Asymptotic expansions for random matrices

Uffe Haagerup, University of Southern Denmark
Steen Thorbjørnsen, University of Aarhus

The Gaussian Unitary Ensemble.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. By $\mathrm{GUE}(n, \sigma^2)$ we denote the set of random $n \times n$ matrices $X = (x_{ij})_{1 \le i,j \le n}$, defined on $(\Omega, \mathcal{F}, P)$, which satisfy the following conditions:

- $\forall i \geq j$: $x_{ij} = \overline{x_{ji}}$.
- the random variables x_{ij}, $1 \le i \le j \le n$, are independent.
- $\forall i < j$: $\mathrm{Re}(x_{ij}), \mathrm{Im}(x_{ij}) \sim \mathrm{i.i.d.}\ N(0, \frac{1}{2}\sigma^2)$.
- $\forall i$: $x_{ii} \sim N(0, \sigma^2)$.

The spectral distribution of a GUE random matrix.

Let X_n be a random matrix in $\mathrm{GUE}(n, \frac{1}{n})$. For any continuous, polynomially bounded function $f\colon \mathbb{R} \to \mathbb{R}$ we then have

$$\mathbb{E}\{\mathrm{tr}_n(f(X_n))\} = \int_{\mathbb{R}} f(x) h_n(x)\,\mathrm{d}x,$$

where the function $h_n\colon \mathbb{R} \to \mathbb{R}$ is given by

$$h_n(x) = \frac{1}{\sqrt{2n}} \sum_{j=0}^{n-1} \varphi_j\big(\sqrt{\tfrac{n}{2}}x\big)^2,$$

and where

- $\varphi_0, \varphi_1, \varphi_2, \ldots,$ is the sequence of Hermite functions:

$$\varphi_k(x) = \tfrac{1}{(2^k k! \sqrt{\pi})^{1/2}} H_k(x) \exp(-\tfrac{x^2}{2}), \qquad (k \in \mathbb{N}_0),$$

- $H_0, H_1, H_2, \ldots,$ are the Hermite polynomials:

$$H_k(x) = (-1)^k \exp(x^2) \cdot \Big(\frac{d^k}{dx^k} \exp(-x^2)\Big).$$

Wigner's semi-circle law.

For each n in $\mathbb{N}$, let X_n be a random matrix in $\mathrm{GUE}(n, \frac{1}{n})$. Then for any continuous bounded function $f\colon \mathbb{R} \to \mathbb{R}$ we have

$$\mathbb{E}\{\mathrm{tr}_n(f(X_n))\} = \int_{\mathbb{R}} f(x)h_n(x)\,\mathrm{d}x = \frac{1}{2\pi}\int_{-2}^{2} f(x)\sqrt{4-x^2}\,\mathrm{d}x + R_n(f),$$

where

$$R_n(f) \to 0 \quad \text{as } n \to \infty.$$

In addition

$$\lim_{n\to\infty} \lambda_{\max}(X_n) = 2, \quad \text{almost surely}$$

and

$$\lim_{n\to\infty} \lambda_{\min}(X_n) = -2, \quad \text{almost surely.}$$

Asymptotic expansion for GUE random matrices

For any function f in $C_b^\infty(\mathbb{R})$ and any k in $\mathbb{N}$ we have

$$\int_{\mathbb{R}} f(x)h_n(x)\,\mathrm{d}x =$$

$$\frac{1}{2\pi}\int_{-2}^{2} f(x)\sqrt{4-x^2}\,\mathrm{d}x + \frac{\alpha_1(f)}{n^2} + \frac{\alpha_2(f)}{n^4} + \cdots + \frac{\alpha_k(f)}{n^{2k}} + O(n^{-2k-2}),$$

for suitable (uniquely determined) constants $\alpha_1(f), \dots, \alpha_k(f)$.
More precisely

$$\alpha_j(f) = \frac{1}{2\pi}\int_{-2}^{2} [T^j f](x)\sqrt{4-x^2}\,\mathrm{d}x,$$

for a certain linear operator $T\colon C_b^\infty(\mathbb{R}) \to C_b^\infty(\mathbb{R})$, and the mapping

$$f \mapsto \alpha_j(f)\colon C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R}),$$

is a Schwarz distribution of order 3j-1 with support $[-2, 2]$.

The Harer-Zagier recursion formulae

For any p in $\mathbb{N}_0$ and n in $\mathbb{N}$ put

$$\gamma(p,n) = \mathbb{E}\{\mathrm{tr}_n(X_n^{2p})\} = \int_{\mathbb{R}} x^{2p} h_n(x)\,\mathrm{d}x.$$

Then for any p in $\mathbb{N}$,

$$\frac{-p(4p^2-1)}{n^2}\gamma(p-1,n) - (4p+2)\gamma(p,n) + (p+2)\gamma(p+1,n) = 0.$$

Claim

For any function g in $C_b^\infty(\mathbb{R})$ there is a unique function f in $C_c^\infty(\mathbb{R})$ such that

$$g(x) = \frac{1}{2\pi}\int_{-2}^{2} g(t)\sqrt{4-t^2}\,\mathrm{d}t + (x^2-4)f'(x) - 3xf(x).$$

A differential equation for the Cauchy transform of h_n

The Cauchy transform of $h_n(x)\,\mathrm{d}x$ is given by

$$G_n(\lambda) = \int_{\mathbb{R}} \frac{1}{\lambda - x} h_n(x)\,\mathrm{d}x, \qquad (\lambda \in \mathbb{C} \setminus \mathbb{R}).$$

Setting

$$f_\lambda(x) = \frac{1}{\lambda - x}, \qquad (x \in \mathbb{R},\ \lambda \in \mathbb{C} \setminus \mathbb{R}),$$

it follows from previous calculations that

$$0 = \int_{\mathbb{R}} \big[-n^{-2} f_\lambda'''(x) - (4 - x^2) f_\lambda'(x) + 3x f_\lambda(x) \big] h_n(x)\,\mathrm{d}x$$

$$= n^{-2} G_n'''(\lambda) + (4 - \lambda^2) G_n'(\lambda) + \lambda G_n(\lambda) - 2,$$

so that

$$n^{-2} G_n'''(\lambda) + (4 - \lambda^2) G_n'(\lambda) + \lambda G_n(\lambda) = 2.$$

The asymptotic expansion for the Cauchy transform of h_n.

Combining the asymptotic expansion

$$\int_{\mathbb{R}} \frac{1}{\lambda - x} h_n(x)\,\mathrm{d}x$$

$$= H_0(\lambda) + \frac{H_1(\lambda)}{n^2} + \frac{H_2(\lambda)}{n^4} + \cdots + \frac{H_k(\lambda)}{n^{2k}} + O(n^{-2k-2})$$

with the differential equation

$$n^{-2} G_n'''(\lambda) + (4 - \lambda^2) G_n'(\lambda) + \lambda G_n(\lambda) = 2.$$

we obtain a sequence of differential equations:

$$(\lambda^2 - 4) H_k'(\lambda) - \lambda H_k(\lambda) = H_{k-1}'''(\lambda), \qquad (k \in \mathbb{N}).$$

These can be solved successively using that

$$H_0(\lambda) = \frac{1}{2\pi} \int_{-2}^{2} \frac{1}{\lambda - x} \sqrt{4 - x^2}\,\mathrm{d}x = \frac{\lambda}{2} - \frac{1}{2}(\lambda^2 - 4)^{1/2}.$$

Explicit formulae

Using the technique described above we obtain

$$H_1(\lambda) = \frac{1}{(\lambda^2 - 4)^{5/2}},$$

$$H_2(\lambda) = \frac{21(\lambda^2 + 1)}{(\lambda^2 - 4)^{11/2}},$$

$$H_3(\lambda) = \frac{1738 + 6138\lambda + 1485\lambda^2}{(\lambda^2 - 4)^{17/2}},$$

and generally

$$H_k(\lambda) = \frac{a_0^{(k)} + a_1^{(k)}\lambda^2 + \cdots + a_{k-1}^{(k)}\lambda^{2k-2}}{(\lambda^2 - 4)^{3k-\frac{1}{2}}},$$

where the coefficients $a_j^{(k)}$ are given by the recursion formula.....

$$
\begin{aligned}
a_t^{(k+1)} =& \frac{1}{2} \sum_{j=(t-1)\vee 0}^{k-1} (-4)^{j+1-t} a_j^{(k)} \Bigg[\\
& - 2j(2j-1)(2j-2) \sum_{r=(j+1-t)\vee 3}^{j+1} \frac{\binom{j-2}{r-3}\binom{r}{j+1-t}}{\binom{3k}{r-2}(r-2)} \\
& + 3(6k-1)(2j)^2 \sum_{r=(j+1-t)\vee 2}^{j+1} \frac{\binom{j-1}{r-2}\binom{r}{j+1-t}}{\binom{3k+1}{r-1}(r-1)} \\
& - 3(6k-1)(6k+1)(2j+1) \sum_{r=(j+1-t)\vee 1}^{j+1} \frac{\binom{j}{r-1}\binom{r}{j+1-t}}{\binom{3k+2}{r} r} \\
& + 3(6k-1)(6k+1)(6k+3) \sum_{r=(j+1-t)\vee 0}^{j+1} \frac{\binom{j+1}{r}\binom{r}{j+1-t}}{\binom{3k+3}{r+1}(r+1)} \Bigg]
\end{aligned}
$$

Wishart matrices.

An $n \times n$ complex Wishart matrix with parameter $\sigma^2 > 0$ is a random matrix W_n in the form

$$W_n = B_n^* B_n,$$

where $B_n = (b_{ij})_{1 \le i,j \le b}$ is an $n \times n$ random matrix satifying that

- the random variables b_{ij}, $1 \le i,j \le n$, are independent.
- $\forall i,j \colon \mathrm{Re}(b_{ij}), \mathrm{Im}(b_{ij}) \sim \text{i.i.d. } N(0, \frac{1}{2}\sigma^2)$.

The spectral distribution of a Wishart matrix

Let W_n be a Wishart matrix with parameter $1/n$. Then for any continuous, polynomially bounded function $f\colon [0,\infty) \to \mathbb{R}$ we have

$$\mathbb{E}\{\mathrm{tr}_n(f(W_n))\} = \int_0^\infty f(x)\rho_n(x)\,\mathrm{d}x,$$

where the function $\rho_n\colon [0,\infty) \to \mathbb{R}$ is given by

$$\rho_n(x) = \sum_{j=0}^{n-1} \psi_j(nx)^2, \qquad (x \in [0,\infty)),$$

and where

- $\psi_0(x), \psi_1(x), \psi_2(x), \ldots$ is the sequence of Laguerre functions:

$$\psi_j(x) = L_j(x)\mathrm{e}^{-x/2} \quad (j \in \mathbb{N}_0, x \in [0,\infty)).$$

- $L_0(x), L_1(x), L_2(x), \ldots$ are the Laguerre polynomials:

$$L_j(x) = \frac{1}{j!}\mathrm{e}^{x} \cdot \frac{d^j}{dx^j}(x^j\mathrm{e}^{-x}), \qquad (j \in \mathbb{N}_0).$$

The Marchenko-Pastur Law

For each n in $\mathbb{N}$ let W_n be a Wishart matrix with parameter $1/n$. Then for any continuous bounded function $f\colon [0,\infty) \to \mathbb{R}$ we have

$$\mathbb{E}\{\mathrm{tr}_n(f(W_n))\} = \int_0^\infty f(x)\rho_n(x)\,\mathrm{d}x = \int_0^4 f(x)\frac{\sqrt{x(4-x)}}{2\pi x}\,\mathrm{d}x + R_n(f),$$

where

$$R_n(f) \to 0 \quad \text{as } n \to \infty.$$

In addition

$$\lim_{n\to\infty} \lambda_{\max}(W_n) = 4, \quad \text{almost surely} \qquad \text{[Geman]}$$

and

$$\lim_{n\to\infty} \lambda_{\min}(W_n) = 0, \quad \text{almost surely} \qquad \text{[Silverstein]}$$

Asymptotic expansions: $\rho_n(x)$ vs. $x\rho_n(x)$

Put

$$\sigma_n(x) = x\rho_n(x), \qquad (x \in [0,\infty)),$$

and for a C^∞-function $g\colon [0,\infty) \to \mathbb{C}$ define

$$f(x) = \begin{cases} \frac{g(x)-g(0)}{x}, & (x > 0), \\ f'(0), & (x = 0). \end{cases}$$

Then $g(x) = g(0) + xf(x)$, and hence

$$\int_0^\infty g(x)\rho_n(x)\,\mathrm{d}x = g(0) + \int_0^\infty f(x)\sigma_n(x)\,\mathrm{d}x.$$

Therefore, it suffices to find asymptotic expansions for $\int_0^\infty f(x)\sigma_n(x)\,\mathrm{d}x$. Note here that

$$\int_0^\infty f(x)\sigma_n(x)\,\mathrm{d}x = \int_0^4 f(x)\frac{\sqrt{x(4-x)}}{2\pi}\,\mathrm{d}x + R_n(f),$$

where $R_n(f) \to 0$ as $n \to \infty$.

Asymptotic expansion for $\int_0^\infty f(x)\sigma_n(x)\,\mathrm{d}x$

For any function $f \in C^\infty_{\mathrm{pol}}(\mathbb{R})$ and for any k in $\mathbb{N}$ we have

$$\int_0^\infty f(x)\sigma_n(x)\,\mathrm{d}x =$$

$$\int_0^4 f(x)\frac{\sqrt{x(4-x)}}{2\pi}\,\mathrm{d}x + \frac{\beta_1(f)}{n^2} + \frac{\beta_2(f)}{n^4} + \cdots + \frac{\beta_k(f)}{n^{2k}} + O(n^{-2k-2}),$$

for suitable (uniquely determined) constants $\beta_1(f), \ldots, \beta_k(f)$.
More precisely

$$\beta_j(f) = \int_0^4 [T^j f](x)\frac{\sqrt{x(4-x)}}{2\pi}\,\mathrm{d}x,$$

for a certain linear operator $T\colon C^\infty_{\mathrm{pol}}(\mathbb{R}) \to C^\infty_{\mathrm{pol}}(\mathbb{R})$.

Sketch of proof.

Show first that $\sigma_n(x)$ satisfies the differential equation

$$\frac{1}{n^2}\left(x\sigma_n''(x) + x^2\sigma_n'''(x)\right) + x(4-x)\sigma_n'(x) + (x-2)\sigma_n(x) = 0,$$

in the *distribution sense*, i.e.,

$$\int_0^\infty \Big[\frac{1}{n^2}\Big(\frac{d^2}{dx^2}(x\phi(x)) - \frac{d^3}{dx^3}(x^2\phi(x))\Big) - \frac{d}{dx}(x(4-x)\phi(x)) + (x-2)\phi(x)\Big]\sigma_n(x)\,\mathrm{d}x = 0, \tag{1}$$

for any *Schwarz-function* ϕ.

Sketch of proof (continued)

This follows by showing that

$$\mathcal{F}\left[n^{-2}\left(x\sigma_n''(x)+x^2\sigma_n'''(x)\right)+x(4-x)\sigma_n'(x)+(x-2)\sigma_n(x)\right]=0,$$

which is a consequence of the fact that

$$\widehat{\sigma}_n(s)=\int_0^\infty \mathrm{e}^{-\mathrm{i}sx}\sigma_n(x)\,\mathrm{d}x=\frac{F\left(1-n,1-n,2,-(s/n)^2\right)}{(1+(\mathrm{i}s/n))^{2n}},$$

where the function

$$w(u)=\frac{F(1-n,1-n,2,u^2)}{(1-u)^{2n}},$$

satisfies the differential equation

$$u(1-u^2)w''(u)+(3-4nu-5u^2)w'(u)-(6n+4u)w(u)=0.$$

Sketch of proof (continued)

By an extension argument the formula

$$\int_0^\infty \Big[\frac{1}{n^2}\Big(\frac{d^2}{dx^2}(x\phi(x)) - \frac{d^3}{dx^3}(x^2\phi(x))\Big) - \frac{d}{dx}(x(4-x)\phi(x)) + (x-2)\phi(x)\Big]\sigma_n(x)\,\mathrm{d}x = 0,$$

actually holds for all ϕ in $C^\infty_{\mathrm{pol}}(\mathbb{R})$, and hence

$$\int_0^\infty \big[(3x-6)\phi(x) - x(4-x)\phi'(x)\big]\sigma_n(x)\,\mathrm{d}x = \frac{1}{n^2}\int_0^\infty \big[4\phi'(x) + 5x\phi''(x) + x^2\phi'''(x)\big]\sigma_n(x)\,\mathrm{d}x, \tag{2}$$

for all ϕ in $C^\infty_{\mathrm{pol}}(\mathbb{R})$.

Asymptotic expansion for the Cauchy transform of σ_n

Consider the Cauchy transform

$$S_n(\lambda) = \int_0^\infty \frac{1}{\lambda - x}\sigma_n(x)\,\mathrm{d}x, \qquad (\lambda \in \mathbb{C} \setminus \mathbb{R}).$$

Then we have the asymptotic expansion

$$S_n(\lambda) = l_0(\lambda) + \frac{l_1(\lambda)}{n^2} + \frac{l_2(\lambda)}{n^4} + \cdots + \frac{l_k(\lambda)}{n^{2k}} + O(n^{-2k-2}),$$

and it follows from (2) that

$$\frac{1}{n^2}\left[\lambda^2 S_n'''(\lambda) + \lambda S_n''(\lambda)\right] + \lambda(4-\lambda)S_n'(\lambda) + (\lambda - 2)S_n(\lambda) = 2.$$

Concrete formulae

Combining the two previous formulae we obtain

$$I_0(\lambda) = \int_0^4 \frac{1}{\lambda - x} \frac{\sqrt{x(4-x)}}{2\pi}\, \mathrm{d}x = \frac{\lambda}{2} - 1 - \frac{1}{2}(\lambda(\lambda-4))^{1/2},$$

$$I_1(\lambda) = \frac{\sqrt{\lambda}}{4(\lambda-4)^{5/2}} - \frac{\sqrt{\lambda}}{16(\lambda-4)^{3/2}} + \frac{\sqrt{\lambda}}{64(\lambda-4)^{1/2}} - \frac{\sqrt{\lambda-4}}{64\sqrt{\lambda}}$$

$$\begin{aligned} I_2(\lambda) = {} & \frac{105\sqrt{\lambda}}{16(\lambda-4)^{11/2}} + \frac{7\sqrt{\lambda}}{32(\lambda-4)^{9/2}} - \frac{5\sqrt{\lambda}}{256(\lambda-4)^{7/2}} \\ & - \frac{\sqrt{\lambda}}{256(\lambda-4)^{5/2}} + \frac{13\sqrt{\lambda}}{4096(\lambda-4)^{3/2}} - \frac{11\sqrt{\lambda}}{8192(\lambda-4)^{1/2}} \\ & + \frac{9\sqrt{\lambda-4}}{4096\sqrt{\lambda}} + \frac{11\sqrt{\lambda-4}}{8192\sqrt{\lambda}} \end{aligned}$$

General Formula for $I_k(\lambda)$

In general $I_k(\lambda)$ seems to be of the form

$$I_k(\lambda) = \sqrt{\lambda}\sum_{j=1}^{3k} C_{j,k}(\lambda-4)^{j-\frac{1}{2}} + \sqrt{4-\lambda}\sum_{j=1}^{k} D_{j,k}\lambda^{j-\frac{1}{2}}$$

$$= \frac{\sqrt{\lambda(\lambda-4)}P(\lambda)}{\lambda^k(\lambda-4)^{3k}},$$

for some polynomial $P(\lambda)$.

A formula of Pastur and Scherbina

Let X_n be a random matrix in $\mathrm{GUE}(n, \frac{1}{n})$. Then for any functions f, g in $C_b^\infty(\mathbb{R})$ we have

$$\begin{aligned}&\mathrm{Cov}\big\{\mathrm{Tr}_n(f(X_n)), \mathrm{Tr}_n(g(X_n))\big\}\\&\qquad = \int_{\mathbb{R}^2} \Big(\frac{f(x)-f(y)}{x-y}\Big)\Big(\frac{g(x)-g(y)}{x-y}\Big)\rho_n(x,y)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

where

$$\rho_n(x,y) = \frac{n}{4}\Big(\varphi_n\big(\sqrt{\tfrac{n}{2}}x\big)\varphi_{n-1}\big(\sqrt{\tfrac{n}{2}}y\big) - \varphi_{n-1}\big(\sqrt{\tfrac{n}{2}}x\big)\varphi_n\big(\sqrt{\tfrac{n}{2}}y\big)\Big)^2.$$

The two-dimensional Cauchy transform

For λ, μ in $\mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu$, we have

$$G_n(\lambda, \mu) := \mathrm{Cov}\{\mathrm{Tr}_n[(\lambda - X_n)^{-1}], \mathrm{Tr}_n[(\mu - X_n)^{-1}]\}$$

$$= \int_{\mathbb{R}^2} (\mu - \lambda)^{-2} \Big(\frac{1}{\lambda - x} - \frac{1}{\mu - x}\Big)\Big(\frac{1}{\lambda - y} - \frac{1}{\mu - y}\Big) \rho_n(x, y) \,\mathrm{d}x \,\mathrm{d}y$$

Then observe that

$$\rho_n(x, y) = \frac{1}{4}\Big(\tilde{h}_n(x)\tilde{h}_n(y) - 4h_n'(x)h_n'(y) + \frac{1}{n^2} h_n''(x)h_n''(y)\Big),$$

where

$$\tilde{h}_n(x) = h_n(x) - xh_n'(x).$$

The two-dimensional Cauchy transform (continued)

From the above formulae we obtain that

$$G_n(\lambda,\mu) = -\frac{1}{2(\lambda-\mu)^2}\Big[\tilde{G}_n(\lambda)\tilde{G}_n(\mu) - \hat{G}_n(\lambda)\hat{G}_n(\mu) + 1 - \frac{1}{n^2}G_n''(\lambda)G_n''(\mu)\Big]$$

where

$$\tilde{G}_n(\lambda) = G_n(\lambda) - \lambda G_n'(\lambda)$$

$$\hat{G}_n(\lambda) = 2G_n'(\lambda) - 1.$$

By inserting the asymptotic expansions for $G_n(\lambda)$ and $G_n(\mu)$ we then obtain

$$G_n(\lambda,\mu) = G(\lambda,\mu) + \sum_{j=1}^{k}\frac{J_j(\lambda,\mu)}{n^{2j}} + O(n^{-2k-2}).$$