Maltusian model (1766-1834)

$$\frac{\partial p}{\partial t} + (\mu - \beta)p = 0 \text{ in } (0, T),$$

$$p(0) = p_0.$$

Solution

$$p(t) = p_0 \exp[-(\mu - \beta)t]$$

$\mu =$ mortality rate
$\beta =$ fertility rate
$\mu \geq 0, \beta \geq 0$
$\mu > \beta$ exponential extinction
$\beta > \mu$ exponential growth

Logistic model (Verhulst 1838)

$$\frac{\partial p}{\partial t} + \alpha p \left(1 - \frac{p}{K}\right) = 0 \text{ in } (0, T),$$

$$p(0) = p_0.$$

Solution:

$$p(t) = \frac{K p_0 \exp(\alpha t)}{K + p_0[\exp(\alpha t) - 1]}$$

$$\lim_{t\to\infty} p(t) = K.$$

$\alpha =$ growth rate
$K =$ carrying capacity

The linear model of Lotka-McKendrick (1925)

▸Age-structure

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} + \mu_0(a)p + \mu(a)p = f \text{ in } (0,T) \times (0,a^+),$$

$$p(0,a) = p_0(a),$$

$$p(t,0) = \int_0^{a^+} \beta(a)p(t,a)da.$$

$\mu_0 =$ (specific) mortality rate
$\mu =$ (supplementary) mortality rate
$\beta =$ fertility rate

The nonlinear model of Gurtin-MacCamy (1973)

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} + \mu_0(a)p + \mu(a, S(t))p = f \text{ in } (0, T) \times (0, a^+),$$

$$p(0, a) = p_0(a),$$

$$p(t, 0) = \int_0^{a^+} \beta(a, S(t))p(t, a)da,$$

$$S(t) = \int_0^{a^+} \gamma(a)p(t, a)da.$$

The nonlinear Gurtin-MacCamy model with diffusion

(first version in 1974)

$\Omega = (0, a^+) \times (y_0, y_L) \qquad (y_0, y_L) = spatial\ domain$ ($habitat$)

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} + \mu_0(a,y)p + \mu(a,y,S(t,y))p - \frac{\partial}{\partial y}(K(a,y)\frac{\partial p}{\partial y}) = f \text{ in } (0,T) \times \Omega, \tag{1}$$

$$p(0,a,y) = p(a,y) \text{ in } \Omega, \tag{2}$$

$$K(a,y)\frac{\partial p}{\partial y} = 0 \text{ on } (0,T) \times \Gamma_{y_0} \text{ and on } (0,T) \times \Gamma_{y_L} \tag{3}$$

$$p(t,0,y) = \int_0^{a^+} \beta(a,y,S(t,y))p(a,t,y)da, \tag{4}$$

$$S(t,y) = \int_\Omega \gamma(a,y,z)p(t,a,z)dzda. \tag{5}$$