > N \end{cases}$$

and

$$F_N(u) = \begin{cases} F(u) & \text{for} \quad \|u\|_{H_\Omega} \leq N \\ F\left(\dfrac{Nu}{\|u\|_{H_\Omega}}\right) & \text{for} \quad \|u\|_{H_\Omega} > N \end{cases}$$

Existence, uniqueness, estimates and all that

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- so that for $N > R$

$$A_N p_N(t) = A p_N(t), \qquad p_N(t,0,y) = \int_0^{a_\dagger} F(p_N(t))(a,y)da,$$

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- and $p_N(t)$ is actually a solution to the original problem