SURFACES WITH $p_g = 0$

MARGARIDA MENDES LOPES

1. Foreword

The present text is the written version (slightly expanded) of a course given in February 2003 at the institute "Simon Stoilow" of the Romanian Academy, under the program Eurrommat. The purpose of this course was explaining a set of recent classification results for surfaces of general type with $p_g = q = 0$, obtained by considering the bicanonical system, and this text is focused on these results. As such it omits, a fortiori, many of the known facts about this type of surfaces, and so neither this text neither the bibliography should be regarded as an exhaustive survey on surfaces of general type with $p_g = 0$.

I want to thank the direction of the Institute "Simon Stoilow" for the invitation to give this course and for the warm hospitality.

2. Introduction

A rational surface, i.e. a surface birationally equivalent to $\mathbb{P}^2$, satisfies $p_g = q = 0$ and Kodaira dimension $\kappa = -\infty$. Castelnuovo, around 1894, started studying algebraic surfaces with $p_g = q = 0$, conjecturing that any such surface was rational. This conjecture was of course inspired on fact that, for non-singular curves C, $p_g = 0$ implies that C is isomorphic to $\mathbb{P}^1$.

However the conjecture was soon shown to be false with the discovery by Enriques of the first example (nowadays appropriately called Enriques surface) of a non rational surface satisfying $p_g = q = 0$. In 1896 Castelnuovo proved that the conditions $p_2 = q = 0$ characterize completely the rational surfaces and found other examples of non rational surfaces satisfying $p_g = q = 0$. Enriques' examples have Kodaira dimension 0 whilst Castelnuovo's have Kodaira dimension 1.

It was only in 1931 that the first example of a surface of general type (i.e. with Kodaira dimension 2) and $p_g = 0$ appeared. This example was constructed by Lucien Godeaux and has $K^2 = 1$. Almost at the same time Luigi Campedelli obtained other examples of surfaces of general type with $p_g = 0$ using double covers of $\mathbb{P}^2$. Godeaux constructed

2000 Mathematics Subject Classification: 14J29.

his example as the quotient of the Fermat quintic in $\mathbb{P}^3$ by the free action of a group of automorphisms of order 5 whilst Campedelli's surfaces which satisfy $K^2 = 1, 2$ were constructed as the desingularization of double covers of $\mathbb{P}^2$ branched on a curve of degree 10 with certain singularities.

Nowadays many more examples of surfaces of general type with $p_g = q = 0$ are known but a detailed classification is still lacking, despite much progress in the theory of algebraic surfaces.

In here some classification results for these surfaces, obtained by considering the bicanonical map, are presented.

3. Preliminaries

A *surface* S is an irreducible algebraic projective surface over $\mathbb{C}$ and usually non-singular, unless otherwise specified.

Remark. A complex irreducible non-singular projective surface can also be characterized as a *complex connected compact manifold of (complex) dimension* 2 *which can be embedded in some* $\mathbb{P}^n$.

Let us remark that contrarily to what happens in dimension 1, where it is well known that every compact connected complex manifold of dimension 1 is an projective algebraic curve, not every compact connected complex surface is algebraic. (see, e.g., [BPV]).

Important sheaves on a surface S

$\mathcal{O}_S$= the sheaf of regular functions on S;
Ω^1_S = the sheaf of regular 1-forms;
$\omega_S = \mathcal{O}_S(K_S)$= the sheaf of regular $2-$forms, i.e. the *canonical sheaf*;
$\omega_S^{\otimes m} = \mathcal{O}_S(mK_S)$ the m-canonical sheaf.

Notation As it is usual, $h^i(S, \mathcal{F})$ denotes $\dim_{\mathbb{C}} H^i(S, \mathcal{F})$, for a sheaf $\mathcal{F}$. There will be no distinction made between line bundles, invertible sheaves and divisors and additive and multiplicative notation will be used interchangeably. Numerical equivalence betwween divisors is denoted by $\sim$ and linear equivalence by $\equiv$. A *surface* S is an irreducible algebraic projective surface over $\mathbb{C}$ and usually non-singular, unless otherwise specified. A *curve* will be an effective non zero divisor on a surface.

A divisor on a surface is *nef* if $DC \geq 0$ for any irreducible curve C on S. A divisor is nef and *big* if it is nef and $D^2 > 0$.

Kodaira dimension $\kappa(S)$

An important invariant of S is the Kodaira dimension $\kappa(S)$, which can be defined as being the maximum of the dimensions of the image of S by the maps φ_{mK_S}, (here by convention $\kappa(S) = -\infty$ if $h^0(S, mK_S) = \{0\}, \forall m$).

Surfaces of general type

A surface S is of *general type* if it has maximal Kodaira dimension 2. For a surface of general type one has always $p_g \geq q$ and so $\chi(\mathcal{O}_S) \geq 1$.

Remark. The notion of Kodaira dimension and of surface of general type can be defined for any connected complex compact surface. Any connected complex compact surface of general type is necessarily algebraic (see [BPV]).

Numbers

Associated to the sheaves a surface S carries, one obtains various numbers which are important for the characterization of S.

- $p_g := h^0(S, \omega_S)$ the *geometrical genus* of S
- $p_m := h^0(S, \omega_S^{\otimes m})$ the *m-th plurigenus* of S;
- $q := h^0(S, \Omega^1_S) = h^1(S, \mathcal{O}_S)$ the *irregularity* of S.
- $\chi(\mathcal{O}_S) := 1 - q + p_g$ the *characteristic* of $\mathcal{O}_S$;
- K_S^2 the *self intersection* of a canonical divisor;
- c_2 the *Euler characteristic* of S.

Recall that whilst p_g, p_m, q and $\chi(\mathcal{O}_S)$ are birational invariants, K_S^2 and c_2 are not.

Relations between invariants for surfaces of general type

- $c_2(S) = 2 - 2b_1 + b_2$ (b_i are the Betti numbers of S);
- $b_1 = 2q$ (from Hodge theory);
- $K_S^2 + c_2(S) = 12\chi(\mathcal{O}_S)$ (Noether's formula).
- $K_S^2 \leq 3c_2$ (Miyaoka-Yau inequality);
- $K_S^2 \geq 2p_g - 4$ (Noether's inequality, (see, e.g., [BPV])).

Remark. Noether's inequality can also be expressed

- $K_S^2 \geq 2\chi(\mathcal{O}_S) - 6$, for K_S^2 even;
- $K_S^2 \geq 2\chi(\mathcal{O}_S) - 5$ for K_S^2 odd.

There are several other properties for invariants of specific types of surfaces. Here we mention two that will be used later:

Proposition 3.1 ([De], Theorem 6.1)**.** *Any irregular minimal surface of general type S satisfies the inequality $K_S^2 \geq 2p_g$.*

Proposition 3.2 ([Be1], Corollary 5.8)**.** *A minimal surface S of general type such that $K_S^2 \leq 2\chi(\mathcal{O}_S)$ and $q = 0$ has no irregular étale covers.*

We recall also:

Theorem 3.3 ([X1])**.** *Let S be a minimal surface of general type. Then $|2K_S|$ is not composed with a pencil (i.e. the image of S via its bicanonical map is a surface).*

Surfaces have been studied since the XIXth century and, as such, there is a considerable number of tools that have been developed. Amongst these we have the:

Theorem 3.4 (Algebraic index theorem or Hodge index theorem)**.** *(see, e.g.,* [BPV]*).*

Let D, E be $\mathbb{Q}-$divisors on the surface S. If $D^2 > 0$ and $DE = 0$ then $E^2 \leq 0$ and $E^2 = 0$ if and only if E is homologous to 0 in rational homology.

The following formulation of Hodge index theorem is extremely useful.

Corollary 3.5. *Let S be a surface and D a $\mathbb{Q}-$divisor such that $D^2 > 0$. Then for any decomposition of D as $D = A + B$ with $A, B, \mathbb{Q}-$divisors, $A^2B^2 - (AB)^2 \leq 0$ and if equality holds then there exist $m, n \in \mathbb{Q}$ such that mA is homologous to nB.*

Proof. Write $D = A + B$ with $A, B\mathbb{Q}-$divisors, and let $p = DA, q = DB$. Then $D(qA - pB) = 0$ and therefore by Hodge index theorem either qA is homologous to pB and $(qA - pB)^2 = 0$ or $(qA - pB)^2 < 0$. Now since $q = B^2 + AB$ and $p = A^2 + AB$, $(qA - pB)^2 = q^2A^2 - 2pqAB + p^2B^2 = (B^2)^2A^2 + 2(AB)A^2B^2 + (AB)^2A^2 - 2(A^2B^2 + A^2(AB) + B^2(AB) + (AB)^2)(AB) + (A^2)^2B^2 + 2(AB)A^2B^2 + (AB)^2B^2 = (B^2)^2A^{+}2(AB)A^2B^2 - (AB)^2A^2 - (AB)^2B^2 - 2(AB)^3 = B^2(A^2B^2 - (AB)^2) + 2(AB)(A^2B^2 - (AB)^2) + A^2(A^2B^2 - (AB)^2) = (A^2 + 2AB + B^2)(A^2B^2 - (AB)^2) = D^2(A^2B^2 - (AB)^2)$.

So the inequality $(qA - pB)^2 \leq 0$ can be written as $D^2(A^2B^2 - (AB)^2) \leq 0$ and therefore because $D^2 > 0$, we obtain the statement. □

We recall now a notion that was first introduced by Franchetta and much used by Bombieri (see [Bo]). Recall that by a *curve* we mean an effective non zero divisor on S.

Definition Let D be a curve on a surface. The curve D is *m-connected* if $AB \geq m$ for any decomposition $D = A + B$ as the sum of two curves.

Proposition 3.6. *Let D be a curve on a surface S such that $D^2 \geq 1$ and D is nef. Then:*

(i) every $D' \in |D|$ is 1-connected.

(ii) If $D' = A + B$ is a decomposition of D with A, B curves such that $A \cdot B = 1$ then only the following possibilities can occur:

- (p_1) $A^2 = -1$ *or* $B^2 = -1$
- (p_2) $A^2 = 0$ *or* $B^2 = 0$
- (p_3) $A^2 = B^2 = 1$, $A \sim B$ *and* $D^2 = 4$.

(iii) If $D' = A + B$ is a decomposition of M with A, B curves such that $A \cdot B = 2$ then only the following possibilities can occur:

- (q_1) $A^2 = -2$ *or* $B^2 = -2$
- (q_2) $A^2 = -1$ *or* $B^2 = -1$
- (q_3) $A^2 = 0$ *or* $B^2 = 0$
- (q_4) $1 \leq A^2$ *and* $1 \leq B^2$ *and* $D'^2 \leq 9$

Furthermore in case (q_4) only the following cases occur:

	A^2	B^2	D'^2	
(C_1)	1	1	6	
(C_2)	1	2	7	
(C_3)	1	3	8	
(C_4)	1	4	9	$2A \sim B$
(C_5)	2	2	8	$A \sim B$

Proof. Suppose we have a decomposition $M = A + B$ with $A \cdot B \leq 0$. Since $M^2 > 0$ and M is nef we must have, say, $A^2 \geq -AB \geq 0$ and $B^2 > -AB \geq 0$. By the index theorem the only possibility would be $A^2 = A \cdot B = 0$ giving that $A \sim B$ (mod $\mathbb{Q}$). But this contradicts $D^2 > 0$ and so D' is 1-connected.

The remainder of the proof follows also by application of the index theorem, using the hypothesis that D is nef. □

The following corollary of Hodge index theorem is also useful.

Proposition 3.7. *Let S be an algebraic surface and let μ be either an element of* $\mathrm{Pic}^0(S) - \{0\}$ *or a non trivial torsion invertible sheaf and let C be a curve on S such that $C^2 > 0$. Then $\mu_{|C}$ is non-trivial.*

Proof. We argue by contradiction. Suppose that μ is a non trivial torsion invertible sheaf such that $\mu_{|C} \equiv \mathcal{O}_C$. Let $\pi : S' \to S$ be the irreducible étale cover such that $\pi^*(\mu) \simeq \mathcal{O}_{S'}$, whose degree we denote by n. Then $\pi^*(C)$ is the disjoint union of n curves $C_1, ..., C_n$ such that $C_i^2 = C^2$ for all $i = 1, ..., n$. Since $C_i \cdot C_j = 0$ for all $i \neq j$, we find a contradiction to the Hodge index theorem.

To prove the statement for $\mu \in \mathrm{Pic}^0(S) - \{0\}$ it suffices to remark that if $\mu_{|C} \equiv \mathcal{O}_C$ then the kernel of the restriction map

$$r : \mathrm{Pic}^0(S) \to \mathrm{Pic}^0(C)$$

is not zero. So there exists a non-zero torsion element ν in $\mathrm{Ker}(r)$ and we can exclude that $\mu_{|C} \equiv \mathcal{O}_C$ as in the preceding paragraph. □

We want also to recall Reider's theorem.

Proposition 3.8 ([Re], Theorem 1)**.** *Let S be a surface and L a nef divisor on S. Then, if $L^2 \geq 5$ and $P \in S$ is a base point of $|K_S + L|$, then there exists an effective divisor E passing through P such that either $LE = 0$, $E^2 = -1$ or $LE = 1$, $E^2 = 0$. If $L^2 \geq 10$ and two points $P, Q \in S$ are not separated by $|K_S + L|$, then there exists an effective divisor E passing through P and Q such that either $LE = 0$ and $E^2 = -1$ or $E^2 = -2$, or $LE = 1$ and $E^2 = -1$ or $E^2 = 0$, or $LE = 2$ and $E^2 = 0$.*

The following variant of Reider's theorem is sometimes very useful.

Proposition 3.9 ([BS])**.** *Let L be a nef divisor on a surface S. Assume that $L^2 \geq 4k + 1$. Given any 0-dimensional scheme Z of length k on S, then either the natural restriction map*

$$H^0(S, \mathcal{O}_S(K + L)) \to H^0(S, \mathcal{O}_Z(K + L))$$

is surjective, or there exist an effective divisor D on S and a not empty subscheme Z' of Z of length $k' \leq k$, such that:
(i) the map

$$H^0(S, \mathcal{O}_S(K + L)) \to H^0(S, \mathcal{O}_{Z'}(K + L))$$

is not surjective;
(ii) Z' is contained in D and there is an integer m such that $m(L-2D)$ is effective;
(iii) one has

$$L \cdot D - k' \leq D^2 < \frac{L \cdot D}{2} < k'$$

If $k = 1$ then either $L \cdot D = 0$ and $D^2 = -1$ or $L \cdot D = 1$ and $D^2 = 0$ and in either case D is 1-connected.

Finally we also recall:

Theorem 3.10 ([Mi]). *Let S be a smooth surface such that K_S is nef and let $C_1, ..., C_r$ be disjoint rational irreducible curves with self-intersection numbers $-n_1, ..., -n_r$, $(n_j \geq 2)$. Then*

$$\sum \frac{(n_j+1)^2}{n_j} \leq 3c_2(S) - K_S^2.$$

4. Covers

An important tool for the construction of examples is the use of double covers or $\mathbb{Z}_2 \times \mathbb{Z}_2$ covers. Here we recall the main facts about these covers. For more details see, e.g., [BPV], [Pa1], [Ca].

4.1. **Double covers.** Let S be a smooth complex surface, $D \subset S$ a curve (possibly empty) with at worst ordinary double points, and M a line bundle on S with $2M \equiv D$. It is well known that there exists a normal surface Y and a finite degree 2 map $\pi \colon Y \to S$ branched over D such that $\pi_* \mathcal{O}_Y = \mathcal{O}_S \oplus M^{-1}$. The singularities of Y are A_1 points and occur precisely above the singular points of D; thus it makes sense to speak of the canonical divisor, the geometric genus, the irregularity and the Albanese map of Y. The surface Y is the *double cover defined by the relation* $2M \equiv D$. One has $K_Y = \pi^*(K_S + M)$ and the invariants of Y are:

$$\begin{aligned} K_Y^2 &= 2(K_S+M)^2; \\ \chi(\mathcal{O}_Y) &= 2\chi(\mathcal{O}_S) + \frac{1}{2}M(K_S+M); \\ p_g(Y) &= p_g(S) + h^0(S, K_S+M). \end{aligned} \tag{4.1}$$

4.2. $\mathbb{Z}_2 \times \mathbb{Z}_2$**- covers.** Similarly, given a smooth surface S and three effective smooth divisors D_1, D_2, D_3, such that $D := D_1 + D_2 + D_3$ is a normal crossing divisor, and two line bundles L_1, L_2 satisfying $2L_1 \equiv D_2 + D_3$, $2L_2 \equiv D_1 + D_3$, there exists a smooth surface Y and a finite degree 4 map, with Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\pi \colon Y \to S$.

Set $L_3 := L_1 + L_2 - D_3$. Then $2L_3 = D_1 + D_2$. One has $\pi_*(\mathcal{O}_Y) = \mathcal{O}_S \oplus L_1^{-1} \oplus L_2^{-1} \oplus L_3^{-1}$ and $2K_Y = \pi^*(2K_S + D)$. In particular $4K_Y^2 = 4(2K_S + D)^2$. The invariants of Y are

$$\begin{aligned} K_Y^2 &= (2K_S+D)^2; \\ \chi(\mathcal{O}_Y) &= 4\chi(\mathcal{O}_S) + \sum \frac{1}{2}L_i(K_S+L_i); \\ p_g(Y) &= p_g(S) + \sum h^0(S, K_S+L_i). \end{aligned} \tag{4.2}$$

5. Properties of surfaces of general type with $p_g = 0$

Let S be a smooth *minimal* projective surface of general type with $p_g = 0$ over $\mathbb{C}$. Then

- $q = 0$ and $\chi(\mathcal{O}_S) = 1$;
- $1 \leq K_S^2 \leq 9$ (comes from Noether's formula and Miyaoka-Yau's inequality);
- $p_2 :=$dim $H^0(S, \mathcal{O}_S(2K_S)) = K_S^2 + 1$;
- $b_2 = \rho(S) = 10 - K_S^2$, ($b_2 := \mathrm{rk} H^2(S, \mathbb{Z})$ and $\rho(S) :=$rankPic(S)).

Proposition 5.1. *Let S be a minimal surface of general type with $p_g = q = 0$. If $K_S^2 \geq 3$, S has no fibration $f : S \to \mathbb{P}^1$ with fibres of genus 2.*

Proof. Assume otherwise. Horikawa proved in [Ho] that given an algebraic surface S with a genus 2 fibration (i.e. a proper surjective morphism $f : S \to B$ with B an algebraic curve of genus b, for which the general fibre is a smooth irreducible genus 2 curve), then, if f is relatively minimal,

$$K_S^2 = 2\chi(\mathcal{O}_S) - 6 + 6b + \sum \nu_i$$

where b is the genus of the curve b and ν_i is a non-negative number associated to each fibre F_i of f which is 1-connected and not 2-connected. For one such fibre denote by l_i the number of irreducible components of F_i. It turns out that for each such fibre one has $\nu_i \leq (l_i - 1)$ (see [Ho] for a precise description of the fibres).

On the other hand it is well known that $\sum(l_i - 1) \leq \rho(S) - 2$, where $\rho(S)$ is the rank of the Picard group of S.

For a surface S with $p_g(S) = q(S) = 0$ one has $\rho(S) = b_2(S) = 10 - K_S^2$, and Horikawa's formula yields

$$\sum \nu_i = K_S^2 + 4 \leq \sum (l_i - 1) \leq \rho(S) - 2 = 8 - K_S^2.$$

Hence $2K_S^2 \leq 4$, i.e., $K_S^2 \leq 2$. □

If $p_g(S) = q(S) = 0$, the existence of a double cover $\pi \colon Y \to S$ with $q(Y) > 0$ forces the existence of a fibration $f \colon S \to \mathbb{P}^1$ such that π^{-1} of the general fibre of f is disconnected. More precisely we have:

Theorem 5.2 (De Franchis)**.** *Let S be a smooth surface with $p_g(S) = q(S) = 0$ and $\pi \colon Y \to S$ a double cover with at most A_1 points; if $q(Y) > 0$, then*

(i) *the Albanese image of Y is a curve B;*

(ii) *let* $\alpha\colon Y \to B$ *be the Albanese fibration. Then there exists a fibration* $g\colon S \to \mathbb{P}^1$ *and a degree* 2 *map* $p\colon B \to \mathbb{P}^1$ *such that* $p \circ \alpha = g \circ \pi$.

Proof. Since $q(S) = 0$, the involution $\sigma\colon Y \to Y$ induced by π acts on the Albanese variety of Y as multiplication by -1. For any $\eta_1, \eta_2 \in H^0(Y, \mathrm{Hom}^1_Y)$, $\theta = \eta_1 \wedge \eta_2$ is a global 2-form on Y invariant under σ, and so it induces an element $\theta' \in H^0(S, K_S)$. Since $p_g(S) = 0$, θ' vanishes identically, and hence so does θ. Therefore the Albanese image of Y is a curve B. The involution σ acts on Y and on B in a compatible way, and thus the fibration $\alpha\colon Y \to B$ induces a fibration $g\colon S \to B/\langle\sigma\rangle$. Finally, the quotient curve $B/\langle\sigma\rangle$ is isomorphic to $\mathbb{P}^1$, since $q(S) = 0$. □

After constructing such a double cover, we can sometimes reach a contradiction either by showing that the restriction of M to the general fibre of the pencil $g\colon S \to \mathbb{P}^1$ is nontrivial, and therefore the inverse image via π of a general fibre of f is connected, or by using the following:

Corollary 5.3. *Let* S *be a minimal surface of general type with* $p_g(S) = q(S) = 0$ *and* $K_S^2 \geq 3$, *and* $\pi\colon Y \to S$ *a double cover with at most* A_1 *points. Then* $K_Y^2 \geq 16(q(Y) - 1)$.

Proof. The statement holds trivially if $q(Y) \leq 1$, so we assume that $q(Y) \geq 2$. By theorem 5.2, the Albanese map of Y is a pencil $\alpha\colon Y \to B$, and there exists $g\colon S \to \mathbb{P}^1$ such that $g \circ \pi$ is composed with α. If f is the genus of a smooth fibre of α (and thus of g), then $K_Y^2 \geq 8(q(Y) - 1)(f - 1)$ by [Be2], p. 344. If the inequality in the statement does not hold, then $f \leq 2$. Since S is of general type, we must have $f = 2$. On the other hand, by proposition 5.1, a surface of general type S with $p_g(S) = 0$, $K^2 \geq 3$ has no genus 2 pencil and thus we have a contradiction. □

Finally, one can also exploit this construction to show the existence of a fibration of S with multiple fibres:

Remark 5.4. *Let* S *be a smooth surface and* $\pi\colon Y \to S$ *a smooth double cover; suppose that* $g\colon S \to \mathbb{P}^1$ *is a fibration such that the general fibre of* $g \circ \pi$ *is not connected, so that there is a commutative diagram:*

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & S \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{\pi} & \mathbb{P}^1 \end{array} \tag{5.1}$$

where B is a smooth curve of genus b and $\overline{\pi}$ a double cover; by commutativity of the diagram, the double cover $\pi\colon Y \to S$ is obtained from $\overline{\pi}$ by base change and normalization. Thus the image via g of the branch locus of π is a finite set of cardinality, say, k, contained in the branch locus of $\overline{\pi}$. It follows that at least $2b+2-k$ fibres of g are divisible by 2. In particular, if π is unramified, then g has at least $2b+2$ fibres that are divisible by 2.

6. The proof of Theorem 3.3 for surfaces with $p_g = 0$

Here we give an alternative proof of theorem 3.3 for the case $p_g = 0$. Note that for $p_g \geq 1$ the theorem is a consequence of the fact that $|2K_S|$ is base point free (see [Ci1] for the appropriate references), which was however only completely proved after Xiao Gang's presented his proof of theorem 3.3.

Theorem 6.1. *Let S be a minimal surface of general type with $p_g = 0$. Then $|2K_S|$ is not composed with a pencil (i.e. the image of S via its bicanonical map is a surface) if and only if $K_S^2 > 1$.*

Proof. Since $\chi(\mathcal{O}_S) = 1$, one has $h^0(S, 2K_S) = K_S^2 + 1$ and therefore if $K_S^2 = 1$, $|2K_S|$ is a pencil.

Suppose now that $K_S^2 > 1$ and that $|2K_S|$ is composed with a pencil with general fibre F. Since $q = 0$, $|F|$ is a rational pencil and therefore we can write $2K_S \equiv dF + Z$, where $d = K_S^2$ and Z is an effective divisor possibly 0.

Now, because K_S is nef we have $2K_S^2 \equiv dK_SF + KZ \geq dK_SF$ and therefore $K_SF \leq 2$. By the index theorem we have $K_S^2F^2 \leq (K_SF)^2$ where if equality holds then for some $a, b \in \mathbb{Q}$, $aK_S \sim bF$. Also by the adjunction formula K_SF and F^2 have the same parity. Since we are assuming $K_S^2 > 1$ we obtain the following the numerical possibilities:

i) $K_SF = 2$, $F^2 = 2$ and $K_S \sim F$;

ii) $K_SF = 2$, $F^2 = 0$.

We start by showing that case i) does not occur. Suppose otherwise. Then we have $2K_S \equiv 2F$ and $\eta := K_S - F$ is a 2-torsion divisor. The étale double cover $p : Y \to S$ associated to η satisfies $\chi(\mathcal{O}_Y) = 2$ and $p_g(Y) = h^0(S, K_S) + h^0(S, K_S + \eta) = h^0(S, K_S) + h^0(S, F) = 2$. So Y is irregular, contradicting proposition 3.2. So case i) is excluded.

For case ii) notice that, anyway, $K_S^2 = 2$, because in this case the pencil $|F|$ is a genus 2 fibration and therefore $K_S^2 < 3$, by proposition 5.1.

Then one has $2K_S = 2F + Z$, where $Z > 0$ satisfies $Z^2 = -8, KZ = 0$ and Z is made-up of -2-curves. We then can write $Z = 2Z_0 + Z_1$ where

Z_1 is a reduced effective divisor. If $Z_1 = 0$, then $\eta := K_S - F - Z_0$ is a 2-torsion divisor and the same argument as above leads again to a contradiction. Suppose now that $Z_1 \neq 0$. Since $Z_1 = 2(K_S - F - Z_0)$, θZ_1 is even for any irreducible component θ of Z_1. On the other hand the dual graph of the configuration of curves in Z_1 is a union of trees and thus, because Z_1 is reduced, necessarily Z_1 is a disjoint union of p irreducible -2-curves $\theta_1, ..., \theta_p$. We can then consider the double cover $p : Y' \to S$ branched on Z_1 and defined by the relation $Z_1 = 2(K_S - F - Z_0)$.

The standard double cover formulas yield

$\chi(\mathcal{O}_{Y'}) = 2\chi(\mathcal{O}_S) + \frac{1}{2}(K_S - F - Z_0)(2K_S - F - Z_0) = 2 - \frac{p}{4}$;

$K^2_{Y'} = 2(2K_S - F - Z_0)^2 = 4 - p$;

$p_g(Y') = h^0(S, \mathcal{O}_S(2K_S - F - Z_0)) + h^0(S, \mathcal{O}_S(K_S)) = 2.$

Since the surface Y' is of course of general type, we conclude that $p = 4$ and $\chi(\mathcal{O}_{Y'}) = 1$. Then $p_g(Y') = h^0(S, F + Z_0 + Z_1) = 2$ implies $q = 2$ and so, by De Franchis theorem 5.2, Y' is not of Albanese general type. On the other hand the minimal model Y of Y' is obtained contracting the 4 exceptional curves which are the inverse image on Y' of the -2-curves of Y and so $K^2_Y = 4$. We have then a contradiction to Arakelov's inequality $K^2_Y \geq 8(f-1)(q-1)$ (see, e.g. [Be2]). So also this case does not occur. □

Remark. By the previous theorem we see that the minimal surfaces with $K^2_S = 1$ and $p_g = 0$, the *numerical Godeaux* are in a class of their own. We just mention here that there is intensive work in progress on this subject by F. Catanese and R. Pignatelli and by Y. Lee, using in particular the bicanonical fibration. For facts on numerical Godeaux see the paper [CP], which has also a very complete list of references.

7. Base points of the bicanonical system

As, already mentioned, if a surface of general type S satisfies $p_g(S) > 0$, the bicanonical map is defined at every point of S ([Bo], [Re], [F], [CC2], cf. [Ci1]). For surfaces of general type with $p_g = 0$ the situation is diferent. For a minimal surface of general type S satisfying $p_g(S) = 0$ and $K^2_S = 1$, one has $h^0(S, 2K_S) = 2$ and so the linear system $|2K_S|$, being a pencil with self-intersection positive, has base points.

If instead S as above satisfies $K^2_S \geq 5$, Reider's theorem 3.8 implies that the linear system $|2K_S|$ has no base points.

For the remaining cases, i.e. $2 \leq K^2_S \leq 4$, it is still unknown whether $|2K_S|$ has base points. As far as it is known, for all the examples of minimal surfaces of general type with $2 \leq K_S^2 \leq 4$ and $p_g = 0$ the bicanonical map is a morphism.

There are several partial results about possible base points or fixed components for $|2K_S|$ when $p_g = 0$ and $2 \leq K_S^2 \leq 4$. Here we just quote some of these results.

For $K_S^2 = 4$, Lin Weng ([W]) has proven that the base locus of the bicanonical system contains no $-2-$curve. This result has later been improved by Langer:

Theorem 7.1. (Langer, [La]) *Let S be a minimal surface of general type with $K_S^2 = 4$ and $p_g(S) = 0$. Then the system $|2K_S|$ has no fixed component.*

Still in the case $K_S^2 = 4$, F. Catanese and F. Tovena ([CT]) and D. Kotschick ([Ko]) have related the existence of base points of the bicanonical system to properties of the fundamental group of the surface. Since the statements are quite technical, let us just quote here the following consequence of their results:

Theorem 7.2. (Catanese-Tovena, [CT], Kotschick, [Ko]) *Let S be a minimal surface of general type with $K_S^2 = 4$ and $p_g(S) = 0$. If $H^2(\pi_1(S), \mathbb{Z}_2) = 0$, then the bicanonical system $|2K_S|$ is base point free.*

8. The first theorem on the degree of the bicanonical map

Theorem 8.1. *Let S be a minimal surface of general type with $p_g = 0$, $K_S^2 \geq 3$, such that ϕ_{2K_S} is a morphism. Then degree $\phi_{2K_S} \leq 4$ and $X :=$ Im ϕ_{2K_S} is a surface of degree bigger or equal to n in $\mathbb{P}^n$.*

Proof. By 6.1, ϕ_{2K_S} is generically finite. Let d, m be respectively the degrees of ϕ_{2K_S} and X. Since, by assumption, $|2K_S|$ is basepoint free, one has $d \cdot m = (2K_S)^2 = 4K_S^2$.

Because $p_2 = K_S^2 + 1$ and X is a non-degenerate surface in $\mathbb{P}^{K_S^2}$, deg $X \geq K_S^2 - 1$. An easy calculation yields then that $d > 4$ can hold only in the following two cases:
(a) $K_S^2 = 5$ and $d = 5$
or
(b) $K_S^2 = 3$ and $d = 6$.

Notice that in these cases (and exactly in these) we have deg $X = K_S^2 - 1$. Now to prove the theorem we are going to show that neither case can occur.

We start with case (a). Suppose that $K_S^2 = 5$ and $d = 5$. Then X is a surface of degree 4 in $\mathbb{P}^5$ and so as such either it is ruled in lines or the Veronese surface (see, e.g.,[Na]). Since X is the bicanonical image of S, X must be the Veronese surface, because otherwise the pull back of a ruling would be a curve C such that $2K_SC = 5$, an odd number.

So suppose that X is the Veronese surface and denote by L the image on X of a line of $\mathbb{P}^2$ and let F be the inverse image of L on S. Notice that, because $h^0(S, \mathcal{O}_S(F) \geq 3$ and $\chi(\mathcal{O}_S) = 1$, $h^1(S, \mathcal{O}_S(F) \geq 2$. Since $2F$ is linearly equivalent to $2K_S$, $\eta := K_S - F$ is a 2-torsion element of $\mathrm{Pic}(S)$.

Let $g : \tilde{S} \to S$ be the étale double cover associated to η. One has $K_{\tilde{S}}^2 = 10$, $\chi(\mathcal{O}_{\tilde{S}}) = 2$ and $q(\tilde{S}) = q(S) + h^1(S, \mathcal{O}_S(F) \geq 2$. Because $K_{\tilde{S}}^2 = 10$, we have a contradiction to corollary 5.3.

Therefore the case $K_S^2 = 5$, $d = 5$ does not occur.

(b) If $K^2 = 3$ and deg $\phi_{2K_S} = 6$, the image of ϕ_{2K_S} is necessarily the quadric cone in $\mathbf{P}^3$. Otherwise the image of ϕ_{2K_S} would be the non-singular quadric in $\mathbb{P}^3$ and thus $2K_S \equiv D_1 + D_2$, with $K_S \cdot D_i = 3$, which is not congruent to $D_i^2 = 0 \pmod 2$. So the image of ϕ_{2K_S} is a quadric cone and therefore we have $2K_S \equiv 2D + G$, where $|D|$ is a rational pencil without fixed part, satisfying $K_S \cdot D = 3$ and G is an effective divisor, possibly 0, such that $K_S \cdot G = 0$. Notice that, if $G \neq 0$, every irreducible component θ of G is a curve such that $\theta^2 = -2$, $K_S \cdot \theta = 0$. We can write G uniquely as $G = 2G' + \Gamma$, where either $\Gamma = 0$ or $\Gamma = \theta_1 + ... + \theta_r$, with $\theta_1, ..., \theta_r$ distinct irreducible curves.

Suppose $\Gamma \neq 0$. The same argument as in the proof of 6.1 can be used to show that Γ is the union of 4 disjoint -2 irreducible curves and that the double cover Y' of S branched on Γ and determined by the relation $\Gamma = 2(K_S - D - G')$ satisfies $\chi($
$OO_{Y'}) = 1$ and $q(Y') \geq 2$. Since the minimal model Y of Y' verifies $K_Y^2 = 6$, we have a contradiction to corollary 5.3.

If $\Gamma = 0$, $\gamma := K_S - D - G'$ is a 2-torsion non zero divisor. Let $\pi : Y' \to S$ be the associated étale double cover. The surface Y' is minimal and $\chi(\mathcal{O}_{Y'}) = 2$, $K_{Y'}^2 = 6$. Since $p_g(Y') = h^0(S, 2K_S - D - G') = h^0(S, D + G') = 2$, $q(S') = 1$.

Let us consider now the Albanese fibration of S', $\alpha : Y' \to E = Alb(Y')$. The fixed point free involution i induced by $\pi : Y' \to S$ acts as (-1) on E and therefore there are 4 fibres of the Albanese pencil which are stable under i. Since π is étale, this implies that the pencil induced on S by the Albanese pencil has 4 double fibres $2F_1, ..., 2F_4$ (cf. remark 5.4). The existence of these 4 double fibres implies in turn, (see [BPV], lemma 8.3, pg. 91), the existence in $Pic(S)$ of seven distinct non-zero 2-torsion divisors, namely $\eta_{ij} = F_i - F_j$, $i, j \in \{1, 2, 3, 4\}, i < j$ and $\eta = F_1 + F_2 - F_3 - F_4$. Notice that, given a non-trivial 2-torsion divisor μ on S, by the Riemann-Roch theorem one has always $h^0(S, \mathcal{O}_S(K_S + \mu)) \geq 1$.

Consider now the pencil $|D|$. Since $K_S \cdot D = 3$, D^2 is an odd number bigger than 0 and either $G \neq 0$, $D^2 = 1$ and $g(D) = 3$, or $G = 0$, $D^2 = 3$ and $g(D) = 4$. In either case a general curve D' in $|D|$ is smooth. Furthermore, by proposition 3.7 the seven linear systems $|K_S + \eta_{ij}|$, $i, j \in \{1, 2, 3, 4\}, i < j$ and $|K_S + \eta|$ cut on D' seven distinct effective divisors N_{ij} and N of degree 3.

Consider $\operatorname{Im} r : H^0(S, \mathcal{O}_S(2K_S)) \to H^0(D', \mathcal{O}_{D'}(2K_S))$. Since $\phi_{2K_S}(D')$ is a line and $|2K_S|$ is basepoint free, $\operatorname{Im} r$ is a g^1_6 without base points. Now $2N_{ij}$, for $i, j \in \{1, 2, 3, 4\}, i < j$ and $2N$ belong to $\operatorname{Im} r$ and each of these divisors gives a contribution of at least 3 for the degree of the ramification divisor R of the 6-1 morphism $D' \to \mathbf{P}^1$. Therefore $\deg R \geq 21$, and so by the Hurwitz formula one has $2g(D') - 2 \geq 6(-2) + 21 = 9$, i.e. $g(D') \geq 6$, which contradicts $g(D') \leq 4$. Thus also the case $K_S^2 = 3$, $d = 6$ cannot occur. □

For "high" values of K_S^2 one can be more precise as we will see in the next section.

9. Effective theorems on the degree of the bicanonical map

This section will be devoted to proving the following:

Theorem 9.1. *Let S be a minimal surface of general type defined over $\mathbb{C}$ with $p_g(S) = 0$, and let $\varphi \colon S \to \Sigma \subset \mathbb{P}^{K_S^2}$ its bicanonical map.*

- (i) *If $K_S^2 = 9$ then φ is birational;*
- (ii) *if $K_S^2 = 7, 8$ then φ has degree ≤ 2;*
- (iii) *if $K_S^2 = 5, 6$ then φ has degree 2 or 4.*

Remark The inequalities in the theorem above are effective, (see §13) and the cases with $K_S^2 \geq 6, d > 1$ can be characterized (see theorems 11.2, 11.3 below).

9.1. **Proof of theorem 9.1,(i),(ii).** Under the assumptions of Theorem 9.1, the image of the bicanonical map is a surface Σ, by 6.1, and the bicanonical map is a morphism, by Reider's theorem 3.8. Moreover, since $4K_S^2 = \deg\varphi \deg\Sigma$ and Σ is a nondegenerate surface in $\mathbb{P}^{K_S^2}$, the possible values of $\deg\varphi$ are $1, 2, 4$ for $K_S^2 = 7, 8$ and $1, 2, 3, 4$ for $K_S^2 = 9$.

We prove the theorem by analysing separately the cases $K_S^2 = 7, 8, 9$. In each case we argue by contradiction.

9.1.1. *The case $K_S^2 = 7$.* By the above remark, it is enough to show that $\deg \varphi = 4$ does not occur. Assume that φ has degree 4. The bicanonical image Σ is a linearly normal surface of degree 7 in $\mathbb{P}^7$ and its nonsingular model has $p_g = q = 0$. By [Na, Theorem 8], Σ is the image of the blowup $\widehat{\mathbb{P}}$ of $\mathbb{P}^2$ at two points P_1, P_2 under its anticanonical map $f\colon \widehat{\mathbb{P}} \hookrightarrow \mathbb{P}^7$. If $P_1 \neq P_2$, then f is an embedding, while if P_2 is infinitely near to P_1 (say) then Σ has an A_1 singularity. In either case, the hyperplane section of Σ can be written as $H \equiv 2l + l_0$, where l is the image on Σ of a general line of $\mathbb{P}^2$ and l_0 is the image on Σ of the strict transform of the line through P_1 and P_2. Notice that l_0 is contained in the smooth part of Σ. Thus we have $2K_S \equiv 2L + L_0$, where $L = \varphi^* l$ and $L_0 = \varphi^* l_0$.

Lemma 9.2. *L_0 satisfies one of the following possibilities:*

(i) *there exists an effective divisor D on S such that $L_0 = 2D$; or*
(ii) *L_0 is a smooth rational curve with $L_0^2 = -4$; or*
(iii) *there exist smooth rational curves A and B with $A^2 = B^2 = -3$, $AB = 1$, and $L_0 = A + B$.*

Proof. Remark first that $K_S L_0 = 2$, $L_0^2 = -4$, and $L_0 = 2(K_S - L)$ is divisible by 2 in $\operatorname{Pic} S$. Let θ be a -2-curve of S; then θ is contracted by φ and thus $L\theta = L_0\theta = 0$. Since L and L_0 are independent elements of the 3-dimensional space $H^{1,1}(S)$, S contains at most one -2-curve. We write $L_0 = C + a\theta$, where C is the strict transform of L_0, θ is a -2-curve and $a \geq 0$ (we set $a = 0$ if S has no -2-curve). The equalities $\theta L_0 = 0$ and $L_0^2 = -4$ imply

$$\theta C = 2a, \quad \text{and} \quad C^2 = -4 - 2a^2. \tag{9.1}$$

If C is irreducible, then $K_S C = 2$ implies $C^2 \geq -4$ and thus $a = 0$ and case (ii) holds. If C is reducible, then $C = A + B$, with A and B irreducible curves such that $K_S A = K_S B = 1$. If $A = B$, then $AL_0 = 2A^2 + a\theta A = 2A^2 + a^2$ is even, because L_0 is divisible by 2, and thus a is even and we are in case (i). If $A \neq B$, then $AB \geq 0$ and $A^2, B^2 \geq -3$; by parity considerations and (9.1) we get $A^2 = B^2 = -3$ and either $AB = 1$, $a = 0$ or $AB = 0$, $a = 1$. The first case corresponds to (iii), while the second does not occur. In fact the intersection matrix of A, B, θ would be negative definite, contradicting the index theorem, since $h^{1,1}(S) = 3$. □

In cases (ii) or (iii) of lemma 9.2, let $\pi\colon Y \to S$ be the double cover given by $2(K_S - L) \equiv L_0$; then the standard formulas for double covers give $\chi(\mathcal{O}_Y) = 2$ and $K_Y^2 = 16$. Since the bicanonical map φ maps L onto a twisted cubic, $h^0(S, \mathcal{O}_S(2K_S - L)) = 4$ and thus $p_g(Y) = p_g(S) + h^0(S, \mathcal{O}_S(2K_S - L)) = 4$; we thus obtain $q(Y) = 3$, contradicting corollary 5.3.

In case (i) of lemma 9.2, consider the étale double cover $\pi\colon Y \to S$ given by $2(K_S - L - D) \equiv 0$; arguing as above, we get that the invariants of Y are

$$K_Y^2 = 14, \quad \chi(\mathcal{O}_Y) = 2, \quad p_g(Y) = p_g(S) + h^0(S, \mathcal{O}_S(2K_S - L - D)) = 3,$$

so that $q(Y) = 2$ and we again obtain a contradiction to corollary 5.3.

Hence $\deg \varphi \neq 4$ and we have proved Theorem 9.1 in case $K_S^2 = 7$.

9.1.2. *The case* $K_S^2 = 8$. As in case $K_S^2 = 7$, it is enough to show that $\deg \varphi = 4$ does not occur. If φ has degree 4, then the bicanonical image Σ is a linearly normal surface of degree 8 in $\mathbb{P}^8$ whose nonsingular model has $p_g = q = 0$. By [Na, Theorem 8], Σ is either the Veronese embedding in $\mathbb{P}^8$ of a quadric $Q \subset \mathbb{P}^3$ or the image of the blowup $\widehat{\mathbb{P}}$ of $\mathbb{P}^2$ at a point P under its anticanonical map $f\colon \widehat{\mathbb{P}} \hookrightarrow \mathbb{P}^8$.

In the first case $2K_S \equiv 2A$, where A is the hyperplane section of Q. Then $\eta = K_S - A$ is a nontrivial 2-torsion element in $\operatorname{Pic} S$, since $p_g(S) = 0$. The étale double cover $\pi\colon Y \to S$ given by $2\eta \equiv 0$ has invariants $\chi(\mathcal{O}_Y) = 2$, $K_Y^2 = 16$. Moreover, $p_g(Y) = p_g(S) + h^0(S, \mathcal{O}_S(A)) = 4$, so that $q(Y) = 3$. Since $K_Y^2 = 16$, this contradicts corollary 5.3, and therefore Σ is not the Veronese embedding of a quadric.

If the bicanonical image Σ is the image of $\widehat{\mathbb{P}}$ via the map induced by $|-K_{\widehat{\mathbb{P}}}|$, then the hyperplane section of Σ can be written as $H \equiv 2l + l_0$, where l is the image on Σ of a general line of $\mathbb{P}^2$ and l_0 is the image on Σ of the strict transform of a general line through P. Thus $2K_S \equiv 2L + L_0$, where $L = \varphi^* l$ and $L_0 = \varphi^* l_0$, and $L_0 = \varphi^* l_0$ is smooth by Bertini's theorem. Consider now the double cover $\pi\colon Y \to S$ given by $2(K_S - L) \equiv L_0$; then one has $\chi(\mathcal{O}_Y) = 3$ and $K_Y^2 = 24$. Since $p_g(Y) = p_g(S) + h^0(S, \mathcal{O}_S(2K_S - L)) = 0 + h^0(S, \mathcal{O}_S(L + L_0)) = 5$, we get $q(Y) = 3$, contradicting corollary 5.3. Thus Σ is also not the image of $\widehat{\mathbb{P}}$.

Hence $\deg \varphi \neq 4$ and the proof of theorem 9.1, (ii) is complete.

9.1.3. *The case* $K_S^2 = 9$. If $K_S^2 = 9$, then by Poincaré duality, $H^2(S, \mathbb{Z})$ is generated up to torsion by the class of a line bundle L with $L^2 = 1$; thus every divisor on S is numerically a multiple of L, and in particular $K_S \sim 3L$.

Assume by contradiction that φ is not birational; then by Reider's theorem (3.9), for every pair of points $x_1, x_2 \in S$ with $\varphi(x_1) = \varphi(x_2)$ there exists an effective divisor C containing x_1, x_2 such that $K_S C - 2 \leq C^2 < \frac{1}{2} K_S C < 2$. Since $K_S \sim 3L$, the only possibility is that $C \sim L$. We can assume that, as x_1 and x_2 vary, the divisor C varies in an irreducible system of curves, which is linear by the regularity of S. Every curve of $|C|$ is irreducible, since the class of C generates $H^2(S, \mathbb{Z})$ up to torsion, and the general curve of $|C|$ is smooth by Bertini's theorem, since $C^2 = 1$. Therefore $|C|$ is a linear pencil of curves of genus 3 with one base point. For a general $C \in |C|$ we consider the exact sequence:

$$(9.2) \qquad 0 \to \mathcal{O}_S(2K_S - C) \to \mathcal{O}_S(2K_S) \to \mathcal{O}_C(2K_S) \to 0.$$

Since $2K_S - C \sim K_S + 2L$, Kodaira vanishing gives $H^1(S, \mathcal{O}_S(2K_S - C)) = 0$, and the map $H^0(S, \mathcal{O}_S(2K_S)) \to H^0(C, \mathcal{O}_C(2K_S))$ induced by the sequence (9.2) is surjective. So the map $f\colon C \to \mathbb{P}^3$ given by $|\mathcal{O}_C(2K_S)|$ is not birational; it follows that f maps C two-to-one onto a twisted cubic, and thus C is hyperelliptic. If we denote by Δ the g^1_2 of C, then $2K_S|_C \equiv 3\Delta$ and also, by the adjunction formula, $K_S + C|_C \equiv 2\Delta$. So $\eta \equiv 4K_S - (3K_S + 3C) \equiv K_S - 3C$ is trivial when restricted to C. Now $\eta \sim 0$ and so η is a torsion element of $\operatorname{Pic} S$. Furthermore, since $p_g(S) = 0$, η is nonzero and we find a contradiction to proposition 3.7.

This proves theorem 9.1, (i). □

9.2. **Proof of Theorem 9.1, (iii).** Assume that $K_S^2 = 5$ or 6. Then, since the bicanonical map φ is a morphism and degree $\varphi \leq 4$ by theorem 8.1, to prove theorem 9.1, (iii) we want to exclude the possibility $\deg \varphi = 3, K_S^2 = 6$. In this case the bicanonical image Σ is a linearly normal rational surface of degree 8 in $\mathbb{P}^6$. We now establish some properties of linearly normal rational surfaces Υ of degree 8 in $\mathbb{P}^6$. Notice that surfaces of degree 8 in $\mathbb{P}^6$ have been studied classically by Castelnuovo and later by P. Ionescu ([Io]), in the smooth case, and by E. Halanay ([Hl]), in the normal case.

Proposition 9.3. *Let Υ be a linearly normal rational surface of degree 8 in $\mathbb{P}^6$, let $\rho\colon X \to \Upsilon$ be the minimal desingularization of Υ and let $H := \rho^* \mathcal{O}_\Upsilon(1)$.*

Then Υ has isolated singularities and one of the following occurs:

(i) *$-K_X$ is nef and big, $H = -2K_X$ and $h^0(X, -K_X) = 3$;*
(ii) *X has a pencil $|C|$ of rational curves such that $HC = 2$;*
(iii) *X has a pencil $|C|$ of rational curves such that $HC = 3$.*

Proof. We break the proof into steps:

Step 1: *The general hyperplane section of Υ is smooth of genus 3.*

Let $H \in |H|$ be general and set $H' := \rho(H)$, so that $H \to H'$ is the normalization map. The curve $H' \subset \mathbb{P}^5$ has degree 8, hence by Castelnuovo's theorem (cf. [Ci2]) its geometrical genus $g(H)$ is lesser than or equal to 3. Since $q(X) = 0$, the restriction map $H^0(X, H) \to H^0(H, H)$ is surjective and therefore $h^0(H, H) = 6$. On the other hand, Riemann–Roch gives $6 = h^0(H, H) \geq 8 + 1 - g(H)$, and so $g(H) \geq 3$. Therefore $g(H) = 3$ and by Castelnuovo's theorem, H' is smooth and Υ has only isolated singularities.

Step 2: *The divisor $K_X + H$ is nef.*

Since $q(X) = p_g(X) = 0$, the restriction map $H^0(X, K_X + H) \to H^0(H, K_H)$ is an isomorphism. Since $g(H) = 3$ by Step 1, it follows $h^0(X, K_X + H) = 3$. Write $|K_X + H| = |M| + D$, where $|M|$ is the moving part and D is the fixed part. Since $|K_X + H|$ cuts out the complete linear system $|K_H|$, which is free for general H, necessarily we have $H\Gamma = 0$ for every component Γ of D. Then by the index theorem one has $\Delta^2 < 0$ for every divisor Δ whose support is contained in the support of D.

Assume that $K_X + H$ is not nef and let θ be an irreducible curve such that $(K_X + H)\theta < 0$. Since $\theta M \geq 0$, necessarily $\theta D < 0$ and so θ is a component of D. Therefore $H\theta = 0$ and $\theta^2 < 0$. The conditions $\theta(K_X + H) < 0$ and $\theta H = 0$ imply $\theta K_X < 0$, so that θ is a $-1-$curve. Now, because $H\theta = 0$, this is a contradiction to the assumption that $\rho\colon X \to \Upsilon$ is the minimal desingularization of Υ.

Since $H^2 = 8$, by the adjunction formula we conclude that $K_X H = -4$. We have $MH = MH + DH = (K_X + H)H = 4$. By the index theorem this implies $(K_X + H)^2 \leq 2$ and, by the nefness of $K_X + H$, $M^2 \leq (K_X + H)^2 \leq 2$. On the other hand $M^2 \geq 0$ since $|M|$ has no fixed component. Notice that either $D = 0$ or $DM > 0$, since $K_X + H$ is nef and $D^2 < 0$ if $D \neq 0$.

Step 3: *If $M^2 = 2$, then $D = 0$ and $H = -2K_X$. In particular $-K_X$ is nef and $h^0(X, -K_X) = 3$.*

In this case we have $2M \sim H$ by the index theorem. Thus $DM = 0$ and by the above remark it follows that $D = 0$. Furthermore, because X is a rational surface, $2M \sim H$ implies $2M \equiv H$ and so the equality $K_X + H = M$ yields $M = -K_X$, $H = -2K_X$.

Step 4: *If $|M|$ is composite with a pencil $|C|$, then the general curve C is rational and $HC = 2$.*

Since $h^0(X, M) = 3$, in this case we have $|M| = |2C|$. Thus $2 \geq M^2 = 4C^2$, hence $M^2 = C^2 = 0$. Since $HM = 4$, one has $HC = 2$. Since $K_X + H$ is nef by Step 1, we get $2 \geq (K_X + H)^2 \geq 2C(K_X + H)$, namely $K_X C \leq -1$. Since the general C is irreducible and $C^2 = 0$, one has $K_X C = -2$ and C is a smooth rational curve.

Step 5: *If $M^2 = 1$, then there is a pencil $|C|$ on X such that the general C is rational and $HC = 3$.*

In this case ϕ_M is a birational morphism $X \to \mathbb{P}^2$ by the proof of Step 4. Since a general curve M in $|M|$ is smooth rational, we have $-2 = K_X M + M^2$ and so $K_X M = -3$. From $MH = 4$ we conclude $M(K_X + H) = 1 = M^2$ and so $MD = 0$, implying $D = 0$. Now the equalities $1 = (K_X + H)^2 = K_X^2 - 8 + 8$ mean that $K_X^2 = 1$ and thus X is $\mathbb{P}^2$ blown-up in 8 points, possibly infinitely near. Let $E_1, ..., E_8$ be the corresponding exceptional divisors. Since $K_X = -3M + E_1 + \cdots + E_8$, we have $H = 4M - E_1 - \cdots - E_8$ and there is at least a pencil $|C|$ of rational curves on X (corresponding to the lines in $\mathbb{P}^2$ passing through one of the blown-up points of $\mathbb{P}^2$) such that $HC = 3$. □

We need also the following technical result.

Lemma 9.4. *Let S be a minimal surface of general type with $p_g(S) = 0$, $K_S^2 = 6$ and let $|F|$ be a rational pencil on S such that $F^2 = 1$, $K_S F = 3$. Then $h^1(S, 2K_S - F) = 0$.*

Proof. We argue by contradiction. Assume that $h^1(S, 2K_S - F) \neq 0$. Then by the Riemann-Roch theorem we conclude that $h^0(S, 2K_S - F) \geq 4$. Considering now the long exact sequence obtained from

$$(9.3) \quad 0 \to \mathcal{O}_S(2K_S - 2F) \to \mathcal{O}_S(2K_S - F) \to \mathcal{O}_F(2K_S - F) \to 0$$

and using the fact that, by the Riemann-Roch theorem for curves, one has $h^0(F, \mathcal{O}_F(2K_S - F)) = 3$, we see that $h^0(S, 2K_S - 2F) \geq 1$.

Since $(K_S - F)^2 = 1$ and we are assuming that $h^1(S, 2K_S - F) \neq 0$, by the Kawamata-Viehweg's vanishing theorem we conclude that $K_S - F$ is not nef. Let θ be an irreducible curve such that $(K_S - F)\theta < 0$. Notice that in particular $F\theta > 0$. Then for any effective divisor $G \in |2K_S - 2F|$ the curve θ is a component of G and $G\theta \leq -2$. Write $G = \theta + A$. Since F is nef and $FG = 4$, we have $F\theta \leq 4$ and $FA = 4 - F\theta \leq 3$. Furthermore, because $4 = G^2 = \theta G + AG$, we have $AG = 4 - \theta G \geq 6$.

We now show that this does not occur by examining the various possibilities for $F\theta$.

If $F\theta = 4$, then $K_S\theta \leq 3$ and so, because θ is an irreducible curve, $\theta^2 \geq -5$. On the other hand by the index theorem $A^2 < 0$, because

$F^2 = 1$ and $FA = 0$. Since $6 \leq AG = A^2 + A\theta$, one has $A\theta \geq 7$. Then $-2 \geq \theta G = \theta^2 + A\theta$ implies $\theta^2 \leq -9$, a contradiction.

If $F\theta = 3$, then $K\theta \leq 2$, and as above $\theta^2 \geq -4$. Since $FA = 1$ by the index theorem $A^2 \leq 1$ and as above we obtain a contradiction to $\theta^2 \geq -4$.

If $F\theta = 2$, as before we conclude that $\theta^2 \geq -3$, $A^2 \leq 4$, implying that $A\theta \geq 2$ which leads us to the same contradiction.

Finally, if $F\theta = 1$ then $K_S\theta = 0$ and θ is a $-2-$curve. In this case an easy calculation shows that $A^2 = 6$ and that $A \sim K_S$. We can write the equality of $\mathbb{Q}-$divisors: $K_S - F = \frac{1}{2}A + \frac{1}{2}\theta$. Since θ is a normal crossings divisor and $\frac{1}{2}A = \frac{1}{2}K_S$ is nef and big, by the vanishing theorem of Kawamata-Viehweg we obtain $h^1(S, K_S + K_S - F) = 0$, contradicting our assumption. □

We are finally ready to prove theorem 9.1 (iii).

Proof of Theorem 9.1 (iii). We prove the theorem by excluding all the possibilities for the bicanonical image Σ described in proposition 9.3.

In the first place notice that case (iii) is trivially impossible. In fact if Σ contains a pencil of rational curves of degree 3, then the pull back $|F|$ to S of this pencil satisfies $2K_SF = 9$, which is impossible.

Now we consider case (ii), i.e. Σ contains a pencil of rational curves of degree 2. This pencil gives rise to a pencil $|F|$ in S such that $2K_SF = 6$, i.e. $K_SF = 3$. Hence $F^2 \geq 0$ is odd, and so by the index theorem $F^2 = 1$ and $g(F) = 3$. Since $2K_SF = 6$ and the image of F is a conic, we conclude that the restriction map $H^0(S, 2K_S) \to H^0(F, \mathcal{O}_F(2K_S))$ is not surjective, hence $h^1(S, 2K_S - F) \neq 0$, contradicting lemma 9.4.

So we are left with case (i), namely $H = -2K_X$. Consider the Stein factorization $X \xrightarrow{\eta} \overline{X} \xrightarrow{\nu} \Sigma$ of $\rho\colon X \to \Sigma$. Since $-K_X = \frac{1}{2}H$ is nef, the map $\eta\colon X \to \overline{X}$ contracts only $-2-$curves. Hence $\overline{X}$ is a normal surface whose singularities are rational double points. In particular $\overline{X}$ is Gorenstein and $K_X = \eta^* K_{\overline{X}}$. By the normality of $\overline{X}$, the bicanonical map $\varphi\colon S \to \Sigma$ induces a morphism $\overline{\varphi}\colon S \to \overline{X}$ such that $2K_S = \overline{\varphi}^*(-2K_{\overline{X}})$. Hence $\xi := \overline{\varphi}^*(-K_{\overline{X}}) - K_S$ is a non trivial $2-$torsion element of $\mathrm{Pic}(S)$ and $h^0(S, K_S + \xi) \geq 3$ by proposition 9.3 (i). Let $Y \to S$ be the étale double cover given by ξ. The standard formulae for double covers yield:

$$\chi(\mathcal{O}_Y) = 2, \quad K_Y^2 = 12, \quad q(Y) \geq 2.$$

This is a contradiction to corollary 5.3, and so also this possibility does not occur.

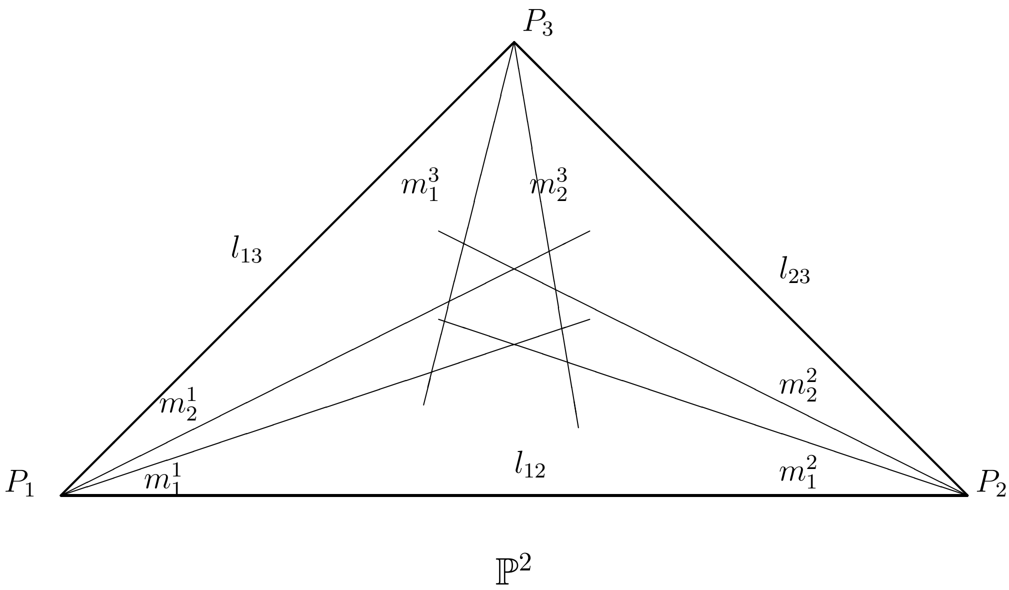
□

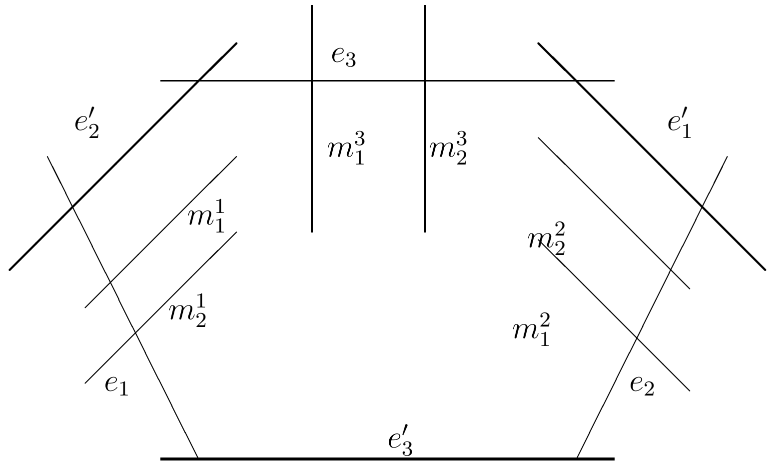
10. Surfaces with $K_S^2 = 6$ and degree $\varphi = 4$

The bounds for the degree of the bicanonical map found above are sharp. It turns out that both the surfaces with $K_S^2 = 7, 8$ and degree $\varphi = 2$ and the surfaces with $K_S^2 = 6$ and degree $\varphi = 4$ can be characterized. We start by explaining one example.

10.1. **The Burniat example.** We explain the construction of Burniat surfaces with $K^2 = 6$ (see [Pe] and [Bu]), describing their bicanonical map.

Let $\Sigma \to \mathbb{P}^2$ be the blowup at three distinct noncollinear points P_1, P_2, P_3. We denote by l the pullback of a line in $\mathbb{P}^2$, by e_i the exceptional curve corresponding to P_i, by $f_i \equiv l - e_i$ for $i = 1, 2, 3$ the strict transform of a general line through P_i and by e'_i the strict transform of the line joining P_j and P_k, where $\{i, j, k\} = \{1, 2, 3\}$; we often take the subscripts modulo 3. The e'_i are disjoint -1-curves that also arise as the exceptional curves of a blowup map $\Sigma \to \mathbb{P}^2$, with the two blowups related by the standard quadratic transformation of $\mathbb{P}^2$ centered at P_1, P_2, P_3. The Picard group of Σ is the free Abelian group generated by the classes of l, e_1, e_2, e_3; the anticanonical class $-K_\Sigma \equiv 3l - e_1 - e_2 - e_3 \equiv f_1 + f_2 + f_3$ is very ample, and $|-K_\Sigma|$ embeds Σ as a smooth del Pezzo surface of degree 6 in $\mathbb{P}^6$.





$$\Sigma := Bl(\mathbb{P}^2)_{P_1 P_2 P_3}$$

The Burniat surfaces are $\mathbb{Z}_2 \times \mathbb{Z}_2$-covers of Σ. Denote by $\gamma_1, \gamma_2, \gamma_3$ the nonzero elements of $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ and by $\chi_i \in \Gamma^*$ the nontrivial character orthogonal to γ_i; to define a smooth Γ-cover $\pi\colon S \to \Sigma$ (see [Pa1], Propositions 2.1 and 3.1), we specify:

(i) smooth divisors D_i for $i = 1, 2, 3$ such that $D = D_1 + D_2 + D_3$ is a normal crossing divisor, and
(ii) line bundles L_1, L_2 satisfying $2L_1 \equiv D_2 + D_3$, $2L_2 \equiv D_1 + D_3$.

The branch locus of π is D. More precisely, D_i is the image of the divisorial part of the fixed locus of γ_i on S. We have

$$\pi_* \mathcal{O}_S = \mathcal{O}_\Sigma \oplus L_1^{-1} \oplus L_2^{-1} \oplus L_3^{-1}$$

where $L_3 = L_1 + L_2 - D_3$, and Γ acts on L_1^{-1} via the character χ_i.

To construct a Burniat surface S with $K_S^2 = 6$, for each $i = 1, 2, 3$, take two smooth divisors $m_1^i, m_2^i \in |f_i|$, such that no three of the m_j^i have a point in common, and set:

$$\begin{aligned} D_1 &= e_1 + e_1' + m_1^2 + m_2^2, \\ D_2 &= e_2 + e_2' + m_1^3 + m_2^3, \\ D_3 &= e_3 + e_3' + m_1^1 + m_2^1, \end{aligned} \quad \text{and} \quad \begin{aligned} L_1 &= 3l - 2e_1 - e_3, \\ L_2 &= 3l - 2e_2 - e_1. \end{aligned}$$

By the above discussion, there exists a smooth Γ-cover $\pi\colon S \to \Sigma$ corresponding to this choice of data, with $L_3 = 3l - 2e_3 - e_2$. The bicanonical divisor $2K_S = \pi^*(2K_\Sigma + D) = \pi^*(-K_\Sigma)$ is ample, as the pullback of an ample divisor, and thus S is a minimal surface of general type and $K_S^2 = \frac{1}{4} \cdot 4K_\Sigma^2 = 6$. The invariants of S are: $\chi(\mathcal{O}_S) =$

$\chi(\pi_*\mathcal{O}_S) = 1$, $p_g(S) = \sum h^0(\Sigma, K_\Sigma + L_i) = 0$ and thus $q(S) = 0$, since S is of general type.

Proposition 10.1. *Let S be a Burniat surface with $K_S^2 = 6$; its bicanonical map is the composite of the degree 4 cover $\pi\colon S \to \Sigma$ with the anticanonical embedding of Σ in $\mathbb{P}^6$ as the smooth Del Pezzo surface of degree 6.*

Proof. Since $p_2(S) = 1 + K_S^2 = 7$, the system $\pi^*|-K_\Sigma|$ is complete, so that $|2K_S| = \pi^*|-K_\Sigma|$. □

10.2. **Surfaces with $K_S^2 = 6$ and deg $\varphi = 4$.** Now we want to classify all the surfaces S satisfying $K_S^2 = 6, d = 4$. In this case the following holds:

Theorem 10.2. *Let S be a minimal surface of general type with $p_g(S) = 0$, $K_S^2 = 6$ and bicanonical map of degree 4. Then S is a Burniat surface.*

This result is also somewhat surprising, since the Burniat construction is apparently very special, and one would not expect it to include all the possible examples. Theorem 10.2 also gives us a good understanding of the moduli of the surfaces we are studying. In fact, using natural deformations of $\mathbb{Z}_2 \times \mathbb{Z}_2$-covers (see [Pa1], Section 5 and [FP]), one is able to to prove:

Theorem 10.3. *Minimal surfaces S with $p_g(S) = 0$, $K_S^2 = 6$ and bicanonical map of degree 4 form an irreducible connected component $\mathcal{Y}$ of the moduli space of surfaces of general type. $\mathcal{Y}$ is unirational of dimension 4.*

For the proof of theorem 10.3 see [MP2].

The proof of theorem 10.2 is somewhat intricate and consists of several steps. The first steps consists of showing that the image of the bicanonical map is smooth and that K_S is ample.

Theorem 10.4. *Let S be a smooth minimal surface of general type with $K_S^2 = 6$, $p_g(S) = q(S) = 0$ and let $\varphi\colon S \to \Sigma = \varphi(S) \subset \mathbb{P}^6$ be the bicanonical map. If $\deg \varphi = 4$ then Σ is the smooth Del Pezzo surface of degree 6 in $\mathbb{P}^6$.*

Proof. Since φ is a morphism the bicanonical image Σ is a linearly normal surface of degree 6. By [Na], theorem 8, it is the image of $\psi\colon \widehat{\mathbb{P}} \to \mathbb{P}^6$, where $\widehat{\mathbb{P}}$ is the blowup of $\mathbb{P}^2$ at points P_1, P_2, P_3 such that $|-K_{\widehat{\mathbb{P}}}|$ has no fixed components, and ψ is given by the system $|-K_{\widehat{\mathbb{P}}}|$.

Thus the P_i can be infinitely near, but it is not possible that 2 of them are distinct and both infinitely near to the third. In other words, the length 3 scheme P_1, P_2, P_3 is a curvilinear scheme. We denote by l the pullback to $\widehat{\mathbb{P}}$ of a general line in $\mathbb{P}^2$, by e_i the exceptional divisor over P_i, and by l_i a general line through P_i, if P_i is not an infinitely near point; moreover we write L, L_i for the strict transform on S of l, respectively l_i. Σ is smooth if and only if P_1, P_2, P_3 are distinct and noncollinear, that is, if and only if $\widehat{\mathbb{P}}$ contains no -2-curves; in all the other cases, ψ contracts to rational double points the -2-curves of $\widehat{\mathbb{P}}$, either components of the e_i or possibly the strict transform of a line containing all the P_i. The proof of theorem 10.4 is a case by case discussion of the possible configurations of the P_i giving rise to singular Σ. In each case, we consider the pullback of a hyperplane section of Σ through one of the singular points, use it to construct an irregular double cover $\pi\colon Y \to S$ and then obtain a contradiction using theorem 5.2 or corollary 5.3 or remark 5.4.

Case A:. The points P_1, P_2, P_3, not necessarily all distinct, lie on a line m. Note that ψ maps the strict transform of the line m on $\widehat{\mathbb{P}}$ to a point $x \in \Sigma$.

In this case we claim that $2K_S \equiv 2D$ for some divisor D with $h^0(D) \geq 3$. We first show that this leads to a contradiction. Write $\pi\colon Y \to S$ for the unramified double cover given by $2(K_S - D) \equiv 0$. Then the formulas (4.1) give $\chi(\mathcal{O}_Y) = 2$, $K_Y^2 = 12$ and $p_g(Y) = h^0(S, 2K_S - D) = h^0(S, D) \geq 3$, so that $q(Y) \geq 2$. This contradicts Corollary 5.3.

We prove the claim in terms of hyperplane sections H of Σ through x, corresponding to cubics in $\mathbb{P}^2$ containing m. The pullback to S of H can be written $2K_S = 2L + Z$, where $h^0(S, L) \geq 3$, and Z is effective with $K_S Z = 0$, so it consists only of -2-curves. Write $Z = 2Z' + Z''$, with Z'' reduced; clearly, Z'' is divisible by 2 in $\operatorname{Pic} S$, so that $(Z'')^2 \equiv 0 \mod 8$. Thus if $Z'' \neq 0$, it contains at least 4 irreducible -2-curves. On the other hand, S contains at most 3 irreducible -2-curves, since $h^{1,1}(S) = 4$. Therefore $Z'' = 0$; this proves the claim. Thus Case A cannot occur.

Case B:. There is no line containing all the P_i.

Assume that P_3 is infinitely near to P_2. There are two subcases, according to whether P_2 is infinitely near to P_1.

Case B1: P_2 is not infinitely near to P_1

The linear system $|-K_{\widehat{\mathbb{P}}}|$ contains $|l_1| + |2l_2| + e_2'$, where $|l_1|$ and $|l_2|$ are free pencils and e_2' is the strict transform of the blowup of P_2. Pulling back the corresponding hyperplane sections of Σ, we can write

$2K_S = 2L_2 + L_1 + Z$, where Z is an effective divisor disjoint from L_1 with $K_SZ = 0$; an argument similar to that of Case A shows that $Z = 2Z'$. Let $\pi\colon Y \to S$ be the double cover branched over a general L_1 defined by $2(K_S - L_2 - Z') \equiv L_1$; formulas (4.1) give

$$\chi(\mathcal{O}_Y) = 3, \quad p_g(Y) = h^0(S, 2K_S - L_2 - Z') = h^0(S, L_1 + L_2 + Z') \geq 4,$$

and thus $q(Y) \geq 2$. By theorem 5.2, the Albanese image of Y is a curve and there exists a pencil $g\colon S \to \mathbb{P}^1$ such that $\pi \circ g$ factors through the Albanese pencil. Since π is branched over L_1, g must be the map given by $|L_1|$ and so by remark 5.4 g has at least 5 fibres divisible by 2. Let $D = \varphi^*(\psi(e_1))$ and write $\overline{D}$ for the strict transform of $\psi(e_1)$, so that $D = \overline{D} + Z$ with Z effective and $K_SZ = 0$; we have $D^2 = -4$, $DK_S = \overline{D}K_S = 2$. Note that $\overline{D}$ is nonreduced if and only if $\varphi\colon S \to \Sigma$ is branched over $\psi(e_1)$.

Write R for the ramification divisor of φ. Then $K_S = R + \varphi^*K_\Sigma$ by adjunction, so that $R \equiv 3K_S$. Since $|l_1|$ on Σ has no double fibres, if the double fibres of g are $2M_i$ for $i = 1, \dots, 5$ then $R \geq \sum_i M_i$. Assume that $\overline{D}$ is reduced, and thus has no common component with R. Then

$$2 = K_S\overline{D} = \frac{1}{3}R\overline{D} \geq \frac{1}{3}\overline{D}\sum_i M_i \geq \frac{5}{6}\overline{D}L_1 = \frac{10}{3},$$

a contradiction. Thus $\overline{D}$ is nonreduced and so, because $\overline{D}K_S = 2$, we have $\overline{D} = 2E$ with E irreducible such that $K_SE = 1$; in this case $L_1E = 2$ and so $M_iE = 1$ for every i, the point $M_i \cap E$ is smooth for E and is a ramification point of the degree 2 map $\varphi|_E\colon E \to \psi(e_1)$. Thus $p_a(E) \geq 2$ by the Hurwitz formula. On the other hand, $0 = ZD = 2EZ + Z^2$ and $-4 = D^2 = (2E + Z)^2 = 4E^2 - Z^2$ and thus $E^2 \leq -1$, $p_a(E) \leq 1$. Therefore Case B1 does not occur.

Case B2: P_2 is infinitely near to P_1.

In this case the linear system $|-K_{\widehat{\mathbb{P}}}|$ contains $|3l_1| + 2e'_1 + e'_2$, where e'_1 and e'_2 are the strict transforms of the blowup of P_1, respectively P_2. Pulling back the corresponding hyperplane sections of Σ, we can write $2K_S = 3L_1 + Z$, where Z is effective with $K_SZ = 0$. Since $K_SL_1 = 4$, the index theorem gives either:

(a) $L_1^2 = 0$, or
(b) $L_1^2 = 2$.

Assume first that case (a) holds: then $8 = 2K_SL_1 = 3L_1^2 + L_1Z$ implies $L_1Z = 8$. Taking squares, we get $24 = 4K_S^2 = 9L_1^2 + 6L_1Z + Z^2$ and thus $Z^2 = -24$. The irreducible components of Z are -2-curves and there are at least two of them, since $-Z^2/2$ is not a square. On the other hand, notice that the classes L and $\varphi^*(\psi(e_3))$ span a 2-dimensional subspace V in $H^2(\Sigma)$, since they are both effective and satisfy $L^2 = 4$ and $\varphi^*(\psi(e_3))L = 0$. Since V is orthogonal to the span of the classes of the -2 curves of S and $h^2(S) = 4$, it follows that S contains precisely two irreducible -2–curves, say θ_1 and θ_2. So we may write $Z = a_1\theta_1 + a_2\theta_2$, with $a_1 \geq a_2 > 0$. Observe that $\theta_1\theta_2 \neq 0$, since otherwise we would have integral solutions of $a_1^2 + a_2^2 = 12$. Thus $\theta_1\theta_2 = 1$, since the intersection form is negative definite on the span of θ_1 and θ_2. The equality $Z^2 = -24$ can be rewritten as $(a_1 - a_2)^2 + a_1a_2 = 12$, and has $a_1 = 4$, $a_2 = 2$ as the only solution. In particular, we have $L_1\theta_1 = 2$, $L_1\theta_2 = 0$. Let $\pi\colon Y \to S$ be the double cover branched over a general L_1 and given by the relation $2(K_S - L_1 - 2\theta_1 - \theta_2) \equiv L_1$; we have $\chi(\mathcal{O}_Y) = 3$, $p_g(Y) = h^0(S, 2K_S - L_1 - 2\theta_1 - \theta_2) = h^0(S, 2L_1 + 2\theta_1 + \theta_2) \geq 3$ and thus $q(Y) = 1$. So we argue as in Case A, and we see that the pencil $|L_1|$ on S is induced by the Albanese pencil of Y. The curve $\Delta = \pi^*\theta_1$ is not contained in a fibre of the Albanese pencil of Y since $\theta_1L_1 = 2$, it is smooth irreducible, since L_1 is general, and it has genus zero by the Hurwitz formula. Thus we have a contradiction and case (a) is ruled out.

Consider now case (b): arguing exactly as in case (a), one shows that $2K_S = 3L_1 + 2\theta_1 + \theta_2$, where θ_1, θ_2 are irreducible -2–curves such that $\theta_1\theta_2 = 1$, $L_1\theta_1 = 1$, $L_1\theta_2 = 0$. So we consider the double cover $\pi\colon Y \to S$ branched over $L_1 + \theta_2$ for general L_1, given by the relation $2(K_S - L_1 - \theta_1) \equiv L_1 + \theta_2$; Y is smooth and, as usual, $\chi(\mathcal{O}_Y) = 3$, $p_g(Y) = h^0(S, 2K_S - L_1 - \theta_1) = h^0(S, 2L_1 + \theta_1 + \theta_2) \geq 3$ and thus $q(Y) \geq 1$. As in the previous cases, the Albanese image of Y is a curve and the Albanese pencil induces a base point free linear pencil $|F|$ on S, that satisfies $L_1F = 0$; the index theorem applied to L_1, F gives a contradiction, and the proof is complete. □

Proposition 10.5. *The canonical divisor K_S of S is ample and φ is finite.*

Proof. By theorem 10.4, we have $h^2(\Sigma) = h^2(S) = 4$. Hence the pullback $\varphi^*\colon H^2(\Sigma) \to H^2(S)$, which is injective, is an isomorphism over $\mathbb{Q}$, that multiplies the intersection form by 4. If a curve C were contracted by φ, its class in $H^2(S)$ would be in the kernel of the intersection form on $H^2(S)$, contradicting Poincaré duality. Thus φ is finite and K_S is ample. □

Now that we know that Σ is smooth, we carry out a detailed study of the pullbacks via φ of the exceptional curves and the free pencils of Σ, producing a subgroup $G \simeq \mathbb{Z}_2^3$ of $\operatorname{Pic} S$ that plays an important role in the proof of Theorem 10.2.

We start with:

Lemma 10.6. *Let $C \subset \Sigma$ be a -1-curve. Then either:*

(i) *φ^*C is a smooth rational curve with self-intersection -4; or*
(ii) *$\varphi^*C = 2E$, where E is an irreducible curve with $E^2 = -1$, $K_S E = 1$.*

Proof. We have $(\varphi^*E)^2 = -4$, $K_S\varphi^*E = 2$. If φ^*E is irreducible, it is smooth rational and we are in case (i). Since K_S is ample, if φ^*E is reducible then $\varphi^*E = A + B$, with A, B irreducible and $K_S A = K_S B = 1$. If $A \neq B$, then $AB \geq 0$, $A^2 + B^2 + 2AB = -4$ and so, by parity considerations, either $A^2 = B^2 = -3$, $AB = 1$ or, say, $A^2 = -3$, $B^2 = -1$, $AB = 0$. In either case, the matrix $\left(\begin{smallmatrix} A^2 & AB \\ AB & B^2 \end{smallmatrix}\right)$ is negative definite, and thus the classes of A, B span a 2-dimensional subspace V of $H^2(S)$.

Now V and $\varphi^*(\langle E\rangle^\perp)$ are orthogonal subspaces by the projection formula. By Poincaré duality, $H^2(\Sigma) = \langle E\rangle \oplus^\perp \langle E\rangle^\perp$ and thus $H^2(S) = \varphi^*\langle E\rangle \oplus^\perp \varphi^*(\langle E\rangle^\perp)$, since, as remarked in the proof of proposition 10.5, φ^* is an isomorphism multiplying the intersection form by 4. Thus $V \subseteq \varphi^*\langle E\rangle$, contradicting the fact that V has dimension 2. So we must have $A = B$ and we are in case (ii). □

Lemma 10.7. *If S is as above, then S does not contain 2 smooth disjoint rational curves with self-intersection -4.*

Proof. Suppose that S contains r disjoint smooth rational curves D with $D^2 = -4$; by theorem3.10 we have the inequality $r\frac{25}{4} \leq 3c_2(S) - K_S^2 = 12$, that is, $r \leq 1$. □ □

Proposition 10.8. *Define $e_i, e_i' \subset \Sigma$ for $i = 1, 2, 3$ as in Section 10.1. Then for $i = 1, 2, 3$ there exist irreducible curves $E_i, E_i' \subset S$ such that $\varphi^*e_i = 2E_i$, $\varphi^*e_i' = 2E_i'$ and $E_i^2 = (E_i')^2 = -1$, $K_S E_i = K_S E_i' = 1$.*

Proof. By lemmas 10.6 and 10.7, we may assume that there exist irreducible curves E_2, E_3, E_1', E_3' on S such that $E_i^2 = (E_i')^2 = -1$, $K_S E_i = K_S E_i' = 1$ and $\varphi^*e_2 = 2E_2$, $\varphi^*e_3 = 2E_3$, $\varphi^*e_1' = 2E_1'$, $\varphi^*e_3' = 2E_3'$, and moreover, that φ^*e_1, φ^*e_2' are either of the same type, or are smooth rational curves. So assume that $\varphi^*e_1 = R$ is a smooth rational curve. Writing $F_i = \varphi^*f_i$ for $i = 1, 2, 3$, we get

$$2K_S \equiv F_1 + F_2 + F_3 \equiv F_1 + R + 2E_3' + 2E_1' + 2E_2 \equiv R + 2F_1 + 2E_1'.$$

Let $\pi\colon Y \to S$ be the double cover defined by $2(K_S - F_1 - E_1') \equiv R$; then Y is a smooth surface with $\chi(\mathcal{O}_Y) = 2$, $K_Y^2 = 14$, $p_g(Y) = h^0(S, 2K_S - F_1 - E_1') = 3$ (see formulas (4.1)). The last equality follows because φ maps F_1 and E_1' to a conic and a line intersecting transversally at one point. Therefore we have $q(Y) = 2$ and the result follows from corollary 5.3. The proof for E_2' is similar. □

Notation 10.9. By Theorem 10.4, Σ is the blowup of $\mathbb{P}^2$ at three non-colllinear points and we use the notation of Section 10.1 for divisors on Σ; in addition, we set $F_i = \varphi^* f_i$ and write $g_i \colon S \to \mathbb{P}^1$ for the morphism given by $|F_i|$, for $i = 1, 2, 3$. We often take the subscripts modulo 3. For instance, the pencil g_i has two reducible double fibers, that we write as $2E_{i+1}+2E'_{i+2}$ and $2E_{i+2}+2E'_{i+1}$. We set $\eta_i = E_{i+1}+E'_{i+2}-E_{i+2}-E'_{i+1}$, for $i = 1, 2, 3$, and $\eta = K_S - (\sum_j E_j + \sum E'_j)$.

Proposition 10.10. *Let $\eta, \eta_1, \eta_2, \eta_3 \in \operatorname{Pic} S$ be defined as in 10.9 and let G be the subgroup of $\operatorname{Pic} S$ generated by these elements. Then*

$$G = \{0, \eta_1, \eta_2, \eta_3, \eta, \eta + \eta_1, \eta + \eta_2, \eta + \eta_3\},$$

with $\eta_1 + \eta_2 + \eta_3 = 0$, and $G \simeq \mathbb{Z}_2^3$.

Proof. Obviously from the definitions, $2\eta = 2\eta_i = 0$ and $\eta_1+\eta_2+\eta_3 = 0$. In addition, $\eta = K_S - \sum_j(E_j + E'_j) \neq 0$ and

$$\eta + \eta_i = K_S - (E_i + E'_i + 2E'_{i+1} + 2E_{i+2}) \neq 0,$$

because $p_g(S) = 0$. Finally, $\eta_i \neq 0$, $i = 1, 2, 3$ by [BPV], Chap. III, Lemma (8.3). So G consists precisely of the 8 elements listed above. □

Lemma 10.11. *With the notation as above, then:*

(i) $h^0(S, K_S+\eta) = h^0(S, K_S+\eta_i) = 1$, $h^1(S, K_S+\eta) = h^1(S, K_S+\eta_i) = 0$ *for* $i = 1, 2, 3$;

(ii) $h^0(S, K_S + \eta + \eta_i) = 2$, $h^1(S, K_S + \eta + \eta_i) = 1$ *for* $i = 1, 2, 3$;

(iii) *if* $\tau \in \operatorname{Pic} S$ *is such that* $2\tau = 0$ *and* $h^0(S, K_S + \tau) \geq 2$, *then* $\tau = \eta + \eta_i$ *for some* $1 \leq i \leq 3$.

Proof. First, if $\tau \in \operatorname{Pic} S$ satisfies $2\tau = 0$, $\tau \neq 0$, then $1 = \chi(K_S+\tau) = h^0(K_S + \tau) - h^1(K_S + \tau)$, and therefore $K_S + \tau$ is effective. Now let $\tau \in \operatorname{Pic} S$ be such that $2\tau = 0$ and $h^0(S, K_S + \tau) \geq 2$, and write $|K_S + \tau| = Z + |M|$, where Z and $|M|$ are the fixed and the moving part, respectively. The curves $2Z + 2M$ belong to the bicanonical system $|2K_S| = \varphi^*|-K_\Sigma|$, and thus $|M| = \varphi^*|N|$, where $|N|$ is a linear system of Σ without fixed components such that $-K_\Sigma-2N$ is effective. The only possibility is $|N| = |f_i|$ for some $i = 1, 2, 3$. In turn, this corresponds to $\tau = \eta + \eta_i$, since $K_S + \eta + \eta_i = F_i + E_i + E'_i$ and $h^0(S, 2(E_i + E'_i)) = 1$. In particular, $h^0(S, K_S + \eta + \eta_i) = 2$. □

Lemma 10.12. *The pencils $|F_1|$, $|F_2|$ and $|F_3|$ are the only irreducible base point free pencils of S.*

Proof. Let D be the cohomology class of a base point free pencil of S. Then D lies in the nef cone $\mathrm{NE}(S) \subset H^2(S, \mathbb{R})$ and satisfies $D^2 = 0$. Conversely, given $D \in \mathrm{NE}(S)$ with $D^2 = 0$ there is at most one irreducible pencil of S whose class is proportional to D.

As we saw in the proof of corollary 10.5, $\varphi^*\colon H^2(\Sigma) \to H^2(S)$ is an isomorphism multiplying the intersection form by 4; in addition, integral classes both on S and on Σ are algebraic because $p_g(S) = p_g(\Sigma) = 0$, and therefore $\mathrm{NE}(S) = \varphi^* \mathrm{NE}(\Sigma)$. Now $\mathrm{NE}(\Sigma)$ is spanned by the classes of f_1, f_2, f_3, l, l', where l' is the pullback of a conic in $\mathbb{P}^2$ through the fundamental points P_1, P_2, P_3, and so D is equal to the class of f_1, f_2 or f_3. □

Lemma 10.13. *Let $g_i\colon S \to \mathbb{P}^1$ be as in Notation 10.9, $i = 1, 2, 3$; then:*

- (i) *the multiple fibres of g_i are double fibers and their number is ≥ 2 and ≤ 4;*
- (ii) *if g_i has 4 double fibres, then E_i and E_i' are smooth elliptic curves.*

Proof. We recall that g_i has at least 2 double fibres, namely $2E_{i+1} + 2E'_{i+2}$ and $2E'_{i+1} + 2E_{i+2}$, (see Proposition 10.8 and Notation 10.9). Let $mD \in |F_i|$, with $m > 1$; since $E_iF_i = E_i'F_i = 2$, we have $m = 2$ and D intersects both E_i and E_i' transversally at smooth points. φ maps the irreducible curves E_i and E_i' of arithmetic genus 1 2-to-1 onto the smooth rational curves e_i and e_i', and the maps $E_i \to e_i$ and $E_i' \to e_i'$ are ramified at the point DE_i, respectively DE_i'. So, by the Hurwitz formula, there are at most 4 double fibres, and in that case E_i and E_i' are smooth. □

Proposition 10.14. *Let S be as above and for $i = 1, 2, 3$, let $F_i \in |F_i|$ be general curves; if $i \neq j$, then $F_j|_{F_i} = K_{F_i}$.*

Proof. We show that $F_3|_{F_1} = F_2|_{F_1} = K_{F_1}$. Notice that

$$2K_S = F_1 + F_2 + F_3 = F_1 + 2E_3' + 2E_1 + 2E_2' + 2E_1,$$

and consider the double cover $\pi\colon Y \to S$ branched over a smooth F_1 and given by $2(K_S - 2E_1 - E_3' - E_1') \equiv F_1$; by the formulas (4.1), the invariants of Y are $\chi(\mathcal{O}_Y) = 3$, $K_Y^2 = 20$, $p_g(Y) = h^0(S, 2K_S - 2E_1 - E_3' - E_2')$. To give a lower bound for $p_g(Y)$, we observe that

$$\begin{aligned} |2K_S - 2E_1 - E_3' - E_2'| &= |(F_1 + 2E_1) + E_2' + E_3'| \\ &= |\varphi^* l + E_2' + E_3'| \supseteq \varphi^*|l| + E_2' + E_3' \end{aligned}$$

(in the notation of Section 10.1) and thus $p_g(Y) = h^0(S, 2K_S - 2E_1 - E_3' - E_2') \geq 3$ and $q(Y) \geq 1$. By theorem 5.2, the Albanese pencil on Y

is the pullback of a pencil $|F|$ on S such that π^*F is disconnected for F general. Since π is branched over a curve of $|F_1|$, it follows that $FF_1 = 0$ and therefore $|F| = |F_1|$. In addition, if F_1 is general then π^*F_1 is the unramified double cover of F_1 given by $2(K_S - 2E_1 - E_3' - E_2')|_{F_1} \equiv 0$; since π^*F_1 is disconnected, the line bundle

$$(K_S-2E_1-E_3'-E_2')|_{F_1} = (K_S-2E_1)|_{F_1} = (K_S-F_3)|_{F_1} = (K_S-F_2)|_{F_1}$$

is trivial. □

Proposition 10.15. *For $i = 1, 2, 3$, let $F_i \in |F_i|$ be a general curve and let $G_i = \{\tau \in G \colon \tau|_{F_i} = 0\}$. Then $G_i = \{\eta_i, \eta + \eta_{i+1}, \eta + \eta_{i+2}\}$.*

Proof. We prove the lemma for G_1. We have $\eta_1 \in G_1$ by definition. Moreover, using Lemma 10.14, it is easy to show that $\eta|_{F_1} = \eta_2|_{F_1} = \eta_3|_{F_1} = (E_1 - E_1')|_{F_1}$, so we only need to show $\eta|_{F_1} \neq 0$. Notice that $K_S + F_1 + \eta + \eta_1 = 2F_1 + E_1' + E_1 = 2K_S - E_1 - E_1'$. Therefore $H^0(S, K_S+F_1+\eta+\eta_1)$ is isomorphic to the kernel of the restriction map $H^0(S, 2K_S) \to H^0(E_1 + E_1', 2K_S|_{E_1+E_1'})$. Since $|2K_S|$ embeds $E_1 + E_1'$ as a pair of skew lines, it follows that $h^0(S, K_S+F_1+\eta+\eta_1) = 3$. Next we restrict $K_S + F_1 + \eta + \eta_1$ to F_1 and get $0 \to H^0(S, K_S + \eta + \eta_1) \to H^0(S, K_S + F_1 + \eta + \eta_1) \to H^0(F_1, K_{F_1}(\eta)) \to H^1(S, K_S + \eta + \eta_1)$. Using Lemma 10.11, it follows that $h^0(F_1, K_{F_1}(\eta)) \leq 2$ and so $\eta|_{F_1}$ is nontrivial. □

We have now the ingredients we need for the:
Proof of Theorem 10.2. Since the proof is long, we break it into four steps. We use the notations introduced in Sections 10.1. In addition, we denote by $\pi_i \colon Y_i \to S$ the unramified double cover given by $\eta + \eta_i$, for $i = 1, 2, 3$. By the formulas (4.1) and lemma 10.11, $p_g(Y_i) = 2$, $q(Y_i) = 1$; we write $\alpha_i \colon Y_i \to B_i$ for the Albanese pencil.
Step 1: *Up to a permutation of $\{1, 2, 3\}$, the pencil $g_{i-1} \circ \pi_i \colon Y_i \to \mathbb{P}^1$ is composed with $\alpha_i \colon Y_i \to B_i$.*

By theorem 5.2, the Albanese pencil $\alpha_i \colon Y_i \to B_i$ arises in the Stein factorization of $g \circ \pi_i$ for some base point free pencil $g \colon S \to \mathbb{P}^1$. By lemma 10.12, $g = g_{s_i}$ for some $s_i \in \{1, 2, 3\}$. Notice that $s_i \neq i$, since by proposition 10.15 the general curve of $\pi_i^*|F_j|$ is connected if and only if $i = j$. To prove the claim, we have to show that $i \mapsto s_i$ is a permutation of $\{1, 2, 3\}$. Assume by contradiction that, say, $s_2 = s_3 = 1$ and denote by $p \colon Z \to S$ the unramified $\mathbb{Z}_2 \times \mathbb{Z}_2$-cover with data $L_1 = \eta_1$, $L_2 = \eta + \eta_2$, $L_3 = \eta + \eta_3$ (see Section 4.1, or [Pa1], proposition 2.1). We have $q(Z) = \sum_i h^1(S, L_i^{-1}) = 2$ by lemma 10.11; we denote by $\alpha \colon Z \to A$ the Albanese map. If σ_i is the element of $\mathbb{Z}_2 \times \mathbb{Z}_2$ that acts trivially on L_i^{-1}, then, for $i = 2, 3$, the surface $Z/\langle\sigma_i\rangle$ can be naturally identified with

Y_i; we denote by $p_i \colon Z \to Y_i$ the projection map and by $p_{i*} \colon A \to B_i$ the homomorphism induced by p_i. Notice that $p_{2*} \times p_{3*} \colon A \to B_2 \times B_3$ is an isogeny, since

$$\begin{aligned} H^1(Z, \mathcal{O}_Z) &\simeq H^1(S, \eta + \eta_2) \oplus H^1(S, \eta + \eta_3) \\ &\simeq p_2^* H^1(Y_2, \mathcal{O}_{Y_2}) \oplus p_3^* H^1(Y_3, \mathcal{O}_{Y_3}). \end{aligned}$$

Since the pencil $g_1 \circ p$ is composed with both $p_{2*} \circ \alpha$ and $p_{3*} \circ \alpha$, the Albanese image of Z is a curve B of genus 2 and $g_1 \circ p = \overline{p} \circ \alpha$, where $\overline{p} \colon B \to \mathbb{P}^1$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-cover. By the Hurwitz formula, $\overline{p}$ is branched exactly over 5 points of $\mathbb{P}^1$, since in a $\mathbb{Z}_2 \times \mathbb{Z}_2$-cover of smooth curves the inverse image of a branch point consists of 2 simple ramification points. Arguing as in the proof of remark 5.4, we see that the fibres of g_1 over the branch points of $\overline{p}$ are double, but this contradicts lemma 10.13.

Step 2: *The general F_i is hyperelliptic for $i = 1, 2, 3$.*

We show that the general F_1 is hyperelliptic. We have seen that the pencil $g_1 \circ \pi$ is composed with the Albanese map $\alpha_2 \colon Y_2 \to B_2$ and that $g_3 \circ \pi_2$ also has disconnected fibres. The Stein factorization of $g_3 \circ \pi_2$ is $Y_2 \xrightarrow{g} C \xrightarrow{\psi} \mathbb{P}^1$ where g has connected fibres, C is a smooth curve and $\deg \psi = 2$. Notice that $C \cong \mathbb{P}^1$, since $q(Y_2) = 1$ and g is not the Albanese pencil. Denote by $\widetilde{F}_1$ a general fibre of α and by $\widetilde{F}_3$ a general fibre of g. From $F_1 F_3 = 4$ it follows that $\widetilde{F}_1 \widetilde{F}_3 = 2$. So the linear system $|\widetilde{F}_3|$ cuts out a g^1_2 on the general $\widetilde{F}_1$, and thus the general F_1 is hyperelliptic.

Step 3: *The Galois group Γ of $\varphi \colon S \to \Sigma$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

For $i = 1, 2, 3$, denote by γ_i the involution on S that induces the hyperelliptic involution on the general F_i; the γ_i are regular maps, since S is minimal, and they belong to Γ by proposition 10.14. Consider the involution $\tilde{\gamma}_1 \colon Y_2 \to Y_2$ inducing the hyperelliptic involution on the general $\widetilde{F}_1$: by construction $\tilde{\gamma}_1$ maps each $\widetilde{F}_3$ to itself, and the restriction of α to $\widetilde{F}_3$ identifies $\widetilde{F}_3 / \langle \tilde{\gamma}_1 \rangle$ with B_2. Since $\pi_2|_{\widetilde{F}_i} \colon \widetilde{F}_i \to \pi_2(\widetilde{F}_i) \in |F_i|$ is an isomorphism compatible with the action of $\tilde{\gamma}_1$ and γ_1 for $i = 1, 3$, this implies that $\gamma_1 \neq \gamma_3$. We prove in a similar way that $\gamma_i \neq \gamma_j$ for $i \neq j$, and thus $\Gamma = \{1, \gamma_1, \gamma_2, \gamma_3\}$.

Step 4: *S is a Burniat surface.*

By Step 1, for each $i = 1, 2, 3$ the map $g_i \circ \pi_{i+1}$ is composed with the Albanese pencil $\alpha_{i+1} \colon Y_{i+1} \to B_{i+1}$ and thus, by remark 5.4 and lemma 10.13, g_i has precisely 4 double fibres. The double fibres are $2(E_{i+1} + E'_{i+2})$, $2(E'_{i+1} + E_{i+2})$, and $2M^i_1 = \varphi^* m^i_1$, $2M^i_2 = \varphi^* m^i_2$, where $m^i_1, m^i_2 \in |f_i|$. If we denote by D the total branch locus of φ, then $D \supseteq D_0 = \sum_i (e_i + e'_i + m^i_1 + m^i_2)$. By [Pa1] proposition 3.1, D is a normal crossing divisor, since S is smooth, and therefore no three of the m^i_j have a common point. Applying the Hurwitz formula to a general bicanonical curve yields $-K_\Sigma D = 18 = -K_\Sigma D_0$ and thus $D = D_0$, since $-K_\Sigma$ is ample. As in Section 10.1, we denote by D_i the image of the divisorial part of the fix locus of γ_i, so that $D = D_1 + D_2 + D_3$. By [Pa1] proposition 3.1, D_i is smooth for every $i = 1, 2, 3$, so there is a permutation $i \mapsto s_i$ of $\{1, 2, 3\}$ such that $D_i \supset m^{s_i}_1 + m^{s_i}_2$; in addition, the quotient of a general F_i by γ_i is rational and therefore $D_i f_i = 4$. We conclude that for $i = 1, 2, 3$ $D_i = e_i + e'_i + m^{s_i}_1 + m^{s_i}_2$ and $s_i \neq i$. Finally, the quotient of a general F_{i+2} by γ_i is the elliptic curve B_{i+1} (cf. Step 3) and thus $D_i f_{i+2} = 2$. So we get $s_i = i+1$ and S is obtained precisely as explained in Section 10.1.

Now that the case $d = 4$, $K^2_S = 6$ is completely charaterized, we want to state the characterization theorems for the cases $6 \leq K^2_S \leq 8$ and deg $\varphi = 2$.

11. The characterization theorems for $6 \leq K^2_S \leq 8$ and $d = 2$

Assume that $d = 2$. In this situation we have in general:

Theorem 11.1 ([X2], [MP3])**.** *Let S be a minimal surface of general type with $p_g = 0$ such that the bicanonical map $\varphi\colon S \to \mathbb{P}^{K^2_S}$ is a morphism of degree 2 onto a surface Σ. Then either Σ is a rational surface or $K^2_S = 3$ and Σ is birationally an Enriques sextic in $\mathbb{P}^3$.*

For the proof of this theorem see the above cited papers. In particular for the cases $K^2_S = 6, 7, 8$ and deg $\varphi = 2$, the bicanonical image is a rational surface, but more can be said:

Theorem 11.2. *Let S be a minimal smooth complex surface of general type with $p_g(S) = 0$ and $K^2_S = 7$ or 8 for which the bicanonical map is not birational. Then:*

- i) *K_S is ample;*
- ii) *there is a fibration $f\colon S \to \mathbb{P}^1$ such that the general fibre F of f is hyperelliptic of genus 3;*
- iii) *the bicanonical involution of S induces the hyperelliptic involution on F.*

Furthermore

- iv) *if $K^2_S = 8$, then f is an isotrivial fibration with 6 double fibres;*
- v) *if $K^2_S = 7$, then f has 5 double fibres and it has precisely one fibre with reducible support, consisting of two components.*

For $K_S^2 = 6$ the result is similar:

Theorem 11.3. *Let S be a minimal surface of general type with $p_g(S) = 0$ and $K_S^2 = 6$ for which the bicanonical map φ has degree 2. Then:*

(i) *there is a fibration $f\colon S \to \mathbb{P}^1$ such that the general fibre F of f is hyperelliptic of genus 3 and f has 4 or 5 double fibres;*
(ii) *the bicanonical involution of S induces the hyperelliptic involution on F.*

For the proof of theorem 11.2 see [MP4] and for the proof of theorem 11.3 see [MP5]. Section 13 exhibits examples of some of the cases described above. For other examples see [MP1] and [MP5].

Remark R.Pardini classified all the surfaces with $p_g = 0$, $K^2 = 8$ which are double covers of rational surfaces. It turns out that all these surfaces are obtained as a quotient of a product of curves $C \times D$ by a free group action. As such they are all obtained by the method of construction proposed by Beauville (see §13 or [BPV], Ch.VII).

In [Pa2], both the curves and the groups are completely described. Her main results are as follows:

Theorem 11.4 ([Pa2])**.** *Let S be a minimal complex projective surface of general type with $K_S^2 = 8$ and $p_g(S) = 0$.*

There exists an automorphism σ of S of order 2 such that S/σ is rational if and only if there exist a curve C, an hyperelliptic curve F of genus 3 or 5 and a finite group G such that:

a) *G acts faithfully on F and C and the diagonal action of G on $F \times C$ is free;*
b) *$|G| = (g(F) - 1)(g(C) - 1)$;*
c) *C/G and F/G are rational curves;*
d) *$S = (F \times C)/G$ and σ is the involution induced by $\tau \times Id$, where τ denotes the hyperelliptic involution of F.*

There are 5 types of such surfaces with the following numerical invariants:

Ia: $g(F) = 3$, $g(C) = 5$, $G = \mathbb{Z}_2^3$;
Ib: $g(F) = 3$, $g(C) = 9$, $G = \mathbb{Z}_2 \times D_4$;
Ic: $g(F) = 3$, $g(C) = 13$, $G = S_4$;
Id: $g(F) = 3$, $g(C) = 25$, $G = \mathbb{Z}_2 \times S_4$;
II: $g(F) = 5$, $g(C) = 16$, $G = A_5$.

Furthermore a minimal surface of general type with $p_g = 0$, $K_S^2 = 8$ has non birational bicanonical map if and only if S is of type Ia, Ib, Ic or Id.

Furthermore:

Corollary 11.5 ([Pa2]). *Denote by $\mathcal{M}$ be the moduli space of surfaces of general type with $p_g = 0$ and $K^2 = 8$. The set of surfaces of $\mathcal{M}$ with non birational bicanonical map is the union of 4 irreducible connected components of $\mathcal{M}$ of respective dimensions 5, 4, 3 and 3.*

12. Nodal curves, codes and involutions

The theorems 11.2 and 11.3 are proved making use of binary codes and the theory for involutions on a surface. For the precise proofs see the papers mentioned above.

Here we just give a brief explanation of these techniques.

12.1. **Involutions on surfaces with $p_g = 0$.** Let S be a smooth surface. An *involution* of S is an automorphism σ of S of order 2. The fixed locus of σ is the union of a smooth curve R and of k isolated points $P_1 \dots P_k$.

We denote by $\pi\colon S \to \Sigma := S/\sigma$ the projection onto the quotient, by B the image of R, by Q_i the image of P_i, $i = 1 \dots k$. The surface Σ is normal and $Q_1 \dots Q_k$ are ordinary double points, which are the only singularities of Σ. In particular, the singularities of Σ are canonical and the adjunction formula gives $K_S = \pi^* K_\Sigma + R$.

Let $\varepsilon\colon V \to S$ be the blow-up of S at $P_1 \dots P_k$ and let E_i be the exceptional curve over P_i, $i = 1 \dots k$. It is easy to check that σ induces an involution $\tilde\sigma$ of V whose fixed locus is the union of $R_0 := \varepsilon^{-1}R$ and of $E_1 \dots E_k$. Denote by $\tilde\pi\colon V \to W := V/\tilde\sigma$ the projection onto the quotient and set $B_0 := \tilde\pi(R_0)$, $C_i := \tilde\pi(E_i)$, $i = 1 \dots k$. The surface W is smooth and the C_i are smooth rational curves of self–intersection -2. Denote by $\eta\colon W \to \Sigma$ the map induced by ε. The map η is the minimal resolution of the singularities of Σ and there is a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\varepsilon} & S \\ {\scriptstyle\tilde\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ W & \xrightarrow{\eta} & \Sigma \end{array} \tag{12.1}$$

The map $\tilde\pi$ is a flat double cover branched on $B_0 + \sum C_i$, hence there exists a line bundle L on W such that $2L \equiv B_0 + \sum C_i$, $\tilde\pi_* \mathcal{O}_V = \mathcal{O}_W \oplus L^{-1}$ and σ acts on L^{-1} as the multiplication by -1.

We recall the following well–known formulas:

(Holomorphic Fixed Point Formula) (see [AS], pg.566):

$$\sum_{i=0}^{2}(-1)^i \mathrm{Trace}(\sigma|H^i(S,\mathcal{O}_S)) = \frac{k - R\cdot K_S}{4}$$

(Topological Fixed Point Formula) (see [Gr], (30.9)):

$$\sum_{i=0}^{4}(-1)^i \mathrm{Trace}(\sigma|H^i(S,\mathbb{C})) = k + e(R),$$

where $e(R) = -R^2 - R\cdot K_S$ is the topological Euler characteristic of R.

For surfaces with $p_g(S) = q(S) = 0$, as shown in [DMP] these formulas yield:

Lemma 12.1. *Let S be a surface with $p_g(S) = q(S) = 0$ and let σ be an involution of S. Let R be the divisorial part of the fixed locus of σ, let k be the number of isolated fixed points of σ and let t be the trace of $\sigma_{|H^2(S,\mathbb{C})}$. Then:*

$$k = K_S\cdot R + 4;\quad t = 2 - R^2.$$

Furthermore if V is the blow-up of the k isolated fixed points of σ, and $W = V/<\sigma>$ one has

$$\rho(S) + t = 2\rho(W) - 2k$$

.

We say that a map $\psi\colon S \to W$ is composed with σ if $\psi\circ\sigma = \psi$.

Proposition 12.2. *Let S be a minimal surface of general type with $p_g = q = 0$ and let σ be an involution of S. Then, with the above notation, $2K_W + B_0$ is nef and big.*

Furthermore:

i) $K_W L + L^2 = -2$;
ii) $K_W^2 + K_W L \geq 0$;
iii) $h^0(W, 2K_W + L) = K_W^2 + K_W L$;
iv) $k = K_S^2 + 4 - 2h^0(W, 2K_W + L)$;
v) $(2K_W + B_0)^2 = 2K_S^2$.

Proof. We have $\chi(\mathcal{O}_S) > 0$, since S is of general type. Since $p_g(S) = 0$ by assumption, one has $q(S) = 0$ and, as a consequence, $p_g(\Sigma) = p_g(W) = 0$ and $q(\Sigma) = q(W) = 0$. We also have $p_2(S) = \chi(\mathcal{O}_S) + K_S^2 = 1 + K_S^2$.

By standard double cover formulas, we have $\chi(\mathcal{O}_V) = 2\chi(\mathcal{O}_W) + \frac{1}{2}(L^2 + K_W L)$, thus statement i) follows from $p_g(W) = q(W) = 0$.

By the adjunction formula and commutativity of diagram (12.1), we have $\tilde{\pi}^*(2K_W + B_0) = 2K_V - 2\sum E_i = \varepsilon^*(2K_S)$. Then statement v) is obvious and $2K_W + B_0$ is nef and big because $2K_S$ is also nef and big.

We have the equality of $\mathbb{Q}-$divisors:

$$K_W + L = \frac{1}{2}(2K_W + B_0) + \frac{1}{2}\sum C_i.$$

The divisor $\frac{1}{2}(2K_W+B_0) = \frac{1}{2}\eta^*(2K_\Sigma+B)$ is nef and big and the divisor $\frac{1}{2}\sum C_i$ is effective with normal crossings support and zero integral part. Thus $h^i(W, 2K_W + L) = 0$ for $i > 0$ by Kawamata–Viehweg vanishing and so:

$$h^0(W, 2K_W + L) = \chi(2K_W + L) = 1 + K_W^2 + \frac{3}{2}K_W L + \frac{1}{2}L^2. \tag{12.2}$$

Since by statement i) one has $K_W L + L^2 = -2$, we obtain $h^0(W, 2K_W + L) = K_W^2 + K_W L$, and therefore statements ii) and iii).

Now for statement iv) it suffices to remember that $k = K_S^2 - K_V^2$ and $K_V^2 = 2(K_W + L)^2$ and use the equalities in statements i), iii). □

Corollary 12.3. *Let S be a minimal surface of general type with $p_g(S) = 0$ and let σ be an involution of S. If σ has $K_S^2 + 4$ isolated fixed points, the bicanonical map $\varphi\colon S \to \mathbb{P}^{K^2}$ is composed with σ. If in addition $|2K_S|$ has no fixed component, then φ is composed with σ iff σ has $K_S^2 + 4$ isolated fixed points.*

Proof. The adjunction formula gives

$$2K_V = \tilde{\pi}^*(2K_W + B_0 + \sum C_i)$$

and the projection formulas for double covers give

$$H^0(V, 2K_V) = H^0(W, 2K_W + L) \oplus H^0(W, 2K_W + B_0 + \sum C_i).$$

The bicanonical map φ is composed with σ iff either $H^0(W, 2K_W+L) = 0$ or $H^0(W, 2K_W+B_0+\sum C_i) = 0$. So the first statement is immediate from proposition 12.2, iv). For the second statement remark that the elements of $H^0(W, 2K_W + L)$ pull–back on V to the sections of $2K_V$ which are anti–invariant under σ, and they vanish on R_0. On the other hand, one has:

$$|2K_V| = \varepsilon^*|2K_S| + 2\sum E_i,$$

hence the fixed part of $|2K_V|$ is supported on $\sum E_i$, since by assumption $|2K_S|$ has no fixed component. It follows that $h^0(W, 2K_W + B_0 + \sum C_i) \neq 0$, and φ is composed with σ iff $h^0(W, 2K_W + L) = 0$. Now, again by 12.2, iv), the statement follows. □

Corollary 12.4. *Let S be a minimal surface of general type with $p_g = 0$ and σ an involution of S. If k is the number of isolated fixed points of S then $4 \leq k \leq K_S^2 + 4$. Furthermore:*

i) *if $k = 4$, then K_S^2 is even;*
ii) *If $k = K_S^2+4$ then φ is composed with σ and $|2K_S| = |\pi^*(2K_\Sigma + B|$*
iii) $K_W^2 \geq K_V^2$
iv) *$K_S R \leq K_S^2$ and equality holds iff $k = K_S^2 + 4$.*

Proof. By lemma 12.1, one has $k \geq 4$, whereas $k \leq K_S^2 + 4$ follows by proposition 12.2, iv).

Part i) follows by proposition 12.2, iv). Part ii) follows from 12.2, iv) and corollary 12.3. Statement iii) follows by the injection of $H^2(W, \mathbb{C})$ into $H^2(V, \mathbb{C})$ determined by $\tilde{\pi}$. Part iv) follows from lemma 12.1 and $k \leq K_S^2 + 4$. □

12.2. **Codes.** Given a smooth projective surface Y and k disjoint nodal curves $C_1, \ldots, C_k$ of Y, (recall that a nodal curve C is an irreducible curve satisfying $C^2 = -2, K_Y C = 0$), we define the binary code V associated to $C_1, \ldots, C_k$. Consider the map $\psi\colon \mathbb{Z}_2^k \to \operatorname{Pic}(Y)/2\operatorname{Pic}(Y)$ defined by $(x_1, \ldots, x_k) \mapsto \sum x_i[C_i]$, where $[D]$ denotes the class of a divisor D in $\operatorname{Pic}(Y)/2\operatorname{Pic}(Y)$. We define V to be the kernel of ψ and we denote by r its dimension.

The vector $v = (x^1 \ldots x^k) \in \mathbb{Z}_2^k$ is in V if and only if there exists $L_v \in \operatorname{Pic}(Y)$ such that $2L_v \equiv \sum x^i C_i$ (when it is convenient, we identify $0, 1 \in \mathbb{Z}_2$ with the integers $0, 1$). Notice that $K_Y \cdot L_v = 0$ and thus L_v^2 is even by the adjunction formula. The weight $w(v)$ of an element $v = (x_1, \ldots, x_k) \in V$ is the number of indices i such that the coordinate x_i is non zero. For any $v \in V$ $w(v)$ is equal to $-2L_v^2$ and so it is divisible by 4. Notice that L_v is uniquely determined by v if and only if ${}_2\operatorname{Pic}(Y) = 0$.

We say that a curve C_i appears in V if there exists $v = (x_1, \ldots, x_k) \in V$ with $x_i \neq 0$. We denote by m the number of curves C_i appearing in V. Let $\alpha\colon Y \to \Sigma$ be the contraction of the k nodal curves. The surface Σ is a normal surface with k singular points of type A_1.

This situation has been studied in detail in [DMP]. In particular, if we let G be the abelian group $\operatorname{Hom}(V, \mathbb{C}^*)$ then there exists a $G-$cover $p\colon Z \to \Sigma$ branched precisely over the nodes of Σ corresponding to the curves that appear in V. The numerical invariants of Z can be computed explicitly in terms of r, m and of the numerical invariants of Y. One has

$$\kappa(Z) = \kappa(Y);$$

$$K_Z^2 = 2^r K_Y^2;$$
$$\chi(\mathcal{O}_Z) = 2^r \chi(\mathcal{O}_Y) - m2^{r-3}.$$

It is sometimes possible to determine V by studying the properties of Z, and viceversa. For instance, if Y is a rational surface with $b_2(Y) \geq 5$ and the number k of disjoint $-2-$curves is the maximum possible ($= b_2(Y) - 2$), then this technique is used in [DMP] to show that b_2 is even and the code V is the code of "doubly even" vectors $DE(s)$, where $k = 2s$. Recall the definition of $DE(s)$: given the code of even vectors $W = \{\sum x_i = 0\} \subset \mathbb{Z}_2^s$, $DE(s)$ is the image of W via the injection $\mathbb{Z}_2^s \to \mathbb{Z}_2^{2s}$ defined by $(x_1, \ldots, x_s) \mapsto (x_1, x_1, \ldots, x_s, x_s)$.

In [DMP] it is shown that every rational surface Y with $b_2 - 2 \geq 3$ nodal curves can be obtained in the following way:

Example Consider a relatively minimal ruled rational surface $\mathbf{F}_e := \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$, $e \geq 0$, and a point $y \in \mathbf{F}_e$. If one blows up y, then the total transform of the ruling of $\mathbf{F}_e$ containing y is the union of two $(-1)-$curves E and E' that intersect transversely in a point y_1. If one blows up also y_1, then the strict transforms of E and E' are disjoint nodal curves. By repeating this procedure n times at points lying on different rulings of $\mathbf{F}_e$, one obtains a rational surface Y containing $2n$ disjoint nodal curves. One has $\rho(Y) = 2n + 2$ and it is easy to check that the code V associated to this collection of curves is $DE(n)$.

Remark Consider a double cover X of the surface Y described above, and assume this double cover is branched on the nodal curves and some smooth divisor not passing through the nodes. Consider the surface X' obtained from X by contracting the exceptional curves which are the pull back to X of the nodal curves. The fibration on Y corresponding to the ruling of Y has double fibres lying above the blown-up fibres of $\mathbf{F}_e$. This explains partly the statement of theorem 11.2.

13. **Some examples**

Here we present some examples that show that there exist surfaces as described in theorems 11.2 and 11.3.

Example 1. We start by describing an example with $K^2 = 7$ in the conditions of theorem 11.2. This example with $K_S^2 = 7$ is due to Inoue [In, remark 6], who constructed it as a quotient of a complete intersection in the product of four elliptic curves by a free action of $\mathbb{Z}_2^5$. Here an alternative description as a $\mathbb{Z}_2 \times \mathbb{Z}_2$-cover of a singular rational surface is given. With this new description it is possible to describe the bicanonical map and compute its degree.

Starting from the quadrilateral $P_1P_2P_3P_4$ in $\mathbb{P}^2$ of Figure 1, let P_5 be the intersection point of the lines P_1P_2 and P_3P_4 and P_6 the intersection point of P_1P_4 and P_2P_3. Write $\Sigma \to \mathbb{P}^2$ for the blowup of $P_1, \dots, P_6$, and e_i for the exceptional curves of Σ over P_i. Denote by l the pullback of a line.

Write $S_1, \dots, S_4$ for the strict transforms on Σ of the sides P_iP_{i+1} of the quadrilateral $P_1P_2P_3P_4$ (we take subscripts modulo 4); these are the only -2-curves of Σ. The morphism $f\colon \Sigma \to \mathbb{P}^3$ given by $|-K_\Sigma|$ has image a cubic surface $V \subset \mathbb{P}^3$, and f is an isomorphism on $\Sigma \setminus \bigcup S_i$, and contracts each S_i to an A_1 point.

If $A \subset \{P_1, \dots, P_6\}$ consists of 4 points no three of which are collinear, then the linear system of conics through the points of A gives rise to a free pencil on Σ; we denote by f_1 the strict transform of a general conic through $P_2P_4P_5P_6$, by f_2 that of a general conic through $P_1P_3P_5P_6$ and by f_3 that of a general conic through $P_1P_2P_3P_4$.

Finally, we introduce the "diagonals" of the quadrilateral $P_1P_2P_3P_4$, writing $\Delta_1, \Delta_2, \Delta_3$ for the strict transform of P_1P_3, P_2P_4 and P_5P_6. The divisors we have introduced satisfy the following relations:

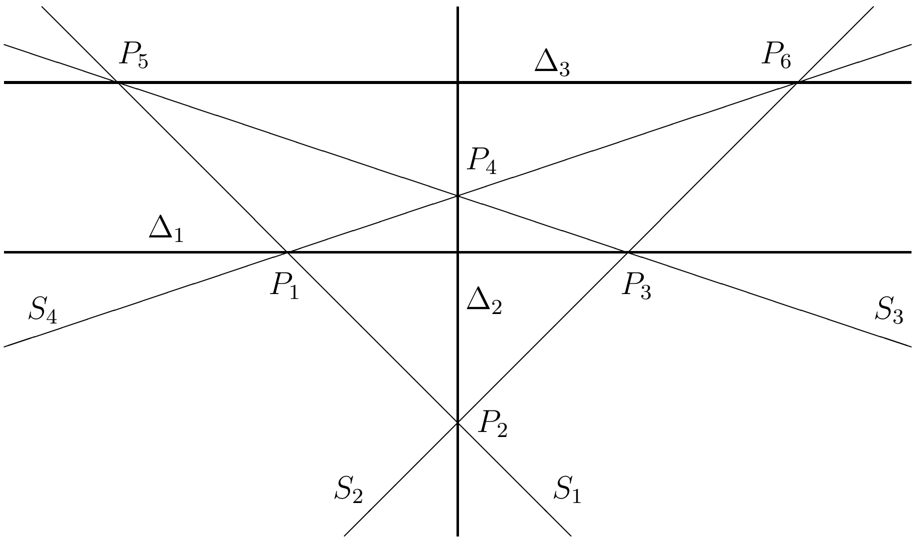


FIGURE 1. The quadrilateral $P_1P_2P_3P_4$ in $\mathbb{P}^2$

(i) $-K_\Sigma \equiv \Delta_1 + \Delta_2 + \Delta_3$;
(ii) $f_i \equiv \Delta_{i+1} + \Delta_{i+2}$ for all $i \in \mathbb{Z}_3$;
(iii) $\Delta_i S_j = 0$ for all i, j;
(iv) $\Delta_i f_j = 2\delta_{ij}$ for $1 \le i, j \le 3$.

To build a $\mathbb{Z}_2 \times \mathbb{Z}_2$ cover set:

(I) $D_1 = \Delta_1+f_2+S_1+S_2$, $D_2 = \Delta_2+f_3$, $D_3 = \Delta_3+f_1+f_1'+S_3+S_4$; where $f_1, f_1' \in |f_1|$, $f_2 \in |f_2|$, $f_3 \in |f_3|$ are general curves;

(II) $L_1 = 5l - e_1 - 2e_2 - e_3 - 3e_4 - 2e_5 - 2e_6$, and
$L_2 = 6l - 2e_1 - 2e_2 - 2e_3 - 2e_4 - 3e_5 - 3e_6$

and we obtain $L_3 = 4l-2e_1-2e_2-2e_3-e_4-e_5-e_6$. For $i = 1, \dots, 4$, the (set theoretic) inverse image of S_i in X is the disjoint union of two -1-curves E_{i1}, E_{i2}; contracting these 8 exceptional curves on X and contracting the S_i on Σ, we obtain a smooth $\mathbb{Z}_2 \times \mathbb{Z}_2$-cover $p\colon S \to V$. The map p is branched on the four singular points of V and on the image $\overline{D}$ of D, which is contained in the smooth locus of V. The bicanonical divisor $2K_X$ is equal to $\pi^*(2K_\Sigma+D) = \pi^*(-K_\Sigma+f_1+S_1+S_2+S_3+S_4) = \pi^*(-K_\Sigma+f_1)+2\sum E_{ij}$, and thus the bicanonical divisor $2K_S$ is equal to $\pi^*(-K_V+\overline{f}_1)$, where $\overline{f}_1$ is the image of f_1 in V. So $2K_S$ is ample, since it is the pullback of an ample line bundle by a finite map, S is minimal and of general type, and $K_S^2 = \frac{1}{4}4(K_V+\overline{f}_1)^2 = 7$.

To compute the geometric genus of S, recall that $p_g(X) = p_g(\Sigma) + \sum h^0(\Sigma, K_\Sigma+L_i)$ (see 4.2). We have

$$K_\Sigma + L_1 = 2l - e_2 - 2e_4 - e_5 - e_6,$$
$$K_\Sigma + L_2 = 3l - e_1 - e_2 - e_3 - e_4 - 2e_5 - 2e_6,$$
$$K_\Sigma + L_3 = l - e_1 - e_2 - e_3.$$

We show that $h^0(\Sigma, K_\Sigma+L_2) = 0$. Assume by contradiction that there exists $D \in |K_\Sigma+L_2|$ and consider the image C of D in $\mathbb{P}^2$; C is a cubic containing $P_1, \dots, P_6$ which has a double point at P_5 and P_6. By Bezout's theorem, Δ_3 is contained in C and thus $C = \Delta_3+Q$, where Q is a conic containing $P_1, \dots, P_6$, which is impossible. By similar (easier) arguments, one shows that $h^0(\Sigma, K_\Sigma+L_1) = h^0(\Sigma, K_\Sigma+L_3) = 0$, and thus $p_g(S) = p_g(X) = 0$. By the projection formula for a finite flat morphism the space $H^0(X, 2K_X)$ decomposes as

$$H^0(\Sigma, -K_\Sigma+f_1+\textstyle\sum S_j) \oplus \Big(\bigoplus_i H^0(\Sigma, -K_\Sigma+f_1+\sum S_j - L_i)\Big),$$

and Γ acts on $H^0(\Sigma, -K_\Sigma+f_1+\sum S_j-L_i)$ via the character χ_i. We have $h^0(\Sigma, -K_\Sigma+f_1+\sum S_j) = h^0(\Sigma, -K_\Sigma+f_1)$, since

$$S_j(-K_\Sigma+f_1+\sum S_i) = -2 \quad \text{for } i = 1, \dots, 4;$$

in addition, $h^0(\Sigma, -K_\Sigma+f_1) = 7$, since Σ is rational, $2f_1+f_2+f_3$ has arithmetic genus 7, and $-K_\Sigma+f_1 = K_\Sigma+2f_1+f_2+f_3$. Since $p_2(S) = 8$, there is a value $i \in \{1,2,3\}$ such that $h^0(\Sigma, -K_\Sigma+\sum S_j+f_1-L_i) = 1$ and $h^0(\Sigma, -K_\Sigma+\sum S_j+f_1-L_k) = 0$ for $k \neq i$. Actually, an argument similar to that used for computing $p_g(S)$ shows that

$$h^0(-K_\Sigma + \sum S_j + f_1 - L_1) = h^0(\sum S_j + e_4) = 1,$$
$$h^0(-K_\Sigma + \sum S_j + f_1 - L_2) = h^0(3l - e_1 - 2e_2 - e_3 - 2e_4 - e_5 - e_6) = 0,$$
$$h^0(-K_\Sigma + \sum S_j + f_1 - L_3) = h^0(5l - e_1 - 2e_2 - e_3 - 3e_4 - 3e_5 - 3e_6) = 0.$$

It follows that the bicanonical map $\varphi\colon S \to \mathbb{P}^7$ is composed with the involution γ_1 but not with γ_2 and γ_3. Since $|2K_S| \supset \pi^*|-K_\Sigma|$ and the map $\Sigma \to \mathbb{P}^3$ induced by $|-K_\Sigma|$ is birational, it follows that φ has degree 2. The linear system $|f_1|$ induces a free pencil F of hyperelliptic curves of genus 3. The bicanonical involution restricts to the hyperelliptic involution on the general F. The pencil $|F|$ has 5 double fibres, corresponding to the pull backs of f_1, f_1', $\Delta_2 + \Delta_3$ and of the two fibres of $|f_1|$ containing the $-2-$curves.

Example 2. Next we describe an example for theorem 11.3. This is obtained as a specialization of the above example, and has also been obtained by Inoue as a specialization of his construction ([In]).

In the same set-up as in the example above assume that f_1, f_2 and f_3 all pass through a general point P and that f_i and f_j intersect transversely at P for $i \neq j$. In other words, in the terminology of [Ca] we let the branch locus D acquire a $(1,1,1)$ point. Denote by $p_0\colon S_0 \to V$ the corresponding $\mathbb{Z}_2^2-$cover. The surface S_0 has a singularity of type $\frac{1}{4}(1,1)$ over the image P' of P in V. This singularity can be solved by taking base change with the blow up $\hat{V} \to V$ at P' and normalizing. Let $p\colon S \to \hat{V}$ be the cover thus obtained. The exceptional divisor of $S \to S_0$ is a smooth rational curve with self-intersection -4. The surface S is smooth of general type with $p_g(S) = 0$ and $K_S^2 = 6$ (see [Ca]). A computation very similar to the one in the example above shows that the bicanonical map of S has degree 2 and that the bicanonical involution coincides with γ_1. As before, the linear system $|f_1|$ induces on S a free pencil $|F|$ of hyperelliptic curves of genus 3 such that the bicanonical involution restricts to the hyperelliptic involution on the general F. Now the pencil $|F|$ has 4 double fibres, corresponding to the pull backs of f_1', of $\Delta_2 + \Delta_3$ and of the two fibres of $|f_1|$ containing the $-2-$curves. Notice that in this case the pull back of f_1 contains with multiplicity 1 the exceptional curve of the resolution $S \to S_0$, hence it is not a multiple fibre.

Example 3. This is a new example, again for theorem 11.3 appearing in [MP5]. With the above notation we consider the point $P_7 = \Delta_2 \cap \Delta_3$ and denote by Σ' the blow-up of Σ at P_7 and by e_7 the corresponding exceptional divisor.

We denote by the same letter the line bundles/divisors on Σ and their pull backs to Σ'. Denote by $\overline{\Delta}_2$ and $\overline{\Delta}_3$ the strict transforms of Δ_2 and Δ_3 and set:

1) $D_1 = C + S_1 + S_2$,
$D_2 = f_3$,
$D_3 = f_1 + f_1' + \overline{\Delta}_2 + \overline{\Delta}_3 + S_3 + S_4$;

where $f_1, f_1' \in |f_1|$, $f_3 \in |f_3|$ are general curves and $C \in |f_2 + f_3 - 2e_7|$ is also general;

2) $L_1 = 5l - e_1 - 2e_2 - e_3 - 3e_4 - 2e_5 - 2e_6 - e_7$, and
$L_2 = 7l - 2e_1 - 3e_2 - 2e_3 - 3e_4 - 3e_5 - 3e_6 - 2e_7$

and we obtain $L_3 = 4l - 2e_1 - 2e_2 - 2e_3 - e_4 - e_5 - e_6 - e_7$.

Since $f_2 + f_3 = 4l - 2e_1 - e_2 - 2e_3 - e_4 - e_5 - e_6$, it is not difficult to show, for instance by applying a Cremona transformation centered at P_1, P_3, P_7, that the general $C \in |f_2 + f_3 - 2e_7|$ is irreducible. Since $p_a(C) = 0$, the general C is also smooth. Thus we obtain a smooth $\mathbb{Z}_2^2$−cover $\pi\colon X \to \Sigma'$. To compute the geometric genus of X, recall that $p_g(X) = p_g(\Sigma') + \sum h^0(\Sigma', K_{\Sigma'} + L_i)$ (cf. [Pa1, lemma 4.2]). We have

$$\begin{aligned}
K_{\Sigma'} + L_1 &= 2l - e_2 - 2e_4 - e_5 - e_6,\\
K_{\Sigma'} + L_2 &= 4l - e_1 - 2e_2 - e_3 - 2e_4 - 2e_5 - 2e_6 - e_7,\\
K_{\Sigma'} + L_3 &= l - e_1 - e_2 - e_3.
\end{aligned}$$

Clearly both $h^0(\Sigma', K_{\Sigma'} + L_1)$ and $h^0(\Sigma', K_{\Sigma'} + L_3)$ vanish.

Now we show that $h^0(\Sigma', K_{\Sigma'} + L_2) = 0$. Assume otherwise and let $\Gamma' \in |K_{\Sigma'} + L_2|$. The image Γ of Γ' in $\mathbb{P}^2$ is a quartic containing $P_1, \ldots, P_6, P_7$ which has double points at P_2, P_4, P_5, P_6. By Bezout's theorem, the lines in $\mathbb{P}^2$ corresponding to S_1 and S_2 are contained in Γ and thus $\Gamma' = S_1 + S_2 + Q'$, where Q' is the strict transform of a conic Q containing P_4, P_5, P_6, P_7 and having a double point at P_4. But obviously there is no such Q because P_5, P_6, P_7 all lie on the line Δ_3, which does not contain P_4. Hence $p_g(X) = 0$.

For every $i = 1, \ldots, 4$, the (set–theoretic) inverse image of S_i in X is the disjoint union of two −1−curves E_{i1}, E_{i2}. Also the inverse image of $\overline{\Delta}_2$ is the disjoint union of two −1−curves E_1, E_2. The bicanonical divisor $2K_X$ is equal to $\pi^*(2K_{\Sigma'} + D) = \pi^*(-K_{\Sigma'} + f_1 + \bar{\Delta}_2 + S_1 + S_2 + S_3 + S_4) = \pi^*(-K_{\Sigma'} + f_1) + 2E_1 + 2E_2 + 2\sum E_{ij}$.

The system $|-K_{\Sigma'}|$ gives a degree 2 morphism $\Sigma' \to \mathbb{P}^2$. Hence $-K_{\Sigma'} + f_1$ is nef and big and it is easy to check that the linear system $|-K_{\Sigma'} + f_1|$ is birational of (projective) dimension 5. It follows that the surface S obtained from X by contracting E_1, E_2 and the E_{ij} is minimal of general type and the rational map $S \to \Sigma'$ is composed with the bicanonical map φ of S. We denote by the same letter the involutions of S induced by γ_1, γ_2, γ_3.

Since $2K_X = \pi^*(-K_{\Sigma'} + f_1) + 2E_1 + 2E_2 + 2\sum E_{ij}$ one has $K_S^2 = \frac{1}{4}(2K_S)^2 = \frac{1}{4}4(-K_{\Sigma'} + f_1)^2 = 6$.

The space $H^0(X, 2K_X)$ decomposes as:

$$H^0(\Sigma', -K_{\Sigma'}+f_1+\overline{\Delta}_2+\sum S_j)\oplus(\oplus_i H^0(\Sigma', -K_{\Sigma'}+f_1+\overline{\Delta}_2+\sum S_j-L_i)),$$

where $\mathbb{Z}_2^2$ acts on $H^0(\Sigma', -K_{\Sigma'} + f_1 + \overline{\Delta}_2 + \sum S_j - L_i)$ via the character χ_i. Since $P_2(S) = 7$ and $h^0(\Sigma', -K_{\Sigma'}+f_1+\overline{\Delta}_2+\sum S_j) = h^0(\Sigma', -K_{\Sigma'}+f_1) = 6$, it follows that $h^0(\Sigma', -K_{\Sigma'}+f_1+\overline{\Delta}_2+\sum S_j-L_i)$ is equal to 1 for one of the indices $i_0 \in \{1, 2, 3\}$ and it is equal to 0 for the remaining two, so that the bicanonical map has degree 2 and the bicanonical involution is γ_{i_0}. A computation shows $h^0(\Sigma', -K_{\Sigma'} + f_1 + \overline{\Delta}_2 + \sum S_j - L_1)) = h^0(\Sigma', e_4+\overline{\Delta}_2+S_1+S_2+S_3+S_4) = 1$, hence the bicanonical involution of S coincides with γ_1. The linear system $|f_1|$ induces on S a free pencil $|F|$ of hyperelliptic curves of genus 3 such that the bicanonical involution restricts to the hyperelliptic involution on the general F. Now the pencil $|F|$ has 5 double fibres, corresponding to the pull backs of f_1, f_1', of the two fibres of $|f_1|$ containing the curves $S_1, \ldots, S_4$ and of the fibre $\overline{\Delta}_2 + \overline{\Delta}_3 + 2e_7$. Let $2A$ be this last fibre of $|F|$ on S. The support of A is the union of an elliptic curve with self–intersection -2, corresponding to e_7, and of a -2–curve, corresponding to $\overline{\Delta}_3$ (recall that the inverse image of $\overline{\Delta}_2$ in X has been contracted in S). The two components of A meet at two points.

Example 4. This is in fact a specialization of Example 3, obtained by letting D_2 contain the curve $\overline{\Delta}_2$. In this case the cover $\pi\colon X \to \Sigma'$ is not normal. The normalization X' of X is again a $\mathbb{Z}_2^2$–cover of Σ' with branch divisors:

$D_1 = C + \overline{\Delta}_2 + S_1 + S_2$,

$D_2 = \Delta_1 + e_7$,

$D_3 = f_1 + f_1' + \overline{\Delta}_3 + S_3 + S_4$ (cf. [Ca]).

The minimal model S of X' is a surface with the same properties as before, but the strict transform of $\overline{\Delta}_2$ is now a -2–curve. Furthermore, if we denote again by $2A$ the reducible double fibre of the pencil $|F|$ on S, then $A = \theta_1 + \theta_2 + 2E$, where θ_1 and θ_2 are disjoint -2–curves, corresponding to $\overline{\Delta}_2$ and $\overline{\Delta}_3$ and E is an elliptic curve with $E^2 = -1$, corresponding to e_7. One has $\theta_1 E = \theta_2 E = 1$.

Remark Notice that for Examples 2 and 3 the divisor K_S is not ample, in contrast with the case of Burniat surfaces and of surfaces with $\deg\varphi = 2$ and $K_S^2 \geq 7$.

More examples Other examples are obtained using the following construction due to Beauville (see [Be3, p. 123, Ex. 4] and cf. [Do]). Let C_1, C_2 be curves of genus g_1, g_2, and assume that a group G of order $(g_1-1)(g_2-1)$ acts on C_1, C_2 so that C_i/G is isomorphic to $\mathbb{P}^1$ for $i=1,2$; write $p_i\colon C_i \to \mathbb{P}^1$ for the projections onto the quotients and $p\colon C_1\times C_2 \to \mathbb{P}^1\times\mathbb{P}^1$ for the product of p_1 and p_2. Thus p is a Galois cover with group $G\times G$. Assume in addition that there exists an automorphism $\psi\in \operatorname{Aut} G$ whose graph $\Gamma=\Gamma_\psi\subset G\times G$ acts freely on $C_1\times C_2$. Then set $S=(C_1\times C_2)/\Gamma$ and denote by $q\colon C_1\times C_2\to S$ the quotient map and by $\pi\colon S\to\mathbb{P}^1\times\mathbb{P}^1$ the map induced by p. If G is Abelian, then π is a G-cover. The surface S is minimal and of general type since $C_1\times C_2$ is minimal of general type and q is étale. Since Γ acts freely, $\chi(\mathcal{O}_{C_1\times C_2})=|G|\chi(\mathcal{O}_S)$ and $K^2_{C_1\times C_2}=|G|K^2_S$, namely $\chi(\mathcal{O}_S)=1$, $K^2_S=8$. The irregularity $q(S)$ equals the dimension of the Γ-invariant subspace of $H^0(C_1\times C_2,\Omega^1_{C_1\times C_2})\cong H^0(C_1,\omega_{C_1})\oplus H^0(C_2,\omega_{C_2})$. Since C_1/G and C_2/G are both rational and ψ is an automorphism, it follows that $q(S)=0$, and thus $p_g(S)=0$. For some specific examples of this method of construction see, in addition to the above cited references, [MP1] and [Pa2].

14. Some open problems

Here we point out some questions that arise naturally from the results explained in the previous sections.

Question 1 (cf. §7) Is the bicanonical map φ of surfaces with $p_g=0$ a morphism also for $2\le K^2_S\le 4$? If not can one characterize the possible base points and fixed components of $|2K_S|$?

Question 2 (cf. Theorem 8.1) Is there a surface with $p_g=0$, $K^2_S=3$ or $K^2_S=4$ and $\deg\varphi=5$?

Notice that for such a surface φ cannot be a morphism.

Question 3 (cf. theorems 8.1, 9.1) Is there a surface with $p_g=0$, $K^2_S=3$ for which the bicanonical map is a morphism of degree 3?

Notice that if so it would be the unique case in which φ is a morphism of *odd* degree.

Question 4 (cf. theorem 10.2)

Is it possible to characterize surfaces with $K^2_S=5$, $p_g=0$ and $\deg\varphi=4$?

Let us point out that in adition to the surfaces obtained by degeneration of the Burniat construction, recently also F. Catanese gave new examples of surfaces with $K_S^2 = 5$ (see [Ca]) and it would be interesting to calculate the degree of the bicanonical map of these surfaces.

Question 5 (cf. §13) Are the Inoue surfaces the only surfaces with $p_g = 0$, $K_S^2 = 7$ and non birational bicanonical map?

Question 6 (cf. theorems 10.3, 11.5) For $K_S^2 = 6, 7$ do the surfaces with degree of the bicanonical map = 2 fill up one (or more) components of the moduli space or can they be deformed to surfaces with birational bicanonical map?

References

[AS] M.F. Atiyah, I.M. Singer, *The index of elliptic operators: III*, Ann. of Math. **87** (1968), 546–604.

[Be1] A. Beauville, *L'application canonique pour les surfaces de type général*, Invent. Math., **55** (1979), 121–140.

[Be2] A. Beauville, *L'inégalité $p_g \geq 2q - 4$ pour les surfaces de type générale*, Bull. Soc. Math. de France, **110** (1982), 343–346.

[Be3] A. Beauville, *Complex Algebraic Surfaces*, Cambridge University Press (1983).

[Bo] E. Bombieri, *Canonical models of surfaces of general type*, Publ. IHES **42** (1973), 447–495.

[BPV] W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **3.** Folge, Band **4**, Springer-Verlag, Berlin (1984).

[BS] M. Beltrametti, A. Sommese *Zero-cycles and kth order embeddings of smooth projective surfaces*, in Problems in the theory of surfaces and their classification (F. Catanese, C. Ciliberto, M. Cornalba ed.), Symposia Math. **32** (1991), 33-48.

[Bu] P. Burniat, *Sur les surfaces de genre $P_{12} > 0$*, Ann. Mat. Pura Appl., (4) 71 (1966).

[Ca] F. Catanese, *Singular bidouble covers and the construction of interesting algebraic surfaces* in Algebraic Geometry: Hirzebruch 70, A.M.S. Contemporary Mathematics, vol. 241 (1999).

[CC1] F. Catanese, C. Ciliberto, *On the irregularity of cyclic coverings of algebraic varieties*, in *Geometry of Complex Projective Varieties - Cetraro*, (Italy), June 1990, (A. Lanteri, M. Palleschi, D. C. Struppa eds.), Mediterranean Press (1993), 89–116.

[CC2] F. Catanese, C. Ciliberto, *Surfaces with $p_g = q = 1$*, Symposia Mathematica I.N.D.A.M., vol. XXXII, Academic Press (1991), 49–79.

[CP] F. Catanese, R. Pignatelli, *On simply connected Godeaux surfaces*, Complex analysis and algebraic geometry, de Gruyter, Berlin (2000), 117–153.

[CT] F. Catanese, F. Tovena, *Vector bundles, linear systems and extensions of π_1*, *Complex algebraic varieties* (Bayreuth, 1990), Lecture Notes in Math., **1507**, Springer, Berlin (1992), 51–70.

[Ci1] C. Ciliberto, *The bicanonical map for surfaces of general type*, Proceedings of Symposia in Pure Mathematics **62.1** (1997), 57–84.

[Ci2] C. Ciliberto, *Alcuni aspetti della classificazione delle famiglie di curve in uno spazio proiettivo*, Boll. Un. Mat. Ital., A (7) **10** (1996), no. 3, 491–535.

[Co] F. Cossec, *Projective models of Enriques surfaces*, Math. Ann. **265** (1983), 283-334.

[CD] F. Cossec, I. Dolgachev, *Enriques surfaces I*, Progress in Mathematics **76**, Birkhäuser (1989).

[DF] M. De Franchis, *Sugl'integrali di Picard relativi a una superficie doppia*, Rend. Circ. Mat. Palermo, XX (1905), 331–334.

[De] O. Debarre, *Inégalités numériques pour les surfaces de type général*, Bull. Soc. Math. de France, **110** (1982), 319–342; *Addendum*, Bull. Soc. Math. France **111** (1983), no. 3, 301–302.

[DMP] I. Dolgachev, M. Mendes Lopes, R. Pardini, *Rational surfaces with many nodes*, Compositio Math., **132** (2002), no. 3, 349–363.

[Do] I. Dolgachev, *Algebraic surfaces with* $p = p_g = 0$, in *Algebraic surfaces* (C.I.M.E. 1977 Varenna), Liguori Napoli (1981), 97–215.

[Eb] W. Ebeling, *Lattices and Codes*, Adv. Lectures in Mathematics, Vieweg, 1994.

[FP] B. Fantechi, R. Pardini, *Automorphisms and moduli spaces of varieties with ample canonical class via deformations of Abelian covers*, Comm. in Alg., **25** (1997), 1413–1441.

[F] P. Francia, *On the base points of the bicanonical system*, Symposia Mathematica I.N.D.A.M., vol. XXXII, Academic Press (1991), 141–150.

[Gr] M. Greenberg, *Algebraic Topology: A First Course*, W. A. Benjamin Publ., Reading, Mass. 1981.

[Hl] E. Halanay, *Normal surfaces of degree* 8 *in* $\mathbb{P}^n$, Ann. Univ. Ferrara Sez. VII (N.S.), **41** (1995), 131–145.

[Ho] E. Horikawa, *On algebraic surfaces with pencils of curves of genus* 2 in *Complex analysis and algebraic geometry*, pp. 79–90. Iwanami Shoten, Tokyo, 1977.

[In] M. Inoue, *Some new surfaces of general type*, Tokyo J. of Math., Vol. 17 No. 2 (1994), 295–319.

[Io] P. Ionescu, *Embedded projective varieties of small invariants*, in *Algebraic geometry, Bucharest 1982*, Lecture Notes in Math., **1056**, Springer, Berlin, (1984), 142–186.

[Ko] D. Kotschick, *On the pluricanonical maps of Godeaux and Campedelli surfaces*, Internat. J. Math. **5**, no. 1 (1994), 53–60.

[La] A. Langer, *Pluricanonical systems on surfaces with small* K^2, International J. of Math. **11** (3) (2000), 379–392.

[Mi] Y. Miyaoka, *The maximal number of quotient singularities on surfaces with given numerical invariants*, Math. Ann., **268** (1973), 159–171.

[MP1] M. Mendes Lopes, R. Pardini, *The bicanonical map of surfaces with* $p_g = 0$ *and* $K^2 \geq 7$, Bull. London Math. Soc. **33** (2001), 1–10.

[MP2] M. Mendes Lopes, R. Pardini, *A connected component of the moduli space of surfaces of general type with* $p_g = 0$, Topology **40** (5) (2001), 977–991.

[MP3] M. Mendes Lopes, R. Pardini, *Enriques surfaces with eight nodes*, Math. Z., **241** (2002), no. 4, 673–683.

[MP4] M. Mendes Lopes, R. Pardini, *The bicanonical map of surfaces with* $p_g = 0$ *and* $K^2 \geq 7$*, II*, Bull. London Math. Soc. (to appear).

[MP5] M. Mendes Lopes, R. Pardini, *The bicanonical mapclassification of surfaces with $p_g = 0$, $K_S^2 = 6$ and non birational bicanonical map*, preprint 2003,math.AG/0301138.

[Na] M. Nagata, *On rational surfaces I.*, Mem. Coll. Sci., U. of Kyoto, ser. A, vol. XXXII, Mathematics No. 3 (1960), 351–370.

[Nai] D. Naie, *Surfaces d'Enriques et une construction de surfaces de type général avec $p_g = 0$*, Math. Z. **215** (2) (1994), 269–280.

[Pa1] R. Pardini, *Abelian covers of algebraic varieties*, J. reine angew. Math. **417** (1991), 191–213.

[Pa2] R. Pardini, *The classification of double planes of general type with $p_g = 0$ and $K^2 = 8$*, preprint Pisa, (2001) (math.AG/0107100), to appear in J.Algebra.

[Pe] C. Peters, *On certain examples of surfaces with $p_g = 0$ due to Burniat*, Nagoya Math. J., Vol.166 (1977), 109–119.

[Re] I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. of Math., **127** (1988), 309–316.

[SGA5] A. Grothendieck, L. Illusie, *Exposé III* in *Cohomologie ℓ-adique et fonctions L. (Seminaire de Géométrie Algébrique du Bois-Marie 1965-66 SGA 5).* Lecture Notes in Mathematics, 589, Springer-Verlag.

[W] L. Weng, *A result on bicanonical maps of surfaces of general type*, Osaka J. Math., **32** (1995), no. 2, 467–473.

[X1] G. Xiao, *Finitude de l'application canonique des surfaces de type général*, Bull. Soc. Math. France, **113** (1985), 23–51.

[X2] G. Xiao, *Degree of the bicanonical map of a surface of general type*, Amer. J. of Math., **112** (5) (1990), 713–737.

Margarida Mendes Lopes
CMAF
Universidade de Lisboa
Av. Prof. Gama Pinto, 2
1649-003 Lisboa, PORTUGAL
mmlopes@ptmat.ptmat.fc.ul.pt