Koszul algebras and modules

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ABSTRACT. In this article we give a survey on some recent developments in the theory of Koszul algebras.

Introduction

These notes reflect the content of the lecture on Koszul algebras given at the "International Conference on Algebra and Geometry" held in Hyderabad, December 7-12, 2001. It is the aim of this text to explain also some of the background of the theory and to point at some open problems. The definition and first properties of Koszul algebras are given in Section 1, while in Section 2 we consider the Koszul property for semigroup rings. A central result in this context is the theorem of Laudal and Sletsjøe which implies a characterization of Koszul semigroup rings by its divisor posets. We describe some of its implications which leads to shelling and sequential conditions for Koszulness. Sequential conditions for Koszulness are the main topic of Section 3. This includes the notions strongly Koszul and universally Koszul algebras. In Section 4 we study when a pure subring of a Koszul algebra is again Koszul and describe some known cases. Finally the last Section 5 is devoted to Koszul modules, a concept that has been introduced by Iyengar and the author.

As general references I recommend [**16**] and [**10**], and for questions regarding infinite free resolutions I suggest to consult [**3**].

1. Koszul algebras

Let K be a field, $S = K[x_1, \dots, x_n]$ a polynomial ring and $A = S/I$ a standard graded K-algebra with graded maximal ideal $\mathfrak{m}$. Unless otherwise stated we will always assume that the A-modules under consideration are graded and finitely generated. Let M be an A-module, and let F be its graded minimal free A-resolution. Then $F_i = \bigoplus_j A(-j)^{\beta_{ij}}$. We call the numbers β_{ij} the *graded Betti numbers of* M. Notice that the vector spaces $\operatorname{Tor}_i^A(K; M)$ are naturally graded and that $\beta_{ij} = \dim_K \operatorname{Tor}_i^A(K, M)_j$ for i and j.

We say that M *has a d-linear resolution*, if $\beta_{ij} = 0$ for all $j \neq i + d$. This is equivalent to say that M is generated in degree d and all matrices describing the differentials of the free resolution F of M have entries of linear forms.

DEFINITION 1.1. A is a *Koszul algebra* if the A-module $K = A/\mathfrak{m}$ has linear resolution.

The polynomial ring is the simplest example of a Koszul algebra. In this case the Koszul complex provides a linear resolution of K. Less obvious, but also simple, is the following example: Assume that I is generated by a regular sequence of

Partially supported by DFG.

quadrics. Then $A = S/I$ is a Koszul algebra. In fact, the algebra is a complete intersection and so the Tate resolution yields immediately the desired conclusion, cf. [**3**, Theorem 6.1.8].

1.1. First properties of Koszul algebras. Let $A = S/I$ be a standard graded K-algebra. Consider the following properties:

(1) A is a Koszul algebra;
(2) I is quadratically generated;
(3) I has a quadratic Gröbner basis for some term order.

Then (3) $\Rightarrow$ (1) $\Rightarrow$ (2), and non of the implications can be reversed in general.

The implication (1) $\Rightarrow$ (2) can be easily seen: Let $f_1, \dots, f_m$ be a minimal homogeneous set of generators of I and write $f_i = \sum_{j=1}^n f_{ij}x_j$. Let $\epsilon\colon \bigoplus_{i=1} Ae_i \to \mathfrak{m}$ be the epimorphism with $\epsilon(e_i) = x_i$ for $i = 1, \dots, n$. Then besides of the linear Koszul relations $x_ie_j - x_je_i$, $\operatorname{Ker}(\epsilon)$ is generated by the relations $\sum_{j=1}^n \bar{f}_{ij}e_j$ for $i = 1, \dots, n$, where $\bar{g} = g + I$. Thus we conclude that $\beta_{2i}(K) = 0$ for $i \neq 2$ if and only if I is generated by quadrics.

The implication (3) $\Rightarrow$ (1) is based on two facts: Let $<$ be a term order, and denote by $\operatorname{in}(I)$ the initial ideal with respect to this term order. Then a deformation argument (or a spectral sequence argument) shows that $\beta_{ij}^{S/I}(K) \leq \beta_{ij}^{S/\operatorname{in}(I)}(K)$ for all i and j. The other fact needed is the following result [**19**]:

THEOREM 1.2 (Fröberg). *If I is generated by quadratic monomials, then $A = S/I$ is Koszul.*

The implication (3) $\Rightarrow$ (1) is one of the techniques used in many papers to show that an algebra is Koszul.

On the other hand, there are examples (see Remark 2.4) of Koszul algebras which have no quadratic Gröbner basis for any term order.

We notice another fundamental property of Koszul algebras: Denote by $H_A(t) = \sum_i \dim_K A_it^i$ the *Hilbert series* and by $P_A(t) = \dim_K \operatorname{Tor}_i^A(K,K)t^i$ the *Poincaré series of* A, and let

$$\cdots \longrightarrow A(-2)^{\beta_2} \longrightarrow A(-1)^{\beta_1} \longrightarrow A \longrightarrow K \longrightarrow 0$$

be the graded minimal free resolution of the Koszul algebra A. The additivity of the Hilbert function implies that

$$1 = H_A(t) - \beta_1 tH_A(t) + \beta_2t^2H_A(1) - \cdots = H_A(t)P_A(-t),$$

so that

$$P_A(t) = H_A(-t)^{-1} = \frac{(1+t)^d}{Q(t)},$$

where $d = \dim A$ and $Q(t)$ is a polynomial. In particular, $P_A(t)$ is a rational function of a particular nature.

2. Affine semigroup rings and the Koszul property

A finitely generated additive subsemigroup $S \subset \mathbb{Z}^n$ containing 0 is called an *affine semigroup.* We will assume that S is positive, that is, S has no invertible elements. Then S can be embedded into $\mathbb{N}^n$.

Let K be a field. For $a \in \mathbb{N}^n$, $a = (a_1, \dots, a_n)$, we set $x^a = \prod_{i=1}^n x_i^{a_i}$. Then $K[S] = \bigoplus_{a\in S} Kx^a \subset K[x_1, \dots, x_n]$ is called the *semigroup ring of* S. Suppose that S is generated by $a_1, \dots, a_m$. Then $K[S] = K[x^{a_1}, \dots, x^{a_m}]$. In the following we will also assume that $K[S]$ is standard graded which is equivalent to say that the generators $a_1, \dots, a_m$ of S lie in a hyperplane.

There is an interesting combinatorial characterization of affine semigroup rings which are Koszul, due to Laudal and Sletsjøe [**24**]. Define a partial order $\leq$ on S:

$$a_1 \leq a_2 \quad \text{if and only if there exists } b \in S \text{ such that } a_2 = a_1 + b.$$

For each $a \in S$ the open interval

$$(0, s) = \{a \in S \colon 0 < b < a\}$$

with the order $\leq$ restricted to $(0, a)$ is a finite poset. Let Δ_a be the order complex of $(0, a)$. Recall that the faces of Δ_a are just the chains (= totally ordered subsets of $(0, a)$). The simplicial complexes Δ_a are called *divisor posets of* S.

THEOREM 2.1 (Laudal-Sletsjøe). *Let $A = K[S]$, then all the K-vector spaces $\operatorname{Tor}_i^A(K, K)$ are $\mathbb{Z}^n$-graded, and for the graded components we have*

$$\operatorname{Tor}_i^A(K, K)_a = \begin{cases} 0 & \text{if } a \notin S \\ \tilde{H}_{i-2}(\Delta_a; K) & \text{if } a \in S, \end{cases}$$

where $\tilde{H}(\Gamma; K)$ denotes the reduced simplicial homology of a simplicial complex Γ.

A simplicial complex Γ is called *Cohen-Macaulay over K* if the Stanley Reisner ring $K[\Gamma]$ is Cohen-Macaulay, see [**10**]. Using the above Theorem 2.1 and the Stanley-Reisner criterion for Cohen-Macaulayness of simplicial complexes (cf. [**10**, Corollary, 5.3.9]) one obtains

COROLLARY 2.2. *If $A = K[S]$ is standard graded, then A is Koszul if and only if Δ_a is Cohen-Macaulay for $a \in S$.*

We write $K[S] = K[x_1, \dots, x_n]/I_S$. The ideal I_S (which is generated by binomial ideals) is called the *toric ideal of S*. The property that I_S has a quadratic Gröbner basis is reflected by the divisor complexes [**23**].

THEOREM 2.3 (Herzog-Reiner-Welker). *If I_S has a quadratic Gröbner basis, then each divisor complex Δ_a is homotopic to wedge of spheres.*

In general it is hard to see whether Δ_a is Cohen-Macaulay. On the other hand, if Δ_a is shellable, then Δ_a is Cohen-Macaulay over any field.

Recall that Δ_a is *shellable* if

(1) all maximal chains $C_1, \dots, C_r$ of the closed intervals $[0, a]$ have the same length; (This is the case if $K[S]$ is standard graded)
(2) there exists an order of the maximal chains $C_1 < C_2 < \cdots < C_r$ such that for all $i < j$, there exists $x \in C_j \setminus C_i$ and k with $i < k < j$ such that $C_j \setminus C_k = \{x\}$.

REMARKS 2.4. (1) There is no affine Koszul semigroup known whose divisor posets are not shellable.
(2) If all divisor posets of S are shellable, then $K[S]$ is Koszul for any base field K.
(3) There is no affine semigroup ring known whose Koszulness depends on the characteristic of K.
(4) Let $K[S] = K[x_1x_2x_3, x_1x_3x_4, x_1x_4x_5, x_1x_2x_5, x_2x_3x_6, x_4x_5x_6, x_2x_5x_7]$. Hibi and Ohsugi [**27**] have shown that all divisor posets of S are shellable, while I_S has no quadratic Gröbner basis.

The following theorem [**28**] provides a sufficient condition for the shellability of divisor posets.

THEOREM 2.5 (Peeva-Reiner-Sturmfels). *Suppose there exists a term order $<$ such that the initial ideal* $\text{in}_<(I_S)$ *is generated by squarefree monomials of degree* 2 *satisfying the following condition:*

$$\text{if } x_ix_j \in \text{in}_<(I_S), \text{ and } i < l < j, \text{ then } x_ix_l \text{ or } x_lx_j \in \text{in}_<(I_S).$$

Then all divisor posets Δ_a *are shellable.*

Not so many explicit examples of semigroups with shellable divisor posets are known. We close this section with a nice class of such semigroups. In the next section we shall see a few more examples.

Suppose $A = K[S]$ is standard graded. We say that A is a *monomial algebra with straightening law* (ASL) if there exists a partial order on the set $P = \{u_1, \dots, u_n\}$ of monomial generators of A such that:

(ASL-1) the standard monomials , i.e. the monomials $u_{i_1} \cdots u_{i_p}$ with $u_{i_1} \leq \cdots \leq u_{i_p}$ form a K-basis of A;

(ASL-2) if $u, v \in P$ are incomparable, and $uv = \sum_i a_i u_{j_i} u_{k_i}$ with $a_i \in K$, $a_i \neq 0$ and $u_{j_i} \leq u_{k_i}$. Then $u_{j_i} \leq u$ and $u_{k_i} \leq v$ for all i.

Given a finite lattice L. The ring

$$K[L] = K[x_\alpha \colon \alpha \in L]/(x_\alpha x_\beta - x_{\alpha\wedge\beta} x_{\alpha\vee\beta})_{\alpha,\beta\in L}$$

is called the *Hibi ring of* L.

THEOREM 2.6 (Hibi). *If L is distributive, then $K[L]$ is a domain and a monomial ASL.*

We have [**2**]

THEOREM 2.7 (Aramova-Herzog-Hibi). *The divisor posets of a monomial ASL are shellable.*

3. Sequential conditions

Let A be a standard graded K-algebra. The following definition is due to Herzog, Hibi and Restuccia [**21**]

DEFINITION 3.1. The algebra A is *strongly Koszul* if the graded maximal ideal of A admits a system of homogeneous generators $u_1, \dots, u_n$ such that for all subsequences $u_{i_1}, \dots, u_{i_k}$ of $u_1, \dots, u_n$ with $i_1 < i_2 < \cdots < i_k$ and for all $j = 1, \dots, k-1$ the colon ideal $(u_{i_1} \dots, u_{i_{j-1}}) : u_{i_j}$ is generated by a subset of $\{u_1, \dots, u_n\}$.

A basic and easy to prove property of strongly Koszul algebras is that any ideal generated by a subset of $u_1, \dots, u_n$ has a linear resolution. In particular, a strongly Koszul algebra is Koszul.

Let P be a finite poset, and let $u, v \in P$. One says that v *covers* u if $u < v$, and if there is no $w \in P$ such that $u < w < v$.

The poset is called *locally upper semimodular* (or *wonderful*) if whenever v_1 and v_2 cover u and $v_1, v_2 < v$ for some $v \in P$, then there is $w \in P$ with $w \leq v$ such that w covers each v_1 and v_2.

One has the following nice result, see [**8**] or [**10**, Theorem 5.1.12].

THEOREM 3.2 (Björner). *The order complex of a wonderful poset which has a least and greatest element is shellable.*

Now affine semigroup rings which are strongly Koszul can be characterized as follows

THEOREM 3.3. *Let $A = K[S]$ be a standard graded affine semigroup ring. Let $a_1, \dots, a_n$ be the generators of S and $u_i = x^{a_i}$, $i = 1, \dots, n$, be the corresponding generators of the K-algebra A. Then the following conditions are equivalent:*

(a) *A is strongly Koszul with respect to the sequence $u_1, \dots, u_n$,*
(b) *The divisor posets of S are wonderful.*

Together with the theorem of Björner it follows that a strongly Koszul algebra has shellable divisor posets.

It is an easy exercise to see that a K-algebra with quadratic monomial relations is strongly Koszul, thus providing a proof of 1.2. Furthermore tensor products, Segre products, fiber products and Veronese subrings of strongly Koszul algebras are strongly Koszul. However, the strongly Koszul property is rather strict and there are not so many examples. In [**2**] there is given a more flexible sequential condition, called *extendable sequentially Koszul*, and it is shown that the divisor posets whose affine semigroup rings are extendable sequentially Koszul are CL-shellable.

Koszul filtrations introduced by Conca, Valla and Trung [**15**] are a natural extension of the above concepts.

DEFINITION 3.4. Let A be a standard graded K-algebra. A family $\mathcal{F}$ of ideals is called a *Koszul filtration* if

(1) every ideal $I \in \mathcal{F}$ is generated by linear forms,
(2) $(0), \mathfrak{m} \in \mathcal{F}$,
(3) for every $I \in \mathcal{F}$, $I \neq 0$, there exists $J \subset I$, $J \in F$ such that I/J is cyclic and $J : I \in \mathcal{F}$.

For example if A is strongly Koszul with respect to the generators $u_1, \dots, u_n$, then $\mathcal{F} = \{I \colon I = (u_{i_1}, \dots, u_{i_k})\}$ is a Koszul filtration.

It is easy to see that all ideals $I \in \mathcal{F}$ have linear resolutions. There are two extreme cases of Koszul filtrations:

(1) A has a Gröbner flag, i.e. $\mathcal{F} = \{(0), (x_1), (x_1, x_2), \dots, (x_1, \dots, x_n) = \mathfrak{m}\}$;
(2) A is universally Koszul, i.e. $\mathcal{F}$ is the set of all ideals generated generated by linear forms.

Algebras with Gröbner flags can be characterized as follows

THEOREM 3.5 (Blum, Conca-Rossi-Valla). *The following conditions are equivalent:*

(a) *A has a Gröbner flag (with respect to the generators $x_1, \dots, x_n$);*
(b) *Let $<$ be the reverse lexicographical order induced by the total order $x_n > x_{n-1} > \cdots > x_1$. Then $\mathrm{in}_<(I)$ is generated by monomials of degree 2, and if $x_i x_j \in \mathrm{in}_<(I)$ with $i \leq j$, then $x_i x_k \in \mathrm{in}_<(I)$ for all k with $i \leq k \leq j$.*

A full classification of universally Koszul domains is given by Conca [**12**].

THEOREM 3.6. *Assume* $\operatorname{char} K = 0$*, and that the standard graded K-algebra A is a domain. Then the following conditions are equivalent:*

(a) *A is universally Koszul;*
(b) *A is isomorphic to either*
    (1) *a hypersurface ring defined by a quadric;*
    (2) *the coordinate ring of a rational normal scroll of type $(a_1, \dots, a_k)$ with $k = 1$ or $k = 2$ and $a_1 = a_2$;*
    (3) *the Veronese ring $K[x, y, z]^{(2)}$ (which is the coordinate ring of the Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^5$).*

4. Subrings of Koszul algebras

Let A be a standard graded K-algebra. Let B be a finitely generated K-subalgebra B of A. The K-subalgebra B is called a *pure subalgebra* of A if for every

A-module M, the natural map $M = M \otimes_B B \to M \otimes_B A$ is injective. For example, if B is a direct summand of A, then B is a pure subring.

In the sequel we will always assume that B is generated over K by homogeneous elements of A, and that B can be given again a standard grading.

QUESTION 4.1. Suppose that A is Koszul and B is a pure subring of A. Is then B again Koszul?

There are several special results which indicate that this might be true. The oldest result in this direction is [**7**]

THEOREM 4.2 (Barcanescu-Manolache). *Segre products of Veronese subrings of polynomial rings are Koszul.*

Later this result was generalized [**6**] as follows

THEOREM 4.3 (Backelin-Fröberg). *Veronese subrings and Segre products of Koszul algebras are Koszul.*

Let $R = \bigoplus_{(i,j)\in\mathbb{N}^2} R_{(i,j)}$ be a bigraded K-algebra. We say R is *standard graded* if $R_{(0,0)} = K$ and if it is generated as a K-algebra by $R_{(1,0)}$ and $R_{(0,1)}$.

The (c,e)-diagonal R_Δ of R is the pure subalgebra $\bigoplus_{s\in\mathbb{N}} R_{(sc,se)}$ of R. For example if A and B are standard graded K-algebras, then the (c,e)-diagonal of $A \otimes_K B$ is the Segre product of the dth Veronese subring $A^{(d)}$ of A and the eth Veronese subring $B^{(e)}$ of B.

The following theorem [**9**] generalizes all the above results

THEOREM 4.4 (Blum). *Let R be a standard bigraded Koszul algebra. Then every diagonal subalgebra of R is again Koszul.*

There is a rather strong version of pure subrings in the context of toric rings. Let $K[S] = K[x^{a_1},\dots,x^{a_m}] \subset K[x_1,\dots,x_n]$ be a semigroup ring, and let T be a subset of $\{1,\dots,n\}$. In [**26**] the algebra $K[S_T] = K[S] \cap K[\{x_i\colon i \in T\}]$ is called a *combinatorial pure subring of* $K[S]$, and it is shown that combinatorial pure subrings of Koszul algebras are Koszul.

Backelin [**5**] introduced the rate of a standard graded K-algebra. The rate measures how much the algebra deviates from being Koszul. Using this concept he proved

THEOREM 4.5 (Backelin). *Let R be a standard graded K-algebra. Then for $d \geq rate(R)$, the dth Veronese algebra $R^{(d)}$ is Koszul.*

So the theorem says that even if R is far from being Koszul, the Veronese subring $R^{(d)}$ is Koszul for $d \gg 0$. There are results of the same spirit, for example, the result by Eisenbud, Reeves and Totaro [**18**] which gives a bound c in terms of the regularity of A for which $A^{(d)}$ is Koszul for $d \geq c$.

Similar results hold for "large" diagonals [**13**].

THEOREM 4.6. *Let A be a bigraded K-algebra. Then there exist c_0 and e_0 (depending on the bigraded shifts of the free resolution A over its polynomial presentation) such that the (c,e)-diagonal algebra A_Δ is Koszul for all $c \geq c_0$ and $e \geq e_0$.*

This theorem has the following interesting consequence

COROLLARY 4.7. *Let I be a graded ideal in the polynomial ring $K[x_1,\dots,x_n]$, and let d be the highest degree of a generator of I. Then there exist integers c_0 and e_0 such that the K-subalgebra $K[(I^e)_{ed+c}]$ of $K[x_1,\dots,x_n]$ is Koszul for all $c \geq c_0$ and $e \geq e_0$.*

5. Koszul duals and Koszul modules

In this section I discuss extensions of a theorem of Eisenbud, Floystad and Schreyer [**17**], based on a joint paper with Iyengar [**22**].

Let K be field, V a finite dimensional K-vector space, $E = \bigwedge V$ the exterior algebra and M a finitely generated graded E-module. Let F be the graded minimal free resolution of M. The linear part F^{lin} of F is obtained from F by erasing all terms of degree > 1 from the matrices representing the differentials of F. Notice that F^{lin} is a complex.

THEOREM 5.1 (Eisenbud-Schreyer). *The linear part F^{lin} of F predominates eventually, i.e.*

$$H_i(F^{lin}) = 0 \quad \textit{for} \quad i \gg 0.$$

We define

$$\operatorname{lpd}(M) = \inf\{i\colon H_i(F^{lin}) \neq 0\}.$$

This is the smallest number from which on the linear part predominates.

There is no global bound for lpd. In fact, consider the free E-resolution F of K. This is known to be linear. Therefore $\operatorname{lpd}(K) = 0$. Since E is injective, dualizing into E yields an exact sequence

$$0 \longrightarrow K^* \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow F_2^* \longrightarrow \cdots$$

Suppose $\dim V = n$, then $K^* \cong K(-n)$. Thus, composing F and F^* we get the exact complex graded

$$\cdots \longrightarrow F_1(-n) \longrightarrow F_0(-n) \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow \cdots$$

Let M_i be the ith syzygy module of K. Then the truncated complex

$$\cdots \longrightarrow F_1(-n) \longrightarrow F_0(-n) \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow \cdots \longrightarrow F_{i-2}^* \longrightarrow F_{i-1}^* \longrightarrow 0$$

is a graded minimal free resolution of M_i^*. It follows that $\operatorname{lpd}(M_i^*) = i - 1$.

The question arises to what extend Theorem 5.1 is also true for other algebras. We will concentrate our attention to commutative rings and algebras. So let $(R, \mathfrak{m}, K)$ be either a local ring or a standard graded K-algebra with graded maximal ideal $\mathfrak{m}$, and M a finitely generated R-module (which should be graded if R is standard graded).

Let F be the (graded) minimal free resolution of M. The linear part can be described as follows: We define a filtration $\mathcal{F}$ of F, setting $\mathcal{F}_j F_i = \mathfrak{m}^{j-i} F_i$ for all i, j. Then F is a filtered complex and for the associated graded complex we have

$$\operatorname{gr}_{\mathcal{F}}(F) \cong F^{lin}.$$

If $\operatorname{lpd}(M) = 0$, then F^{lin} is a minimal (linear) graded A-resolution of $\operatorname{gr}_{\mathfrak{m}}(M)$, where $A = \operatorname{gr}_{\mathfrak{m}}(R)$.

DEFINITION 5.2. The R-module M is *Koszul* if $\operatorname{lpd}(M) = 0$, that is, if F^{lin} is acyclic.

For the ring R itself we say that R is Koszul if $K = R/\mathfrak{m}$ is a Koszul R-module. We notice that a standard graded K-algebra is Koszul in the classical sense of Section 1 if and only if it is Koszul in this new sense. Moreover, if R is Koszul, then $A = \operatorname{gr}_{\mathfrak{m}}(R)$ is Koszul as well.

Suppose $\operatorname{lpd}(K) < \infty$. In case R is standard graded, it is easy to see that this implies that the regularity $\operatorname{reg}(K)$ of K is finite. Now a theorem of Avramov and Peeva [**4**] implies that R is Koszul. Thus we have

THEOREM 5.3. *Let R be a standard graded K-algebra. The following conditions are equivalent:*

(a) $\operatorname{lpd}(K) < \infty$,
(b) *R is Koszul.*

In particular, if $\operatorname{lpd}(M) < \infty$ *for all graded R-modules, then R is Koszul.*

We do not know if the same result holds in the local case. We say that R is a *finite lpd-ring* if all finitely generated (graded) R-modules have finite lpd.

PROPOSITION 5.4. *Suppose* $\operatorname{lpd}(M) < \infty$. *Then M has a rational Poincaré series of the form*

$$P_M(t) = Q_M(t) + \frac{H_{\operatorname{gr}_{\mathfrak{m}}(M)}(-t)}{H_A(-t)},$$

where $Q_M(t)$ is a polynomial.

The proposition implies in particular that for finite lpd rings all finitely generated R-modules M have a rational Poincaré series with a denominator not depending on M but only on A.

EXAMPLE 5.5 (Roos). Let $R = K[x_1, x_2, x_3, x_4]/(x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_3x_4)$. Then R is Koszul by Theorem 1.2 of Fröberg. However, the graded R-module $M = R/(x_1 - x_3, x_1 - x_4)$ has no rational Poincaré series. In particular, R is not a finite lpd-ring.

For further discussions we need a characterization of Koszul algebras, due to Löfwall ([**25**]): $\operatorname{Ext}_R(K, K)$ with the Yoneda product is an associative, graded K-algebra. We denote by $A^\perp$ the K-subalgebra of $\operatorname{Ext}_R(K, K)$ generated by $\operatorname{Ext}_R^1(K, K)$. The following conditions are equivalent:

(a) R is Koszul;
(b) $\operatorname{Ext}_R(K, K) = \operatorname{Ext}_A(K, K)$ and $A^\perp = \operatorname{Ext}_A(K, K)$.

If A is Koszul, then $A^\perp$ is called the *Koszul dual of* A and one has $(A^\perp)^\perp = A$.

Let M be a finitely generated (graded) R-module, $x_1, \ldots, x_n$ a minimal system of generators of $\mathfrak{m}$, and $y_i \colon \mathfrak{m}/\mathfrak{m}^2 \to R/\mathfrak{m}$ the K-linear map with $y_i(x_j) = \delta_{ij}$. Then $y_1, \ldots, y_n$ is a K-basis of $(\mathfrak{m}/\mathfrak{m}^2)^* = \operatorname{Ext}_R^1(K, K)$, and hence a system of generators of $A^\perp$. There is a natural action

$$\operatorname{Ext}_R^1(K, K) \times \operatorname{Tor}_i^R(M, K) \to \operatorname{Tor}_{i-1}^R(M, K),$$

making $\operatorname{Tor}^R(M, K)$ a left $A^\perp$-module. We define a complex

$$L(M) = A \otimes_K \operatorname{Tor}^R(M, K)$$

of free A-modules with differential

$$A \otimes \operatorname{Tor}_i^R(M, K) \longrightarrow A \otimes \operatorname{Tor}_{i-1}^R(M, K), \quad 1 \otimes \xi \mapsto \sum_{i=1}^n x_i \otimes y_i \xi.$$

We have the following result (similar to Theorem 4.3 of Eisenbud, Floystad and Schreyer, see [**17**])

THEOREM 5.6. *Let F be a minimal free R-resolution of M. Then*

$$F^{lin} \cong L(M).$$

From this description of the linear part one deduces

COROLLARY 5.7. *For all i and j one has*

$$H_i(F^{lin})_j \cong \operatorname{Ext}_{A^\perp}^{j-i}(\operatorname{Ext}_A(M, K), K))_j.$$

Let N be a graded $A^{\perp}$-module (for example, $\operatorname{Ext}_A(M,K)$). Then

$$\operatorname{reg}_{A^{\perp}}(N) = \sup\{j\colon \operatorname{Ext}^i_{A^{\perp}}(N,K)_{-i-j} \neq 0 \text{ } quad\text{for some integer } i\}.$$

Thus Corollary 5.7 implies

COROLLARY 5.8. $\operatorname{lpd}(M) = \operatorname{reg}_{A^{\perp}}(\operatorname{Ext}_A(M,K))$.

THEOREM 5.9 (Backelin, Roos). *Let S be a local complete intersection, $S \to R$ a Golod homomorphism and M a finitely generated R-module. Then*

$\operatorname{Ext}_{\operatorname{Ext}_R(K,K)}(\operatorname{Ext}_R(M,K),K)$ *is a finitely generated* $\operatorname{Ext}_{\operatorname{Ext}_R(K,K)}(K,K)$*-module.*

Assume that in addition to the hypotheses of 5.9, R is a Koszul ring. Then $\operatorname{Ext}_{A^{\perp}}(\operatorname{Ext}_A(M,K),K)$ is a finitely generated A-module, and hence

$$\operatorname{reg}_{A^{\perp}} \operatorname{Ext}_A(M,K) < \infty, \quad \text{and} \quad \operatorname{lpd}_R(M) < \infty.$$

Therefore we get

COROLLARY 5.10. *Koszul rings which are Golod quotients of complete intersections are finite lpd rings.*

5.1. Global bounds. Consider the K-algebra $A = K[x_1,\dots,x_n]/(x_1^2,\dots,x_n^2)$. Then A is a finite lpd ring. However, the same argument as in the case of an exterior algebra (cf. Section 3) shows that there is no global bound for linear dominance.

Let A be a standard graded K-algebra. We set

$$\operatorname{gllpd}(A) := \sup\{\operatorname{lpd}_A(M)\colon M \text{ is a finitely generated graded } A\text{-module}\}.$$

Similarly we define $\operatorname{gllpd}(R)$ for a local ring R.

The question is whether there exist algebras with finite global lpd. It would be interesting to have full classification of such algebras.

Suppose that K is infinite, A is Cohen-Macaulay and Koszul. Then it is easily seen that A is Golod if and only if A has minimal multiplicity, or equivalently, $\mathfrak{m}^2 = (x_1,\dots,x_d)\mathfrak{m}$ for any system of parameters of $\mathbf{x} = x_1,\dots,x_d \in A_1$. In other words, if $\bar{\mathfrak{m}}$ is the graded maximal ideal of $\bar{A} = A/(\mathbf{x})A$, then $\bar{\mathfrak{m}}^2 = 0$. From this one concludes that if M is the first syzygy module of a maximal graded Cohen-Macaulay A-module, then $\mathfrak{m}(M/(\mathbf{x})M) = 0$. In particular, $M/(\mathbf{x})M$ is isomorphic to a direct sum of copies of $K(a)$, and so has finite lpd over $\bar{A}$. This property can be lifted to M. Hence, since every $(d+1)$th syzygy module of some graded A-module is a first syzygy module of graded maximal Cohen-Macaulay A module, we have

THEOREM 5.11. *Let A be a standard graded Cohen-Macaulay Koszul K-algebra. If A is Golod, then* $\operatorname{gllpd} A \le \dim A + 1$.

If we skip the Cohen-Macaulay hypothesis we still get a global, but slightly weaker bound for lpd. The precise statement is

THEOREM 5.12. *Let R be local ring which is Koszul and Golod. Then*

$$\operatorname{gllpd} R \le 2 \operatorname{embdim} R.$$

Among the Gorenstein Koszul algebras only the hypersurface rings have finite global lpd.

THEOREM 5.13. *Let A be a standard graded Gorenstein Koszul k-algebra. Then the following conditions are equivalent:*

(1) $\operatorname{gllpd} A < \infty$;
(2) *A is a hypersurface ring.*

References

[1] A. Aramova, J. Herzog and T. Hibi, *Finite lattices and lexicographic Grbner bases*, European J. Combin. **21** (2000), no. 4, 431–439.

[2] A. Aramova, J. Herzog and T. Hibi, *Shellability of semigroup rings*, to appear in Nagoya J. Math. 2002.

[3] L. L. Avramov, *Infinite free resolutions*, Six Lectures on Commutative Algebra; Barcelona 1997 (J. Elias, J.M. Giral, R.M. Miró-Roig, S. Zarzuela, eds.), Progress in Mathematics **166**, Birkhäuser, 1998, pp. 1–118.

[4] L. L. Avramov, I. Peeva, *Finite regularity and Koszul algebras*, Amer. J. Math. **123** (2001), 275–281.

[5] J. Backelin, *On the rate of growth of the homologies of Veronese subrings*, Algebra, Algebraic Topology and their Interactions (J.E. Roos, ed.), Springer Lecture notes in Math. **1183**, Springer 1986, pp. 79–100.

[6] J. Backelin, R. Fröberg, *Koszul algebras, Veronese subrings, and rings with linear resolutions*, Rev. Roumaine Math. Pures Appl. **30** (1985), 549–565.

[7] S. Barcanescu, N. Manolache, *Betti-numbers of Segre-Veronese singularities*, Rev. Roumaine Math. Pures Appl. **26** (1982), 549–565.

[8] A. Björner, *Shellable and Cohen-Macaulay partially ordered sets*, Trans. Amer. Math. Soc. **269** (1980), 159–183.

[9] S. Blum, *Initially Koszul algebras*, Beiträge Alg. Geom. **41** (2000), no. 2, 455–467.

[10] W. Bruns, J. Herzog, *Cohen-Macaulay Rings*, Cambridge Stud. Adv. Math., no. 39, Revised version, Cambridge University Press, 1998.

[11] J. Backelin, J.-E. Roos, *When is the double Yoneda Ext-algebra of a local Noetherian ring again Noetherian?*, Algebra, algebraic topology and their interactions; Stockholm, 1983 (J.-E. Roos, ed.), Lecture Notes Math., no. 1183, Springer, Berlin, 1986, pp. 101–119.

[12] A. Conca, *Universally Koszul algebras*, Math. Ann. **317** (2000), no. 2, 329–346.

[13] A. Conca, J. Herzog, N.V. Trung, G. Valla, *Diagonal subalgebrs of bigraded algebras and embeddings of blow-ups of projective spaces*, Amer. J. Math. **119** (1997), 859–901.

[14] A. Conca, M.E. Rossi, G. Valla, *Gröbner flags and Gorenstein Algebras*, Compositio Math. **129** (2001), 95–121.

[15] A. Conca, N.V. Trung, G. Valla, *Koszul property for points in projective spaces*, Math. Scand. **89** (2001), no. 2, 201–216.

[16] D. Eisenbud, *Commutative Algebra with a view towards algebraic geometry*, Graduate Texts Math. no. 150, Springer, 1995.

[17] D. Eisenbud, G. Floystad, F.-O. Schreyer, *Sheaf cohomology and resolutions over the exterior algebra*, preprint 2001.

[18] D. Eisenbud, A. Reeves, B. Totaro, *Initial ideals, Veronese subrings, and rates of algebras*, Adv. Math. **109** (1994), 168–187.

[19] R. Fröberg, *Determination of a class of Poincaré series*, Math. Scand. **37** (1975), 29–39.

[20] J. Herzog, T. Hibi, *Componentwise linear ideals*, Nagoya Math. J. **153** (1999), 141–153.

[21] J. Herzog, T. Hibi, G. Restuccia, *Strongly Koszul algebras*, Math. Scand. **86** (2000), no. 2, 161–178.

[22] J. Herzog, S. Iyengar, *Koszul modules*, work in progress.

[23] J. Herzog, V. Reiner, V. Welker, *The Koszul property in affine semigroup rings*, Pacific J. Math. **186** (1998), no. 1, 39–65.

[24] O.A. Laudal, A. Sletsjøe, *Betti numbers of monoid algebras. Applications to 2-dimensional torus embeddings*, Math. Scand. **56** (1985), 145–162.

[25] C. Löfwall, *On the subalgebra generated by the one-dimensional elements in the Yoneda Ext-algebra*, Algebra, algebraic topology and their interactions; Stockholm, 1983 (J.-E. Roos, ed.), Lecture Notes Math. no. 1183, Springer, Berlin, 1986, pp. 291–338.

[26] H. Ohsugi, J. Herzog, T. Hibi, *Combinatorial pure subrings*, Osaka J. Math. **37** (2000), 745–757.

[27] H. Ohsugi, T. Hibi, *Toric Ideals generated by Quadratic Binomials*, J. Alg. **218** (1999), 509–527.

[28] I. Peeva, V. Reiner, B. Sturmfels, *How to shell a monoid*, Math. Ann. **310** (1998), no. 2, 379–393.

[29] S. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39–60.

[30] J.-E. Roos, *Relations between Poincaré-Betti series of loop spaces and of local rings*, Séminaire d'Algèbre Paul Dubreil; Paris, 1977–78 (M.-P. Malliavin, ed.), Lecture Notes in Math. no. 740, Springer, Berlin, 1979; pp. 285–322.

[31] J.-E. Roos, *Sur l'algèbre Ext de Yoneda d'un anneau local de Golod*, C. R. Acad. Sci. Paris Sér. A-B 286 (1978), A9–A12.

[32] T. Römer, *On minimal graded free resolutions*, Dissertation, Essen 2001.

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