These are preliminary notes from a series of lectures given by Cirto Ciliberto at IMAR Bucharest, in February 2003. They refer to recent work in collaboration by Ciliberto and Francesco Russo. The proof of Enriques' theorem (see section **??**) is simplified in a separate file.

1. The tangent cone to a higher secant variety

We describe the tangent cone to the variety S^kX of $k+1$ secant $\mathbb{P}^k$, $k \geq 1$, at a general point of S^lX, $0 \leq l < k$, where $X \subset \mathbb{P}^N$ is an irreducible algebraic variety. In characteristic zero the classical Terracini Lemma describes the tangent space to S^kX at a general point of it. By $S(Y,Z) \subset \mathbb{P}^N$ we indicate the join of the irreducible varieties $Y, Z \subset \mathbb{P}^N$. With this notation, $S^kX = S(X, S^{k-1}X) = S(S^lX, S^hX)$ if $l + h = k-1$, $l \geq 0$, $h \geq 0$, $S^0X := X$. The embedded projective tangent space to a variety X at a point $x \in X$ is indicated by T_xX, while the tangent cone (scheme) to X at a point $x \in X$ by C_xX. The dimension of S^lX, $l \geq 1$, is $s_l(X)$. We suppose that $S^kX \subsetneqq \mathbb{P}^N$ to avoid trivialities.

Our first result is the following.

Proposition 1.1. *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety and let $h, l \in \mathbb{N}$ be such that $h + l = k - 1$. If $z \in S^hX$ is a general point, then $S(T_zS^hX, S^lX)$ is an irreducible component of $(C_zS^kX)_{\mathrm{red}}$. Moreover, $mult_zS^kX \geq \deg(S(T_zS^hX, S^lX)) = \deg(X^{h+1})$, where $X^{h+1} \subset \mathbb{P}^{N-s_h(X)-1}$ is the projection of X from the linear space T_zS^hX.*

Proof. The scheme C_zS^kX is of pure dimension $s_k(X)$. By Terracini Lemma and by the generality of $z \in S^hX$, we get $\dim(S(T_zS^hX, S^lX)) = \dim(S(S^hX, S^lX)) = s_k(X)$. Since $S(T_zS^hX, S^lX)$ is irreducible, it is enough to prove the inclusion $S(T_zS^hX, S^lX) \subseteq C_zS^kX$.

Let $w \in S^lX$ be a general point. We claim that $w \notin T_zS^hX$. By definition of h, we have $S^hX \neq \mathbb{P}^N$ so that

$$Vert(S^hX) := \bigcap_{y \in S^hX} T_yS^hX$$

is either empty or a proper linear subspace of $\mathbb{P}^N$. If the general point of S^lX is contained in $Vert(S^hX)$, then $X \subseteq S^lX \subseteq Vert(S^hX)$ and X would be degenerated, contrary to our assumption.

Since $w \notin T_zS^hX$, z is a smooth point of the cone $S(w, S^hX)$. ¿From Terracini Lemma we deduce that

$$< w, T_zS^hX >= T_zS(w, S^hX) = C_zS(w, S^hX) \subseteq C_zS(S^lX, S^hX) = C_zS^kX.$$

By the generality of $w \in S^lX$ we finally have $S(T_zS^hX, S^lX) \subseteq C_zS^kX$. $\square$

We collect some easy remarks on the behavior of higher secant varieties under projection from a general point of a variety or from the tangent space to a higher secant variety.

Lemma 1.2. *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety and let $k \geq 0$ be an integer such that $S^kX \subsetneq \mathbb{P}^N$. Then the general point $x \in X$ does not belong to $Vert(S^kX)$. Moreover, if $\pi_x : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$ is the projection from a point $x \in X \setminus Vert(S^kX)$, then $\pi_x(S^kX) = S^k(\pi_x(X))$, $\dim(S^kX) = \dim(S^k\pi_x(X))$ so that $\deg(S^kX) = \deg(\pi_{x_{|S^kX}})\deg(S^k\pi_x(X)) + \operatorname{mult}_x S^kX \geq \deg(S^k\ i_x(X)) + \operatorname{mult}_x S^kX$. In particular if $\deg(S^kX) = \deg(S^k\pi_x(X)) + \operatorname{mult}_x S^kX$, then $\pi_{x_{|S^kX}} : S^kX \dashrightarrow S^k\pi_x(X)$ is birational.*

Proof. If the general point of X belongs to $Vert(S^kX)$, then $Vert(S^kX)$ would be a proper linear subspace containing X and X would be degenerated, proving the first assertion of the lemma.

If $x \in X \setminus Vert(X)$, then $\dim(\pi_x(X)) = \dim(X)$ and

$$\deg(X) = \deg(\pi_{x_{|X}})\deg(\pi_x(X)) + \operatorname{mult}_x X \geq \deg(\pi_x(X)) + \operatorname{mult}_x X.$$

So the assertion is true for $k = 0$ and we proceed by induction on k. By definition $S^k(\pi_x(X)) = S(\pi_x(X), S^{k-1}\pi_x(X)))$ and by induction $S(\pi_x(X), S^{k-1}\pi_x(X)) = S(\pi_x(X), \pi_x(S^{k-1}X))$. Moreover we have that $\pi_x(S^kX) = \pi_x(S(X, S^{k-1}X)) \subseteq S(\pi_x(X), \pi_x(S^{k-1}X))$ and both are irreducible. To conclude it is enough to prove that they have the same dimension. Recall that by the first part of the lemma $x \notin Vert(S^lX)$ for every $l \geq 0$ such that $S^lX \subsetneq \mathbb{P}^N$ so that π_x restricted to X and to $S^{k-1}X$ is generically smooth. For general $y \in X$ and for general $z \in S^{k-1}X$ we get by Terracini lemma that $x \notin < T_yX, T_zS^{k-1}X >$ because $x \notin Vert(S^kX)$. The conclusion follows by generically smoothness and by Terracini lemma applied to $S(\pi_x(X), \pi_x(S^{k-1}X))$: if $y' = \pi_x(y)$ and $z' = \pi_x(z)$ are the projections of the general points $y \in X$ and $z \in S^{k-1}X$, then $T_{y'}\pi_x(X)$, respectively $T_{z'}\pi_x(X)$, are the projections of T_yX and $T_zS^{k-1}X$ by generically smoothness and the restriction of π_x to $< T_yX, T_zS^{k-1}X >$ is an isomorphism.

□

Let us recall the following notation. Let $S^k_X \subset \mathbb{G}(k, N) \times \mathbb{P}^N$ be the *abstract symmetrized join* of X and let $p^k_X : S^k_X \to S^kX \subseteq \mathbb{P}^N$ be the projection onto the last factor. If $\dim(S^kX) = \dim(S^k_X) = (k+1)n + k$, then p^k_X is a generically finite morphism, whose degree equals the number of $(k+1)$-secant $\mathbb{P}^k$ passing through the general point of S^kX.

Lemma 1.3. *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety and let $k \geq 0$ be an integer such that $S^kX \subsetneq \mathbb{P}^N$. Assume that $\dim(S^kX) = (k+1)n + k$, i.e. that p^k_X is generically finite. Let $x \in X \setminus Vert(S^kX)$ and let π_x be as above. If $\pi_x : S^kX \dashrightarrow S^k\pi_x(X)$ is birational, then $0 < \deg(p^k_{\pi_x(X)}) = \deg(p^k_X)$.*

Proof. Letting notations as in the previous lemma, we have that π_x induces a commutative diagram of rational maps

$$\begin{tikzcd}
S^k_X \arrow[r, dashed, "\pi_x"] \arrow[d, "p^X_k"'] & S^k_{\pi_x(X)} \arrow[d, "p^{\pi_x(X)}_k"] \\
S^kX \arrow[r, dashed, "\pi_x"] & S^k\pi_x(X)
\end{tikzcd}$$

and the conclusion follows. □

2. Minimal degree of a higher secant variety

It is well known that an irreducible non-degenerate variety $X \subset \mathbb{P}^N$ has degree greater or equal to $\mathrm{codim}(X)+1$. Varieties whose degree is equal to $\mathrm{codim}(X)+1$ are called varieties of minimal degree. The degree of higher secant varieties satisfies a stronger bound.

Let us indicate by X^h as above the projection of X from the tangent space to $S^{h-1}X$ at a general point on it. In particular X^1 is the projection of X from a general tangent space to X. By X_h we will indicate the projection of X from h general points on it.

Theorem 2.1. *Let $X \subset \mathbb{P}^{s_k(X)+h}$, $h = \mathrm{codim}(S^kX)$, be an irreducible non-degenerate variety. Then*

$$\deg(S^kX) \geq \binom{\mathrm{codim}(S^kX)+k+1}{k+1}.$$

If equality holds and if $h \geq 1$, then:

i) *$C_xS^kX = S(T_xX, S^{k-1}X)$ for $x \in X$ a general point and $\mathrm{mult}_x S^kX = \binom{k+h}{k}$;*
ii) *the projection of X from m points on it, X_m, $1 \leq m \leq h$, are irreducible varieties for which $\deg(S^kX_m) = \binom{h-m+k+1}{k+1}$;*
iii) *for every $1 \leq j \leq h$ the projections $\pi_x : S^kX_{j-1} \dashrightarrow S^kX_j$ are birational;*
iv) *the projections from the general tangent space to $S^{m-1}X$, X^m, $1 \leq m \leq k$, are irreducible varieties for which $\deg(S^{k-m}X) = \binom{h+k-m+1}{k-m+1}$ so that X^k is a variety of minimal degree;*
v) *if moreover X is not k-defective, then $0 < \deg(p^k_{X_m}) = \deg(p^k_X)$ for every $1 \leq m \leq h$ so that $\nu_k(X_h) = \deg(p^k_X)$.*
vi) *if moreover X is not k-defective and if $\sigma_k : X \dashrightarrow X^k \subset \mathbb{P}^{n+h}$ is the projection from $T_zS^{k-1}X$, $z \in S^{k-1}X$ general, then $0 < \deg(\sigma_k) \leq \deg(p^k_X)$.*

Proof. By induction on k and h. For $k = 0$ we have the known bound for the minimal degree of an algebraic variety, while for $h = 0$ it is obvious for every k. Let us project X and S^kX from a general point $x \in X$. Since $\mathrm{codim}(X_1) = h-1$ and $\mathrm{codim}(S^{k-1}X^1) = h$, by lemma 1.2 and by proposition 1.1 and by induction we get

$$\deg(S^kX) \geq \deg(S^kX_1) + \mathrm{mult}_x(S^kX) \geq$$

$$\geq \deg(S^kX_1) + \deg(S^{k-1}X^1) \geq$$

k+h$\big(_{k+1+\binom{k+h}{k}=\binom{k+h+1}{k+1}.}\big)$

Moreover if equality holds, by iterating the process we get all the assertions with the exception of v) and vi). The conclusion of v) follows from lemma 1.3, while the conclusion of vi) by $\deg(p^k_X) = \nu_k(X_h) \geq \delta_k(X_h) = \deg(\sigma_k)$ by commutativity of projections from linear spaces, by theorem $\nu_k(X) \geq \delta_k(X)$ and by the fact that a variety of codimension at least 2 projects birationally from a point of itself. In other words we have the next commutative diagram of rational maps, whose vertical maps are birational being projections from h general points on a variety of codimension al least h:

$$\begin{tikzcd} X \arrow[r, dashed, "\sigma_k(X)"] \arrow[d] & X^k \arrow[d] \\ X_h \arrow[r, dashed, "\sigma_k(X_h)"] & \mathbb{P}^n \end{tikzcd}$$

$\square$

Definition 1. Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety of dimension n. We say that X *has minimal k-secant degree if* $S^kX \subsetneq \mathbb{P}^N$ *and if* $\deg(S^kX) = \binom{\mathrm{codim}(S^kX)+k+1}{k+1}$. We say that X *is a variety with the minimal number of apparent $(k+1)$-secant* $\mathbb{P}^{k-1}$, briefly $\mathcal{MA}^{k+1}_{k-1}$*-variety, if* $s_k(X) = (k+1)n+k$ *and if* $\nu_k(X) = \binom{\mathrm{codim}(S^kX)+k+1}{k+1}$.

Remark 1. If $X \subset \mathbb{P}^N$ is not k-defective, then $\nu_k(X) = \deg(p^k_X)\deg(S^kX) \geq \deg(S^kX) \geq \binom{\mathrm{codim}(S^kX)+k+1}{k+1}$

We get that a $\mathcal{MA}^{k+1}_{k-1}$-variety is a variety of minimal k-secant degree.

We state an useful corollary of the above theorem and of theorem $\nu_k(X) \geq \delta_k(X)$, whose proof is now immediate for $h \geq 1$.

Corollary 2.2. *Let $X \subset \mathbb{P}^N$ be a $\mathcal{MA}^{k+1}_{k-1}$-variety for a given $k \geq 1$. Let $h = N - (k+1)n - k = N - s_k(X)$. Then $\sigma_k : X \dashrightarrow X^k \subseteq \mathbb{P}^{n+h}$ is birational and X^k is a variety of minimal degree $h+1$. In particular a $\mathcal{MA}^{k+1}_{k-1}$-variety $X \subset \mathbb{P}^N$ is a rational variety such that the mobile part of the linear system of k-tangent hyperplane sections is a rational variety.*

The next corollary is very useful to generate new examples of varieties with $\nu_k(X) = 1$.

Corollary 2.3. *Let $X \subset \mathbb{P}^{(k+1)n+k}$ be a $\mathcal{MA}^k_{k-2}$-variety. Then the projection of X from $n+1$ points of itself is a $\mathcal{OA}^k_{k-2}$-variety $\widetilde{X} = X_{n+1} \subset \mathbb{P}^{kn+k-1}$.*

3. Examples of $\mathcal{OA}^{k+1}_{k-1}$-varieties and of $\mathcal{MA}^{k+1}_{k-1}$-varieties

In this section we construct examples of n-dimensional varieties $X \subset \mathbb{P}^N$, $N \geq (k+1)n+k$, not k-secant defective and such that $\nu_k(X) = \binom{k+\mathrm{codim}(S^kX)+1}{k+1}$.

Example 1. *Rational normal scrolls.* Let $0 \leq a_1 \leq a_1 \leq ... \leq a_n$ be integers and set $N = a_1 + ... + a_n + n - 1$. Recall that a *rational normal scroll* $S(a_1, ..., a_n)$ in $\mathbb{P}^N$ is the image of the projective bundle $\mathbb{P}(a_1, ..., a_n) := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1}(a_n))$ via the linear system $|\mathcal{O}_{\mathbb{P}}(1)|$. The dimension of $S(a_1, ..., a_n)$ is n, its degree is $a_1 + ... + a_n = N - n + 1$ and $S(a_1, ..., a_n)$ is smooth if and only if $a_1 > 0$. Otherwise, if $0 = a_1 = ... = a_i < a_{i+1}$, it is the cone over $S(a_{i+1}, ..., a_n)$ with vertex a $\mathbb{P}^{i-1}$. One uses the simplified notation $S(a_1^{h_1}, ..., a_m^{h_m})$ if a_i is repeated h_i times, $i = 1, ..., m$.

Recall that rational normal scrolls, the (cones over) Veronese surface in $\mathbb{P}^5$, and quadrics, can be characterized as those non-degenerate irreducible varieties in a projective space having minimal degree (see [EH]).

Given positive integers $0 < m_1 \leq ... \leq m_h$ we will denote by $Seg(m_1, ..., m_h)$ the Segre embedding of $\mathbb{P}^{m_1} \times ... \times \mathbb{P}^{m_h}$ in $\mathbb{P}^N$, $N = (m_1+1)\cdots(m_h+1)-1$. We use the shorter notation $Seg(m_1^{k_1}, ..., m_s^{k_s})$ if m_i is repeated k_i times, $i = 1, ..., s$. Recall that $Pic(Seg(m_1, ..., m_h)) \simeq \mathbb{Z}^h$, generated by the line bundles $\xi_i = pr_i^*(\mathcal{O}_{\mathbb{P}^{m_i}}(1))$, $i = 1, ..., h$. A divisor D on $Seg(m_1, ..., m_h)$ is said to be of *type* $(\ell_1, ..., \ell_h)$ if $\mathcal{O}_{Seg(m_1,...,m_h)}(D) \simeq \xi_1^{\ell_1} \otimes ... \otimes \xi_h^{\ell_h}$. The hyperplane divisor of $Seg(m_1, ..., m_h)$ is of type $(1, ..., 1)$.

Let $X = S(a_1, \ldots, a_n) \subset \mathbb{P}^N$ be as above. Then $\dim(S^kX) = (k+1)n + k$ if and only if $a_1 \geq k$ and more precisely we have that

$$\dim(S^kX) = \min\{N, N+k+1 - \sum_{1\leq j\leq n;\ k\leq a_j} (a_j - k)\},$$

by applying Terracini Lemma and induction to projections from general tangent spaces or by writing equations of S^kX, see [Ro] and [CJ1] for this last point of view. To calculate the degree of S^kX, we will generalize Room specialization argument, see [Ro] pg. 257. From the determinantal description of $S^kX \subsetneq \mathbb{P}^N$ as the variety whose ideal is generated by $(k+2)\times(k+2)$ minors of a suitable *one*-generic $(k+2) \times \sum_{1\leq j\leq n;\ k\leq a_j}(a_j - k)$ matrix of linear forms (a suitable *Hankel* matrix of linear forms), see [CJ1], one gets $\operatorname{codim}(S^kX) = \sum_{1\leq j\leq n;\ k\leq a_j}(a_j - k) - k - 1$. Then S^kX is a specialization of the generic $(k+2)\times\sum_{1\leq j\leq n;\ k\leq a_j}(a_j-k)$ matrix of linear forms so that

$$\deg(S^kX) = \binom{\sum_{1\leq j\leq n;\ k\leq a_j}(a_j-k)}{k+1} = \binom{\operatorname{codim}(S^kX)+k+1}{k+1}.$$

This yields $\nu_k(X) = \binom{k+\operatorname{codim}(S^kX)+1}{k+1}$ for every k such that $N > (k+1)n + k = \dim(S^kX)$, i. e. for every k such that $a_1 \geq k$ and $N > (k+1)n+k$. From the above description it also follows that, whenever $S^kX \subsetneq \mathbb{P}^N$, $Sing(S^kX) = S^{k-1}X$, $k \geq 1$.

Let $X = S(a_1, \ldots, a_n) \subset \mathbb{P}^N$ as above with $a_1 \geq k$ and with $N = (k+1)n + k$. We have $S^kX = \mathbb{P}^N$ and we prove that $\nu_k(X) = 1$. Let $H \in |\mathcal{O}_{\mathbb{P}}(1)|$ and let F be a fiber of the structural morphism $\pi : X \to \mathbb{P}^1$. Then $|H - kF|$ is generated by global sections and $h^0(H-kF) = \sum_{i=1}^n (a_i + 1 - k) = k(n+1)+1-n(k-1) = k+n+1$. Let

$$\phi_1 = \phi_{|H-kF|} : X \to \mathbb{P}^{k+n}$$

and let

$$\phi_2 = \phi_{|kF|} : X \to \mathbb{P}^k.$$

Clearly $\phi_1(X) = S(a_1 - k, \ldots, a_n - k)$. Let $\phi = (\phi_1, \phi_2)$. We get the commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\phi=(\phi_1,\phi_2)"] \arrow[d] & \mathbb{P}^k \times \mathbb{P}^{k+n} \arrow[d] \\ \mathbb{P}^N \arrow[r, hook] & \mathbb{P}^{(k+1)(k+n+1)-1} \end{tikzcd}$$

Set $\mathbb{P}^{(k+1)(n+1)-1} := \mathbb{P}_{n,n+k}$. Now let $\psi : \mathbb{P}_{n,n+k} \dashrightarrow \mathbb{G}(k, n+k)$ be the map which associates to a $(k+1)\times(n+k+1)$ matrix its equivalence class under the action of $GL(k+1)$. It immediately follows that the closure of the fibers of ψ are linear spaces of dimension k^2+2k. Let us remark that ψ is given by forms of degree $k+1$ vanishing with order al least k along $Seg(k, n+k) \subset \mathbb{P}_{k,n+k}$ and it is not defined along $S^{k-1}Seg(k, n+k)$. Moreover, a linear fiber through a general point $p \in \mathbb{P}_{k,n+k}$, $\mathbb{P}_p^{k^2+k}$, can be interpreted as the linear span of a $Seg(k, k) = \mathbb{P}^k \times \mathbb{P}_p^k \subset Seg(k, n+k)$, which is the closure of the locus of points of $Seg(k, n+k)$ described by the $(k+1)$-secant $\mathbb{P}^k$ to $Seg(k, n+k)$ passing through p.

Let us indicate by $\widetilde{\psi} : \mathbb{P}^N \dashrightarrow \mathbb{G}(k, n+k)$ the restriction of ψ to $\mathbb{P}^N \subset \mathbb{P}_{k,n+k}$. The morphism $\widetilde{\psi}$ is defined on $\mathbb{P}^N$. Indeed, by the above description it is sufficient to show that a general point $p \in \mathbb{P}^N$ has *rank k+1*, thought as a point of $\mathbb{P}_{k,n+k}$. We have to go back through the commutative diagram above. The point p belongs to a $k+1$-secant $\mathbb{P}^k$, let us say $< p_0, \ldots, p_k >$ with p_i's general on X. Then also their images through ϕ_1 and ϕ_2 will generate a $\mathbb{P}^k$ in the respective space. Modulo a projective change of coordinates in $\mathbb{P}^k$, respectively $\mathbb{P}^{n+k}$, we can suppose $\phi_i(p_j) = (0 : \ldots : 0 : \underbrace{1}_{j} : 0 : \ldots : 0)$. The claim easily follows.Moreover, $\widetilde{\psi}$ is dominant since for a general fiber F of ψ we have $\dim(F \cap \mathbb{P}^N) \geq k^2+k+N-(k+1)(n+k+1)+1 = k$. By the theorem of the dimension of the fibers, the general fiber of $\widetilde{\psi}$ has dimension $k = (k+1)n + k - (k+1)n$ and its closure, being the intersection of two linear spaces, is a $\mathbb{P}^k$, which is $(k+1)$-secant to X. Since $\widetilde{\psi}$ is defined by forms of degree $k+1$ vanishing with order at least k along X, a $(k+1)$-secant $\mathbb{P}^k$ passing through a general point of $\mathbb{P}^N$ is contracted by $\widetilde{\psi}$ so that it coincides with the fiber passing through the general point. Then through a general point of $\mathbb{P}^N$ there passes a unique $(k+1)$-secant $\mathbb{P}^k$, i.e. $\nu_k(X) = 1$.

We now consider the case of some hyperquadric fibrations.

Example 2. *Hyperquadric fibrations.* Let $X = \mathbb{P}(a_1, ..., a_n) := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1}(a_n))$ and let notations be as in the previous section. Let $\phi_1 = \phi_{|H|}(X) = S(a_1, \ldots, a_n) \subset \mathbb{P}^N$, $0 \leq a_1 \leq \ldots \leq a_n$, $N+1 = \sum_{i=1}^n a_i + n$. Suppose also that $\sum a_i \geq 2$, i.e. that $\phi_1(X) \subsetneq \mathbb{P}^N$. Then $|H+F|$ is very ample on X and $\phi_2 = \phi_{|H+F|}(X) = S(a_1+1, \ldots, a_n+1) \subset \mathbb{P}^{N+1}$. Finally let $\phi_3 = \phi_{|2H+F|} : X \to \mathbb{P}^{(n+1)(N+1)}$.

Then we claim that $\phi_3(X) \subset \mathbb{P}^{(n+1)(k+1)-1}$ is a $\mathcal{OA}_{N-1}^{N+1}$-variety. The verification is similar to the case of the rational normal scrolls, since we have a diagram:

$$\begin{tikzcd}
X \arrow[r, "\phi=(\phi_1,\phi_2)"] \arrow[d, "\phi_3"'] & \mathbb{P}^N \times \mathbb{P}^{N+n} \arrow[d] \\
\mathbb{P}^N \arrow[r, hook] & \mathbb{P}^{(N+1)(N+n+1)-1}
\end{tikzcd}$$

Set $\mathbb{P}_{N,n+N} := \mathbb{P}^{(N+1)(N+n+1)-1}$. By restricting to $\mathbb{P}^{(n+1)(N+1)-1}$ the rational map $\psi : \mathbb{P}_{N,n+N} \dashrightarrow \mathbb{G}(N, n+N)$, we obtain $S^N X = \mathbb{P}^{(n+1)(N+1)-1}$ and $\nu_N(\phi_3(X)) = 1$ since the general fiber of the restriction is once again a $(N+1)$-secant $\mathbb{P}^N$ to $\phi_3(X)$.

Example 3. *5-Veronese embedding of $\mathbb{P}^2$ and its tangential projections from 1,2 or 3 points.* We prove that the 5-Veronese embedding of $\mathbb{P}^2$, $X = \nu_5(\mathbb{P}^2) \subset \mathbb{P}^{20}$, and its tangential projections from 1, 2 or 3 points, $X^i \subset \mathbb{P}^{20-3i}$, $i = 1, 2, 3$, are smooth surfaces such that $\nu_6(X) = 1$, respectively $\nu_{6-i}(X^i) = 1$.

We will slightly modify and adapt to our need a construction of N. Shepherd-Barron. Let us first consider the case of $X = \nu_5(\mathbb{P}^2) \subset \mathbb{P}^{20}$. Let $F \in |\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^{2*}}(1,1)|$ and let p_1 and p_2 indicate the projections (or their restrictions to F) of $\mathbb{P}^2 \times (\mathbb{P}^2)^*$. Let us recall that $F = \{(x,l) \in \mathbb{P}^2 \times (\mathbb{P}^2)^* \ : \ x \in l\}$. Let $\phi = \phi_{|\mathcal{O}_F(1,2)|} : F \hookrightarrow \mathbb{P}^{14}$. Since every fiber of $p_2 : F \to (\mathbb{P}^2)^*$ is embedded as a line in $\mathbb{P}^{14}$, we get an isomorphism of $(\mathbb{P}^2)^*$ with a subvariety of $\mathbb{G}(1,14)$. Let $X \subset \mathbb{G}(1,14) \subset \mathbb{P}^{104}$ be the image of $(\mathbb{P}^2)^* \subset \mathbb{G}(1,14)$ under the Plücker embedding of $\mathbb{G}(1,14)$. We claim that X is the 5-Veronese embedding of $\mathbb{P}^2$.

To prove this let us introduce the following Schubert cycles in $\mathbb{G} = \mathbb{G}(1,N)$. $A = \{l \in \mathbb{G} \ : \ l$ lies in a given hyperplane $\}$, $B = \{l \in \mathbb{G} \ : \ l$ meets a given linear space of codimension $3\}$, $C = \{l \in \mathbb{G} \ : \ l$ meets a given linear space of codimension $2\}$. Then C is a hyperplane section of $\mathbb{G}$ in its Plücker embedding and $C^2 = A + B$. Note that $\deg(X) = X \cdot C^2 = X \cdot A + X \cdot B$. The embedding of X is given by a complete linear system, because it is G-equivariant (see [SB]), so that it is enough to prove that $\deg(X) = 25$. By definition in our example, $X \cdot A = \deg(F) = (p_1^*\mathcal{O}(1) + p_2^*\mathcal{O}(2))^3 = 18$. Let $H \subset \mathbb{P}^{14}$ be a general hyperplane and let $S \in |F \cap H|$. Then $X \cdot A$ is equal to the number of fibers of p_2 that lie in H, i.e. the number of exceptional curves contracted by $p_2 : S \to (\mathbb{P}^2)^*$. Then $X \cdot A = 9 - K_S^2 = 7$, since $K_S = \mathcal{O}_S(-1,0)$ and

$K_S^2 = (p_1^*\mathcal{O}(-1))^2 \cdot (p_1^*\mathcal{O}(1) + p_2^*\mathcal{O}(2)) = 2$. Finally $\deg(X) = 18 + 7 = 25$ and the conclusion follows. By the above discussion the linear span of $X \subset \mathbb{P}^{104}$ is $< X >= \mathbb{P}^{20}$.

Let us recall that given a vector space W of odd dimension $2k+1$, there is a natural rational map $\psi : \mathbb{P}(\Lambda^2 W^*) \dashrightarrow \mathbb{P}(W^*)$, associating to a skew 2-form its kernel; recall that a skew 2-form has even rank. Then the general fiber of ψ is a linear space and if $\dim(W) = 2k+1$, then the map is given by forms of degree k vanishing with order al least $k-1$ along $\mathbb{G}(1,2k) \subset \mathbb{P}(\Lambda^2 W^*)$.

For $W = H^0(\mathcal{O}_F(1,2))$ we get a rational map $\psi : \mathbb{P}^{104} \dashrightarrow \mathbb{P}^{14}$ for which the closure of a general fiber F is a $\mathbb{P}^{90}$. In [SB], lemma 12, it is shown that the locus of indetermination of ψ does not contain $S^6X =< X >$, the last equality being well known. The general fiber F will cut $< X >= \mathbb{P}^{20}$ in a linear space of dimension at least 90+20-104=6, so that the restriction of ψ to $< X >$, $\psi : \mathbb{P}^{20} \dashrightarrow \mathbb{P}^{14}$ is dominant and the closure of a general fiber is then a linear space of dimension 6 by the above analysis and by the theorem of the dimension of the fibers. Then a 7-secant $\mathbb{P}^6$ passing through a general point of $\mathbb{P}^{20}$ is contracted by ψ so that it coincides with the general fiber of ψ restricted to $\mathbb{P}^{20}$, i.e. $\nu_6(X) = 1$.

Now we slightly modify the construction to show that the surfaces X^i, $i = 1,2,3$, have $\nu_{6-i}(X^i) = 1$. We will treat the case $i = 1$ for simplicity since the other follows by applying the same construction. Let $p \in (\mathbb{P}^2)^*$ be general and let $l \subset \mathbb{P}^2$ be the corresponding line. We prove that the projection of $X \subset \mathbb{P}^{20}$ from the tangent space at the general point p is isomorphic to the image in (the Plücker embedding of) $\mathbb{G}(1,12)$ of the projection of the scroll F from the line l, i.e. if $\pi_l : \mathbb{P}^{14} \dashrightarrow \mathbb{P}^{12}$ is projection from the line l, if $F' = \pi_l(F)$ and if $X' \subset \mathbb{G}(1,12) \subset \mathbb{P}^{77}$ is the corresponding image of the scroll F', then $X' \simeq Bl_p(\mathbb{P}^2)^*$, $< X' >= \mathbb{P}^{17}$ and X' is embedded by quintics having a double point at p, i.e. $X' = X^1$. In the Plücker embeddings, the natural map $\widetilde{\pi}_l : \mathbb{G}(1,14) \dashrightarrow \mathbb{G}(1,12)$ clearly corresponds to projection from the tangent space to $\mathbb{G}(1,14)$ at the point $l \in \mathbb{G}(1,14)$. This gives that $\pi_l(X) = X'$ is a surface whose hyperplane section in its linear span are represented on X by a linear system of quintics having only a singular point at p. Recall that $T_l\mathbb{G} \cap \mathbb{G} = \{l' \in \mathbb{G} \ : \ l \cap l' \neq \emptyset\}$ so that $T_l\mathbb{G}(1,14) \cap X = \{l' \in X \ : \ l \cap l' \neq \emptyset\} = l$ as sets. Set $\mathbb{P}^2 \times Bl_p((\mathbb{P}^2)^* \supset \widetilde{F} = Bl_{p_2^{-1}(p)} F \to Bl_p(\mathbb{P}^2)^*$ and let $\phi : \widetilde{F} \to \mathbb{P}^{12}$ be the map given by the linear system $|p_1^*(\mathcal{O}(1)) + p_2^*(\mathcal{O}(2)) - \widetilde{E}|$, where p_1 and p_2 are the projections of $\mathbb{P}^2 \times Bl_p((\mathbb{P}^2)^*$ and where $\widetilde{E}$ is the exceptional divisor of $\widetilde{F}$. Then $F' \simeq \phi(\widetilde{F})$ from which it follows $X' \simeq Bl_p(\mathbb{P}^2)^*$. To get the conclusion it is sufficient to show that $\deg(X') = 21$. We have to make exactly the same calculation as before. Now $X' \cdot B = \deg(F') = 15$ and $X' \cdot A = 6$ so that $\deg(X') = 21$. This shows that $\widetilde{\pi}_l$ restricted to X is exactly the projection of X from T_lX, that $< X^1 >= \mathbb{P}^{17} = S^5X$, where the last equality follows from Terracini lemma. Then reasoning as above, now with the corresponding $\psi : \mathbb{P}^{77} \dashrightarrow \mathbb{P}^{12}$ and by restricting to the linear span of $< X^1 >= \mathbb{P}^{17} \subset \mathbb{P}^{77}$, we get a rational map $\psi : \mathbb{P}^{17} \dashrightarrow \mathbb{P}^{12}$, whose general fiber is a 6-secant $\mathbb{P}^5$ to X^1, i.e. $\nu_5(X^1) = 1$.

In the same way we can iterate the projection from tangent space and get smooth surfaces $X^i \subset \mathbb{P}^{20-3i}$ such that $\nu_{6-i}(X^i) = 1$.

Example 4. Let $X \subset \mathbb{P}^N \subset \mathbb{P}^{N+l+1}$ be an irreducible non-degenerate (in $\mathbb{P}^N$) variety of dimension n and let $L = \mathbb{P}^l \subset \mathbb{P}^{N+l+1}$, $l \geq 0$, be such that $L \cap \mathbb{P}^N = \emptyset$. Let $Y = S(L, X)$ be the cone over X with vertex L. Then $\dim(Y) = n+l+1$. More generally for every $m \geq 1$ we have $S^mY = S(L, S^mX)$ so that $s_l(Y) = s_l(X)+l+1$, $h_m(Y) := N + l + 1 - s_m(Y) = N - s_m(X) = h_m(X)$ and $\deg(S^mY) = \deg(S^mX)$ for every $m \geq 1$. In particular if X has minimal k-secant degree, then Y has minimal k-secant degree.

We now slightly modify the above example to motivate the hypothesis we will introduce in our classification theorems in the following sections.

Example 5. Let $C \subset \mathbb{P}^{2k+1} \subset \mathbb{P}^{3k+2}$, $k \geq 1$, be a rational normal curve of degree $2k+1$. Take $L = \mathbb{P}^k \subset \mathbb{P}^{3k+2}$ such that $L \cap \mathbb{P}^{2k+1} = \emptyset$ and a morphism $\phi : C \to C' \subset \mathbb{P}^k$ and take $X = \cup_{p \in C} < p, \phi(p) > \subset \mathbb{P}^{3k+2}$. Then $\nu_k(X) = \nu_k(C) = 1$ and as soon as $k \geq 3$, one can take as ϕ a general projection of C and obtain examples of smooth surfaces $X \subset \mathbb{P}^{3k+2}$, which not linearly normal, i.e. such that the linear system of hyperplane section of X is not complete. Let us remark that such a surface X is k-weakly defective being contained in a cone of vertex a $\mathbb{P}^k$ over the curve C, see [CC].

This example could also be generalized to higher dimensions but we leave the details to the reader.

4. On a theorem of Castelnuovo-Enriques

labelCasEnr In this section we prove a theorem of Enriques, generalising to arbitrary surfaces a result proved by Castelnuovo for rational surfaces, see [En] and [Ca].

Theorem 4.1. (Enriques, [En]) *Let S be a smooth irreducible projective surface and let $D \subset S$ be an irreducible curve of geometric genus $g \geq 2$. Suppose that $r = \dim(|D|) \geq 3g+5$ and suppose there does not exist an irreducible rational curve $F \subset S$ such that $F^2 = 0$ and such that $F \cdot D = 1$, i.e. $\phi_{|D|}(S) \subset \mathbb{P}^r$ is not ruled. Then $r = 3g+5$, $D^2 = 4g+4$ and (S, D) is one of the following:*

(1) *$S \simeq \mathbb{P}^2$ and $D \in |\mathcal{O}_{\mathbb{P}^2}(4)|$;*

(2) *$S \simeq \mathbb{F}_a$, $a \geq 0$, and D is of type $2E + (a+g+1)F$ with $0 \leq a \leq g+1$, where E is a $(-a)$-curve and F a fiber of the ruling. The divisor D is very ample if and only if $0 \leq a \leq g$. If $a + 1 + k \equiv 0$, (mod 2), then $\phi_{|D|}(S)$ is $(k+1)$-defective, while if If $a + k \equiv 0$, (mod 2), then $\phi_{|D|}(S)$ is a $\mathcal{OA}_k^{k+2}$-surface.*

Corollary 4.2. *Let $S \subset \mathbb{P}^r$, $r \geq 3g+5$, $g \geq 2$, be an irreducible non-degenerate surface which is not ruled and having general hyperplane section H of geometric genus g. Then $r = 3g+5$, S has at most one singular point, the general hyperplane section is smooth, S has degree $4g+4$ and it is one of the following:*

(1) *$S \simeq \mathbb{P}^2$ and $D \in |\mathcal{O}_{\mathbb{P}^2}(4)|$;*
(2) *$S \simeq \mathbb{F}_a$, $a \geq 0$, and H is of type $2E + (a+g+1)F$ with $0 \leq a \leq g$, where E is a $(-a)$-curve and F a fiber of the ruling.*
(3) *S is isomorphic to the 2-Veronese embedding of a cone over a rational normal curve of degree a, $S_a \subset \mathbb{P}^{a+1}$, $a \geq 3$ and $g = a - 1$.*

During the proof of this result and of some of its consequences we use a theorem of M. Reid, proved via Mori theory, which we recall here.

Theorem 4.3. (Reid, [Re]) *Let S be a smooth irreducible projective surface and let D be a nef and big divisor on S such that $D \cdot K_S < 0$. Define*

$$\rho = \rho_D = \sup\{m \in \mathbb{Q} \ : \ D + mK_S \geq 0\}.$$

Then ρ, 2ρ or $3\rho \in \mathbb{Z}$, and $D + \rho K_S$ has a Zariski decomposition

$$D + \rho K_S = P + N,$$

such that:

(1) *P is nef and $P^2 = 0$;*
(2) *$N \geq 0$ and $P \cdot C = 0$ for every C in the support of N;*
(3) *the negative part N can be contracted out by a sequence of contractions $S \to S_1 \to \ldots \to S_k = S^\#$, where each step contracts out a single (-1)-curve E_i, with*

$$E_i \cdot D_i = \mu_i < \rho$$

(here D_i is the direct image of D on S_i);

(4) *the surface $S^\#$ is called the $\#$-minimal model of (S, D) and it is either a weak conic bundle, i.e. there exists a morphism $\pi : S^\# \to B$ with B a smooth curve such that $-K_{S^\#}$ is nef and big relatively to π, or $S^\#$ is a weak del Pezzo surface, i.e. $-K_{S^\#}$ is nef and big. Moreover, in the first case, if F is a fiber, then*

$$D^\# \cdot F = 2\rho \in \mathbb{Z},$$

while in the second case

$$D^\# + \rho K_{S^\#} = 0,$$

where $D^\#$ is the image of D on $S^\#$.

Proof. (of theorem 4.1). By eventually blow-up S and by resolving the singularities of a general member of $|D|$, we can suppose that D is smooth. Set $d = D^2$. From the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(D) \to \mathcal{O}_D(D) \to 0,$$

we get $\dim(|D_{|D}|) \geq 3g+4$ so that $D_{|D}$ is non-special by Clifford theorem. From Riemann-Roch theorem we get that $r \geq 3g+5$ gives $3g+5 \leq d+1-g$, i.e $d \geq 4g+4$.

We have $K_S \cdot D = 2g-2-d \leq 2g-2-4g-4 = -2g-6 < 0$ so that S has negative Kodaira dimension. On the other hand,

$$(D+2K_S)\cdot D = d + 2(2g-2-d) = 4(g-1) - d \leq -8,$$

yields $\rho = \rho(D) < 2$. With the notations of theorem 4.3, we have $\mu_i < \rho < 2$ so that $\mu_i = 0, 1$. Then $D^\#$ is smooth and irreducible, $g(D^\#) = g$, and $\dim(|D^\#|) \geq \dim(|D|) \geq 3g+5$ and we can suppose

$S = S^\#$ and $D = D^\#$.

Suppose that S is a weakly del Pezzo surface. Then

$$0 < D^2 = \rho^2 K_S^2,$$

yields $K_S^2 > 0$. On the other hand,

$$2g-2 = (D+K_S)\cdot D = (-\rho+1)K_S \cdot (-\rho)K_S = \rho(\rho-1)K_S^2,$$

so that

$$\rho^2 K_S^2 = d \geq 4g+4 = 2\rho(\rho-1)K_S^2 + 8,$$

i.e.

$$\rho^2 K_S^2 - 2\rho K_S^2 + 8 \leq 0.$$

Finally from

$$0 < (\rho-1)^2 K_S^2 \leq K_S^2 - 8,$$

we deduce $K_S^2 = 9$ and $S \simeq \mathbb{P}^2$. Since $D \in |\mathcal{O}_{\mathbb{P}^2}(a)|$, $a \geq 3$, and since

$$a^2 = d \geq 4g+4 = 2(a-1)(a-2)+4,$$

we necessarily have $a = 3$ and we are in case 1).

Now we treat the case in which S is a weak conic bundle. Since $2\rho = D\cdot F \geq 2$, where the last inequality comes from the hypothesis of the theorem, and since $2 \leq 2\rho < 4$, we have to consider the cases $2\rho = 2, 3$.

Suppose $2\rho = 3$. By theorem 4.3

$$D + \frac{3}{2}K_S = P,$$

and

$$8 \geq K_S^2 = \frac{4}{9}(D-P)^2 = \frac{4}{9}(D^2 - 2D\cdot P) = \frac{4}{9}(D^2 - 2D^2 - 3D\cdot K_S) = -\frac{4}{9}(D^2 + 3D\cdot K_S) =$$

$$= -\frac{4}{9}(d + 3(2g-2-d))) = -\frac{4}{9}(6(g-1)-2g) = \frac{8}{9}(d-3(g-)) \geq \frac{8}{9}(4g+4-3g+3) =$$

$$= \frac{8}{9}(g+7) \geq 8.$$

Then $K_S^2 = 8$, $g = 2$ and $d = 4g+4 = 10$. From

$$D\cdot P = D^2 + \frac{3}{2}(K_S\cdot D) = 10 + \frac{3}{2}(2-10) = -2,$$

we get a contradiction since P is nef and D is an effective divisor.

Suppose now $\rho = 1$. Reasoning as above one gets

$$8 \geq K_S^2 = (D-P)^2 = D^2 - 2D\cdot P = D^2 - 2D^2 - 2D\cdot K_S = d - 4(g-1) \geq 8.$$

Hence $K_S^2 = 8$ and $S \simeq \mathbb{F}_a$, $a \geq 0$. With the standard notations

$$D = 2E + \beta F,$$

with $\beta \geq 2a$ because D is an irreducible smooth curve. Since $K_S = -2E-(a+2)F$ and $K_S + D = (\beta - a - 2)F$, we obtain

$$2g - 2 = 2(\beta - a - 2),$$

i.e.

$$\beta = a + g + 1.$$

Finally from $0 \leq D\cdot E = -2a+\beta = -2a+a+g+1 = g+1-a$ we get the restriction $0 \leq a \leq g+1$. Moreover, if D is very ample, then $D\cdot E > 0$, i.e. $a \leq g$. It is a standard fact in the theory of ruled surfaces, that on the contrary if $0 \leq a \leq g$, the divisor $D = 2E+(a+g+1)F$ is very ample. In any case we have that D is generated by global sections, that $D^2 = 4g+4$ and that $\dim(|D|) = 3g+5$. If $a = g+1$, then $\phi = \phi_{|D|}$ contracts the curve E to a point and it is an isomorphism outside E and $\phi(S) = \widetilde{S}$ has a unique singular point of multiplicity a. The surface $\widetilde{S} \subset \mathbb{P}^{3k+2}$ is the 2-Veronese embedding of the rational normal cone over the rational normal curve of degree a so that it is $(k+1)$-defective. In the same way, we see that if $a+1+k \equiv 0$, $(mod\ 2)$, then $\phi_{|D|}(S)$ is the 2-Veronese embedding of a smooth rational normal scroll so that it is $(k+1)$-defective. If $a+k \equiv 0$, $(mod\ 2)$, then $\phi_{|D|}(S)$ is a $\mathcal{OA}_k^{k+2}$-surface, see example 2. □

Proof. (of corollary 4.2). Let $\pi : \widetilde{S} \to S$ be a desingularisation of S and let $D = \pi^*(H)$. Then D is nef and big, the general element of $|D|$ is irreducible and of geometric genus g and $s = \dim(|D|) \geq r \geq 3g+5$. There does not exists on S an irreducible rational curve F such that $F^2 = 0$ and such that $D \cdot F = 0$, because otherwise S would be ruled. Then by theorem 4.1, we have that $s = r = 3g+5$ and the conclusion easily follows. □

We are ready to give an application of the theorem of Reid and of the theorem of Enriques, which will be very useful for classification in the next section. We refrain for the moment to formulate it in the maximal generality, i.e. for singular surfaces, the adaptation being almost obvious.

Theorem 4.4. *Let S be a smooth surface and $H \subset S$ be an irreducible curve of geometric genus $g \geq 2$, which is an ample divisor. Suppose that $r = \dim(|D|) \geq 3g+\alpha$, $\alpha \geq 2$, and suppose there does not exist an irreducible rational curve $F \subset S$ such that $F^2 = 0$ and such that $F\cdot H = 1$, i.e. $\phi_{|H|}(S) \subset \mathbb{P}^r$ is not ruled. Then S is rational, H is very ample, $\alpha \leq 5$, and (S,H) is one of the following:*

1. *the 4-Veronese embedding of $\mathbb{P}^2$, $g = 3$; or one of its projections from $i = 1,2$ points on it, $g = 3$, $\alpha = 5 - i$;*
2. *S is the linear projection from $\beta = 5-\alpha$, $0 \leq \beta \leq 3$, distinct points of a surface $\widetilde{S} \subset \mathbb{P}^{3g+5}$ of degree $d = 4g+4$ and genus g, i.e. of a surface $\widetilde{S} \simeq \mathbb{F}_a$, $a \geq 0$, whose hyperplane section is of type $2E+(a+g+1)F$ with $0 \leq a \leq g$, where E is the $(-a)$-curve and F a fiber of the ruling of $\mathbb{F}_a$; $\alpha = 5-\beta$.*

(3) *$S \simeq Q \subset \mathbb{P}^3$ and $H \in |\mathcal{O}_Q(3,3)|$, $g = 4$ and $\alpha = 3$.*
(4) *the 5-Veronese embedding of $\mathbb{P}^2$, $g = 6$; or one of its tangential projections from $1, 2$ or 3 point, $g = 6 - j$, $j = 1, 2, 3$, $\alpha = 2$;*

Proof. From $H \cdot K_S = 2g - 2 - 4g - \alpha + 1 = -2g - 1 - \alpha < 0$ we deduce that S has negative Kodaira dimension. Since $d = H^2 \geq 4g+1 \geq 9$, the linear system $|K_S+H|$ is base point free by Reider's theorem, taking into account the hypothesis. We have to consider two cases: either $(K_S+H)^2 > 0$ and the general element $D \in |K_S+H|$ is smooth and irreducible or $(K_S+H)^2 = 0$ and $|K_S+H|$ is composed with a pencil. Let us consider first the last case. Let $\psi : S \to C$ be the Stein factorization of $\phi_{|K_S+H|}$ and let F be a general fiber of ψ. Then F is an irreducible smooth curve such that $F^2 = 0$ and such that $K_S \cdot F = -H \cdot F < 0$, i.e. $F \simeq \mathbb{P}^1$ and $K_S \cdot F = -2 = -H \cdot F$. Since $(H + mK_S) \cdot F = 2 - 2m < 0$ for every $m > 1$, letting notations as in theorem 4.3, $\rho(H) = 1$ so that (S, H) is a weakly conic bundle by theorem 4.3. Since $H \cdot F = 2$, there are at most two irreducible components in the reducible fibers of the fibration and these are necessarily (-1)-curves. From $0 = (K_S + H)^2 = K_S^2 + 2K_S \cdot H + d = K_S^2 + 4g - 4 - d \leq K_S^2 - 3 - \alpha$, we get $K_S^2 \geq \alpha + 3$ so that S is rational, $d = 4g + \alpha - 1$ and $K_S^2 = \alpha + 3$. There at most $0 \leq \beta = 8 - K_S^2 = 5 - \alpha \leq 3$ singular fibers. After contracting β (-1)-curves E_i, one in each reducible fiber, we get a surface $\widetilde{S}$ and a morphism $\varphi : S \to \widetilde{S}$. If $\widetilde{H}$ is the image of the smooth irreducible curve H, then $\widetilde{H}$ is smooth, irreducible and $\dim(|\widetilde{H}|) = 3g + \alpha + \beta = 3g + 5$; recall that $H \cdot E_i = 1$ for each i. Then we apply theorem 4.1 and get case 2 taking into account that since $h^1(\mathcal{O}_S(H)) = 0$, the divisor $\widetilde{H}$ on $\widetilde{S}$ is easily seen to be very ample.

Let us consider the case in which $D \in |K_S+H|$ is a nef and big divisor. We have $3 \leq h^0(\mathcal{O}_S(D)) = h^0(\mathcal{O}_H(K_H)) - h^1(\mathcal{O}_S) \leq h^0(\mathcal{O}_H(K_H)) = g$. Since $(D+K_S) \cdot H = (H + 2K_S) \cdot H = d + 4g - 4 - 2d < 0$ and since $D \cdot K_S = 2g - 2 - d + K_S^2 \leq -2g - \alpha - 1 + K_S^2 < 0$ (if $K_S^2 = 9$ we get $S \simeq \mathbb{P}^2$ and the inequality holds for $g \geq 3$), we get $\rho(D) < 1$ so that $\rho(D) = 1/2, 1/3$ or $2/3$. Let $f : S \to S^\#$ be the reduction to the $\#$-minimal model. In each case $E_i \cdot D_i = 0$ yields that, $D = f^*(D^\#)$ so that $D^\#$ is ample and $\rho(D^\#) = 1/2, 1/3$ or $2/3$. Since $\rho(D^\#)$ is not an integer, the surface $S^\#$ is minimal, i.e. it is, respectively, either a $\mathbb{P}^1$-bundle relatively to $D^\#$, or $S^\# = \mathbb{P}^2$ and $D^\# = \mathcal{O}(1)$ or $\mathcal{O}(2)$. Suppose $S^\#$ is a $\mathbb{P}^1$-bundle over a smooth curve E of genus $g(E) = h^1(\mathcal{O}_S)$. We have $1 \leq (D^\#)^2 = D^2 = K_S^2 + 4g - 4 - 2d + d$, so that $K_S^2 \geq d + 1 - 4g + 4 \geq \alpha + 4 \geq 6$ and a fortiori $K_{S^\#}^2 = 8$, i.e. $h^1(\mathcal{O}_S) = 0$, and once again S is rational. In conclusion in each case $D^\#$ is a smooth rational curve on $S^\#$, which is a very ample divisor. From the rationality of S we deduce $(D^\#)^2 = h^0(\mathcal{O}_{S^\#}(D^\#)) - 2 = h^0(\mathcal{O}_S(D)) - 2 = g - 2$. Hence $D^\#$ embeds $S^\#$ in $\mathbb{P}^{g-1}$ as a smooth surface of minimal degree $g - 2$. Moreover $d = 4g + \alpha - 1$ and $K_S^2 - 3 - \alpha = D^2 = (D^\#)^2 = g - 2$. From $9 \geq K_S^2 = g + \alpha + 1$ we deduce $3 \leq g \leq 8 - \alpha$ and we can now conclude the classification.

Suppose $\alpha = 4$ so that $g = 3$ or 4. If $g = 4$, then $K_S^2 = 9$, $S \simeq \mathbb{P}^2$ and this case is not possible. Suppose $g = 3$. Then $S^\# = \mathbb{P}^2$, $K_S^2 = 8$ so that $S \simeq Bl_p\mathbb{P}^2$ and $H \in |\pi^*(\mathcal{O}_{\mathbb{P}^2}(4)) - E|$.

Suppose $\alpha = 3$, so that $g = 3, 4$ or 5. If $g = 5$, then $K_S^2 = 9$, $S \simeq \mathbb{P}^2$ so this case cannot exist. If $g = 4$, then $S \simeq S^\# \simeq Q \subset \mathbb{P}^3$ and $H \in |\mathcal{O}_Q(3,3)|$. If $g = 3$, then $S^\# = \mathbb{P}^2$, $K_S^2 = 7$ so that $S \simeq Bl_{p_1,p_2}\mathbb{P}^2$ and $H \in |\pi^*(\mathcal{O}_{\mathbb{P}^2}(4)) - E_1 - E_2|$.

Suppose finally $\alpha = 2$.
If $g = 6$, then $K_S^2 = 9$ so that $S = S^\# \simeq \mathbb{P}^2$, $D^\# \in |\mathcal{O}_{\mathbb{P}^2}(2)|$ and $H \in |\mathcal{O}_{\mathbb{P}^2}(5)|$. If $g = 5$, then $K_S^2 = 8$ so that $S = S^\# \simeq Bl_p\mathbb{P}^2$, $D^\# \in |\pi^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E|$ and $H \in |\pi^*(\mathcal{O}_{\mathbb{P}^2}(5)) - 2E|$.

If $g = 4$, then $K_S^2 = 7$ so that $S \simeq Bl_pQ$, where $Q = \mathbb{P}^1 \times \mathbb{P}^1 = S^\#$, $D^\# \in |\mathcal{O}_Q(1,1)|$ and $H \in |\pi^*(\mathcal{O}_Q(3,3)) - E|$. The pair (S, H) is isomorphic to $(Bl_{p_1,p_2}\mathbb{P}^2, \pi^*(\mathcal{O}_{\mathbb{P}^2}(5)) - 2E_1 - 2E_2)$. If $g = 3$, then $K_S^2 = 6$ so that $S \simeq Bl_{p_1,p_2,p_3}\mathbb{P}^2$, because $(S^\#, D^\#) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, and $H \in |\pi^*(\mathcal{O}_{\mathbb{P}^2}(4)) - E_1 - E_2 - E_3|$. The pair $(Bl_{p_1,p_2,p_3}\mathbb{P}^2, \pi^*(\mathcal{O}_{\mathbb{P}^2}(4)) - E_1 - E_2 - E_3)$ is isomorphic to $(Bl_{q_1,q_2,q_3}\mathbb{P}^2, \widetilde{\pi}^*(\mathcal{O}_{\mathbb{P}^2}(5)) - 2\widetilde{E}_1 - 2ldeE_2 - 2\widetilde{E}_3)$, concluding the proof of the theorem. □

5. Some general Lemmas

The two next lemmas are consequences of theorem 1.4 of [CC] we partially recall here.

Theorem 5.1. ([CC], th. 1.4) *Let $X \subset \mathbb{P}^N$ be a non-degenerate irreducible variety of dimension n. If X is not k-weakly defective for a given k such that $N \geq (n+1)(k+1)$, then, given $p_1, \ldots, p_{k+1}$ general points on X, the general $(k+1)$-tangent hyperplane section $H \in |\mathcal{O}(1) - 2p_1 \ldots - 2p_{k+1}|$ is tangent to X only at $p_1, \ldots, p_{k+1}$. Moreover such a hyperplane section H has ordinary double points at $p_1, \ldots, p_{k+1}$.*

The first consequence we are interested in is the following.

Lemma 5.2. *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety, which is not $(k-1)$-weakly defective for a fixed $k \geq 2$ such that $N \geq kn + k$. Let $p_1, \ldots, p_{k-1}$ be general points on X and let $z \in < p_1, \ldots, p_{k-1} >$ be a general point of $S^{k-2}X$. Let $\pi : X \dashrightarrow Y \subset \mathbb{P}^{N-(k-1)n-k+1}$, $N - (k-1)n - k + 1 \geq n + 1$, be the projection from $T_zS^{k-2}X$. Then π is a birational morphism. In particular, if $N \geq 2n + 2$ and X is not 1-weakly defective, then the projection from a general tangent space is birational onto the image.*

Proof. Since X is not $(k-1)$-weakly defective, we have $s_l(X) = (l+1)n + l$ for every $l \leq k-1$ so that by Terracini's lemma $\dim(Y) = s_{k-1}(X) - s_{k-2}(X) - 1 = n$ and π is generically finite. Suppose that given a general $p_k \in X$ there exists a point $q \in X \setminus (T_zS^{k-2}X \cap X)$, $q \neq p_k$ and $q \in \pi^{-1}(\pi(p_k))$. Let $\pi(p_k) = y \in Y$. Then $< T_yY, T_zS^{k-2}X >=< T_{p_k}X, T_zS^{k-2}X >=< T_qX, T_zS^{k-2}X >$ by generic smoothness, from which it follows that the general hyperplane tangent at $p_1, \ldots, p_k$ is also tangent at q. Since X is not $(k-1)$-weakly defective, theorem 5.1 implies $N < kn + k$, contrary to our assumption. □

In the sequel we also need this fact.

Lemma 5.3. *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate surface, which is not $(k-1)$-weakly defective for a fixed $k \geq 1$ such that $N \geq 3k + 2$. Let $B = \sum n_i\Gamma_i$, $n_i \geq 0$, be the fixed part of the linear system of hyperplane sections tangent at $k-1$ general points $p_1, \ldots, p_{k-1}$ and let $|A|$ be its mobile part. If $n_i > 0$ for some i, then $n_i = 1$ and Γ_i is an irreducible smooth curve. Moreover, if $B \neq \emptyset$, then either B is a smooth rational curve passing through $p_1, \ldots, p_{k-1}$ or $B = \sum_{i=1}^{k-1}\Gamma_i$, with B_i a smooth rational curve passing through p_i, $i = 1, \ldots, k-1$, and $\Gamma_i \cap \Gamma_j = \emptyset$ if $i \neq j$. If moreover the general tangent projection from $k-1$ tangent spaces is a surface $Y \subset \mathbb{P}^{N-3k+3}$ with general hyperplane section a smooth rational curve, then A is rational.*

Proof. Let C be a general hyperplane section tangent at the general points $p_1, \dots, p_{k-1}$ and let $|C| = |A| + B$. By a standard analysis and by theorem 5.1, i.e. by the fact that C has only ordinary double points at $p_1, \dots, p_{k-1}$, we get that, if $B \neq \emptyset$, A is irreducible and smooth, that $n_i = 1$ if positive, that each Γ_i is smooth and that only the two above cases are possible. So it suffices to show that each Γ_i is a rational curve. By projecting X from the span of the tangent space to $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{k-1}$, we get an irreducible surface $Y \subset \mathbb{P}^{N-3k+3}$, with $N - 3k + 3 \geq 5$, which is birational to X by lemma 5.2 and which is not 0-weakly defective. Then the general hyperplane section tangent to Y at a general point is necessarily reducible and has only a double point as its singularities. This forces Y to be either the Veronese surface in $\mathbb{P}^5$ or a not developable scroll over a curve. Then either A and each Γ_i are rational curves smooth at p_i, each one being birational to a conic, or the linear system of tangent hyperplane sections has a fixed part consisting of a line and once again Γ_i is rational being birational to a line. The conclusion follows. □

6. Classification of curves with minimal higher secant degree

Let $C \subset \mathbb{P}^N$ be an irreducible non-degenerate curve. Then it is well known that $s_k(C) = \min\{2k+1, N\}$. From now on let us suppose that $k \geq 1$ is such that $2k + 1 \leq N$ and let $h_k = N - 2k - 1 = \mathrm{codim}(S^k X)$. Let us recall that in this case we always have $\deg(p^k_C) = 1$.

Theorem 6.1. *Let $C \subset \mathbb{P}^N$ be an irreducible non-degenerate curve linearly normal curve. Let $k \geq 1$ be such that $2k+1 \leq N$ and let $h_k = N - 2k - 1 = \mathrm{codim}(S^k X)$. Then $\nu_k(C) = \binom{k+h_k+1}{k+1}$ if and only if $C \subset \mathbb{P}^N$ is a rational normal curve of degree N. If $h_k > 0$, then $\deg(S^k(C)) = \binom{k+h_k+1}{k+1}$ if and only if $C \subset \mathbb{P}^N$ is a rational normal curve of degree N.*

Proof. The curve is linearly normal and rational so that the conclusions easily follow. □

7. Classification of $\mathcal{OA}^{k+1}_{k-1}$-surfaces

In this section we furnish the classification of smooth linearly normal surfaces $X \subset \mathbb{P}^{3k+2}$, $k \geq 2$ such that $\nu_k(X) = 1$. It can be easily extended to the classification of such surfaces with at most a finite number of singular points.

Theorem 7.1. (Bronowski, [Br])*Let $X \subset \mathbb{P}^{3k+2}$, $k \geq 2$, be a smooth projective irreducible linearly normal surface such that $\nu_k(X) = 1$. Then X is one of the following:*

(1) *a rational normal scroll $S(a_1, a_2)$ with $k \leq a_1 \leq a_2$, $d = a_1 + a_2 = 3k + 1$ and sectional genus $g(X) = 0$.*
(2) *the image in $\mathbb{P}^{3k+2}$ of a $S(a_1, a_2) \subset \mathbb{P}^k$ with $0 \leq a_1 \leq a_2$, $a_1 + a_2 = k - 1$ by the linear system $|2H + F|$, $H \in |\mathcal{O}_{S(a_1,a_2)}(1)|$, F a fiber of the ruling. In this case $d = 4k$ and $g(X) = k - 1$ and the hyperplane sections of H are hyperelliptic curves.*
(3) *X is the linear projection from 3 distinct points of a surface $\widetilde{X} \subset \mathbb{P}^{3k+5}$ of degree $d = 4k + 4$ and genus k, i.e. of a surface $\widetilde{X} \simeq \mathbb{F}_a$, $a \geq 0$, whose hyperplane section is of type $2E + (a + k + 1)F$ with $0 \leq a \leq k$, where*

E is the $(-a)$-curve and F a fiber of the ruling of $\mathbb{F}_a$. Then $d = 4k+1$, $g(X) = 4k$ and the hyperplane sections are hyperelliptic curves.

(4) *the 5-Veronese embedding of $\mathbb{P}^2$, $k = 6$, or one of its tangential projections from $1, 2$ or 3 point and $k = 6 - j$, $j = 1, 2, 3$. In this case $d = 25 - 4j$ and $g(X) = 6 - j$, $j = 0, 1, 2, 3$.*

Proof. A consequence of the analysis of the hyperplane sections tangent at k points and of theorem 4.4. □

8. Classification of surfaces with the minimal number of apparent double points

We apply theorem 2.1 to the study of surfaces $X \subset \mathbb{P}^N$ such that $SX \subsetneqq \mathbb{P}^N$ is a proper subvariety and such that $\nu(X) = \binom{h+2}{2}$, where $h_1(X) = \mathrm{codim}(SX) \geq 1$, or to secant defective surfaces of minimal secant degree. This immediately yields $N - 1 \geq s_1(X) = s(X) \geq 4$. For simplicity let $h_1(X) = h$.

Theorem 8.1. *Let $X \subset \mathbb{P}^{s(X)+h}$, $h \geq 1$, be an irreducible non-degenerate smooth surface of minimal secant degree $\binom{h+2}{2}$ and with $s(X) = 4$. Then X is one of the following:*

(1) *the Veronese surface in $\mathbb{P}^5$;*
(2) *a singular rational normal scroll;*

Proof. If $s(X) = 4$ by a well known theorem of Severi we have that X is either a cone or the Veronese surface. The Veronese surface has secant variety of degree 3 so that SX is of minimal degree in this case. If X is a cone, then it is a cone over a rational normal curve because its tangential projection is a rational normal curve. A cone over a rational normal curve is a singular rational normal scroll in $\mathbb{P}^N$ and these varieties have minimal secant degree. □

Theorem 8.2. *Let $X \subset \mathbb{P}^{5+h}$, $h \geq 1$, be a smooth non-degenerate surface with $\binom{h+2}{2}$ apparent double points. Then X is one of the following:*

(1) *a smooth rational normal scroll of degree $4 + h$;*
(2) *a del Pezzo surface of degree $5 + h$, $1 \leq h \leq 4$.*

Proof. Such a surface is linearly normal, see for example [CMR]. By corollary 2.2, the projection from a general tangent space to X is birational and maps X onto a rational normal scroll in $\mathbb{P}^{2+h}$. Then a general tangent hyperplane section C of X is rational and has a unique double point as its singularities since X is neither a cone, neither a tangent developable so that it has no dual defect. If C is irreducible, then the general hyperplane section has genus 1, being smooth and of arithmetic genus one, and the claim easily follows. If C is reducible, then X is a not developable scroll and the tangent hyperplane section consists of a line and of a unisecant curve, which is then a rational curve, so that X has as general hyperplane section a smooth rational curve. Then X is a rational normal scroll. □

9. A theorem of Bronowski and the classification of surfaces of minimal k-secant degree

In [Br] Bronowski "obtained" the classification of surfaces $X \subset \mathbb{P}^{3k+2}$ such that $\nu_k(X) = 1$, which we fixed in the previous sections. In the same paper Bronowski also "classifies" surfaces in $\mathbb{P}^{3k+2}$ such that $s_k(X) < 3k+2$ and such that $s_{k-1}(X) = 3(k-1)+2 = 3k-1$, $k \geq 2$. From general facts, it easily follows that $s_k(X) = s_{k-1}(X) + 2 = 3(k-1)+4 = 3k+1$ because $S^kX \subsetneq \mathbb{P}^N$, so that S^kX is an hypersurface. The conclusion of the theorem of Bronowski is clearly false without additional assumptions as the following example shows.

Example 6. Let $C \subset \mathbb{P}^M$, $M \geq 2k+2$, $k \geq 1$, be a smooth non-degenerate curve. Take $\mathbb{P}^{M+k} \supseteq \mathbb{P}^M$ and a $\mathbb{P}^{k-1} = L \subset \mathbb{P}^{M+k}$ such that $L \cap \mathbb{P}^M = \emptyset$. Consider the variety $Y = S(L, C) \subset \mathbb{P}^{M+k}$ of dimension $k+1$. Take a smooth, irreducible, non-degenerate surface X in Y. Then X is a smooth surface such that $S^kX \subseteq S^kY = S(L, S^kC)$. If $M = 2k+2$, the variety $S(L, S^kC)$ is an irreducible hypersurface, because S^kC is. If moreover, S^kX is at least an hypersurface (this always happens for $k=2$, $M=6$ and X smooth), we get $S^kX = S(L, S^kC)$, i.e. S^kX is an hypersurface of degree $\deg(S^kC)$, which is a cone. For arbitrary smooth $C \subset \mathbb{P}^{2k+2}$, for example a rational normal curve of degree $2k+2$, we certainly get examples not contained in Bronowski' s list.

This opens the problem of classifying the surfaces contained in Bronowski' s list. One immediately remarks that all these surfaces are also surfaces of minimal k-secant degree $k+2$ and that, naturally, after the classification of surfaces $X \subset \mathbb{P}^{3k+2}$ with $\nu_k(X) = 1$, it is natural to ask for the classification of k-defective surfaces $X \subset \mathbb{P}^{3k+2}$ of minimal k-secant degree $k+2$ and more generally of $\mathcal{MA}^{k-1}_{k+1}$-surfaces. This is done in the next theorems, which together with theorem 7.1, finally completely clarify the content of Bronowski' s results on page 311 of [Br] and naturally extend them. The conclusion of the "theorem" of Bronowski is got by applying a theorem of Enriques, see our theorem 4.1.

Let us remark that for $k=1$ Severi' s theorem yields that the Veronese surface is the unique 1-defective surface which is not 0-weakly defective, so that Bronowski' s "theorem" can be interpreted as a generalization of this consequence of Severi's theorem.

Theorem 9.1. (Bronowski, [Br]) *Let $X \subset \mathbb{P}^N$, $N \geq 3k+2$, be an irreducible non-degenerate linearly normal surface with at most a finite number of singular points, which is not $(k-1)$-weakly defective and such that $s_k(X) < 3k+2$.*

Then $N = 3k+2$, S^kX is an hypersurface and X is one of the following:

1. *the Veronese surface in $\mathbb{P}^5$ ($\deg(SX) = 3$);*
2. *there exists a desingularization $\phi : X' \to X$ such that $X' \simeq \mathbb{F}_a$, $a \geq 0$, and $\phi^*(H)$ is of type $2E+(a+k)F$ with $0 \leq a \leq k$ and $a+k = 2\beta$ ($\deg(X) = 4k$, $g(H) = k-1$, $\deg(S^k(X)) = k+2$); if $0 < a < k$, then $X' = X$, while if $a = k$, X' has only a singular point, being the 2-Veronese embedding of a rational normal scroll which is a cone;*
3. *the 4-Veronese embedding of $\mathbb{P}^2$ ($\deg(X) = 16$, $g(H) = 3$, $\deg(S^4X) = 6$).*

Proof. ¿From general facts, it easily follows that $s_k(X) = s_{k-1}(X) + 2 = 3(k-1)+4 = 3k+1$ because $S^kX \subsetneq \mathbb{P}^N$ and $s_l(X) = 3l+2$ for $0 \leq l \leq k-1$.

We can suppose $k \geq 2$ by Severi' theorem, which says that the Veronese surface is the unique surface, not a cone, of $\mathbb{P}^N$, $N \geq 5$ such that $\dim(SX) = 4$. Let us recall that cones are 0-weakly defective surfaces. Since X is not $(k-1)$-weakly defective, the projection π from the tangent space to $S^{k-2}X$ is birational by lemma 5.2 and maps X onto an irreducible surface $Y \subset \mathbb{P}^M$, $M \geq 5$, such that $\dim(SY) = 4$, i.e. Y is either the Veronese surface and $M = 5$, i.e. $N = 3k+2$, or Y is a cone. This last case cannot occur because otherwise X would be $(k-1)$-weakly defective. In conclusion, $Y \subset \mathbb{P}^5$ is the Veronese surface, $N = 3k+2$, X has at most a finite number of singular points and SX is an hypersurface. The general element $|A|$ of the mobile part of the linear system of the $(k-1)$-tangent hyperplane sections $|C| = |A| + B$ of X projects birationally onto a rational normal curve of degree 4, so that it is rational.

If $B = \emptyset$, then C is irreducible and having $k-1$ ordinary double points as singularities, it has arithmetic genus $k-1$ and degree $\deg(C) = 4k$. Then the linear system of the hyperplane sections of X has geometric genus $k-1$ and degree $4k$ and by applying the theorem of Enriques, theorem 4.1, we get cases 2 and 3 if $k \geq 3$. For $k = 2$ we get a surface with sectional genus 1 and degree 8, which is necessarily as in case 2 as one easily sees, i.e. it is the 2-Veronese embedding of a quadric surface in $\mathbb{P}^3$.

Let us suppose now $B \neq \emptyset$. If B has a unique irreducible component, necessarily a smooth rational curve, then we have $\dim|B| = k-1$ and $B^2 = k-2$. Moreover, since $\dim(A) = k+4$, we get $A^2 = k+3$; from $A \cdot B = k-1$ we get $\deg(X) = C^2 = k+3+2k-2+k-2 = 4k-1$; moreover we know that the sectional genus of a general hyperplane section is $k-2$. Suppose that $k \geq 3$. Since $3(k-2)+5 = 3k+2$ and since (X, H) is not ruled by lines $H \cdot B = 2k-3 \geq 2$, we see that this case does not exist by theorem 4.1. If $k = 2$ we are in the next case we now discuss.

Finally if B consists of $k-1$ irreducible components, then they belong to a pencil of lines and $C = A + (k-1)F$ is a rational normal scroll of degree $3k+2$. This case does not exists since a rational normal scroll cannot project onto the Veronese surface. □

As we saw it is not possible to eliminate the hypothesis of $(k-1)$-weakly defectiveness in the above theorem. Since $(k-1)$-weakly defective surfaces, which are not $(k-1)$-defective were completely classified in [CC], theorem 1.4, one gets the following result which can be considered the natural generalization of Severi' s theorem.

Theorem 9.2. *Let $X \subset \mathbb{P}^N$, $N \geq 3k+2$, be an irreducible non-degenerate k-defective, not $(k-1)$-defective, linearly normal surface with at most a finite number of singular points and with minimal k-secant degree.*

Then X is one of the following:

1. *the Veronese surface in $\mathbb{P}^5$;*
2. *$N = 3k+2$ and there exists a desingularization $\phi : X' \to X$ such that $X' \simeq \mathbb{F}_a$, $a \geq 0$, and $\phi^*(H)$ is of type $2E + (a+k)F$ with $0 \leq a \leq k$ and $a + k = 2\beta$* ($\deg(X) = 4k$ and $g(H) = k-1$)*;*
3. *the 4-Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^{14}$* ($\deg(X) = 16$ and $g(H) = 3$)*;*
4. *$k \geq 1$ and X is a $(k-1)$-weakly defective surface lying on a $k+1$-dimensional cone over a curve $C \subset \mathbb{P}^M$, $M \geq 2k+2$, and with vertex a linear space of dimension $\mathbb{P}^{k-1}$.*

Finally as a corollary of the above results we get the classification of irreducible k-defective, not $(k-1)$-defective, surfaces in $\mathbb{P}^N$, $N \geq 3k+2$, with minimal k-secant degree.

Corollary 9.3. *Let $X \subset \mathbb{P}^N$, $N \geq 3k+2$, be an irreducible non-degenerate k-defective linearly normal surface with minimal k-secant degree. Then X is one of the following:*

1. *the Veronese surface in $\mathbb{P}^5$;*
2. *there exists a desingularization $\phi : X' \to X$ such that $X' \simeq \mathbb{F}_a$, $a \geq 0$, and $\phi^*(H)$ is of type $2E+(a+k)F$ with $0 \leq a \leq k$ and $a+k = 2\beta$* ($\deg(X) = 4k$ and $g(H) = k-1$)*;*
3. *the 4-Veronese embedding of $\mathbb{P}^2$* ($\deg(X) = 16$ and $g(H) = 3$)*;*
4. *$k \geq 1$ and X is a $(k-1)$-weakly defective surface lying on a $k+1$-dimensional cone over a rational normal curve $C \subset \mathbb{P}^{2k+2}$ and with vertex a linear space of dimension $\mathbb{P}^{k-1}$.*

Once again since $(k-1)$-weakly defective surfaces are completely classified and since, for $N \geq 3k+3$, $\deg(p^k_X) \geq 2$ implies that X is k-weakly defective by theorem 5.1 and by Terracini lemma, in order to conclude the classification of surfaces of minimal k-secant degree it suffices to classify surfaces $X \subset \mathbb{P}^{3k+2+h}$, $h \geq 1$, such that $\nu_k(X) = \binom{h+k+1}{k+1}$, i.e. $\mathcal{MA}^{k+1}_{k-1}$-surfaces. The proof is completely parallel to the classification theorem of $\mathcal{OA}^{k+1}_{k-1}$-surfaces, but in easier since the theorem of Enriques immediately applies and gives that one a prori possible case does not exist.

Theorem 9.4. *Let $X \subset \mathbb{P}^{3k+2+h}$, $h \geq 1$, be an irreducible non-degenerate linearly normal $\mathcal{MA}^{k+1}_{k-1}$-surface with at most a finite number of singular points. Then X is one of the following:*

1. *a rational normal scroll $S(a_1, a_2)$ of degree $d = 3k+1+h$ and type (a_1, a_2) with $k \leq a_1 \leq a_2$;*
2. *a rational surface of degree $5+h$ with elliptic curve sections, $1 \leq h \leq 4$, i.e. a del Pezzo surface with at most a singular point ($k = 1$).*
3. *$1 \leq h \leq 3$ and X is the projection from $3-h$ point of a surface $\widetilde{X} \subset \mathbb{P}^{3k+5}$ for which there exists a desingularization $\phi : \widetilde{X'} \to \widetilde{X}$ such that $\widetilde{X'} \simeq \mathbb{F}_a$, $a \geq 0$, and $\phi^*(H)$ is of type $2E+(a+k+1)F$ with $0 \leq a \leq k+1$ and $a+k = 2\beta$* ($\deg(X) = 4k+1+h$ and $g(H) = k$)*;*
4. *the 4-Veronese embedding of $\mathbb{P}^2$* ($\deg(X) = 16$, $g(H) = 3$, $k = 3$, $h = 3$ and $\nu_3(X) = 35$) *or one of its projections from $3-h$ points on it, $1 \leq h \leq 2$, $k = 3$;*
5. *$S \simeq Q \subset \mathbb{P}^3$ and $H \in |\mathcal{O}_Q(3,3)|$, $g = 4$ and $\alpha = 3$.*
6. *X is the 3-Veronese embedding of $\mathbb{P}^2$* ($k = 2$ and $\nu_2(X) = 4$)*.*
7. *$k \geq 1$ and X is a k-weakly defective surface lying on a $k+1$-dimensional cone over a rational normal curve $C \subset \mathbb{P}^{2k+1+h}$ and with vertex a linear space of dimension $\mathbb{P}^k$.*

Proof. If it is k-weakly defective, being not k-defective, we are in case 6) by the classification theorem of [CC]. So from now on we suppose X is not k-weakly defective.

By corollary 2.2, the surface X projects birationally from a general tangent space to $S^{k-1}X$ onto a surface of minimal degree in $\mathbb{P}^{2+h}$, so that the linear system of hyperplane sections tangent at k general points, $|C| = |A| + B$ has rational mobile part.

If there are not fixed components, i.e. $B = \emptyset$, then $p_a(C) = k$ so that a general hyperplane section has genus k and degree $4k+1+h$. We have that either $k = 1$ and X is a (possible singular) del Pezzo surface as in case 2) or, if $k \geq 2$, by applying the theorem 4.1 that $3k + 2 + h \leq 3k + 5$, i.e. $h \leq 3$. Moreover, if $h = 3$, then X is either as in case 3) or 4) with $h = 3$ by the above mentioned theorem and these cases have the minimal number of apparent $(k+1)$-secant $(k-1)$-planes. By projecting them from 1 or 2 we obtain $\mathcal{MA}_{k-1}^{k+1}$-surface, see theorem 2.1, and it can easily seen that the #-minimal model of the desingularization of a surface with a finite number of singular points in $\mathbb{P}^{3k+2+h}$, $1 \leq h \leq 2$, of degree $4k + 1 + h$ and sectional genus k is either as in case (3) or as in case (4) with $h = 3$.

If B consists of a smooth rational curve passing through the fixed k points , then the sectional genus of a general hyperplane section is $k - 1$ and $A \cdot B = k \geq 2$. If $k \geq 3$, from $3k + 2 + h > 3(k - 1) + 5 = 3k + 2$ and from theorem 4.1, we see that this case cannot occur. Suppose $k = 2$. Then from $\dim |A| = k + 2 + h$ and from $\dim |B| = k$ we get $A^2 = k + 1 + h$ and $B^2 = k - 1$, i.e. $\deg(X) = C^2 = k + 1 + h + 2k + k - 1 = 4k + h = 8 + h$. Then necessarily $h = 1$, $\deg(X) = 9$ and X is the 3-Veronese embedding of $\mathbb{P}^2$, which is a surfaces with the minimal number of apparent 3-secant lines.

If B consists of $k \geq 1$ smooth rational curves which are lines with respect to A, then one immediately sees that we are in case (1). $\square$

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